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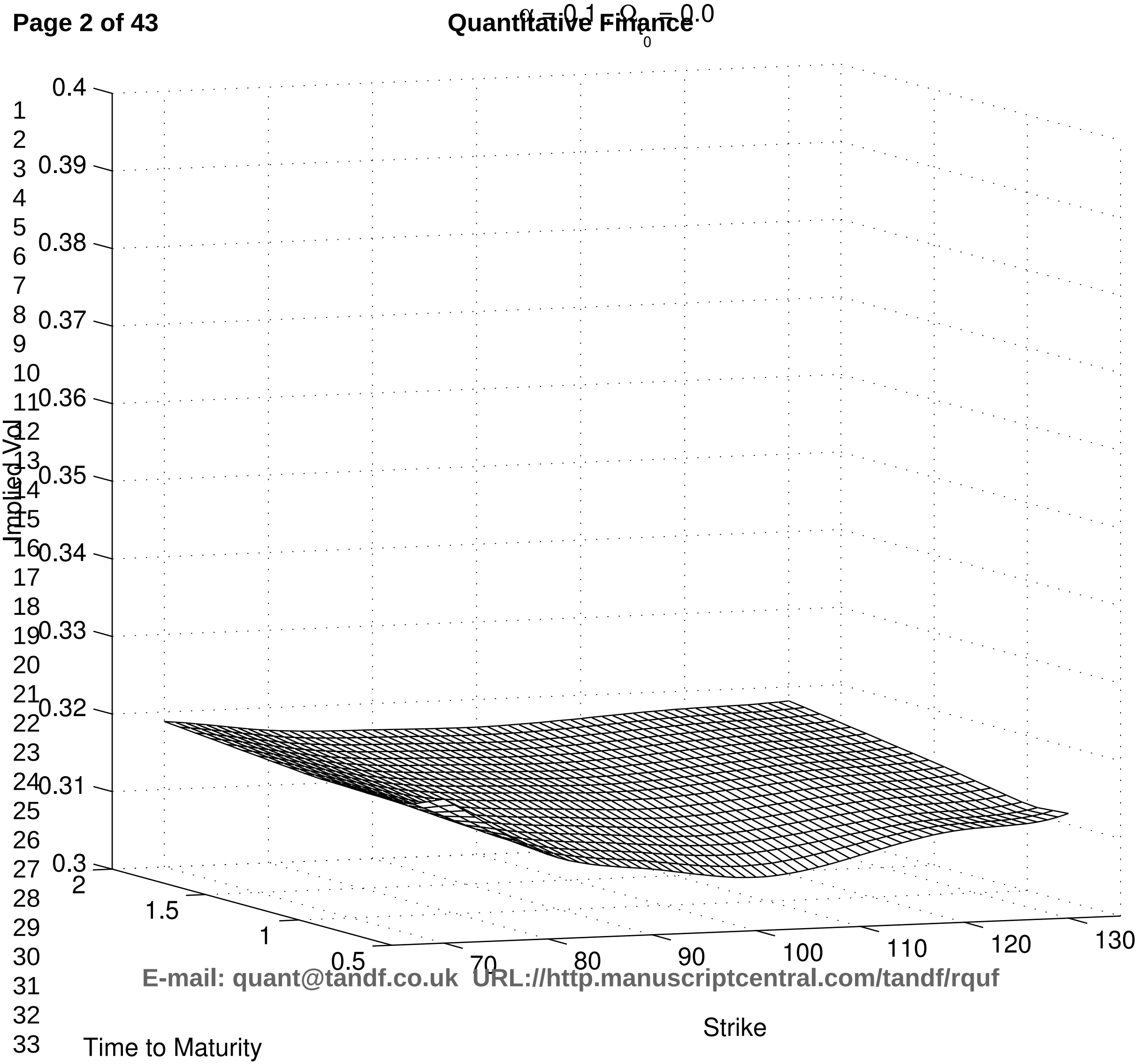
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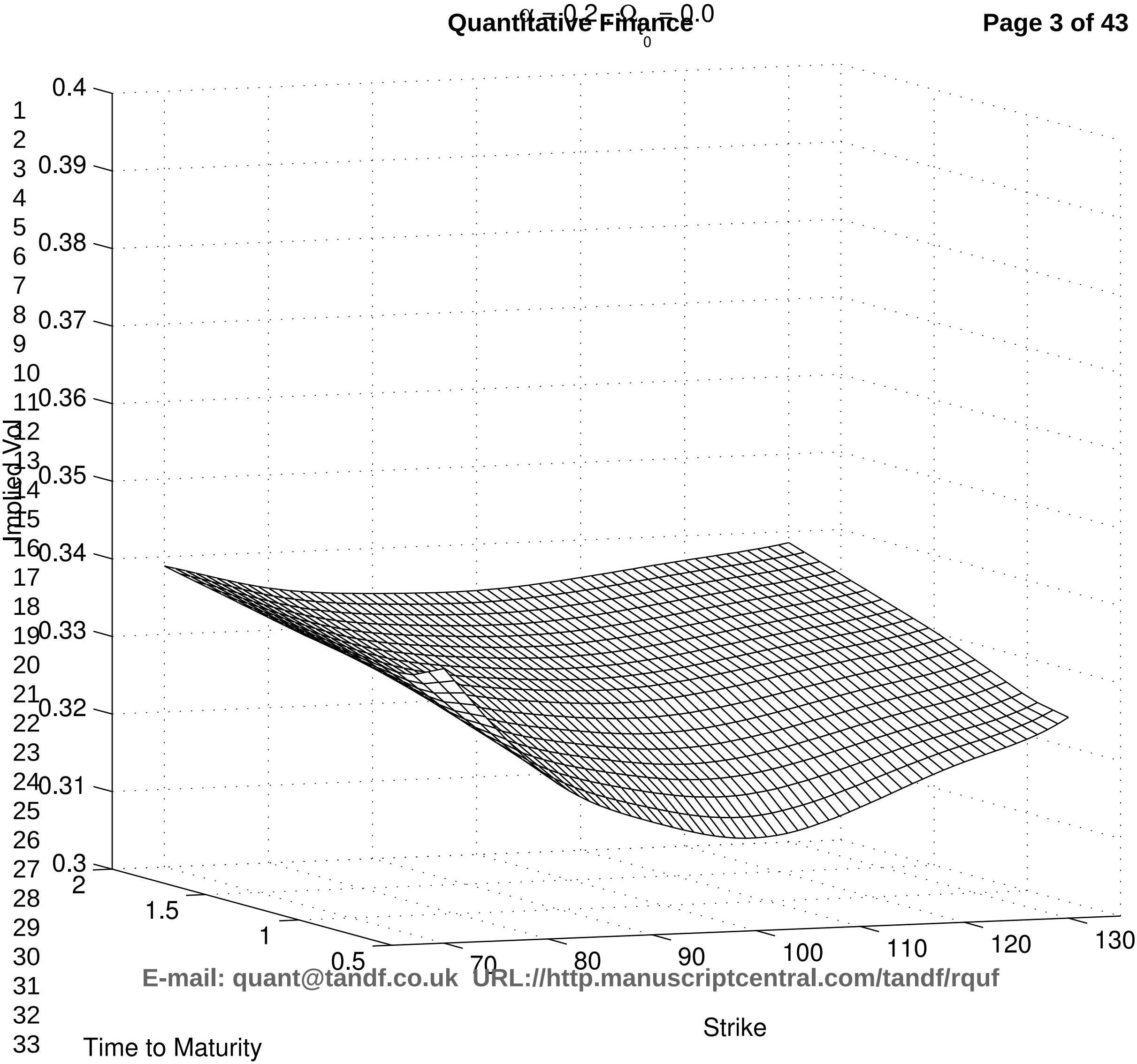
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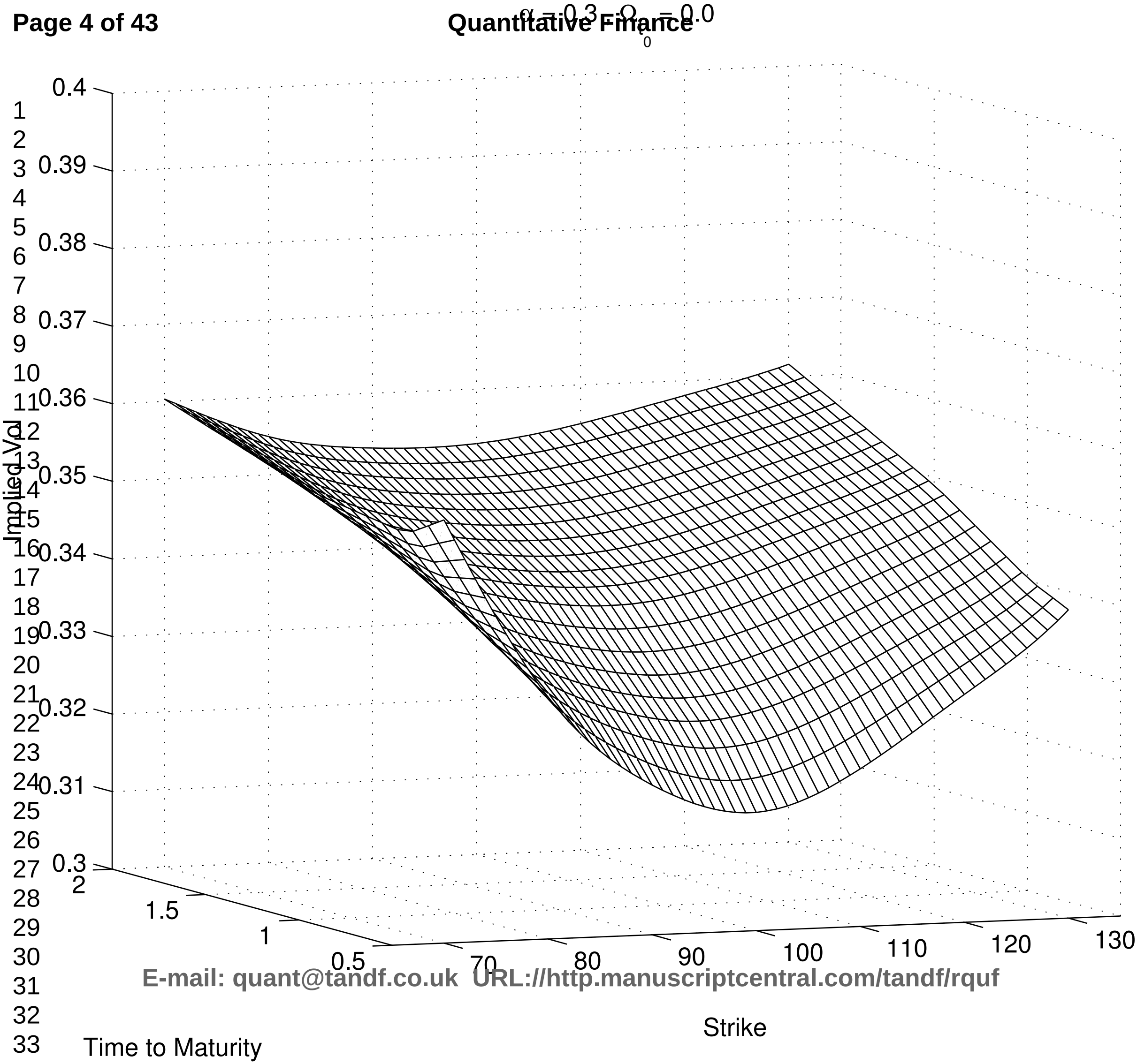
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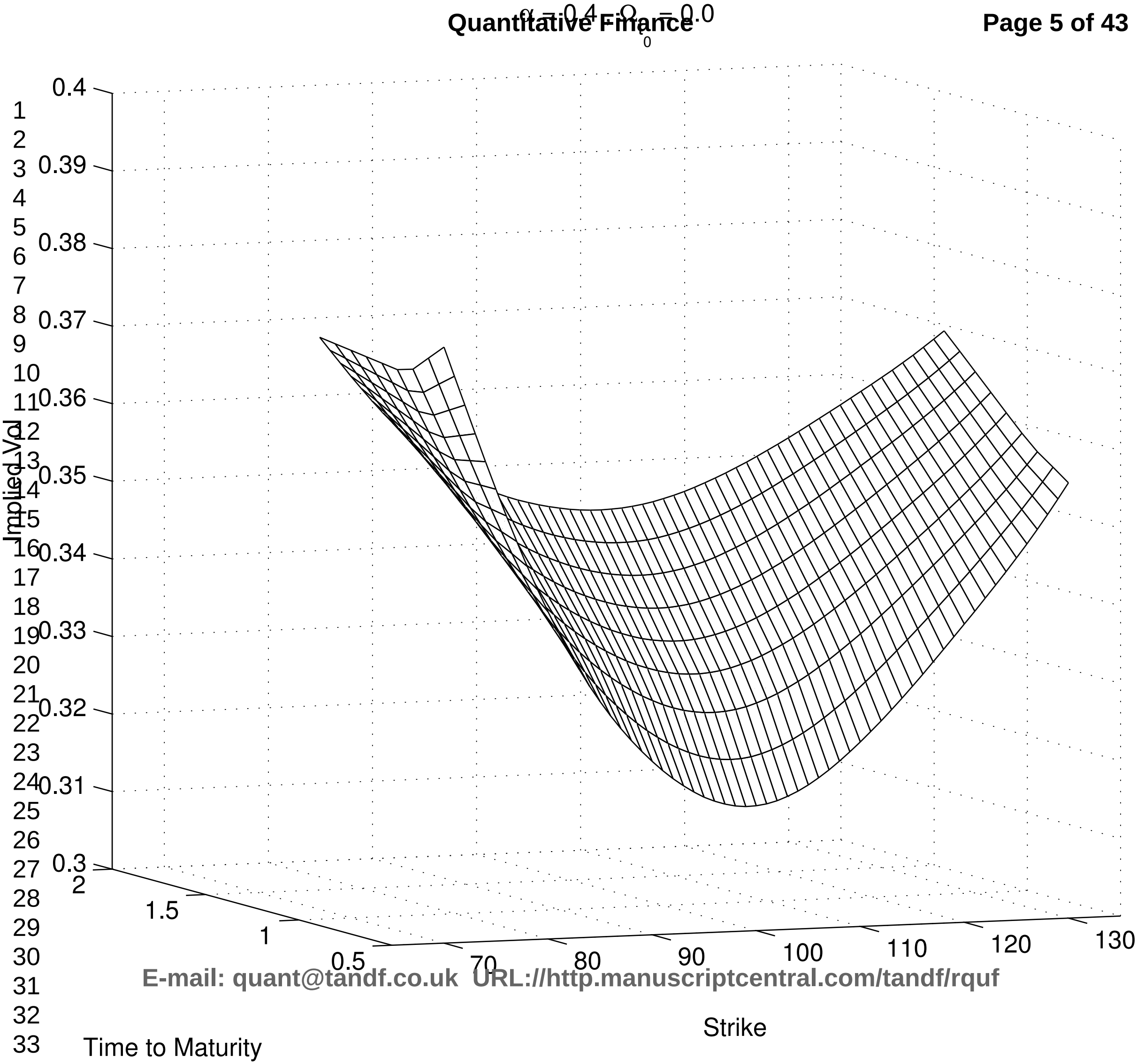
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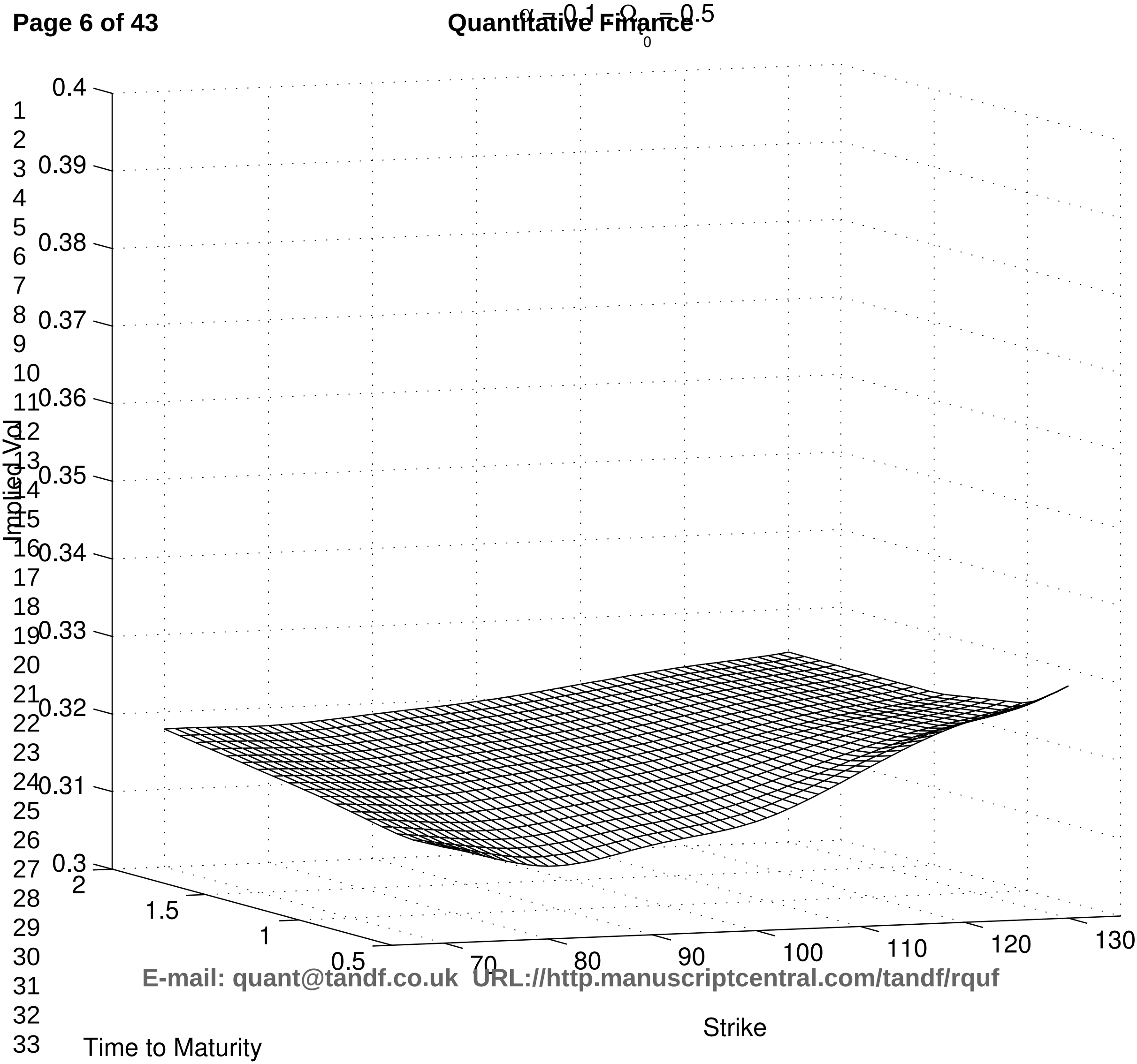
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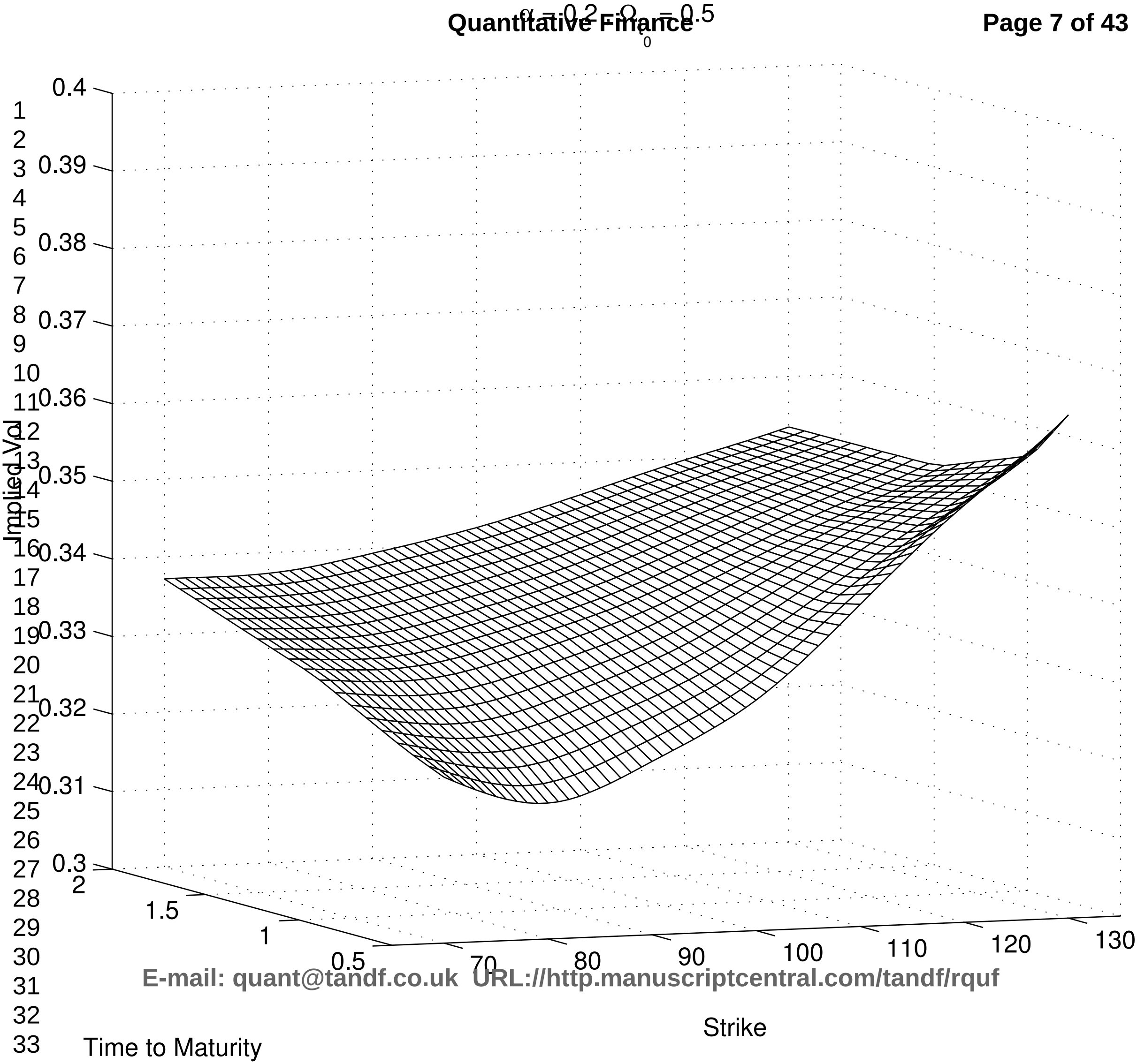
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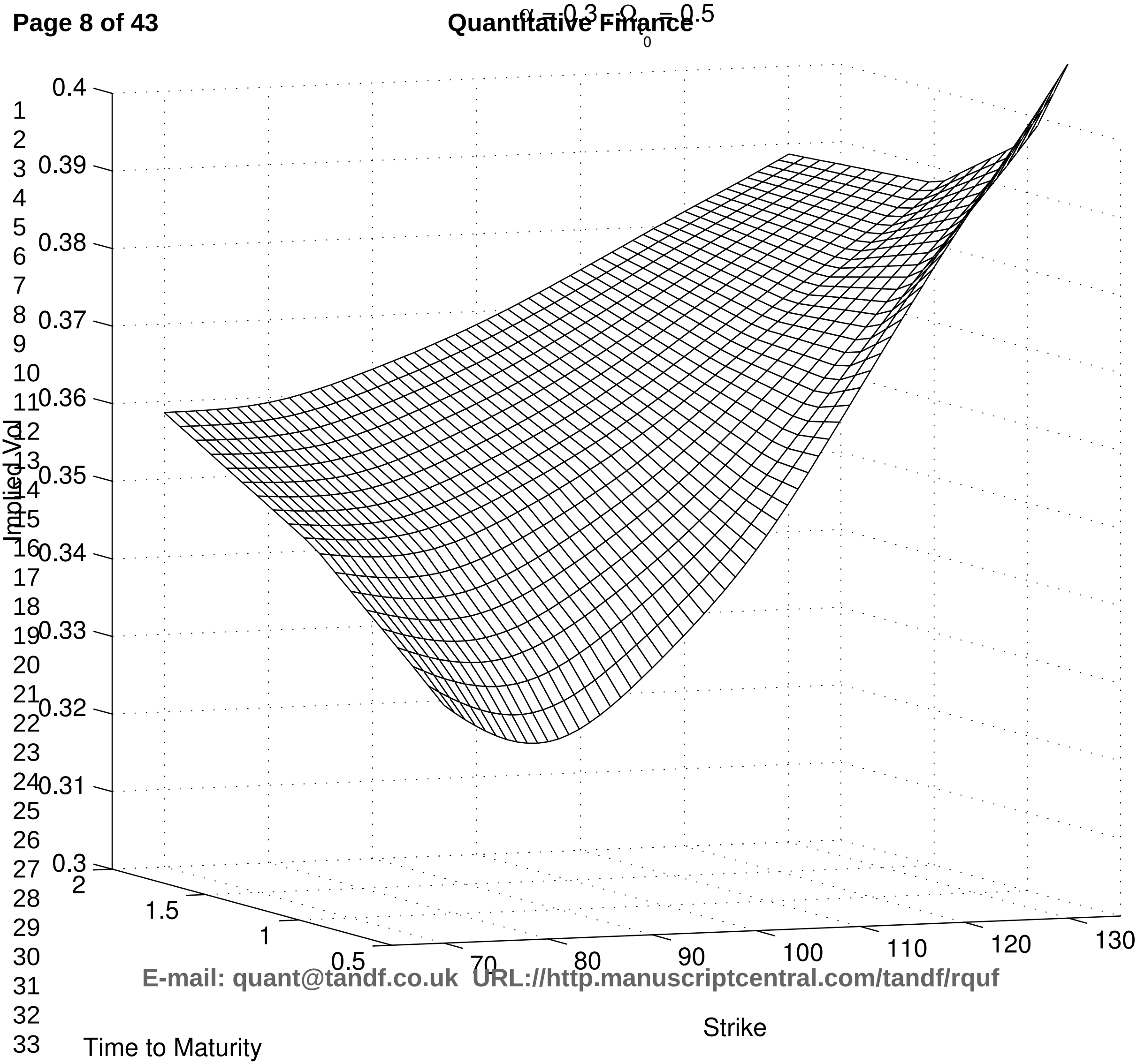
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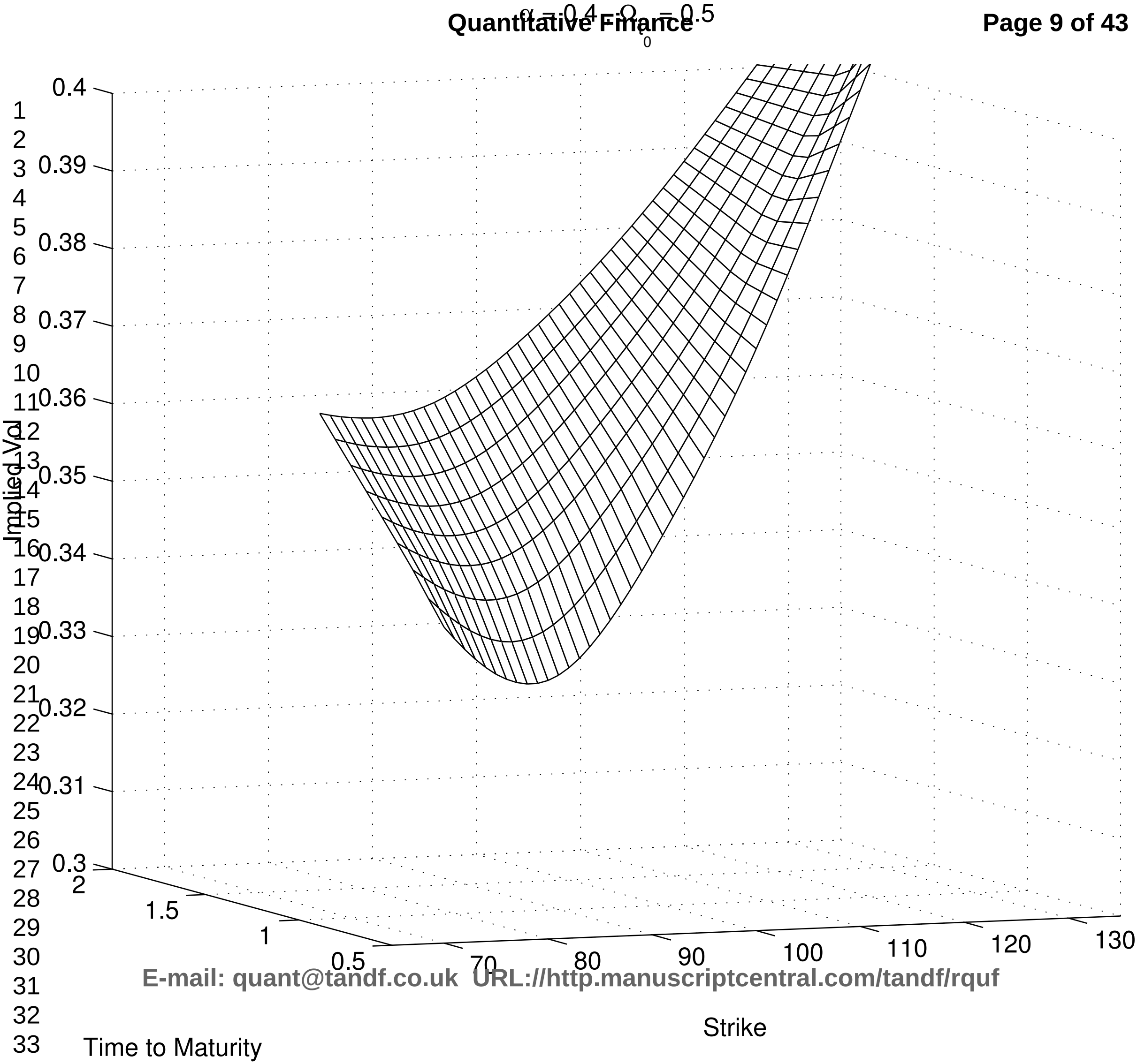
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On Option Pricing Models in the Presence of Heavy Tails*

