Physics 504, Lecture 18
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Darwin and Proca Lagrangians

1 Darwin Lagrangian

Newtonian classical mechanics is based on the idea that motion is described by the positions $\vec{x}_j$ of particles at a given time t, and these evolve according to forces given by the positions of all other particles $\vec{x}_k(t)$ at that instant. When these forces are conservative we may specify the forces by potential energies which are again functions of the several positions $\vec{x}_j(t)$ at that instant. This notion violates the constraint of special relativity that information cannot travel faster than light, so there is no way the particle at $\vec{x}_j(t)$ can feel a force that depends on where particle k is at this instant, unless $\vec{x}_j(t) = \vec{x}_k(t)$. So the only potential consistent with relativity is a delta function!

Nonetheless we know that Lagrangians with potential energies are a very effective way of describing physics if the relevant velocities remain small compared to c.

Let us see what we can do for charged particles interacting electromagnetically. We learned as freshmen how to do the lowest order ($c \to \infty$): $V(\vec{x}_j, \vec{x}_k) = q_j q_k/|\vec{x}_j - \vec{x}_k|$ and $T = \frac{1}{2}\sum m_j \vec{v}_j^2$. This encapsulates the effect of the electric field produced by one charge on the motion of the other. In our relativistic treatment the interaction lagrangian (of charges with *the fields*) $L_{\rm int} = \sum_j q_j \left(-\Phi(\vec{x}_j) + \frac{1}{c}\vec{u}_j \cdot \vec{A}(\vec{x}_j)\right)$, and $\Phi(\vec{x}_j) = \sum_k q_k/|\vec{x}_j - \vec{x}_k|$ is the $c \to \infty$ limit for the scalar potential. Magnetic forces are only produced to next order in v/c, and these produce forces only proportional to the velocity of the second particle, so these only enter to order v^2/c^2. At that order, the expressions for Φ and $\vec{A}$ will depend on the choice of gauge, and it is useful here to use not the Lorenz gauge but the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, because in that gauge[1] $\nabla^2\Phi = -4\pi\rho$, and $\Phi(\vec{r}, t) = \int d^3r' \rho(\vec{r}\,', t)/|\vec{r} - \vec{r}\,'|$ really is instantaneous. From $\partial_\sigma F^{\sigma j} = 4\pi J^j/c$ we have

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} + \vec{\nabla}\left(\frac{1}{c}\frac{\partial}{\partial t}\Phi + \vec{\nabla}\cdot\vec{A}\right) = 4\pi\vec{J}/c.$$

[1] Gaussian units.

The $\vec{\nabla}\cdot\vec{A}$ is zero. Working accurate to order $(v/c)^2$ we may drop the $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}$ term, as $\vec{A}$ is already order $(v/c)^1$. Thus we may take

$$\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J} + \frac{1}{c}\vec{\nabla}\frac{\partial}{\partial t}\Phi.$$

The contribution of particle j to $\vec{J}(\vec{x}\,')$ is $q_j\vec{v}_j\delta^3(\vec{x}\,' - \vec{x}_j)$. It's contribution to $\Phi(\vec{x}\,')$ is $q_j/|\vec{x}\,' - \vec{x}\,'_j|$, so it contributes $q_j\vec{v}_j \cdot (\vec{x}\,' - \vec{x}_j)/|\vec{x}\,' - \vec{x}_j|^3$ to $\partial\Phi/\partial t$. As the Green's function for Laplace's equation is $1/|\vec{x} - \vec{x}\,'|$, we have

$$\begin{aligned}
\vec{A}(\vec{x}) &= \int \frac{d^3x'}{|\vec{x} - \vec{x}\,'|}\left(\frac{1}{c}\vec{J}(\vec{x}\,') - \frac{1}{4\pi c}\vec{\nabla}'\frac{\partial}{\partial t}\Phi(\vec{x}\,')\right) \\
&= \int \frac{d^3x'}{|\vec{x} - \vec{x}\,'|}\left(\frac{q_j v_j}{c}\delta^3(\vec{x}\,' - \vec{x}_j) - \frac{q_j}{4\pi c}\vec{\nabla}'\left(\frac{\vec{v}_j \cdot (\vec{x}\,' - \vec{x}_j)}{|\vec{x}\,' - \vec{x}_j|^3}\right)\right) \\
&= \frac{q_j\vec{v}_j}{c|\vec{x} - \vec{x}_j|} + \frac{q_j}{4\pi c}\int d^3x'\left(\frac{\vec{v}_j \cdot (\vec{x}\,' - \vec{x}_j)}{|\vec{x}\,' - \vec{x}_j|^3}\right)\vec{\nabla}'\frac{1}{|\vec{x} - \vec{x}\,'|}
\end{aligned}$$

where we have integrated by parts and thrown away the surface at infinity. The gradient $\vec{\nabla}' \sim -\vec{\nabla}$ action on a function of $\vec{x} - \vec{x}\,'$, so we can pull $\vec{\nabla}$ out of the integral. Let $\vec{r} = \vec{x} - \vec{x}_j$ and $\vec{y} = \vec{x}\,' - \vec{x}_j$. Then

