Rational Erdős numbers

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The concept of Erdős number is well known among mathematicians and somewhat known more generally (see, for example, [de Castro & Grossman, 1999] or the web site http://www.oakland.edu/ grossman/erdoshp.html). Paul Erdős, who died in late 1996, was a mathematician who was author or coauthor—most often the latter—of nearly 1500 papers and had nearly 500 coauthors. By definition, Erdős had Erdős number zero, his collaborators had Erdős number one, a person who not collaborated with Erdős, but had collaborated with one of his collaborators has Erdős number two and so on. For example, I have written a joint paper with Michael Makkai, who has a joint paper with Erdős and so Makkai has Erdős number one and mine is two.

But there is something not entirely rational about this assignment of numbers. Surely a person who has written, say, five joint papers with Erdős is more closely associated with Erdős than someone who has written only one and therefore ought to have a smaller Erdős number. In fact, it seems entirely reasonable that such a person be assigned the Erdős number 1/5. If a person has written two joint papers with someone whose Erdős number is 1, then his Erdős number ought to be 3/2. If you have written two joint papers with someone who has written three joint papers with Erdős, then your Erdős number should be 5/6 and so on. But what if you have written two joint papers with one Erdős coauthor and three with someone whose Erdős number is 2? This leads to an obvious iteration procedure for calculating these new, rational, Erdős numbers. The procedure must converge, since each iteration can only lower your Erdős number while it must always remain positive.

Two questions remain. First could the iteration converge to 0? Possibly even for everyone who has an Erdős number. This would be a very uninteresting result. The second question is simply this: Is the rational Erdős number really rational?

In this note, we explore one possible approach to these questions. What is interesting is that a bit of physical reasoning is used to show that the procedure converges. More precisely, we use a mathematical idealization of a physical situation.

As a first approximation, we will suppose that there are no papers involving three or more authors. Later, we will consider one possible way of dealing with that eventuality.

To explain the procedure, it is helpful to define a new arithmetic operation, I will call the harmonic sum.

$$x \oplus y = \frac{1}{\frac{1}{x} + \frac{1}{y}} = \frac{xy}{x+y}$$

This is defined at least on positive real numbers, which is all we are interested in. It is not hard to show that this is a commutative, associative operation. The reason for the name is that if x and y represent distances to the bridge on a fret board, then the frequency represented by $x \oplus y$ is the sum of the frequencies represented by x and y. The harmonic mean is so named for the same reason.

Now we describe an iterative procedure that, if it converges, defines a rational number that has the properties we described. For each mathematician M form the harmonic sum, over all collaborators, of 1+ the Erdős numbers of the collaborators. However, this procedure is also carried out for the collaborators and so a second round is necessary to replace the Erdős numbers of the collaborators by their new Erdős numbers. This process can be iterated indefinitely. Does it converge to a positive rational number? Here is a beautiful physical argument that shows it does.

Build a resistive network that has the world's mathematician as nodes and a 1 ohm resistor between two mathematicians for every joint paper between them. Then we know that there is a well-defined resistance between any two nodes. In particular, the resistance between any non-Erdős node and Erdős takes on a definite positive value, which is the rational Erdős number we seek. Moreover, the solutions are solutions to a set of simultaneous linear equations given by Kirchhoff's laws (see, for example, [Bollobás, 1997]), whose coefficients are the resistances. Since they are rational (all 1, in fact), the solutions are all rational too. Of course in the real world, these equations are only a very good approximation, but the solution they give to the purely mathematical question is exact.

There remains the question of whether this computation is feasible. The Kirchhoff equations lead to an $n \times n$ matrix where n is the number of mathematicians who have an Erdős number—over 200,000. On the other hand, it is fairly sparse, so inversion might not be unfeasible. This would also give the distance, in the collaborator metric described here, between any two mathematicians. (Actually, that would require extending it to all mathematicians who have ever published a paper. Or at least those reviewed in Math Reviews, which is not completely comprhensive—at least one very important paper of mine was never reviewed. At any rate that number is well over 300,000, a staggering thought.)

It remains to deal with many author papers. One problem is that the nature of the collaboration changes. In the case of two authors, the authors collaborate with each other. With more than two authors, the connection between any two of the authors ranges between the same as in a two author paper to none at all, since it might consist of two of the authors collaborating with a third. A procedure described here can obviously not take all these possibilities into account. I therefore offer the following suggestion. Build the same resistor network as before for the two author network. For an n-author paper, $n > 2$, add a new node and a resistor of $n/4$ ohm between that node and each of the authors. If this were also done for two author papers, it would have the effect of putting two $1/2$ ohm resistors in series between the authors, which is equivalent to a 1 ohm resistor. In a three author paper, this puts a $3/2$ ohm resistance between any two of them, and so on. The result is still a resistive network with rational resistances and the resistance between any two points will be a positive rational number, as before.

References

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