Michel Vellekoop† and Hans Nieuwenhuis‡

October 19, 2006

Abstract

We propose a modification of the option pricing framework derived by Borland [4, 5] which removes the possibilities for arbitrage within this framework. It turns out that such arbitrage possibilities arise due to an incorrect derivation of the martingale transformation in the non-Gaussian option models which are used in that paper. We show how a similar model can be built for the asset price processes which excludes arbitrage. However, the correction causes the pricing formulas to be less explicit than the ones in the original formulation, since the stock price itself is no longer a Markov process. Practical option pricing algorithms will therefore have to resort to Monte Carlo methods or partial differential equations and we show how these can be implemented. An extra parameter, which needs to be specified before the model can be used, will give market makers some extra freedom when fitting their model to market data.

1 Introduction

Models for equity option pricing in which the underlying asset exhibit tails which are heavier than those of a lognormal distribution have been researched for many years now. A lot of empirical evidence suggests that such heavy-tailed models can provide a better fit for many equity price processes (or indices thereof) and it is therefore only natural that many authors have tried to develop new models which go beyond the standard lognormal assumptions of the celebrated Black and Scholes model [3]. Among the many possible assumptions made by different authors for the distribution of future asset prices are jump-diffusion models [2], level-dependent volatilities [6, 7], or hypergeometric and inverse Gaussian models that are analytically tractable and allow level-dependent volatilities as well [1]. Another class of possible models is characterised by the fact that the asset price process itself is no longer a Markov process. Perhaps the most well-known of the models in this class are the stochastic volatility models, such as those defined by Hull and White [10, 13], Heston [8] and Hobson and Rogers [9].
Recently, a very interesting new approach was proposed in a paper by Borland [4]. In it, the author defines a diffusion process, in the usual form of a stochastic differential equation driven by a Wiener process, which has heavy tails. Its distribution at future times can be characterized explicitly as a Tsallis distribution [14], which implies a probability density function for the logarithm of the assets which is asymptotically equal to $x^{-\gamma}$ for certain values of $\gamma > 3$. In the paper, a riskneutral pricing argument is then used to derive closed-form option pricing formulas for European calls and puts. It is shown that the implied volatility smile observed in practice can be represented well in this model if one chooses the model parameters carefully. The model has a stochastic volatility but it still generates a complete market, since no extra Brownian Motions are introduced for the volatility process and in this way the model resembles the approach taken in Hobson and Rogers

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[9]. Since it is also possible to find a closed-form solution for the future distributions of asset prices, the model therefore seems to provide a very clever combination of analytical tractability (since everything is defined in terms of diffusions, with distributions which can be characterized explicitly) and sufficient complexity to be of use in market practice.

Unfortunately, the model as is stands admits arbitrage. Some mathematical conditions which need to be fulfilled to carry out the Girsanov transformation from the real world measure to the riskneutral measure, are not satisfied for this particular model. The closed-form option formulas defined in the paper are therefore not valid and for options with long maturities they do not even form a useful approximation. This is a pity, since the ideas underlying the model are very interesting and deserving of further analysis. Indeed, the Borland model provides a nice hybrid between the Heston model and the Black-Scholes model. It has a volatility which varies stochastically, but since the volatility is driven by the same Brownian Motion as the asset process itself, the model is still complete.

One of the most important feautures of any practical option pricing model is that it should be guaranteed to be arbitrage-free. Since this is not the case for the original model in [4] or the slighly different approximations given in the later paper [5], we will change the model in such a way that it can be guaranteed to be arbitrage-free and we will show how option prices can still be calculated. Our analysis should not be interpreted as an indication that the Borland model is not useful; we merely try to repair the mathematical problems associated with it. We believe the ideas behind the model to be innovative, and very useful for practical option trading.

In our approach, it is no longer possible to give closed-form formulas of European option prices. However, we show how we can use a partial differential equation (which is totally different from the one used in [4]) to find these prices, and we check the results using Monte Carlo methods. It turns out that many of the nice features of the original model are retained after our modification.

The organization of the paper is as follows. In the next section we will formulate the model used in [4], with a slighly different notation at some places to emphasize some important characteristics of the parameters, and we show why arbitrage occurs in this model. Section 3 derives an alternative model which excludes this arbitrage. Section 4 shows how option prices can be calculated using this model, and we use the methods defined there to show some examples of the option prices in section 5. In the last section we formulate conclusions and possible subjects for further research.

2 The Earlier Model

In [4], the stochastic process driving the rates of return of the stock price process is not Brownian Motion but a continuous Markov process defined as

$$d\Omega_t = f(t,\Omega_t)\,dW_t \quad (1)$$
$$\Omega_0 = 0 \quad (2)$$

where $\{W_t, t \geq 0\}$ is an $\mathcal{F}_t$-adapted Brownian Motion process on a filtered probability space $(\bar{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, and f is defined by

$$\mathcal{P}(t,w) = (1+\beta(t)\alpha w^2)^{-\frac{1}{\alpha}}/Z_t^\alpha, \qquad t>0$$
$$f(t,w) = \begin{cases} \xi\,\mathcal{P}(t,w)^{-\frac{1}{2}\alpha} = \xi\,(Z_t^\alpha)^{\frac{1}{2}\alpha}\sqrt{1+\beta(t)\alpha w^2} & t>0 \\ 0 & t=0 \end{cases}$$

where $\alpha \in]0, \frac{1}{2}[$ is a constant [1]. The starting value for Ω_0 need not be zero, and we will change it later in the paper, but for now we will use this assumption from the Borland model. The timescale

[1] We use a slightly different notation than the one in Borland's paper to emphasize which constants are positive or negative, $\alpha = q - 1$ in the Borland paper. We also take α smaller than $\frac{1}{2}$ instead of Borland's $\frac{2}{3}$ to make sure that the expectation of quadratic variation $\mathbb{E}\langle\Omega,\Omega\rangle_t$ is finite for all $t \in [0,T]$, as shown later. Also note that the constant ξ that we introduce here was taken $\xi = 1$ in the Borland paper.

β is given by

$$\beta(t) \;=\; [\,(1-\alpha)(2-\alpha)\,t\,]^{-\frac{2}{2-\alpha}}$$

and Z_t^α, A_α and ξ are given by

$$Z_t^\alpha \;=\; \int_{\mathbb{R}} (1+\beta(t)\alpha u^2)^{-\frac{1}{\alpha}}du \;=\; \frac{A_\alpha}{\sqrt{\beta(t)}}, \qquad A_\alpha = \frac{\sqrt{\frac{\pi}{\alpha}}\,\Gamma(\frac{1}{\alpha}-\frac{1}{2})}{\Gamma(\frac{1}{\alpha})}, \qquad \xi \;=\; (A_\alpha)^{-\alpha/2}.$$

Note that if $\nu = \frac{2}{\alpha}-1$ happens to be an element of $\mathbb{N}$, then $\sqrt{(2-\alpha)\beta(t)}\Omega_t$ has a Student's t-distribution with ν degrees of freedom. We can write

$$d\Omega_t \;=\; c\,\sqrt{t^{\frac{\alpha}{2-\alpha}} + Kt^{-1}\,\Omega_t^2}\,dW_t$$

where $K \in \mathbb{R}^+$, $c \in \mathbb{R}^+$ are constants which depend on α but not on the time t. The Kolmogorov Forward (or Fokker-Planck) equation

$$\frac{\partial}{\partial t}p(t,z) \;=\; \frac{1}{2}\frac{\partial^2}{\partial z^2}[f^2(t,z)p(t,z)] \qquad (3)$$

then shows that the probability density funcion p of Ω_t, which satisfies

$$\mathbb{P}(\Omega_t \in A) \;=\; \int_A p(t,z)dz$$

for all Borel sets A, is given by the function $\mathcal{P}$ mentioned before: $p = \mathcal{P}$.