$$\begin{aligned}
\vec{A}(\vec{x}) &= \frac{q_j\vec{v}_j}{c|\vec{r}|} - \frac{q_j}{4\pi c}\vec{\nabla}\int d^3y \frac{\vec{v}_j \cdot \vec{y}}{|\vec{y}|^3}\frac{1}{|\vec{y} - \vec{r}|} \\
&= \frac{q_j\vec{v}_j}{c|\vec{r}|} - \frac{q_j}{4\pi c}\vec{\nabla}\int_0^\infty y^2 dy \int_0^\pi d\theta \sin\theta \\
&\qquad \int_0^{2\pi} d\phi \frac{y(\cos\theta v_{jz} + \sin\theta\cos\phi v_{jx})}{y^3}\frac{1}{\sqrt{y^2 + r^2 - 2yr\cos\theta}}
\end{aligned}$$

where we have chosen z in the $\vec{r}$ direction and $\vec{v}_j$ in the xz plane. The ϕ integral kills the v_{jx} term and, writing $v_{jz} = \vec{v}_j \cdot \vec{r}/r$, we have

$$\vec{A}(\vec{r}) = \frac{q_j}{c}\left[\frac{\vec{v}_j}{|\vec{r}|} - \frac{1}{2}\vec{\nabla}\left(\frac{\vec{v}_j \cdot \vec{r}}{r}\right)C\right],$$

where the integral giving C is

$$C = \int_0^\infty dy \int_{-1}^1 du \frac{u}{\sqrt{y^2 + r^2 - 2yru}} = 1,$$

though this integral is not as straightforward as Jackson claims. Then

$$\vec{A}_j(\vec{x}_k) = \frac{q_j}{2c|\vec{x}_j - \vec{x}_k|}\left[\vec{v}_j + \frac{(\vec{x}_k - \vec{x}_j)\vec{v}_j \cdot (\vec{x}_k - \vec{x}_j)}{|\vec{x}_k - \vec{x}_j|}\right].$$

Multiplying by $q_k\vec{v}_k/c$ to get the appropriate contribution to L_{int}, and correcting the free-particle Lagrangian, $-mc^2\gamma^{-1} + mc^2 \approx \frac{1}{2}mv^2 + \frac{1}{8}mv^4/c^2$, we get the Darwin Lagrangian

$$\begin{aligned} L_{\text{Darwin}} &= \frac{1}{2}\sum_j m_j v_j^2 + \frac{1}{8c^2}\sum_j m_j v_j^4 \\ &\quad + \sum_{j\neq k} \frac{q_j q_k}{r_{jk}}\left(-\frac{1}{2} + \frac{1}{4c^2}\left[\vec{v}_j \cdot \vec{v}_k + (\vec{v}_j \cdot \hat{r}_{jk})(\vec{v}_k \cdot \hat{r}_{jk})\right]\right), \end{aligned}$$

where of course $\vec{r}_{jk} := \vec{x}_j - \vec{x}_k$, $r_{jk} := |\vec{r}_{jk}|$, and $\hat{r}_{jk} = \vec{r}_{jk}/r_{jk}$.

We mostly experience slightly relativistic particles in atomic physics, though the electrons are best described by the Dirac formalism, so the velocities are replaced by $\vec{\alpha}$. It is also of use in plasma physics.

2 Proca Lagrangian

As we saw, our lagrangian density for electromagetic fields,

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - \frac{1}{c}J_\mu A^\mu,$$

gives equations of motion which do not completely determine the evolution of the fields A^μ. Let us consider adding a term proportional to A^2:

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \frac{\mu^2}{8\pi}A_\mu A^\mu - \frac{1}{c}J_\mu A^\mu,$$

known as the Proca Lagrangian, which as we shall see describes a field which has quanta of mass μ rather than the massless photons whose classical limit is Maxwell theory. We still mean $F_{\mu\nu}$ to be shorthand for $\partial_\mu A_\nu - \partial_\nu A_\mu$ rather than an independent field, so the homogeneous Maxwell's equations still hold, as they are consequences of $\mathbf{F} = \mathbf{d}A$. The extra term does not contribute to ${P_\alpha}^\mu$, as it depends only on A and not on derivatives thereof, so the extra contribution to the equations of motion is just from $\partial\mathcal{L}/\partial A^\mu = (\mu^2/4\pi)A_\mu$, and

$$\partial^\beta F_{\beta\alpha} + \mu^2 A_\alpha = \frac{4\pi}{c}J_\alpha.$$

One consequence comes from taking the 4-divergence of this equation:

$$\partial^\alpha\partial^\beta F_{\beta\alpha} + \mu^2\partial^\alpha A_\alpha = \frac{4\pi}{c}\partial^\alpha J_\alpha.$$