In the paper, Ω_t is now used to define a stock price process $\{S_t,\, t \geq 0\}$ using

$$\Omega_t \;=\; \frac{\ln S_t/S_0 - \mu t}{\sigma}$$

for strictly positive constants μ, σ. This defines a continuous *Markov* process S since

$$\begin{aligned} d(\ln S_t) \;&=\; \mu dt \;+\; \sigma d\Omega_t \\ &=\; \mu dt \;+\; \sigma f(t,\Omega_t)dW_t \\ &=\; \mu dt \;+\; \sigma f(t, \tfrac{1}{\sigma}(\ln S_t/S_0 - \mu t))dW_t \end{aligned}$$

The distribution of $\ln S$ has tails which are heavier than those for a Gaussian distribution. The risk free rate of return r is assumed to be constant, with $0 < r < \mu$, and $B_t = B_0 e^{rt}$ thus models a bank account. Option prices are then derived for this model, under the assumption that the underlying asset price process S follows the stochastic differential equation given above. The calculations lead to the following option price function $C(S,T)$ for a European call with strike K and time to maturity T which pays $\Phi(S_T) = \max(0, S_T - K)$ at time T:

$$C(s,T) \;=\; e^{-rT}\int_{\mathbb{R}}\left(se^{rT+\sigma w-\frac{1}{2}\gamma\sigma^2 T^{\frac{2}{2-\alpha}}(1+\alpha\beta(T)w^2)} - K\right)^+ \mathcal{P}(T,w)\,dw$$

where $\gamma = \frac{1}{2}(2-\alpha)\,[(2-\alpha)(1-\alpha)]^{-\frac{\alpha}{2-\alpha}}$ is a strictly positive constant. However, this formula cannot be correct.

Theorem 1. *The call option formula given above admits arbitrage.*

Proof.
Since $bx - \frac{1}{2}x^2 a \leq \frac{1}{2}b^2/a$ for $a > 0$, we have that

$$\sigma w - \tfrac{1}{2}w^2\gamma\sigma^2 T^{\frac{2}{2-\alpha}}\alpha\beta(T) \;\leq\; \tfrac{1}{2}(\gamma T^{\frac{2}{2-\alpha}}\alpha\beta(T))^{-1}$$

so

$$se^{rT+\sigma w-\frac{1}{2}\gamma\sigma^2 T^{\frac{2}{2-\alpha}}(1+\alpha\beta(T)w^2)} \quad \leq \quad S^{\max}(T,\alpha)$$

where the value of $S^{\max}(T,\alpha)$ does not depend on w. This means that the price of a European Call with maturity T and strike $K > S^{\max}$ has the value zero. But there is a positive probability that the option ends up in the money because under $\mathbb{P}$ the probability density function of S_T is positive for values higher than $S^{\max}(T,\alpha)$. This clearly constitutes an arbitrage. ■

In the later paper [5] slightly different call option price formulas are given, but one may construct arbitrage opportunities for these formulas in a way similar to the one given above. We will now analyze how the arbitrage arises.

Borland would like to work in a complete and arbitrage-free market, and she therefore wants to construct a measure $\mathbb{Q}$, equivalent with $\mathbb{P}$, such that the discounted process $\tilde{S}_t = S_t/B_t$ becomes a martingale under $\mathbb{Q}$, i.e. $\mathbb{E}^{\mathbb{Q}}[S_t/B_t \mid \mathcal{F}_u] = S_u/B_u$ for all $t \geq u \geq 0$. To find such a measure $\mathbb{Q}$ Borland writes

$$\begin{aligned} d(\ln S_t) &= \mu dt \; + \; \sigma f(t,\Omega_t) dW_t \\ dS_t &= (\mu + \tfrac{1}{2}\sigma^2 f^2(t,\Omega_t)) S_t dt \; + \; \sigma S_t f(t,\Omega_t) dW_t \end{aligned}$$

so the discounted asset price process satisfies

$$\begin{aligned} d\tilde{S}_t &= (\mu - r + \tfrac{1}{2}\sigma^2 f^2(t,\Omega_t))\tilde{S}_t dt \; + \; \sigma \tilde{S}_t f(t,\Omega_t) dW_t \\ &= \sigma \tilde{S}_t f(t,\Omega_t)\,(u_t dt + dW_t) \end{aligned}$$

where

$$u_t \quad = \quad \frac{\mu - r + \frac{1}{2}\sigma^2 f^2(t,\Omega_t)}{\sigma f(t,\Omega_t)}$$

An equivalent measure $\mathbb{Q}$ which makes $\tilde{S}_t$ a martingale must be such that the process $W + \int u dt$ is a Brownian Motion under $\mathbb{Q}$ and to construct such a measure one may try to use the Girsanov Theorem. Define for all $A \in \mathcal{F}$

$$\mathbb{Q}(A) \quad = \quad \int_A \zeta_T(\omega) d\mathbb{P}(\omega), \qquad \zeta_T(\omega) = \exp\left(- \int_0^T u_s(\omega) dW_s(\omega) - \tfrac{1}{2}\int_0^T u_s^2(\omega) ds \right) \tag{4}$$

The Girsanov Theorem states that $W + \int u dt$ is indeed a Brownian Motion under $\mathbb{Q}$ if $\mathbb{E}\zeta_T = 1$. A sufficient condition for this to be true is the *Novikov condition* which is stated in equation (45) of [4] as

$$\exp\left(-\tfrac{1}{2}\int_0^T u_s^2 ds \right) \quad < \quad \infty$$

but which should in fact be

$$\mathbb{E}^{\mathbb{P}} \exp\left(\tfrac{1}{2}\int_0^T u_s^2 ds \right) \quad < \quad \infty \tag{5}$$

Theorem 2. *The Novikov condition (5) is not satisfied for the model proposed above.*

Proof. See Appendix. ■

The arguments given in the Borland paper are therefore not sufficient to conclude that an equivalent martingale measure $\mathbb{Q}$ exists. Since the Novikov condition is a sufficient but not a necessary condition, this does not automatically imply that such a measure $\mathbb{Q}$ does *not* exist. However, it is easy to see from the proof of Theorem 2 that the tails of the Borland model seem to be too heavy to be of practical use anyway. In fact, under our original measure $\mathbb{P}$ we have that

$$S_t = S_0 e^{\mu t + \sigma \Omega_t}$$

and since for $t \in]0, T]$

$$\mathbb{E}^{\mathbb{P}} e^{\sigma \Omega_t} = \frac{1}{Z_t^\alpha} \int_{\mathbb{R}} e^{\sigma w} (1 + \alpha \beta(t) w^2)^{-\frac{1}{\alpha}} dw = \infty$$

this implies that in Borland's model

$$\mathbb{E}^{\mathbb{P}} S_t = \infty$$

for all $t \in]0, T]$. This means that the expectations of the asset price process values are not finite under $\mathbb{P}$, which is a serious limitation for practical use.

At the same time this indicates how we can try to change the model to remove this problem, as we will now show in the next section.

3 A Different Option Pricing Model

The tails of the asset price process S can be made less heavy if we use the model (under the original measure $\mathbb{P}$)

$$dS_t = \mu S_t dt + \sigma S_t d\Omega_t \tag{6}$$
$$d(\ln S_t) = (\mu - \tfrac{1}{2}\sigma^2 f^2(t, \Omega_t))dt + \sigma d\Omega_t \tag{7}$$

instead of the earlier model

$$d(\ln S_t) = \mu dt + \sigma d\Omega_t$$
$$dS_t = (\mu + \tfrac{1}{2}\sigma^2 f^2(t, \Omega_t)) S_t dt + \sigma S_t d\Omega_t$$

The equation (6) is a special case of a class of models proposed in [5], but this special case was assumed to be equivalent to the earlier model, which is not the case.
If the asset price process S defined by (6) exists, then it has a finite expectation at all times: $\mathbb{E}^{\mathbb{P}} S_t = e^{\mu t}$. However, before we can proceed we first have to check whether the stochastic differential equation (1) used as a definition of the process Ω does indeed have a solution. It is by no means clear that this is the case, since standard results on the existence of solutions would assume $f(t, \Omega)$ to be uniformly Lipschitz in its second variable, i.e. $|f(t,x) - f(t,y)| \leq L|x - y|$ for all $t > 0$, with L a constant which does not depend on t. This condition is clearly not satisfied in this case. However, we can still show that this stochastic differential equation admits a strong solution Ω_t on the finite time interval $[0, T]$ (for the proof, see the Appendix).

Theorem 3. *The stochastic differential equation*

$$d\Omega_t = f(t, \Omega_t)\, dW_t$$
$$\Omega_0 = 0,$$

admits a strong solution in the sense[2] *that for all $T > 0$ there exists an a.s. continuous stochastic process X such that*

- *The process X is adapted to the filtration $(\mathcal{F}_t^W)_{t \in [0,T]}$ generated by the Brownian Motion.*

[2]The formulation from the book of Karatzas and Shreve [11] has been used.

- $\mathbb{P}(\int_0^T f^2(t, X_t)dt < \infty) = 1$.
- *We have almost surely, for all* $t \in [0, T]$, *that* $X_t = \int_0^t f(u, X_u)dW_u$.

The probability density funcion p of Ω_t, which satisfies

$$\mathbb{P}(\Omega_t \in A) = \int_A p(t, z)dz$$

for all Borel sets A, can therefore indeed be found using the Fokker-Planck equation:

$$\frac{\partial}{\partial t}p(t, z) = \frac{1}{2}\frac{\partial^2}{\partial z^2}[f^2(t, z)p(t, z)] \tag{8}$$

As mentioned before, the solution[3] to this equation is the earlier defined function $\mathcal{P}$, but it is important to stress that the *conditional* probabilities do *not* follow this Tsallis distribution:

Lemma 1. *The distribution of* Ω *is given by a Tsallis-distribution, in particular we have for* $t > 0$ *for all Borel sets* A

$$\mathbb{P}(\Omega_t \in A) = \frac{1}{Z_t^\alpha}\int_A (1 + \beta(t)\alpha z^2)^{-\frac{1}{\alpha}}dz$$

but the conditional distribution of Ω *is* **not** *given by a Tsallis-distribution, i.e. if* $t > s > 0$ *it is not necessarily true for all Borel sets* A *that*

$$\mathbb{P}(\Omega_t \in A \mid \Omega_s = w) = \frac{1}{Z_{t-s}^\alpha}\int_A (1 + \beta(t-s)\cdot\alpha\cdot(z-w)^2)^{-\frac{1}{\alpha}}dz$$

Proof.
The first result follows from substituting $p(t, z) = \mathcal{P}(t, z)$ in (3), and the second result follows from substituting $p(t, z) = \mathcal{P}(t - s, z - w)$ in that equation. Notice that $p(t, z) = \mathcal{P}(t - s, z - w)$ *does* satisfy the equation

$$\frac{\partial}{\partial t}p(t, z) = \frac{1}{2}\frac{\partial^2}{\partial z^2}[f^2(t - s, z - w)p(t, z)]$$

mentioned in [4], but that is not the correct Fokker-Planck equation for the Ω process defined in (1). ■

Notice that from the above we can conclude in particular that

$$\mathbb{E}\Omega_t = 0, \qquad \mathbb{E}\Omega_t^2 = \frac{1}{(2 - 3\alpha)\beta(t)} \sim t^{\frac{2}{2-\alpha}}. \tag{9}$$

We will from now on work with the model defined in (6). In Appendix A of [4] this model is mentioned as well, and it is argued there that both models give the same option prices since their only difference is a drift term which will be removed in the Girsanov transformation. We now know that this is not the case because the violation of the Novikov condition makes the Girsanov transformation itself impossible. Under the model (6), however, the transformation can be carried out, since we would now like to construct a measure $\mathbb{Q}$ under which $W + \int u dt$ is a Brownian Motion, where this time

$$\begin{aligned} u_t &= \frac{\mu - r}{\sigma f(t, \Omega_t)} = \frac{\mu - r}{\sigma}\frac{(Z_t^\alpha)^{-\frac{1}{2}\alpha}}{\xi\sqrt{1 + \alpha\beta(t)\Omega_t^2}} \\ &\le \frac{\mu - r}{\sigma}\beta(t)^{\frac{\alpha}{4}} \le \frac{\mu - r}{\sigma}[(1 - \alpha)(2 - \alpha)t]^{-\frac{\alpha}{4-2\alpha}} = C\, t^{-\frac{\alpha}{4-2\alpha}} \end{aligned}$$

[3] Note that in Borland's paper, the constant ξ was taken to be one, but then $\mathcal{P}$ will not satisfy the Fokker-Planck equation. We thank the anonymous referee for pointing this out to us.

where the positive constant C is defined in an obvious way, and therefore

$$\mathbb{E}^{\mathbb{P}} \exp\left(\tfrac{1}{2}\int_0^T u_s^2 ds\right) \quad \leq \quad \exp\left(\tfrac{1}{2}C^2 \int_0^T s^{-\frac{\alpha}{2-\alpha}} ds\right)$$

The integral on the righthand side is convergent around zero for $\alpha \in]0, \frac{1}{2}[$, which shows that the Novikov condition can be met and that therefore the construction of $\mathbb{Q}$ as given in (4) is well-defined. Under this new equivalent measure $W^{\mathbb{Q}} = W + \int u dt$ is a Brownian Motion and therefore we have

$$\begin{aligned} dS_t &= \mu S_t dt \;+\; \sigma S_t d\Omega_t \\ &= \mu S_t dt \;+\; \sigma S_t f(t, \Omega_t) dW_t \\ &= r S_t dt \;+\; \sigma S_t f(t, \Omega_t) d(W_t + \int_0^t u_s ds) \\ &= r S_t dt \;+\; \sigma S_t f(t, \Omega_t) dW_t^{\mathbb{Q}} \\ &= r S_t dt \;+\; \sigma S_t d\Omega_t^{\mathbb{Q}} \end{aligned}$$

where

$$\Omega_t^{\mathbb{Q}} \quad = \quad \int_0^t f(s, \Omega_s) dW_s^{\mathbb{Q}}$$

is not just a local $\mathbb{Q}$-martingale but a $\mathbb{Q}$-martingale, since[4] for all $t \in]0, T]$

$$\tfrac{d}{dt}\mathbb{E}\left\langle \Omega, \Omega \right\rangle_t \quad = \quad \mathbb{E} f^2(t, \Omega_t) \;=\; \xi^2 (Z_t^{\alpha})^{\alpha} \mathbb{E}(1 + \beta(t)\alpha\Omega_t^2) \sim \; t^{\frac{\alpha}{2-\alpha}}(1 + \beta(t)\alpha\mathbb{E}\Omega_t^2) \;\sim\; t^{\frac{\alpha}{2-\alpha}}$$

so $\mathbb{E}\left\langle \Omega, \Omega \right\rangle_t$ is finite for all $t \in]0, T]$ by our assumption that $\alpha \in]0, \frac{1}{2}[$.

It seems that we now arrive at the same model as in the Borland paper under the risk-neutral measure $\mathbb{Q}$. But there is an important difference. We have that under $\mathbb{P}$

$$S_t \quad = \quad S_0 \exp\left[\mu t - \tfrac{1}{2}\sigma^2 \int_0^t f^2(s, \Omega_s) ds + \sigma \Omega_t\right] \tag{10}$$

In the original model of Borland we had under $\mathbb{P}$

$$S_t \quad = \quad S_0 \exp\left[\mu t + \sigma \Omega_t\right]$$

so we could always write Ω_t as a function of S_t i.e.

$$\Omega_t \quad = \quad \frac{\ln(S_t/S_0) - \mu t}{\sigma}$$

and the process S_t was therefore a Markov process. However, in our corrected model we lose this property, due to the integral in (10). This integral

$$I_t \quad = \quad \int_0^t f^2(s, \Omega_s) ds$$

depends on the whole history of the Ω process up to time t, and cannot be written in terms of the final value Ω_t alone. In the Borland paper it is suggested that this can be done, indeed it is mentioned in equation (71) of that paper that Ω_s equals

$$\sqrt{\frac{\beta(T)}{\beta(s)}}\Omega_T$$

[4] Local martingales with finite quadratic variation processes are martingales, see Protter [12] II.6 coll. 3.

That equality should of course mean *equality in distribution* but Borland then applies this equality as an *almost sure equality* in her equation (72). This is incorrect, and it explains the arbitrage we found in the option formulas derived in the rest of that paper.
In the second paper [5] another approximation is used for the integral I_T, of the form $g_0(T) + g_2(T)\Omega_T^2$ for certain deterministic functions g_0 and g_2, but this still suggests that the entire path integral can be expressed in terms of Ω_T which is not true. Even if the distributions would be close (which they do not seem to be, witnessing the scale of the errors in figure 11 of the paper) then S_T could still have a very different distribution from its approximation. Even if I_T and $g_0(T) + g_2(T)\Omega_T^2$ were close in distribution, this would not necessarily mean that the term which defines the risk-neutral distribution of $\ln S_T$, i.e. $\sigma\Omega_T + I_T$, is close to $\sigma\Omega_T + g_0(T) + g_2(T)\Omega_T^2$, since correlation plays a role there. As shown before, arbitrage is possible in this approximated model, and we expect the arbitrage possibilities to be even more severe for path-dependent options, such as American or barrier-type options.

The integral I_t in (10) represents the quadratic variation process which on the one hand makes sure that the expectation of the stock price process is now finite under both $\mathbb{P}$ and $\mathbb{Q}$ (and hence that conditional expectations and option prices exist) but on the other hand it causes our process S to lose the Markov property. The stochastic processes (S, Ω) do form a Markov process together, but not S alone. In particular, when we want to price an option at a time $t \in [0, T[$, we should not just observe the stock price S_t at that time but also the stochastic variable Ω_t, since it governs the future quadratic variation of S and it cannot be calculated from S directly. This is no problem at time zero (when $\Omega_0 = 0$) but it will be at later times. There can therefore not be a Black-Scholes like formula $C(S_t, t)$ for the option price C in terms of S and t alone, but instead $C = C(t, S_t, \Omega_t)$.