The first term is identically zero by symmetry, and if the current density still represents a conserved charge, the right hand side is also zero, so the Lorenz condition $\partial^\alpha A_\alpha = 0$ is now a consequence of the equations of motion and not an arbitrary choice. As a consequence, we now have $\partial^\beta F_{\beta\alpha} = \Box A_\alpha$, so

$$\left(\Box + \mu^2\right)A_\alpha = \frac{4\pi}{c}J_\alpha.$$

In the absence of sources, this has solutions as before,

$$\sum_{\vec{k}}\left(A^\mu_{\vec{k}+}e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}}t} + A^\mu_{\vec{k}-}e^{i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}}t}\right),$$

but with $\omega^2 = c^2(\vec{k}^2 + \mu^2)$. Quantum mechanically we know $\vec{p} = -i\hbar\vec{\nabla} \sim \hbar\vec{k}$ and $E = i\hbar\partial/\partial t = \pm\hbar\omega$, so this field represents particles for which $E^2 = P^2c^2 + \mu^2\hbar^2c^2$. Of course quantum field theoriests take $\hbar = 1$ and $c = 1$, so this represents a massive photon with mass μ.

If we consider a point charge at rest and look for the static field it would generate, we need to solve

$$\nabla^2\Phi + \mu^2\Phi = -4\pi q\delta^3(\vec{r})$$

or

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + r^2\mu^2\Phi = -q\delta(r).$$

Away from $r = 0$ this clearly requires $r\Phi(r) = Ce^{-\mu r}$ and Gauss's law tells us

$$-4\pi q = 4\pi R^2\, d\Phi/dr|_R + \mu^2\int_{r<R}\Phi \underset{R\to 0}{\longrightarrow} 4\pi C,$$

so $C = q$ and

$$\Phi(\vec{x}) = q\frac{e^{-\mu r}}{r}, \qquad \text{with} \quad r = |\vec{x}|.$$

This is the well-known Yukawa potential, which nuclear physicists had found was a good fit to the binding of nucleons in a nucleus, leading Yukawa to propose the existance of a massive carrier of the nuclear force, which we now know to be the π meson.

2.1 Superconductors

In the BCS theory of superconductivity, electrons form pairs, and each pair acts like a boson. So the quantum mechanical state that each pair is in can be multiply occupied, and superconductivity occurs when states develop macroscopic occupation numbers, $\gg 1$. The wave function $\psi(\vec{x})$ describing these particles is a complex function, with the density of particles $n(\vec{x}) = \psi^*\psi$, so $\psi = n(\vec{x})e^{i\theta(\vec{x})}$. We may approximate $n(\vec{x})$ as being roughly constant.

The velocity of these particles is related to the canonical momentum by

$$\vec{v} = \frac{1}{m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)$$

which can be viewed as an operator acting between ψ^* and ψ. It is the *canonical momentum* $\vec{P}$ which acts like $-i\hbar\vec{\nabla}$. Thus the current density is

$$\vec{J} = q\psi^*\vec{v}\psi = \frac{nq}{m}\left(\hbar\nabla\theta - \frac{q}{c}\vec{A}\right).$$

If we take the curl of both sides, we get

$$\vec{\nabla}\times\vec{J} = -\frac{nq^2}{mc}\vec{\nabla}\times\vec{A} = -\frac{nq^2}{mc}\vec{B}, \tag{1}$$

as $\vec{\nabla}\times\vec{\nabla}\theta = 0$. This equation doesn't quite say

$$\vec{J} = -\frac{nq^2}{mc}\vec{A}, \tag{2}$$

but it does say, in a simply connected region, that the difference is the gradient of something, and as such a gradient could be added to $\vec{A}$ by a gauge transformation, we might as well assume (2), which is known as the London equation. This gauge is still compatible with Lorenz (which can be viewed as determining A^0), so we have

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J} = \frac{4\pi nq^2}{mc^2}\vec{A},$$

which is the Proca equation with $\mu^2 = 4\pi nq^2/\,mc^2$.

At the boundary of the superconductor, if no current is crossing the boundary, we must have $\vec{n}\cdot\vec{A} = 0$. If we look for a static solution for a

planar boundary $\perp z$, uniform along the boundary, we have $A \propto e^{-\mu z}$. The London penetration depth is

$$\lambda_{\rm L} := \frac{1}{\mu} = \sqrt{\frac{mc^2}{4\pi nq^2}}.$$

With $q = -2e$ and $m = 2m_e$ for the electron pair, and taking n as the density of valence electrons, the penetration depth is of the order of tens of nanometers. As the A field is not penetrating further than that into the medium, any external magnetic field has been excluded.

But magnetic field lines can enter the medium if our assumption of being able to do away with $\vec{\nabla}\cdot\vec{A}$ by a gauge transformation is not correct. That could happen if the region of the superconductor is not simply connected — that is, a flux line could enter and destroy the superconducting region around which θ is incremented by a multiple of 2π. This is called a vortex line, and corresponds to a quantized amount of flux, as

$$\oint \vec{A}\cdot d\ell = 2\pi N\hbar c/q = \int_S \vec{\nabla}\times\vec{A} = \Phi_B,$$

with $N \in \mathbb{Z}$. With $q = -2e$, the quantum of flux is $hc/2e$.