In a sense the model thus resembles a stochastic volatility model since it has an unknown parameter which varies stochastically and the value of which is needed to calculate the price of an option. But there is an important difference too: in stochastic volatility models the stock and the volatility are driven by two Wiener processes, while in this model, there is only one which drives both (which is also the case in the earlier mentioned model by Hobson and Rogers). This is the reason that we can still define a complete and arbitrage free model, even though the quadratic variation processes varies stochastically. And we thus retain the nice feauture of the Borland model that it is a hybrid which lies in between the standard Black-Scholes model and for example the Heston model with stochastic volatility.

However, the downside is that it is not possible to use explicit expressions for option values in terms of S alone, and even when Ω and S are both known, we cannot calculate call option values $C(t, S_t, \Omega_t)$ with strike K in closed form. The discounted asset price process

$$\frac{S_t}{B_t} = \frac{S_0}{B_0} e^{\sigma\Omega_t - \frac{1}{2}\sigma^2 \int_0^t f^2(s,\Omega_s)ds}$$

must be an exponential martingale under the risk-neutral measure $\mathbb{Q}$ and the pricing formula can be written as

$$C(t, s, w) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[(S_t e^{r(T-t) + \sigma(\Omega_T - \Omega_t) - \frac{1}{2}\sigma^2 \int_t^T f^2(s,\Omega_s)ds} - K)^+ \,\middle|\, S_t = s, \, \Omega_t = w \right]$$

but we cannot write this in a closed form, due to the presence of the quadratic variation integral in the exponent. Note that the same was true for the formulas in the Borland paper at any time after $t = 0$, because the closed form solution for the European Call is not valid for $t > 0$, even as an approximation. This can be seen from the second part of Lemma 1, which states that distributions at later times (which are conditional distributions given the information at that time) are no longer Tsallis-distributions.

But we can still calculate option prices with Monte Carlo simulation methods, or by using a finite difference implementation based on a partial differential equation, as we will now show.

4 Calculation of Price Functions

We define the operator $\mathcal{L}$ with domain $D_{\mathcal{L}}$, the set of functions $F : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ with function values $F(S, \Omega, t)$ which are continuously differentiable with respect to t and twice continuously differentiable with respect to S an Ω:

$$\mathcal{L}F = \tfrac{1}{2}\sigma^2 S^2 f^2(t,\Omega)\frac{\partial^2 F}{\partial S^2} + \sigma S f^2(t,\Omega)\frac{\partial^2 F}{\partial S \partial \Omega} + \tfrac{1}{2} f^2(t,\Omega)\frac{\partial^2 F}{\partial \Omega^2} + rS\frac{\partial F}{\partial S} - \lambda\frac{\partial F}{\partial \Omega} - rF$$

where $\lambda = \frac{\mu - r}{\sigma}$ represents a market price of risk parameter.

Theorem 4. *If the partial differential equation*

$$\begin{aligned} \frac{\partial F}{\partial t} + \mathcal{L}F &= 0 \\ F(S,\Omega,T) &= \Phi(S), \qquad (\forall \Omega \in \mathbb{R}) \end{aligned}$$

has a unique solution in $D_{\mathcal{L}}$, then the European-style contingent claim paying $\Phi(S_T)$ at time T can be replicated (using a self-financing strategy in the asset and the bank account) after an initial time $t < T$ from an initial investment $F(S_t, \Omega_t, t)$ at time t.

Proof.
Under the martingale measure $\mathbb{Q}$ we have

$$\begin{aligned} dS_t &= rS_t dt \ + \ \sigma S_t f(t,\Omega_t) dW_t^{\mathbb{Q}} \\ d\Omega_t &= f(t,\Omega_t) dW_t \\ &= f(t,\Omega_t)\cdot[dW_t^{\mathbb{Q}} - \tfrac{\mu - r}{\sigma f(t,\Omega_t)} dt] \\ &= f(t,\Omega_t) dW_t^{\mathbb{Q}} - \lambda dt \end{aligned}$$

Let F be a solution as mentioned in the Theorem. Then we have by Ito's rule,

$$\begin{aligned} dF(S_t,\Omega_t,t) &= (rF(S_t,\Omega_t,t) + \frac{\partial F}{\partial t} + \mathcal{L}F)dt \ + \ (\frac{\partial F}{\partial \Omega}(S_t,\Omega_t,t) + \sigma S_t \frac{\partial F}{\partial S}(S_t,\Omega_t,t)) f(t,\Omega_t) dW_t^{\mathbb{Q}} \\ &= rF(S_t,\Omega_t,t)dt \ + \ (\frac{\partial F}{\partial \Omega}(S_t,\Omega_t,t) + \sigma S_t \frac{\partial F}{\partial S}(S_t,\Omega_t,t)) f(t,\Omega_t) dW_t^{\mathbb{Q}} \end{aligned}$$

so if we define

$$\begin{aligned} \phi_t^S &= \frac{\partial F}{\partial S}(S_t,\Omega_t,t) \ + \ \frac{1}{\sigma S_t}\frac{\partial F}{\partial \Omega}(S_t,\Omega_t,t) \\ \phi_t^B &= (F(t,S_t,\Omega_t) - \phi_t^S S_t)/B_t \end{aligned}$$

then we have that

$$\begin{aligned} dF(S_t,\Omega_t,t) &= \phi_t^S dS_t \ + \ \phi_t^B dB_t \\ F(S_t,\Omega_t,t) &= \phi_t^S S_t \ + \ \phi_t^B B_t \end{aligned}$$

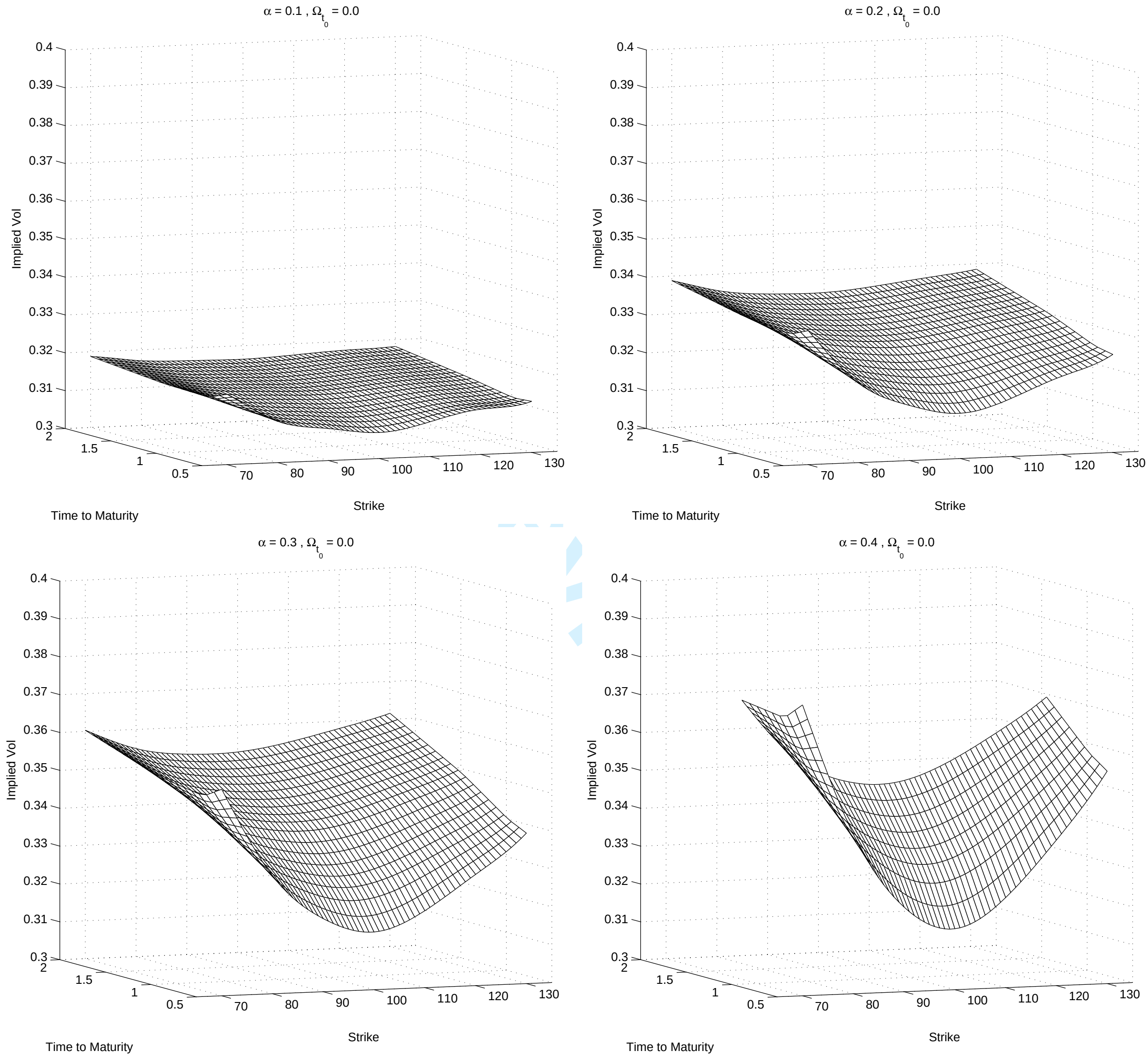
while $F(S_T, \Omega_T, T)$ equals $\Phi(S_T)$, the payoff of the contingent claim. This proves the Theorem. ■

Notice that we have not specified the precise conditions under which a solution to the partial differential equation (with the desired properties) exists. Indeed, the exact conditions which guarantee existence of a classical solution to the Cauchy problem posed here (with nonlinear and time-inhomogeneous terms) will require further study.

It is interesting to see the role played by the market price of risk here. Since the volatility of the underlying asset price process is stochastic, there is a market price of volatility risk. But since the driving noise term of the volatility is the same as the one of the underlying process itself, this market price of risk simply boils down to the market price of risk for the asset process that we find in a standard Black-Scholes model:

$$\lambda = \frac{\mu - r}{\sigma}.$$

This again illustrates how the Borland model represents a tractable alternative for a full stochastic volatilty model such as Heston's, where there is a second Brownian Motion to drive the volatility process which therefore brings with it a new market price of risk which cannot be determined directly but must be estimated from market data.

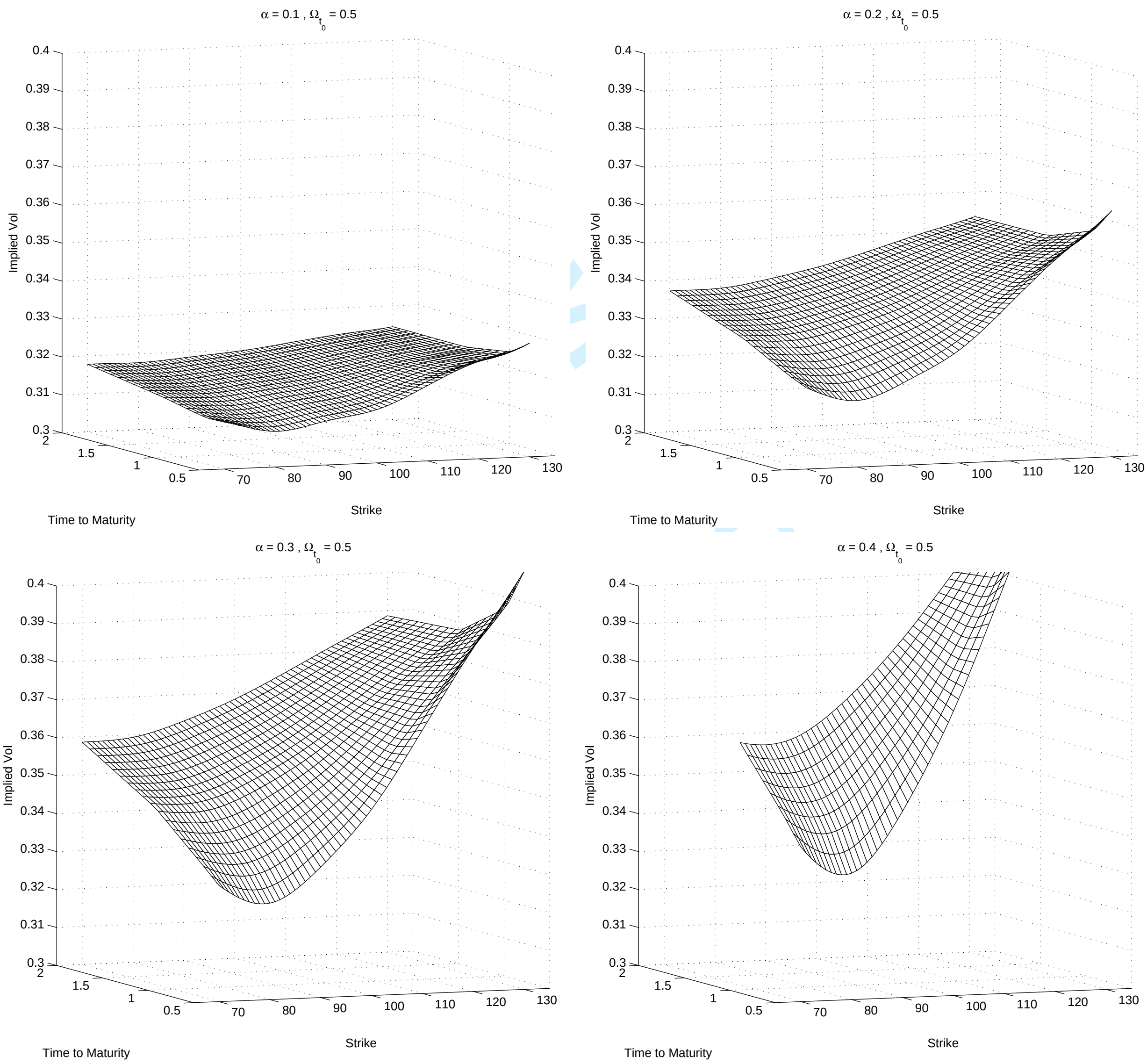


5 Numerical Results

In this section we use the partial differential equation of the previous section to calculate option prices for European call options. We use $S_{t_0} = 100$, $r = 3\%$, $\sigma = 30\%$. The starting value for the model t_0 was taken to be 0.2 to show the influence of different values of Ω at that time, and the option characteristics (strike K and maturity T) are varied in the graphs. In all the numerical results shown here we took $\lambda = 0$. We used an explicit finite difference method with a 100×100

grid for the values of S and Ω and 20000 timesteps. The boundary conditions used for the S and Ω variables were a vanishing second derivative of the option price with respect to S and Ω respectively.

The prices we found are shown as implied volatilities in the Black-Scholes model for European options. We have used the values $\alpha \in \{0.10, 0.20, 0.30, 0.40\}$ and $\Omega_{t_0} \in \{0, 0.50\}$ to show the effect of changing these important model parameters. We checked some option prices using Monte Carlo simulations of the risk-neutral asset price process, and found good agreement. Using 500000 simulations with 1000 timesteps per simulation the maximal relative error we found between prices generated by finite differences and by Monte Carlo simulations was 0.3% for the calls we considered. In all Monte Carlo simulations we used Black-Scholes dynamics to define control variates for the payoffs. Monte Carlo calculation times took 3 to 4 times as much CPU time as finite differences. In figures 1 to 4, we have $\Omega_{t_0} = 0$, while in figure 5 to 8, $\Omega_{t_0} = 0.50$. We notice that we have a clear volatility smile, which is more pronounced for shorter maturities. Also notice that if the current value of Ω is not zero, the steepness of the smile increases and it shifts a little bit as well. The fact that we get different curves for different values of Ω shows that it is essential to include this parameter in the process of fitting the model to market data. In fact, this may provide an interesting opportunity for market makers to use a richer class of possible volatility surfaces instead of the single possibility provided when Ω is just taken to be zero.



6 Conclusions and Future Research

It has been shown that the tails in the Borland model for non-Gaussian option pricing are so heavy that conditional expectations, and hence option prices, do not exist in this model. However, we have shown that a different model can be defined which remedies this by making the tails less heavy, and option prices can then be calculated as soon as an additional parameter (the value of Ω_t, which governs the future quadratic variation of the asset price) has been specified. However, we can no longer find closed-form formulas for Europan vanilla options.

We like to stress that the main innovatove idea of the Borland model, i.e. letting the volatility be stochastic but keeping the completeness of the model, is not changed by our modification. But the option pricing formulas we get are very different indeed, as can be seen by comparing the partial differential equations generated by the two models. We believe that the model provides very interesting perspectives for practical applications, and more particularly for improved fitting of option prices. In future work we also hope to investigate the more general class of models given by

$$\begin{aligned} S_t &= S_0 e^{rt+\sigma\Omega_t-\frac{1}{2}\sigma^2\langle\Omega,\Omega\rangle_t} \\ d\Omega_t &= g(t,\Omega_t,S_t)\,dW_t \end{aligned}$$

for suitably chosen functions g.

To determine how well the model presented in this paper can be fitted in practice, further investigation is needed of the relationship between the observed volatility smiles and the parameter Ω which needs to be specified in the modified model, but which was not present in the original one. As we have pointed out, the extra flexibility that this parameter provides could be an advantage in practical fitting problems. The fact that the model seems to generate volatility smiles which steepen as time to maturity decreases is very promising. Obviously, the types of smile and skew patterns that can be generated within this framework (for example by using different functions g in the equation above) should be researched more extensively.

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Appendix: Proofs of Theorems 3 and 2

Proof of Theorem 3.
Since f is not uniformly Lipschitz in its second variable we will use the following bound:

$$\begin{aligned}
|f(t,x)-f(t,y)| &= \partial_x f(t,\theta)|x-y|, \qquad \text{for some } \theta \in [\min\{x,y\},\ \max\{x,y\}] \\
&= \xi\,(Z_t^{\alpha})^{\frac{1}{2}\alpha}|x-y|\cdot\left|\frac{2\alpha\beta(t)\theta}{2\sqrt{1+\alpha\beta(t)\theta^2}}\right| \\
&\leq \xi\,|x-y|\sqrt{\alpha\beta(t)}\left(\frac{A_\alpha}{\sqrt{\beta(t)}}\right)^{\frac{1}{2}\alpha} = |x-y|\cdot[(1-\alpha)(2-\alpha)t]^{-\frac{1}{2}}\sqrt{\alpha}
\end{aligned}$$

but since $\frac{\alpha}{(1-\alpha)(2-\alpha)} \leq 1$ for all $\alpha \in]0,\frac{1}{2}[$ this gives

$$|f(t,x)-f(t,y)|^2 \leq \frac{|x-y|^2}{t} \tag{11}$$

for all $t>0$ and all $x,y\in\mathbb{R}$.
Now define the sequence of adapted processes

$$X_t^0 \equiv 0, \qquad X_t^{k+1} = \int_0^t f(s,X_s^k)dW_s$$

and let

$$E_t^k = \mathbb{E}|X_t^{k+1}-X_t^k|^2.$$

Notice that

$$X_t^1 = \xi\int_0^t (Z_s)^{\frac{1}{2}\alpha}dW_s = \tilde{A}_\alpha\int_0^t s^{\frac{\alpha}{4-2\alpha}}dW_s, \qquad \tilde{A}_\alpha = [(1-\alpha)(2-\alpha)]^{\frac{\alpha}{4-2\alpha}}$$

Since

$$\sup_{\alpha\in]0,\frac{1}{2}[} \tilde{A}_\alpha \leq 2$$

we have that

$$E_t^0 = \mathbb{E}|X_t^1|^2 \leq 4\int_0^t s^{\frac{\alpha}{2-\alpha}}ds = 4t^{\frac{2}{2-\alpha}}\frac{2-\alpha}{2} \leq 4t^{\frac{4}{3}}$$

which shows that X_t^1 is a well-defined continuous martingale. We will make use of the following Lemma.

Lemma 2. *For all $k\in\mathbb{N}$ and all $t\geq 0$ we have*

$$\begin{aligned}
E_t^k = \mathbb{E}|X_t^{k+1}-X_t^k|^2 &\leq 4\,t^{\frac{2}{2-\alpha}}(1-\tfrac{\alpha}{2})^k \\
\mathbb{E}|X_t^k|^2 &\leq \tfrac{8}{\alpha}\,t^{\frac{2}{2-\alpha}}.
\end{aligned}$$

Proof of Lemma.
We have shown that this claim is true for $k=0$. Assuming the claim to be true for a certain $k\in\mathbb{N}$, we calculate (using the bound on f proven before)

$$\begin{aligned}
E_t^{k+1} &= \mathbb{E}|X_t^{k+2}-X_t^{k+1}|^2 = \mathbb{E}\left(\int_0^t [f(s,X_s^{k+1})-f(s,X_s^k)]dW_s\right)^2 \\
&= \mathbb{E}\int_0^t \left(f(s,X_s^{k+1})-f(s,X_s^k)\right)^2 ds \qquad (12) \\
&\leq \mathbb{E}\int_0^t s^{-1}\left(X_s^{k+1}-X_s^k\right)^2 ds = \int_0^t s^{-1}\mathbb{E}\left(X_s^{k+1}-X_s^k\right)^2 ds \\
&= \int_0^t s^{-1}E_s^k ds \leq 4(1-\tfrac{\alpha}{2})^k\int_0^t s^{-1}s^{\frac{2}{2-\alpha}}ds = 4t^{\frac{2}{2-\alpha}}(1-\tfrac{\alpha}{2})^{k+1}
\end{aligned}$$

so the first claim of the Lemma is proven by induction. From this result we then conclude that

$$\mathbb{E}|X_t^k|^2 \;\leq\; \sum_{m=0}^{k-1} \mathbb{E}|X_t^{m+1} - X_t^m|^2 \;=\; \sum_{m=0}^{k-1} E_t^m \;\leq\; 4t^{\frac{2}{2-\alpha}} \sum_{m=0}^{k-1} (1-\tfrac{\alpha}{2})^m \;=\; \tfrac{8}{\alpha}\, t^{\frac{2}{2-\alpha}} \qquad (13)$$

and we're done. Note that this result implies that $\mathbb{E}\left[\, t^{-1}(X_t^k)^2 \,\right] < \infty$ for all $t > 0$. ■

Continuation of Proof of Theorem 3.
For fixed $k \in \mathbb{N}$ we define

$$V_t^k \;=\; X_t^{k+2} - X_t^{k+1} \;=\; \int_0^t [f(s, X_s^{k+1}) - f(s, X_s^k)] dW_s$$

This process V_t^k is a local martingale and in fact even a martingale since by the Lemma and (12)

$$\begin{aligned} \mathbb{E}\left\langle V^k \right\rangle_t \;&=\; \mathbb{E}\int_0^t \left(f(s, X_s^{k+1}) - f(s, X_s^k)\right)^2 ds \\ &=\; E_t^{k+1} \;\leq\; 4\, t^{\frac{2}{2-\alpha}} (1 - \tfrac{\alpha}{2})^{k+1}, \end{aligned}$$

so V is a local martingale which has finite expected value for its quadratic variation process at all times and hence[5] it is a continuous martingale. We then use the standard martingale inequality which states that

$$\mathbb{E}\left[\max_{s\in[0,T]} |V_s^k|^2\right] \;\leq\; 4\mathbb{E}\left\langle V^k \right\rangle_T$$

to derive that

$$\mathbb{E}\left[\max_{s\in[0,T]} |X_s^{k+2} - X_s^{k+1}|^2\right] \;\leq\; c(1 - \tfrac{\alpha}{2})^k$$

for $c = 16T^{\frac{2}{2-\alpha}}$. The Chebyshev inequality then allows us to conclude that

$$\begin{aligned} \mathbb{P}\left(\max_{s\in[0,T]} |X_s^{k+2} - X_s^{k+1}|^2 \geq (1 - \tfrac{\alpha}{4})^k\right) \;&\leq\; (\tfrac{4}{4-\alpha})^k\, \mathbb{E}\left[\max_{s\in[0,T]} |X_s^{k+2} - X_s^{k+1}|^2\right] \\ &\leq\; c(\tfrac{4}{4-\alpha})^k (\tfrac{2-\alpha}{2})^k \;=\; c(1 - \tfrac{\alpha}{4-\alpha})^k \end{aligned}$$

and since the series on the righthand side converges when we sum over all $k \in \mathbb{N}$ we can use the Borell-Cantelli lemma to conclude that for almost all ω there exists an $N(\omega) \in \mathbb{N}$ such that for all $k \geq N(\omega)$ and all $m \in \mathbb{N}^+$

$$\max_{t\in[0,T]} |X_t^{k+m}(\omega) - X_t^{k+1}(\omega)| \;\leq\; \frac{(1 - \frac{\alpha}{4})^{\frac{1}{2}k}}{1 - \sqrt{1 - \frac{\alpha}{4}}}.$$

This shows that the sequence $\{X_t^k(\omega),\, t \in [0, T]\}_{k\in\mathbb{N}}$ of continuous paths convergences uniformly in the sup-norm and thus has a limit $X_t(\omega) = \lim_{k\to\infty} X_t^k(\omega)$ which is a.s. continuous itself. Clearly this limiting process is adapted, almost surely equal to zero for $t = 0$, and the requirement $\mathbb{P}(\int_0^T f^2(t, X_t) dt < \infty) = 1$ follows from equations (11) and (13) and dominated convergence. To prove the last requirement we let k go to infinity in the equation

$$X_t^{k+1} \;=\; \int_0^t f(s, X_s^k) dW_s$$

[5] See for example the book by Protter [12], II.6 coll. 3.

The lefthand side converges to X_t while the righthand side converges as well since

$$\begin{aligned}\mathbb{E}\left|\int_0^t [f(s,X_s^k)-f(s,X_s)]dW_s\right|^2 &= \mathbb{E}\int_0^t \left|f(s,X_s^k)-f(s,X_s)\right|^2 ds \\ &\leq \int_0^t s^{-1}\mathbb{E}\left|X_s^k - X_s\right|^2 ds.\end{aligned}$$

This last expression goes to zero for $k\to\infty$ by dominated convergence, since $\mathbb{E}|X_t^k|^2 \leq \frac{8}{\alpha}T^{\frac{2}{2-\alpha}} < \infty$ by the Lemma, which implies $\mathbb{E}|X_t|^2 < \infty$ by dominated convergence as well. ■

Proof of Theorem 2.
Since $\mu - r$, σ and the values of the function f are all strictly positive we have that

$$\begin{aligned}\tfrac{1}{2}u_t^2 &\geq \tfrac{1}{2}(\tfrac{1}{2}\sigma f(t,\Omega_t))^2 \\ &\geq \tfrac{1}{8}\sigma^2(1+\beta(t)\alpha\Omega_t^2)\beta(t)^{-\frac{1}{2}\alpha} \\ &\geq \tfrac{1}{8}\sigma^2\beta(T)^{1-\frac{1}{2}\alpha}\alpha\Omega_t^2 \\ &\geq \eta\Omega_t^2\end{aligned}$$

where $\eta = \frac{1}{8}\sigma^2\beta(T)^{1-\frac{1}{2}\alpha}\alpha$ is a positive constant which does not depend on t. Take a $t_1 \in]0,T[$ and define for $t\in[t_1,T]$

$$\begin{aligned}M_t &= \Omega_t - \Omega_{t_1} \\ H_t &= \mathbb{E}^{\mathbb{P}}[M_t^2 \mid \mathcal{F}_{t_1}]\end{aligned}$$

Using elementary properties of the Wiener integral and Fubini's Theorem we now calculate for $t_1 \leq t \leq t_2$

$$\begin{aligned}H_t &= \mathbb{E}^{\mathbb{P}}[M_t^2 \mid \mathcal{F}_{t_1}] = \mathbb{E}^{\mathbb{P}}[\left(\int_{t_1}^t f(s,\Omega_s)dW_s\right)^2 \mid \mathcal{F}_{t_1}] \\ &= \mathbb{E}^{\mathbb{P}}[\int_{t_1}^t f^2(s,\Omega_s)ds \mid \mathcal{F}_{t_1}] = \mathbb{E}^{\mathbb{P}}[\int_{t_1}^t \xi^2(Z_s^\alpha)^\alpha(1+\beta(s)\alpha\Omega_s^2)ds \mid \mathcal{F}_{t_1}] \\ &= \xi^2\int_{t_1}^t (Z_s^\alpha)^\alpha\mathbb{E}^{\mathbb{P}}[(1+\beta(s)\alpha\Omega_s^2)\mid\mathcal{F}_{t_1}]ds = \xi^2\int_{t_1}^t (Z_s^\alpha)^\alpha(1+\alpha\beta(s)H_s+\alpha\beta(s)\Omega_{t_1}^2)ds\end{aligned}$$

Let $K = \xi^2(Z_T^\alpha)^\alpha\max\{1,\alpha\beta(t_1)\}$, then

$$\begin{aligned}H_{t_1} &= 0 \\ \frac{d}{dt}H_t &= \xi^2(Z_t^\alpha)^\alpha(1+\alpha\beta(t)\Omega_{t_1}^2+\alpha\beta(t)H_t) \\ &\leq K(1+\Omega_{t_1}^2+H_t)\end{aligned}$$

and Gronwall's Lemma then gives that

$$H_t \leq (e^{K(t-t_1)}-1)(1+\Omega_{t_1}^2), \qquad t\in[t_1,T]$$

Now take $t_2 \in]t_1,T]$ such that $e^{K(t_2-t_1)}-1 \leq \frac{1}{32}$, then

$$H_{t_2} \leq \tfrac{1}{32}(1+\Omega_{t_1}^2) \tag{14}$$

Let

$$X = \sup_{t_1\leq s\leq t_2} M_s^2$$

and let m be any real number larger than one. The set

$$C = \{\omega : \Omega_{t_1}(\omega) \geq 2m,\, X(\omega) \leq m^2\}$$

is a subset of the set $\{\omega : \inf_{t_1 \leq s \leq t_2} \Omega_t^2(\omega) \geq m\}$, so on C we have that $\frac{1}{2}u_t^2 \geq \frac{1}{2}\eta m^2$. We then bound the expectation in the Novikov condition as follows

$$\begin{aligned} L &= \mathbb{E}^{\mathbb{P}} \exp\left(\tfrac{1}{2}\int_0^T u_s^2 ds\right) \geq \mathbb{E}^{\mathbb{P}} \exp\left(\tfrac{1}{2}\int_{t_1}^{t_2} u_s^2 ds\right) \\ &\geq \mathbb{E}^{\mathbb{P}} \mathbf{1}_{\{C\}} \exp\left(\tfrac{1}{2}\int_{t_1}^{t_2} u_s^2 ds\right) \\ &\geq e^{\frac{1}{2}\eta m^2 (t_2 - t_1)} \mathbb{E}^{\mathbb{P}} \mathbf{1}_{\{C\}} \\ &= e^{\frac{1}{2}\eta m^2 (t_2 - t_1)} \mathbb{E}^{\mathbb{P}} \left(\mathbb{E}^{\mathbb{P}}[\, \mathbf{1}_{\{\Omega_{t_1} \geq 2m\}} \mathbf{1}_{\{X \leq m^2\}} \mid \mathcal{F}_{t_1}]\right) \\ &= e^{\frac{1}{2}\eta m^2 (t_2 - t_1)} \mathbb{E}^{\mathbb{P}} \left(\mathbf{1}_{\{\Omega_{t_1} \geq 2m\}}\, \mathbb{E}^{\mathbb{P}}[\, \mathbf{1}_{\{X \leq m^2\}} \mid \mathcal{F}_{t_1}]\right) \end{aligned}$$

But

$$\mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_{t_1}] = \mathbb{E}^{\mathbb{P}}[\sup_{t_1 \leq s \leq t_2} M_s^2 \mid \mathcal{F}_{t_1}] \leq 4\mathbb{E}^{\mathbb{P}}[M_{t_2}^2 \mid \mathcal{F}_{t_1}] = 4H_{t_2} \leq \tfrac{1}{8}(1 + \Omega_{t_1}^2)$$

where we have used a standard martingale inequality and (14). But then

$$\begin{aligned} m^2 \mathbb{E}^{\mathbb{P}}[\, \mathbf{1}_{\{X \geq m^2\}} \mid \mathcal{F}_{t_1}] &\leq \mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_{t_1}] \leq \tfrac{1}{8}(1 + \Omega_{t_1}^2) \\ \Rightarrow \quad \mathbb{E}^{\mathbb{P}}[\, \mathbf{1}_{\{X \leq m^2\}} \mid \mathcal{F}_{t_1}] &\geq 1 - \tfrac{1}{m^2}\tfrac{1}{8}(1 + \Omega_{t_1}^2) \end{aligned}$$

so

$$\begin{aligned} L &\geq e^{\frac{1}{2}\eta m^2 (t_2 - t_1)}\, \mathbb{E}^{\mathbb{P}} \left(\mathbf{1}_{\{\Omega_{t_1} \geq 2m\}} (1 - \tfrac{1}{m^2}\tfrac{1}{8}(1 + \Omega_{t_1}^2))\right) \\ &= e^{\frac{1}{2}\eta m^2 (t_2 - t_1)}\, (\tfrac{1}{2} - \tfrac{1}{8m^2})\, \mathbb{P}(\Omega_{t_1} \geq 2m) \\ &= e^{\frac{1}{2}\eta m^2 (t_2 - t_1)}\, (\tfrac{1}{2} - \tfrac{1}{8m^2}) \int_{2m}^{\infty} (1 + 4\alpha\beta(t_1)u^2)^{-\frac{1}{\alpha}} du / Z_{t_1}^{\alpha} \end{aligned}$$

Letting m go to infinity now proves the result. ■

On Option Pricing Models in the Presence of Heavy Tails*

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Abstract

We propose a modification of the option pricing framework derived by Borland [4, 5] which removes the possibilities for arbitrage within this framework. It turns out that such arbitrage possibilities arise due to an incorrect derivation of the martingale transformation in the non-Gaussian option models which are used in that paper. We show how a similar model can be built for the asset price processes which excludes arbitrage. However, the correction causes the pricing formulas to be less explicit than the ones in the original formulation, since the stock price itself is no longer a Markov process. Practical option pricing algorithms will therefore have to resort to Monte Carlo methods or partial differential equations and we show how these can be implemented. An extra parameter, which needs to be specified before the model can be used, will give market makers some extra freedom when fitting their model to market data.

1 Introduction

Models for equity option pricing in which the underlying asset exhibit tails which are heavier than those of a lognormal distribution have been researched for many years now. A lot of empirical evidence suggests that such heavy-tailed models can provide a better fit for many equity price processes (or indices thereof) and it is therefore only natural that many authors have tried to develop new models which go beyond the standard lognormal assumptions of the celebrated Black and Scholes model [3]. Among the many possible assumptions made by different authors for the distribution of future asset prices are jump-diffusion models [2], level-dependent volatilities [6, 7], or hypergeometric and inverse Gaussian models that are analytically tractable and allow level-dependent volatilities as well [1]. Another class of possible models is characterised by the fact that the asset price process itself is no longer a Markov process. Perhaps the most well-known of the models in this class are the stochastic volatility models, such as those defined by Hull and White [10, 13], Heston [8] and Hobson and Rogers [9].

Recently, a very interesting new approach was proposed in a paper by Borland [4]. In it, the author defines a diffusion process, in the usual form of a stochastic differential equation driven by a Wiener process, which has heavy tails. Its distribution at future times can be characterized explicitly as a Tsallis distribution [14], which implies a probability density function for the logarithm of the assets which is asymptotically equal to $x^{-\gamma}$ for certain values of $\gamma > 3$. In the paper, a riskneutral pricing argument is then used to derive closed-form option pricing formulas for European calls and puts. It is shown that the implied volatility smile observed in practice can be represented well in this model if one chooses the model parameters carefully. The model has a stochastic volatility but it still generates a complete market, since no extra Brownian Motions are introduced for the volatility process and in this way the model resembles the approach taken in Hobson and Rogers

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[9]. Since it is also possible to find a closed-form solution for the future distributions of asset prices, the model therefore seems to provide a very clever combination of analytical tractability (since everything is defined in terms of diffusions, with distributions which can be characterized explicitly) and sufficient complexity to be of use in market practice.

Unfortunately, the model as is stands admits arbitrage. Some mathematical conditions which need to be fulfilled to carry out the Girsanov transformation from the real world measure to the riskneutral measure, are not satisfied for this particular model. The closed-form option formulas defined in the paper are therefore not valid and for options with long maturities they do not even form a useful approximation. This is a pity, since the ideas underlying the model are very interesting and deserving of further analysis. Indeed, the Borland model provides a nice hybrid between the Heston model and the Black-Scholes model. It has a volatility which varies stochastically, but since the volatility is driven by the same Brownian Motion as the asset process itself, the model is still complete.

One of the most important feautures of any practical option pricing model is that it should be guaranteed to be arbitrage-free. Since this is not the case for the original model in [4] or the slighly different approximations given in the later paper [5], we will change the model in such a way that it can be guaranteed to be arbitrage-free and we will show how option prices can still be calculated. Our analysis should not be interpreted as an indication that the Borland model is not useful; we merely try to repair the mathematical problems associated with it. We believe the ideas behind the model to be innovative, and very useful for practical option trading.

In our approach, it is no longer possible to give closed-form formulas of European option prices. However, we show how we can use a partial differential equation (which is totally different from the one used in [4]) to find these prices, and we check the results using Monte Carlo methods. It turns out that many of the nice features of the original model are retained after our modification.

The organization of the paper is as follows. In the next section we will formulate the model used in [4], with a slighly different notation at some places to emphasize some important characteristics of the parameters, and we show why arbitrage occurs in this model. Section 3 derives an alternative model which excludes this arbitrage. Section 4 shows how option prices can be calculated using this model, and we use the methods defined there to show some examples of the option prices in section 5. In the last section we formulate conclusions and possible subjects for further research.

2 The Earlier Model

In [4], the stochastic process driving the rates of return of the stock price process is not Brownian Motion but a continuous Markov process defined as

$$d\Omega_t = f(t,\Omega_t)\,dW_t \quad (1)$$
$$\Omega_0 = 0 \quad (2)$$

where $\{W_t, t \geq 0\}$ is an $\mathcal{F}_t$-adapted Brownian Motion process on a filtered probability space $(\bar{\Omega}, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, and f is defined by

$$\mathcal{P}(t,w) = (1+\beta(t)\alpha w^2)^{-\frac{1}{\alpha}}/Z_t^\alpha, \qquad t>0$$
$$f(t,w) = \begin{cases} \xi\,\mathcal{P}(t,w)^{-\frac{1}{2}\alpha} = \xi\,(Z_t^\alpha)^{\frac{1}{2}\alpha}\sqrt{1+\beta(t)\alpha w^2} & t>0 \\ 0 & t=0 \end{cases}$$

where $\alpha \in]0, \frac{1}{2}[$ is a constant [1]. The starting value for Ω_0 need not be zero, and we will change it later in the paper, but for now we will use this assumption from the Borland model. The timescale

[1] We use a slightly different notation than the one in Borland's paper to emphasize which constants are positive or negative, $\alpha = q-1$ in the Borland paper. We also take α smaller than $\frac{1}{2}$ instead of Borland's $\frac{2}{3}$ to make sure that the expectation of quadratic variation $\mathbb{E}\langle\Omega,\Omega\rangle_t$ is finite for all $t \in [0,T]$, as shown later. Also note that the constant ξ that we introduce here was taken $\xi = 1$ in the Borland paper.

β is given by

$$\beta(t) \;=\; [\,(1-\alpha)(2-\alpha)\,t\,]^{-\frac{2}{2-\alpha}}$$

and Z_t^α, A_α and ξ are given by

$$Z_t^\alpha \;=\; \int_{\mathbb{R}} (1+\beta(t)\alpha u^2)^{-\frac{1}{\alpha}} du \;=\; \frac{A_\alpha}{\sqrt{\beta(t)}}, \qquad A_\alpha = \frac{\sqrt{\frac{\pi}{\alpha}}\,\Gamma(\frac{1}{\alpha}-\frac{1}{2})}{\Gamma(\frac{1}{\alpha})}, \qquad \xi \;=\; (A_\alpha)^{-\alpha/2}.$$

Note that if $\nu = \frac{2}{\alpha} - 1$ happens to be an element of $\mathbb{N}$, then $\sqrt{(2-\alpha)\beta(t)}\Omega_t$ has a Student's t-distribution with ν degrees of freedom. We can write

$$d\Omega_t \;=\; c\,\sqrt{t^{\frac{\alpha}{2-\alpha}} + Kt^{-1}\,\Omega_t^2}\,dW_t$$

where $K \in \mathbb{R}^+$, $c \in \mathbb{R}^+$ are constants which depend on α but not on the time t. The Kolmogorov Forward (or Fokker-Planck) equation

$$\frac{\partial}{\partial t}p(t,z) \;=\; \frac{1}{2}\,\frac{\partial^2}{\partial z^2}[f^2(t,z)p(t,z)] \tag{3}$$

then shows that the probability density funcion p of Ω_t, which satisfies

$$\mathbb{P}(\Omega_t \in A) \;=\; \int_A p(t,z)dz$$

for all Borel sets A, is given by the function $\mathcal{P}$ mentioned before: $p = \mathcal{P}$.

In the paper, Ω_t is now used to define a stock price process $\{S_t,\, t \geq 0\}$ using

$$\Omega_t \;=\; \frac{\ln S_t/S_0 - \mu t}{\sigma}$$

for strictly positive constants μ, σ. This defines a continuous *Markov* process S since

$$\begin{aligned} d(\ln S_t) &= \mu dt \;+\; \sigma d\Omega_t \\ &= \mu dt \;+\; \sigma f(t,\Omega_t)dW_t \\ &= \mu dt \;+\; \sigma f(t, \tfrac{1}{\sigma}(\ln S_t/S_0 - \mu t))dW_t \end{aligned}$$

The distribution of $\ln S$ has tails which are heavier than those for a Gaussian distribution. The risk free rate of return r is assumed to be constant, with $0 < r < \mu$, and $B_t = B_0 e^{rt}$ thus models a bank account. Option prices are then derived for this model, under the assumption that the underlying asset price process S follows the stochastic differential equation given above. The calculations lead to the following option price function $C(S,T)$ for a European call with strike K and time to maturity T which pays $\Phi(S_T) = \max(0, S_T - K)$ at time T:

$$C(s,T) \;=\; e^{-rT}\int_{\mathbb{R}} \left(s e^{rT+\sigma w - \frac{1}{2}\gamma\sigma^2 T^{\frac{2}{2-\alpha}}(1+\alpha\beta(T)w^2)} - K \right)^+ \mathcal{P}(T,w)\,dw$$

where $\gamma = \frac{1}{2}(2-\alpha)\,[(2-\alpha)(1-\alpha)]^{-\frac{\alpha}{2-\alpha}}$ is a strictly positive constant. However, this formula cannot be correct.

Theorem 1. *The call option formula given above admits arbitrage.*

Proof.
Since $bx - \frac{1}{2}x^2 a \leq \frac{1}{2}b^2/a$ for $a > 0$, we have that

$$\sigma w - \tfrac{1}{2}w^2\gamma\sigma^2 T^{\frac{2}{2-\alpha}}\alpha\beta(T) \;\leq\; \tfrac{1}{2}(\gamma T^{\frac{2}{2-\alpha}}\alpha\beta(T))^{-1}$$

so

$$se^{rT+\sigma w-\frac{1}{2}\gamma\sigma^2 T^{\frac{2}{2-\alpha}}(1+\alpha\beta(T)w^2)} \quad \leq \quad S^{\max}(T,\alpha)$$

where the value of $S^{\max}(T,\alpha)$ does not depend on w. This means that the price of a European Call with maturity T and strike $K > S^{\max}$ has the value zero. But there is a positive probability that the option ends up in the money because under $\mathbb{P}$ the probability density function of S_T is positive for values higher than $S^{\max}(T,\alpha)$. This clearly constitutes an arbitrage. ■

In the later paper [5] slightly different call option price formulas are given, but one may construct arbitrage opportunities for these formulas in a way similar to the one given above. We will now analyze how the arbitrage arises.

Borland would like to work in a complete and arbitrage-free market, and she therefore wants to construct a measure $\mathbb{Q}$, equivalent with $\mathbb{P}$, such that the discounted process $\tilde{S}_t = S_t/B_t$ becomes a martingale under $\mathbb{Q}$, i.e. $\mathbb{E}^{\mathbb{Q}}[S_t/B_t \mid \mathcal{F}_u] = S_u/B_u$ for all $t \geq u \geq 0$. To find such a measure $\mathbb{Q}$ Borland writes

$$\begin{aligned} d(\ln S_t) &= \mu dt \;+\; \sigma f(t,\Omega_t)dW_t \\ dS_t &= (\mu + \tfrac{1}{2}\sigma^2 f^2(t,\Omega_t))S_t dt \;+\; \sigma S_t f(t,\Omega_t)dW_t \end{aligned}$$

so the discounted asset price process satisfies

$$\begin{aligned} d\tilde{S}_t &= (\mu - r + \tfrac{1}{2}\sigma^2 f^2(t,\Omega_t))\tilde{S}_t dt \;+\; \sigma\tilde{S}_t f(t,\Omega_t)dW_t \\ &= \sigma\tilde{S}_t f(t,\Omega_t)\,(u_t dt + dW_t) \end{aligned}$$

where

$$u_t \;=\; \frac{\mu - r + \frac{1}{2}\sigma^2 f^2(t,\Omega_t)}{\sigma f(t,\Omega_t)}$$

An equivalent measure $\mathbb{Q}$ which makes $\tilde{S}_t$ a martingale must be such that the process $W + \int u dt$ is a Brownian Motion under $\mathbb{Q}$ and to construct such a measure one may try to use the Girsanov Theorem. Define for all $A \in \mathcal{F}$

$$\mathbb{Q}(A) \;=\; \int_A \zeta_T(\omega)d\mathbb{P}(\omega), \qquad \zeta_T(\omega) = \exp\left(-\int_0^T u_s(\omega)dW_s(\omega) - \tfrac{1}{2}\int_0^T u_s^2(\omega)ds\right) \tag{4}$$

The Girsanov Theorem states that $W + \int u dt$ is indeed a Brownian Motion under $\mathbb{Q}$ if $\mathbb{E}\zeta_T = 1$. A sufficient condition for this to be true is the *Novikov condition* which is stated in equation (45) of [4] as

$$\exp\left(-\tfrac{1}{2}\int_0^T u_s^2 ds\right) \;<\; \infty$$

but which should in fact be

$$\mathbb{E}^{\mathbb{P}}\exp\left(\tfrac{1}{2}\int_0^T u_s^2 ds\right) \;<\; \infty \tag{5}$$

Theorem 2. *The Novikov condition (5) is not satisfied for the model proposed above.*

Proof. See Appendix. ■

The arguments given in the Borland paper are therefore not sufficient to conclude that an equivalent martingale measure $\mathbb{Q}$ exists. Since the Novikov condition is a sufficient but not a necessary condition, this does not automatically imply that such a measure $\mathbb{Q}$ does *not* exist. However, it is easy to see from the proof of Theorem 2 that the tails of the Borland model seem to be too heavy to be of practical use anyway. In fact, under our original measure $\mathbb{P}$ we have that

$$S_t \;=\; S_0 e^{\mu t + \sigma \Omega_t}$$

and since for $t \in]0, T]$

$$\mathbb{E}^{\mathbb{P}} e^{\sigma \Omega_t} \;=\; \frac{1}{Z_t^\alpha} \int_{\mathbb{R}} e^{\sigma w} (1 + \alpha \beta(t) w^2)^{-\frac{1}{\alpha}} dw \;=\; \infty$$

this implies that in Borland's model

$$\mathbb{E}^{\mathbb{P}} S_t \;=\; \infty$$

for all $t \in]0, T]$. This means that the expectations of the asset price process values are not finite under $\mathbb{P}$, which is a serious limitation for practical use.

At the same time this indicates how we can try to change the model to remove this problem, as we will now show in the next section.

3 A Different Option Pricing Model

The tails of the asset price process S can be made less heavy if we use the model (under the original measure $\mathbb{P}$)

$$dS_t \;=\; \mu S_t dt \;+\; \sigma S_t d\Omega_t \tag{6}$$

$$d(\ln S_t) \;=\; (\mu - \tfrac{1}{2}\sigma^2 f^2(t, \Omega_t))dt \;+\; \sigma d\Omega_t \tag{7}$$

instead of the earlier model

$$d(\ln S_t) \;=\; \mu dt \;+\; \sigma d\Omega_t$$

$$dS_t \;=\; (\mu + \tfrac{1}{2}\sigma^2 f^2(t, \Omega_t))S_t dt \;+\; \sigma S_t d\Omega_t$$

The equation (6) is a special case of a class of models proposed in [5], but this special case was assumed to be equivalent to the earlier model, which is not the case.
If the asset price process S defined by (6) exists, then it has a finite expectation at all times: $\mathbb{E}^{\mathbb{P}} S_t = e^{\mu t}$. However, before we can proceed we first have to check whether the stochastic differential equation (1) used as a definition of the process Ω does indeed have a solution. It is by no means clear that this is the case, since standard results on the existence of solutions would assume $f(t, \Omega)$ to be uniformly Lipschitz in its second variable, i.e. $|f(t, x) - f(t, y)| \leq L|x - y|$ for all $t > 0$, with L a constant which does not depend on t. This condition is clearly not satisfied in this case. However, we can still show that this stochastic differential equation admits a strong solution Ω_t on the finite time interval $[0, T]$ (for the proof, see the Appendix).

Theorem 3. *The stochastic differential equation*

$$d\Omega_t \;=\; f(t, \Omega_t)\, dW_t$$

$$\Omega_0 \;=\; 0,$$

admits a strong solution in the sense[2] *that for all $T > 0$ there exists an a.s. continuous stochastic process X such that*

- *The process X is adapted to the filtration $(\mathcal{F}_t^W)_{t \in [0,T]}$ generated by the Brownian Motion.*

[2] The formulation from the book of Karatzas and Shreve [11] has been used.

- $\mathbb{P}(\int_0^T f^2(t, X_t)dt < \infty) = 1$.
- *We have almost surely, for all* $t \in [0, T]$, *that* $X_t = \int_0^t f(u, X_u)dW_u$.

The probability density funcion p of Ω_t, which satisfies

$$\mathbb{P}(\Omega_t \in A) \quad = \quad \int_A p(t, z)dz$$

for all Borel sets A, can therefore indeed be found using the Fokker-Planck equation:

$$\frac{\partial}{\partial t} p(t, z) \quad = \quad \frac{1}{2} \frac{\partial^2}{\partial z^2}[f^2(t, z)p(t, z)] \tag{8}$$

As mentioned before, the solution[3] to this equation is the earlier defined function $\mathcal{P}$, but it is important to stress that the *conditional* probabilities do *not* follow this Tsallis distribution:

Lemma 1. *The distribution of* Ω *is given by a Tsallis-distribution, in particular we have for* $t > 0$ *for all Borel sets* A

$$\mathbb{P}(\Omega_t \in A) \quad = \quad \frac{1}{Z_t^\alpha} \int_A (1 + \beta(t)\alpha z^2)^{-\frac{1}{\alpha}} dz$$

but the conditional distribution of Ω *is* **not** *given by a Tsallis-distribution, i.e. if* $t > s > 0$ *it is not necessarily true for all Borel sets* A *that*

$$\mathbb{P}(\Omega_t \in A \mid \Omega_s = w) \quad = \quad \frac{1}{Z_{t-s}^\alpha} \int_A (1 + \beta(t-s) \cdot \alpha \cdot (z-w)^2)^{-\frac{1}{\alpha}} dz$$

Proof.
The first result follows from substituting $p(t, z) = \mathcal{P}(t, z)$ in (3), and the second result follows from substituting $p(t, z) = \mathcal{P}(t - s, z - w)$ in that equation. Notice that $p(t, z) = \mathcal{P}(t - s, z - w)$ *does* satisfy the equation

$$\frac{\partial}{\partial t} p(t, z) \quad = \quad \frac{1}{2} \frac{\partial^2}{\partial z^2}[f^2(t - s, z - w)p(t, z)]$$

mentioned in [4], but that is not the correct Fokker-Planck equation for the Ω process defined in (1). ■

Notice that from the above we can conclude in particular that

$$\mathbb{E}\Omega_t = 0, \qquad \mathbb{E}\Omega_t^2 = \frac{1}{(2 - 3\alpha)\beta(t)} \sim t^{\frac{2}{2-\alpha}}. \tag{9}$$

We will from now on work with the model defined in (6). In Appendix A of [4] this model is mentioned as well, and it is argued there that both models give the same option prices since their only difference is a drift term which will be removed in the Girsanov transformation. We now know that this is not the case because the violation of the Novikov condition makes the Girsanov transformation itself impossible. Under the model (6), however, the transformation can be carried out, since we would now like to construct a measure $\mathbb{Q}$ under which $W + \int u dt$ is a Brownian Motion, where this time

$$\begin{aligned} u_t \quad &= \quad \frac{\mu - r}{\sigma f(t, \Omega_t)} \quad = \quad \frac{\mu - r}{\sigma} \frac{(Z_t^\alpha)^{-\frac{1}{2}\alpha}}{\xi\sqrt{1 + \alpha\beta(t)\Omega_t^2}} \\ &\leq \quad \frac{\mu - r}{\sigma} \beta(t)^{\frac{\alpha}{4}} \quad \leq \quad \frac{\mu - r}{\sigma} [(1 - \alpha)(2 - \alpha)t]^{-\frac{\alpha}{4-2\alpha}} \quad = \quad C\, t^{-\frac{\alpha}{4-2\alpha}} \end{aligned}$$

[3] Note that in Borland's paper, the constant ξ was taken to be one, but then $\mathcal{P}$ will not satisfy the Fokker-Planck equation. We thank the anonymous referee for pointing this out to us.

where the positive constant C is defined in an obvious way, and therefore

$$\mathbb{E}^{\mathbb{P}} \exp\left(\tfrac{1}{2}\int_0^T u_s^2 ds\right) \quad \leq \quad \exp\left(\tfrac{1}{2}C^2 \int_0^T s^{-\frac{\alpha}{2-\alpha}} ds\right)$$

The integral on the righthand side is convergent around zero for $\alpha \in]0, \frac{1}{2}[$, which shows that the Novikov condition can be met and that therefore the construction of $\mathbb{Q}$ as given in (4) is well-defined. Under this new equivalent measure $W^{\mathbb{Q}} = W + \int u dt$ is a Brownian Motion and therefore we have

$$\begin{aligned} dS_t &= \mu S_t dt \ + \ \sigma S_t d\Omega_t \\ &= \mu S_t dt \ + \ \sigma S_t f(t,\Omega_t) dW_t \\ &= r S_t dt \ + \ \sigma S_t f(t,\Omega_t) d(W_t + \int_0^t u_s ds) \\ &= r S_t dt \ + \ \sigma S_t f(t,\Omega_t) dW_t^{\mathbb{Q}} \\ &= r S_t dt \ + \ \sigma S_t d\Omega_t^{\mathbb{Q}} \end{aligned}$$

where

$$\Omega_t^{\mathbb{Q}} \quad = \quad \int_0^t f(s,\Omega_s) dW_s^{\mathbb{Q}}$$

is not just a local $\mathbb{Q}$-martingale but a $\mathbb{Q}$-martingale, since[4] for all $t \in]0,T]$

$$\tfrac{d}{dt}\mathbb{E}\left\langle \Omega,\Omega\right\rangle_t \quad = \quad \mathbb{E}f^2(t,\Omega_t) \ = \ \xi^2 (Z_t^{\alpha})^{\alpha} \mathbb{E}(1+\beta(t)\alpha\Omega_t^2) \sim \ t^{\frac{\alpha}{2-\alpha}}(1+\beta(t)\alpha\mathbb{E}\Omega_t^2) \ \sim \ t^{\frac{\alpha}{2-\alpha}}$$

so $\mathbb{E}\left\langle \Omega,\Omega\right\rangle_t$ is finite for all $t \in]0,T]$ by our assumption that $\alpha \in]0, \frac{1}{2}[$.

It seems that we now arrive at the same model as in the Borland paper under the risk-neutral measure $\mathbb{Q}$. But there is an important difference. We have that under $\mathbb{P}$

$$S_t \quad = \quad S_0 \exp\left[\mu t - \tfrac{1}{2}\sigma^2 \int_0^t f^2(s,\Omega_s) ds + \sigma\Omega_t\right] \tag{10}$$

In the original model of Borland we had under $\mathbb{P}$

$$S_t \quad = \quad S_0 \exp\left[\mu t + \sigma\Omega_t\right]$$

so we could always write Ω_t as a function of S_t i.e.

$$\Omega_t \quad = \quad \frac{\ln(S_t/S_0) - \mu t}{\sigma}$$

and the process S_t was therefore a Markov process. However, in our corrected model we lose this property, due to the integral in (10). This integral

$$I_t \quad = \quad \int_0^t f^2(s,\Omega_s) ds$$

depends on the whole history of the Ω process up to time t, and cannot be written in terms of the final value Ω_t alone. In the Borland paper it is suggested that this can be done, indeed it is mentioned in equation (71) of that paper that Ω_s equals

$$\sqrt{\frac{\beta(T)}{\beta(s)}}\Omega_T$$

[4]Local martingales with finite quadratic variation processes are martingales, see Protter [12] II.6 coll. 3.

That equality should of course mean *equality in distribution* but Borland then applies this equality as an *almost sure equality* in her equation (72). This is incorrect, and it explains the arbitrage we found in the option formulas derived in the rest of that paper.
In the second paper [5] another approximation is used for the integral I_T, of the form $g_0(T) + g_2(T)\Omega_T^2$ for certain deterministic functions g_0 and g_2, but this still suggests that the entire path integral can be expressed in terms of Ω_T which is not true. Even if the distributions would be close (which they do not seem to be, witnessing the scale of the errors in figure 11 of the paper) then S_T could still have a very different distribution from its approximation. Even if I_T and $g_0(T) + g_2(T)\Omega_T^2$ were close in distribution, this would not necessarily mean that the term which defines the risk-neutral distribution of $\ln S_T$, i.e. $\sigma\Omega_T + I_T$, is close to $\sigma\Omega_T + g_0(T) + g_2(T)\Omega_T^2$, since correlation plays a role there. As shown before, arbitrage is possible in this approximated model, and we expect the arbitrage possibilities to be even more severe for path-dependent options, such as American or barrier-type options.

The integral I_t in (10) represents the quadratic variation process which on the one hand makes sure that the expectation of the stock price process is now finite under both $\mathbb{P}$ and $\mathbb{Q}$ (and hence that conditional expectations and option prices exist) but on the other hand it causes our process S to lose the Markov property. The stochastic processes (S, Ω) do form a Markov process together, but not S alone. In particular, when we want to price an option at a time $t \in [0, T[$, we should not just observe the stock price S_t at that time but also the stochastic variable Ω_t, since it governs the future quadratic variation of S and it cannot be calculated from S directly. This is no problem at time zero (when $\Omega_0 = 0$) but it will be at later times. There can therefore not be a Black-Scholes like formula $C(S_t, t)$ for the option price C in terms of S and t alone, but instead $C = C(t, S_t, \Omega_t)$.

In a sense the model thus resembles a stochastic volatility model since it has an unknown parameter which varies stochastically and the value of which is needed to calculate the price of an option. But there is an important difference too: in stochastic volatility models the stock and the volatility are driven by two Wiener processes, while in this model, there is only one which drives both (which is also the case in the earlier mentioned model by Hobson and Rogers). This is the reason that we can still define a complete and arbitrage free model, even though the quadratic variation processes varies stochastically. And we thus retain the nice feauture of the Borland model that it is a hybrid which lies in between the standard Black-Scholes model and for example the Heston model with stochastic volatility.

However, the downside is that it is not possible to use explicit expressions for option values in terms of S alone, and even when Ω and S are both known, we cannot calculate call option values $C(t, S_t, \Omega_t)$ with strike K in closed form. The discounted asset price process

$$\frac{S_t}{B_t} = \frac{S_0}{B_0} e^{\sigma\Omega_t - \frac{1}{2}\sigma^2 \int_0^t f^2(s,\Omega_s)ds}$$

must be an exponential martingale under the risk-neutral measure $\mathbb{Q}$ and the pricing formula can be written as

$$C(t,s,w) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_t e^{r(T-t)+\sigma(\Omega_T-\Omega_t)-\frac{1}{2}\sigma^2\int_t^T f^2(s,\Omega_s)ds} - K)^+ \,\middle|\, S_t = s,\, \Omega_t = w \right]$$

but we cannot write this in a closed form, due to the presence of the quadratic variation integral in the exponent. Note that the same was true for the formulas in the Borland paper at any time after $t = 0$, because the closed form solution for the European Call is not valid for $t > 0$, even as an approximation. This can be seen from the second part of Lemma 1, which states that distributions at later times (which are conditional distributions given the information at that time) are no longer Tsallis-distributions.

But we can still calculate option prices with Monte Carlo simulation methods, or by using a finite difference implementation based on a partial differential equation, as we will now show.

4 Calculation of Price Functions

We define the operator $\mathcal{L}$ with domain $D_{\mathcal{L}}$, the set of functions $F : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ with function values $F(S, \Omega, t)$ which are continuously differentiable with respect to t and twice continuously differentiable with respect to S an Ω:

$$\mathcal{L}F = \tfrac{1}{2}\sigma^2 S^2 f^2(t,\Omega)\frac{\partial^2 F}{\partial S^2} + \sigma S f^2(t,\Omega)\frac{\partial^2 F}{\partial S \partial \Omega} + \tfrac{1}{2} f^2(t,\Omega)\frac{\partial^2 F}{\partial \Omega^2} + rS\frac{\partial F}{\partial S} - \lambda\frac{\partial F}{\partial \Omega} - rF$$

where $\lambda = \frac{\mu - r}{\sigma}$ represents a market price of risk parameter.

Theorem 4. *If the partial differential equation*

$$\begin{aligned} \frac{\partial F}{\partial t} + \mathcal{L}F &= 0 \\ F(S,\Omega,T) &= \Phi(S), \qquad (\forall \Omega \in \mathbb{R}) \end{aligned}$$

has a unique solution in $D_{\mathcal{L}}$, then the European-style contingent claim paying $\Phi(S_T)$ at time T can be replicated (using a self-financing strategy in the asset and the bank account) after an initial time $t < T$ from an initial investment $F(S_t, \Omega_t, t)$ at time t.

Proof.
Under the martingale measure $\mathbb{Q}$ we have

$$\begin{aligned} dS_t &= rS_t dt \ + \ \sigma S_t f(t,\Omega_t) dW_t^{\mathbb{Q}} \\ d\Omega_t &= f(t,\Omega_t) dW_t \\ &= f(t,\Omega_t) \cdot [dW_t^{\mathbb{Q}} - \tfrac{\mu - r}{\sigma f(t,\Omega_t)} dt] \\ &= f(t,\Omega_t) dW_t^{\mathbb{Q}} - \lambda dt \end{aligned}$$

Let F be a solution as mentioned in the Theorem. Then we have by Ito's rule,

$$\begin{aligned} dF(S_t,\Omega_t,t) &= (rF(S_t,\Omega_t,t) + \frac{\partial F}{\partial t} + \mathcal{L}F)dt \ + \ (\frac{\partial F}{\partial \Omega}(S_t,\Omega_t,t) + \sigma S_t \frac{\partial F}{\partial S}(S_t,\Omega_t,t)) f(t,\Omega_t) dW_t^{\mathbb{Q}} \\ &= rF(S_t,\Omega_t,t)dt \ + \ (\frac{\partial F}{\partial \Omega}(S_t,\Omega_t,t) + \sigma S_t \frac{\partial F}{\partial S}(S_t,\Omega_t,t)) f(t,\Omega_t) dW_t^{\mathbb{Q}} \end{aligned}$$

so if we define

$$\begin{aligned} \phi_t^S &= \frac{\partial F}{\partial S}(S_t,\Omega_t,t) \ + \ \frac{1}{\sigma S_t}\frac{\partial F}{\partial \Omega}(S_t,\Omega_t,t) \\ \phi_t^B &= (F(t,S_t,\Omega_t) - \phi_t^S S_t)/B_t \end{aligned}$$

then we have that

$$\begin{aligned} dF(S_t,\Omega_t,t) &= \phi_t^S dS_t \ + \ \phi_t^B dB_t \\ F(S_t,\Omega_t,t) &= \phi_t^S S_t \ + \ \phi_t^B B_t \end{aligned}$$

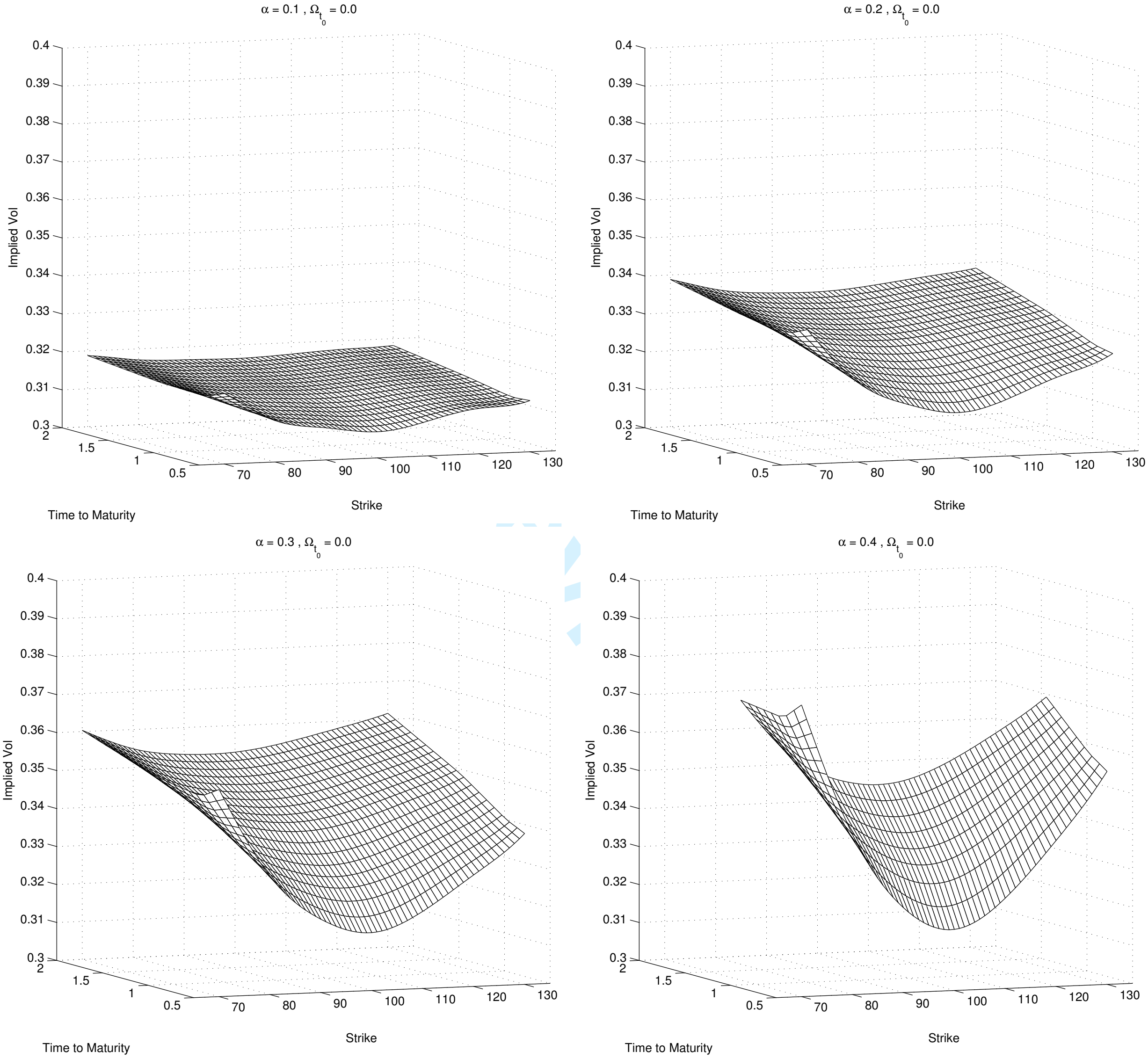
while $F(S_T, \Omega_T, T)$ equals $\Phi(S_T)$, the payoff of the contingent claim. This proves the Theorem. ■

Notice that we have not specified the precise conditions under which a solution to the partial differential equation (with the desired properties) exists. Indeed, the exact conditions which guarantee existence of a classical solution to the Cauchy problem posed here (with nonlinear and time-inhomogeneous terms) will require further study.

It is interesting to see the role played by the market price of risk here. Since the volatility of the underlying asset price process is stochastic, there is a market price of volatility risk. But since the driving noise term of the volatility is the same as the one of the underlying process itself, this market price of risk simply boils down to the market price of risk for the asset process that we find in a standard Black-Scholes model:

$$\lambda \;=\; \frac{\mu - r}{\sigma}.$$

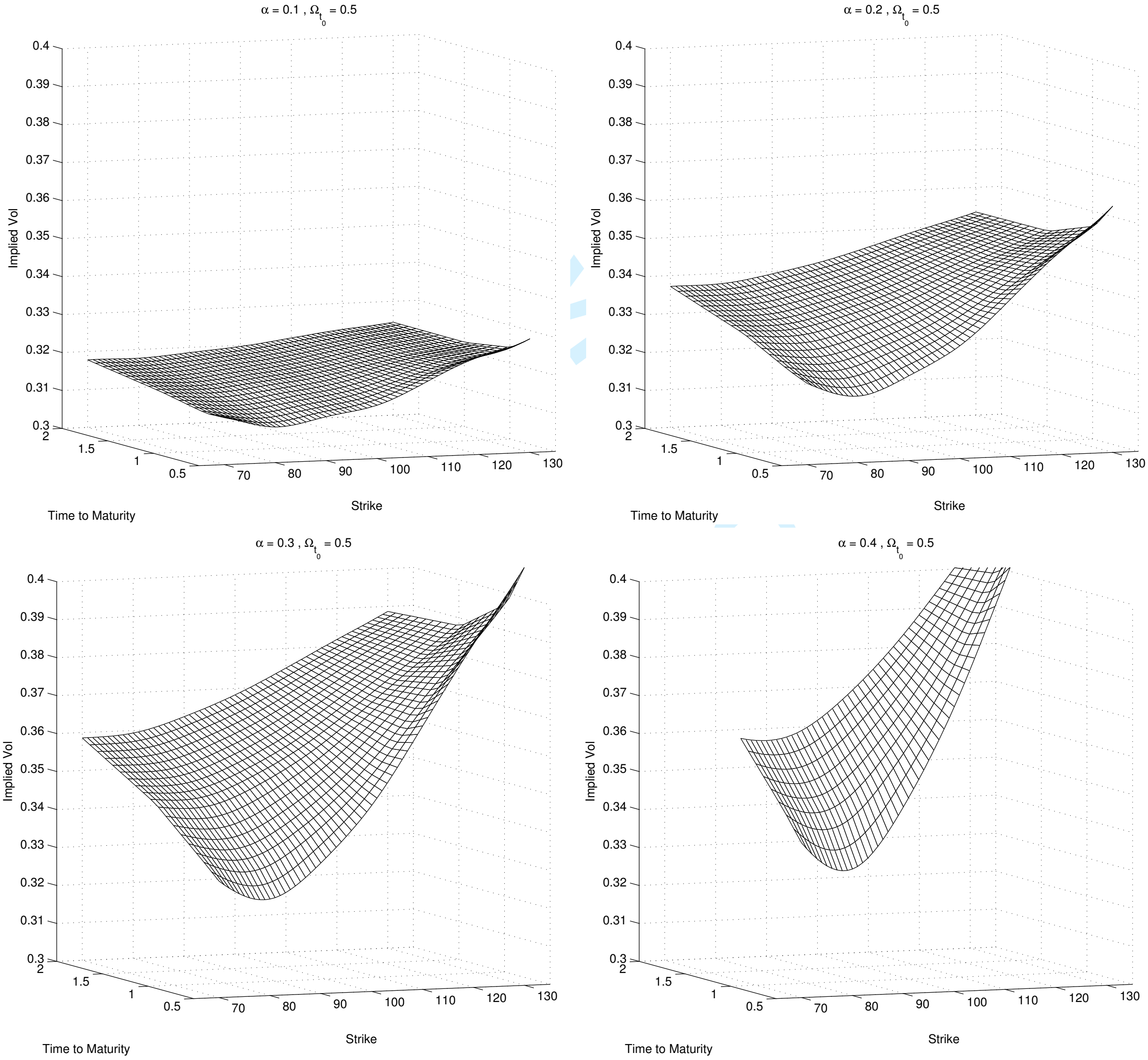
This again illustrates how the Borland model represents a tractable alternative for a full stochastic volatilty model such as Heston's, where there is a second Brownian Motion to drive the volatility process which therefore brings with it a new market price of risk which cannot be determined directly but must be estimated from market data.



5 Numerical Results

In this section we use the partial differential equation of the previous section to calculate option prices for European call options. We use $S_{t_0} = 100$, $r = 3\%$, $\sigma = 30\%$. The starting value for the model t_0 was taken to be 0.2 to show the influence of different values of Ω at that time, and the option characteristics (strike K and maturity T) are varied in the graphs. In all the numerical results shown here we took $\lambda = 0$. We used an explicit finite difference method with a 100×100

grid for the values of S and Ω and 20000 timesteps. The boundary conditions used for the S and Ω variables were a vanishing second derivative of the option price with respect to S and Ω respectively.
The prices we found are shown as implied volatilities in the Black-Scholes model for European options. We have used the values $\alpha \in \{0.10, 0.20, 0.30, 0.40\}$ and $\Omega_{t_0} \in \{0, 0.50\}$ to show the effect of changing these important model parameters. We checked some option prices using Monte Carlo simulations of the risk-neutral asset price process, and found good agreement. Using 500000 simulations with 1000 timesteps per simulation the maximal relative error we found between prices generated by finite differences and by Monte Carlo simulations was 0.3% for the calls we considered. In all Monte Carlo simulations we used Black-Scholes dynamics to define control variates for the payoffs. Monte Carlo calculation times took 3 to 4 times as much CPU time as finite differences. In figures 1 to 4, we have $\Omega_{t_0} = 0$, while in figure 5 to 8, $\Omega_{t_0} = 0.50$. We notice that we have a clear volatility smile, which is more pronounced for shorter maturities. Also notice that if the current value of Ω is not zero, the steepness of the smile increases and it shifts a little bit as well. The fact that we get different curves for different values of Ω shows that it is essential to include this parameter in the process of fitting the model to market data. In fact, this may provide an interesting opportunity for market makers to use a richer class of possible volatility surfaces instead of the single possibility provided when Ω is just taken to be zero.



6 Conclusions and Future Research

It has been shown that the tails in the Borland model for non-Gaussian option pricing are so heavy that conditional expectations, and hence option prices, do not exist in this model. However, we have shown that a different model can be defined which remedies this by making the tails less heavy, and option prices can then be calculated as soon as an additional parameter (the value of Ω_t, which governs the future quadratic variation of the asset price) has been specified. However, we can no longer find closed-form formulas for Europan vanilla options.

We like to stress that the main innovatove idea of the Borland model, i.e. letting the volatility be stochastic but keeping the completeness of the model, is not changed by our modification. But the option pricing formulas we get are very different indeed, as can be seen by comparing the partial differential equations generated by the two models. We believe that the model provides very interesting perspectives for practical applications, and more particularly for improved fitting of option prices. In future work we also hope to investigate the more general class of models given by

$$\begin{aligned} S_t &= S_0 e^{rt+\sigma\Omega_t-\frac{1}{2}\sigma^2\langle\Omega,\Omega\rangle_t} \\ d\Omega_t &= g(t,\Omega_t,S_t)\,dW_t \end{aligned}$$

for suitably chosen functions g.

To determine how well the model presented in this paper can be fitted in practice, further investigation is needed of the relationship between the observed volatility smiles and the parameter Ω which needs to be specified in the modified model, but which was not present in the original one. As we have pointed out, the extra flexibility that this parameter provides could be an advantage in practical fitting problems. The fact that the model seems to generate volatility smiles which steepen as time to maturity decreases is very promising. Obviously, the types of smile and skew patterns that can be generated within this framework (for example by using different functions g in the equation above) should be researched more extensively.

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Appendix: Proofs of Theorems 3 and 2

Proof of Theorem 3.
Since f is not uniformly Lipschitz in its second variable we will use the following bound:

$$\begin{aligned} |f(t,x)-f(t,y)| &= \partial_x f(t,\theta)|x-y|, \qquad \text{for some } \theta\in[\min\{x,y\},\ \max\{x,y\}] \\ &= \xi\,(Z_t^\alpha)^{\frac{1}{2}\alpha}|x-y|\cdot\left|\frac{2\alpha\beta(t)\theta}{2\sqrt{1+\alpha\beta(t)\theta^2}}\right| \\ &\le \xi\,|x-y|\sqrt{\alpha\beta(t)}\left(\frac{A_\alpha}{\sqrt{\beta(t)}}\right)^{\frac{1}{2}\alpha} = |x-y|\cdot[(1-\alpha)(2-\alpha)t]^{-\frac{1}{2}}\sqrt{\alpha} \end{aligned}$$

but since $\frac{\alpha}{(1-\alpha)(2-\alpha)}\le 1$ for all $\alpha\in]0,\frac{1}{2}[$ this gives

$$|f(t,x)-f(t,y)|^2 \le \frac{|x-y|^2}{t} \tag{11}$$

for all $t>0$ and all $x,y\in\mathbb{R}$.
Now define the sequence of adapted processes

$$X_t^0\equiv 0, \qquad X_t^{k+1} = \int_0^t f(s,X_s^k)dW_s$$

and let

$$E_t^k = \mathbb{E}|X_t^{k+1}-X_t^k|^2.$$

Notice that

$$X_t^1 = \xi\int_0^t (Z_s)^{\frac{1}{2}\alpha}dW_s = \tilde{A}_\alpha\int_0^t s^{\frac{\alpha}{4-2\alpha}}dW_s, \qquad \tilde{A}_\alpha=[(1-\alpha)(2-\alpha)]^{\frac{\alpha}{4-2\alpha}}$$

Since

$$\sup_{\alpha\in]0,\frac{1}{2}[}\ \tilde{A}_\alpha\le 2$$

we have that

$$E_t^0 = \mathbb{E}|X_t^1|^2 \le 4\int_0^t s^{\frac{\alpha}{2-\alpha}}ds = 4t^{\frac{2}{2-\alpha}}\frac{2-\alpha}{2} \le 4t^{\frac{4}{3}}$$

which shows that X_t^1 is a well-defined continuous martingale. We will make use of the following Lemma.

Lemma 2. *For all $k\in\mathbb{N}$ and all $t\ge 0$ we have*

$$\begin{aligned} E_t^k = \mathbb{E}|X_t^{k+1}-X_t^k|^2 &\le 4\,t^{\frac{2}{2-\alpha}}(1-\tfrac{\alpha}{2})^k \\ \mathbb{E}|X_t^k|^2 &\le \tfrac{8}{\alpha}\,t^{\frac{2}{2-\alpha}}. \end{aligned}$$

Proof of Lemma.
We have shown that this claim is true for $k=0$. Assuming the claim to be true for a certain $k\in\mathbb{N}$, we calculate (using the bound on f proven before)

$$\begin{aligned} E_t^{k+1} &= \mathbb{E}|X_t^{k+2}-X_t^{k+1}|^2 = \mathbb{E}\left(\int_0^t[f(s,X_s^{k+1})-f(s,X_s^k)]dW_s\right)^2 \\ &= \mathbb{E}\int_0^t\left(f(s,X_s^{k+1})-f(s,X_s^k)\right)^2 ds \qquad (12) \\ &\le \mathbb{E}\int_0^t s^{-1}\left(X_s^{k+1}-X_s^k\right)^2 ds = \int_0^t s^{-1}\mathbb{E}\left(X_s^{k+1}-X_s^k\right)^2 ds \\ &= \int_0^t s^{-1}E_s^k ds \le 4(1-\tfrac{\alpha}{2})^k\int_0^t s^{-1}s^{\frac{2}{2-\alpha}}ds = 4t^{\frac{2}{2-\alpha}}(1-\tfrac{\alpha}{2})^{k+1} \end{aligned}$$

so the first claim of the Lemma is proven by induction. From this result we then conclude that

$$\mathbb{E}|X_t^k|^2 \leq \sum_{m=0}^{k-1} \mathbb{E}|X_t^{m+1} - X_t^m|^2 = \sum_{m=0}^{k-1} E_t^m \leq 4t^{\frac{2}{2-\alpha}} \sum_{m=0}^{k-1} (1-\tfrac{\alpha}{2})^m = \tfrac{8}{\alpha} t^{\frac{2}{2-\alpha}} \qquad (13)$$

and we're done. Note that this result implies that $\mathbb{E}\left[t^{-1}(X_t^k)^2\right] < \infty$ for all $t > 0$. ■

Continuation of Proof of Theorem 3.
For fixed $k \in \mathbb{N}$ we define

$$V_t^k = X_t^{k+2} - X_t^{k+1} = \int_0^t [f(s, X_s^{k+1}) - f(s, X_s^k)] dW_s$$

This process V_t^k is a local martingale and in fact even a martingale since by the Lemma and (12)

$$\begin{aligned} \mathbb{E}\langle V^k \rangle_t &= \mathbb{E}\int_0^t \left(f(s, X_s^{k+1}) - f(s, X_s^k)\right)^2 ds \\ &= E_t^{k+1} \leq 4\, t^{\frac{2}{2-\alpha}} (1-\tfrac{\alpha}{2})^{k+1}, \end{aligned}$$

so V is a local martingale which has finite expected value for its quadratic variation process at all times and hence[5] it is a continuous martingale. We then use the standard martingale inequality which states that

$$\mathbb{E}\left[\max_{s\in[0,T]} |V_s^k|^2\right] \leq 4\mathbb{E}\langle V^k \rangle_T$$

to derive that

$$\mathbb{E}\left[\max_{s\in[0,T]} |X_s^{k+2} - X_s^{k+1}|^2\right] \leq c(1-\tfrac{\alpha}{2})^k$$

for $c = 16T^{\frac{2}{2-\alpha}}$. The Chebyshev inequality then allows us to conclude that

$$\begin{aligned} \mathbb{P}\left(\max_{s\in[0,T]} |X_s^{k+2} - X_s^{k+1}|^2 \geq (1-\tfrac{\alpha}{4})^k\right) &\leq (\tfrac{4}{4-\alpha})^k\, \mathbb{E}\left[\max_{s\in[0,T]} |X_s^{k+2} - X_s^{k+1}|^2\right] \\ &\leq c(\tfrac{4}{4-\alpha})^k (\tfrac{2-\alpha}{2})^k = c(1-\tfrac{\alpha}{4-\alpha})^k \end{aligned}$$

and since the series on the righthand side converges when we sum over all $k \in \mathbb{N}$ we can use the Borell-Cantelli lemma to conclude that for almost all ω there exists an $N(\omega) \in \mathbb{N}$ such that for all $k \geq N(\omega)$ and all $m \in \mathbb{N}^+$

$$\max_{t\in[0,T]} |X_t^{k+m}(\omega) - X_t^{k+1}(\omega)| \leq \frac{(1-\frac{\alpha}{4})^{\frac{1}{2}k}}{1-\sqrt{1-\frac{\alpha}{4}}}.$$

This shows that the sequence $\{X_t^k(\omega),\, t \in [0,T]\}_{k\in\mathbb{N}}$ of continuous paths convergences uniformly in the sup-norm and thus has a limit $X_t(\omega) = \lim_{k\to\infty} X_t^k(\omega)$ which is a.s. continuous itself. Clearly this limiting process is adapted, almost surely equal to zero for $t = 0$, and the requirement $\mathbb{P}(\int_0^T f^2(t, X_t)dt < \infty) = 1$ follows from equations (11) and (13) and dominated convergence. To prove the last requirement we let k go to infinity in the equation

$$X_t^{k+1} = \int_0^t f(s, X_s^k) dW_s$$

[5] See for example the book by Protter [12], II.6 coll. 3.

The lefthand side converges to X_t while the righthand side converges as well since

$$\begin{aligned}\mathbb{E}\left|\int_0^t [f(s,X_s^k)-f(s,X_s)]dW_s\right|^2 &= \mathbb{E}\int_0^t \left|f(s,X_s^k)-f(s,X_s)\right|^2 ds \\ &\le \int_0^t s^{-1}\mathbb{E}\left|X_s^k - X_s\right|^2 ds.\end{aligned}$$

This last expression goes to zero for $k\to\infty$ by dominated convergence, since $\mathbb{E}|X_t^k|^2 \le \frac{8}{\alpha}T^{\frac{2}{2-\alpha}} < \infty$ by the Lemma, which implies $\mathbb{E}|X_t|^2 < \infty$ by dominated convergence as well. ■

Proof of Theorem 2.
Since $\mu - r$, σ and the values of the function f are all strictly positive we have that

$$\begin{aligned}\tfrac{1}{2}u_t^2 &\ge \tfrac{1}{2}(\tfrac{1}{2}\sigma f(t,\Omega_t))^2 \\ &\ge \tfrac{1}{8}\sigma^2(1+\beta(t)\alpha\Omega_t^2)\beta(t)^{-\frac{1}{2}\alpha} \\ &\ge \tfrac{1}{8}\sigma^2\beta(T)^{1-\frac{1}{2}\alpha}\alpha\Omega_t^2 \\ &\ge \eta\Omega_t^2\end{aligned}$$

where $\eta = \frac{1}{8}\sigma^2\beta(T)^{1-\frac{1}{2}\alpha}\alpha$ is a positive constant which does not depend on t. Take a $t_1 \in]0,T[$ and define for $t\in[t_1,T]$

$$\begin{aligned}M_t &= \Omega_t - \Omega_{t_1} \\ H_t &= \mathbb{E}^{\mathbb{P}}[M_t^2 \mid \mathcal{F}_{t_1}]\end{aligned}$$

Using elementary properties of the Wiener integral and Fubini's Theorem we now calculate for $t_1 \le t \le t_2$

$$\begin{aligned}H_t &= \mathbb{E}^{\mathbb{P}}[M_t^2 \mid \mathcal{F}_{t_1}] = \mathbb{E}^{\mathbb{P}}[\left(\int_{t_1}^t f(s,\Omega_s)dW_s\right)^2 \mid \mathcal{F}_{t_1}] \\ &= \mathbb{E}^{\mathbb{P}}[\int_{t_1}^t f^2(s,\Omega_s)ds \mid \mathcal{F}_{t_1}] = \mathbb{E}^{\mathbb{P}}[\int_{t_1}^t \xi^2(Z_s^\alpha)^\alpha(1+\beta(s)\alpha\Omega_s^2)ds \mid \mathcal{F}_{t_1}] \\ &= \xi^2\int_{t_1}^t (Z_s^\alpha)^\alpha\mathbb{E}^{\mathbb{P}}[(1+\beta(s)\alpha\Omega_s^2)\mid\mathcal{F}_{t_1}]ds = \xi^2\int_{t_1}^t (Z_s^\alpha)^\alpha(1+\alpha\beta(s)H_s+\alpha\beta(s)\Omega_{t_1}^2)ds\end{aligned}$$

Let $K = \xi^2(Z_T^\alpha)^\alpha\max\{1,\alpha\beta(t_1)\}$, then

$$\begin{aligned}H_{t_1} &= 0 \\ \frac{d}{dt}H_t &= \xi^2(Z_t^\alpha)^\alpha(1+\alpha\beta(t)\Omega_{t_1}^2+\alpha\beta(t)H_t) \\ &\le K(1+\Omega_{t_1}^2+H_t)\end{aligned}$$

and Gronwall's Lemma then gives that

$$H_t \le (e^{K(t-t_1)}-1)(1+\Omega_{t_1}^2), \qquad t\in[t_1,T]$$

Now take $t_2 \in]t_1,T]$ such that $e^{K(t_2-t_1)}-1 \le \frac{1}{32}$, then

$$H_{t_2} \le \tfrac{1}{32}(1+\Omega_{t_1}^2) \tag{14}$$

Let

$$X = \sup_{t_1\le s\le t_2} M_s^2$$

and let m be any real number larger than one. The set

$$C = \{\omega : \Omega_{t_1}(\omega) \geq 2m,\, X(\omega) \leq m^2\}$$

is a subset of the set $\{\omega : \inf_{t_1 \leq s \leq t_2} \Omega_t^2(\omega) \geq m\}$, so on C we have that $\frac{1}{2}u_t^2 \geq \frac{1}{2}\eta m^2$. We then bound the expectation in the Novikov condition as follows

$$\begin{aligned} L &= \mathbb{E}^{\mathbb{P}} \exp\left(\tfrac{1}{2}\int_0^T u_s^2 ds\right) \geq \mathbb{E}^{\mathbb{P}} \exp\left(\tfrac{1}{2}\int_{t_1}^{t_2} u_s^2 ds\right) \\ &\geq \mathbb{E}^{\mathbb{P}} \mathbf{1}_{\{C\}} \exp\left(\tfrac{1}{2}\int_{t_1}^{t_2} u_s^2 ds\right) \\ &\geq e^{\frac{1}{2}\eta m^2 (t_2 - t_1)} \mathbb{E}^{\mathbb{P}} \mathbf{1}_{\{C\}} \\ &= e^{\frac{1}{2}\eta m^2 (t_2 - t_1)} \mathbb{E}^{\mathbb{P}} \left(\mathbb{E}^{\mathbb{P}}[\, \mathbf{1}_{\{\Omega_{t_1} \geq 2m\}} \mathbf{1}_{\{X \leq m^2\}} \mid \mathcal{F}_{t_1}]\right) \\ &= e^{\frac{1}{2}\eta m^2 (t_2 - t_1)} \mathbb{E}^{\mathbb{P}} \left(\mathbf{1}_{\{\Omega_{t_1} \geq 2m\}}\, \mathbb{E}^{\mathbb{P}}[\, \mathbf{1}_{\{X \leq m^2\}} \mid \mathcal{F}_{t_1}]\right) \end{aligned}$$

But

$$\mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_{t_1}] = \mathbb{E}^{\mathbb{P}}[\sup_{t_1 \leq s \leq t_2} M_s^2 \mid \mathcal{F}_{t_1}] \leq 4\mathbb{E}^{\mathbb{P}}[M_{t_2}^2 \mid \mathcal{F}_{t_1}] = 4H_{t_2} \leq \tfrac{1}{8}(1 + \Omega_{t_1}^2)$$

where we have used a standard martingale inequality and (14). But then

$$\begin{aligned} m^2 \mathbb{E}^{\mathbb{P}}[\, \mathbf{1}_{\{X \geq m^2\}} \mid \mathcal{F}_{t_1}] &\leq \mathbb{E}^{\mathbb{P}}[X \mid \mathcal{F}_{t_1}] \leq \tfrac{1}{8}(1 + \Omega_{t_1}^2) \\ \Rightarrow \quad \mathbb{E}^{\mathbb{P}}[\, \mathbf{1}_{\{X \leq m^2\}} \mid \mathcal{F}_{t_1}] &\geq 1 - \tfrac{1}{m^2}\tfrac{1}{8}(1 + \Omega_{t_1}^2) \end{aligned}$$

so

$$\begin{aligned} L &\geq e^{\frac{1}{2}\eta m^2 (t_2 - t_1)}\, \mathbb{E}^{\mathbb{P}} \left(\mathbf{1}_{\{\Omega_{t_1} \geq 2m\}} (1 - \tfrac{1}{m^2}\tfrac{1}{8}(1 + \Omega_{t_1}^2))\right) \\ &= e^{\frac{1}{2}\eta m^2 (t_2 - t_1)}\, (\tfrac{1}{2} - \tfrac{1}{8m^2})\, \mathbb{P}(\Omega_{t_1} \geq 2m) \\ &= e^{\frac{1}{2}\eta m^2 (t_2 - t_1)}\, (\tfrac{1}{2} - \tfrac{1}{8m^2}) \int_{2m}^{\infty} (1 + 4\alpha\beta(t_1)u^2)^{-\frac{1}{\alpha}} du / Z_{t_1}^{\alpha} \end{aligned}$$

Letting m go to infinity now proves the result. ■