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# Transformation Properties of a Class of Variable Coefficient Boiti–Leon–Manna–Pempinelli Equations

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**Abstract:** We derive the enhanced Lie group classification for a general class of variable coefficient Boiti–Leon–Manna–Pempinelli equations. This task is achieved with the use of the equivalence group admitted by the class. Using the admitted equivalence group, we transform the general class into a much simpler class of equations. Additionally, examples of non-Lie reduction operators are presented.

**Keywords:** Boiti–Leon–Manna–Pempinelli equations; equivalence group; Lie group classification; non-Lie reduction operators

**MSC:** 35A30; 35K55; 58J70

## 1. Introduction

The parameters that appear in physical models may change in time and, therefore, the coefficients in partial differential equations can be functions of time. Usually, such equations describe physical phenomena with more accuracy. In recent decades, many variable coefficient equations appear in the literature that are studied from various points of view. Obviously, the study of these equations is more difficult than the study of the corresponding equations with constant coefficients. Before considering a variable coefficient equation, it will be useful if we can find a similar simpler equation. We call two differential equations similar if they are connected by a point transformation. In fact, many variable coefficient equations that appear in the literature are similar to constant coefficient equations and, in most cases, are similar to simpler equations.

The task of the simplification of variable coefficient equations can be achieved in most cases with the employment of the equivalence transformations. These are non-degenerate point transformations that preserve the differential structure of the equation, might change only the coefficient functions (arbitrary elements), and also form a group. There exist four kinds of equivalence groups. The simplest is the usual equivalence group where the point transformations of the dependent and independent variables do not depend on arbitrary elements [1]. If the transformations of the dependent and independent variables depend on arbitrary elements, then it is called generalized equivalence group [2,3]. The extended equivalence group consists of transformations that include nonlocalities with respect to arbitrary elements [4]. The generalized extended equivalence group has the properties of both generalized and extended equivalence groups.

In the present work, we consider the variable coefficient nonlinear partial differential equation

$$a(t)u_{tx} + b(t)u_{ty} + c(t)u_{xy} + d(t)u_{yy} + k(u_x u_y)_x + u_{xxxy} = 0, \quad (1)$$

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where $a(t),\ b(t),\ c(t),\ d(t)$ are smooth functions and $k$ is an arbitrary constant. This class of equations appeared recently in the literature [5–7] as a generalization of the Boiti–Leon–Manna–Pempinelli equation

$$u_{ty} - 3u_x u_{xy} - 3u_y u_{xx} + u_{xxxy} = 0,$$

which was introduced as a generalization of a two-dimensional KdV equation [8]. In [5], the problem of Lie group classification for (1) was considered; in [6], wave-type solutions were derived; and in [7] Bäcklund transformations were constructed. We use equivalence groups to simplify Equation (1). We show that it is similar to an equation with only one variable coefficient. Initially, we perform the Lie group classification for the simplified equation and, using the equivalence transformations, we obtain the corresponding results for the general class (1). The Lie group classification of class (1) that has been derived here completes the results in [5]. Furthermore, we present a number of non-Lie reduction operators (nonclassical symmetries) for the simplified equation. Finally, some remarks are pointed out for the original Boiti–Leon–Manna–Pempinelli equation.

## 2. Equivalence Transformations and Their Applications

We derive the equivalence transformations of class (1) with the ultimate goal to derive a mapping that transforms it into a simpler form. The usual equivalence group $G$ consists of non-degenerate point transformations in the space $(t, x, y, u, a, b, c, d, k)$ that are projected on the space of $(t, x, y, u)$. That is, they have the form

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (T^t, T^x, T^y, T^u)(t, x, y, u),$$

$$(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{k}) = (T^a, T^b, T^c, T^d, T^k)(t, x, y, u, a, b, c, d, k)$$

and map any differential equation from class (1) with dependent function $u(t, x, y)$ and arbitrary elements $(a, b, c, d, k)$ to a differential equation from the same class with dependent variable $\tilde{u}(\tilde{t}, \tilde{x}, \tilde{y})$ and arbitrary elements $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{k})$.

In the following two theorems, we present the equivalence transformations for class (1). We state that equivalence transformations have been derived using the direct method [9]. We give a brief sketch of the proofs. Most of the calculations were performed with the assistance of the algebraic manipulation package REDUCE.

**Theorem 1.** *The usual equivalence group $G$ admitted by class (1) is formed by the non-degenerate point transformations*

$$\tilde{t} = Q(t),\ \ \tilde{x} = \beta_1 x + \beta_5,\ \ \tilde{y} = \beta_2 y + \psi(t),\ \ \tilde{u} = \beta_3 u + \beta_4 x + \eta(t),$$

$$\tilde{a}(\tilde{t}) = \frac{Q'(t)a(t)}{\beta_1^2\beta_2},\ \ \tilde{b}(\tilde{t}) = \frac{Q'(t)b(t)}{\beta_1^3},\ \ \tilde{d}(\tilde{t}) = \frac{\beta_2 d(t)}{\beta_1^3} + \frac{\psi'(t)b(t)}{\beta_1^3},$$

$$\tilde{c}(\tilde{t}) = \frac{c(t)}{\beta_1^2} + \frac{\psi'(t)a(t)}{\beta_1^2\beta_2} - \frac{\beta_4 k}{\beta_1^2\beta_3},\ \ \tilde{k} = \frac{k}{\beta_1\beta_3},$$

*where $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ are arbitrary constants and $Q(t), \psi(t), \eta(t)$ are arbitrary smooth functions of $t$. For non-degenerate transformations, we require that $\beta_1\beta_2\beta_3 Q'(t) \neq 0$.*

**Proof.** We consider the point transformation

$$\tilde{t} = Q(t, x, y, u),\ \ \tilde{x} = P(t, x, y, u),\ \ \tilde{y} = S(t, x, y, u),\ \ \tilde{u} = R(t, x, y, u),$$

which connects the class

$$\tilde{a}(\tilde{t})\tilde{u}_{\tilde{t}\tilde{x}} + \tilde{b}(\tilde{t})\tilde{u}_{\tilde{t}\tilde{y}} + \tilde{c}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{y}} + \tilde{d}(\tilde{t})\tilde{u}_{\tilde{y}\tilde{y}} + \tilde{k}(\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}})_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{y}} = 0 \qquad (2)$$

with (1). We assume that this transformation is non-degenerate, $\frac{\partial(Q,P,S,R)}{\partial(t,x,y,u)} \neq 0$. We use the formulaes of the derivatives of $\tilde{u}$ in terms of the derivatives of $u$ [9] and we substitute in (2). We eliminate $u_{xxxy}$ from (1) and Equation (2) takes the form

$$E(t,x,y,u,Q,P,S,R,u_t,u_x,\dots) = 0,$$

where $E$ is a multi-variable polynomial in the derivatives of $u$. Because the connected classes are polynomials in the derivatives of $u$ and $\tilde{u}$, the transformation is of the restricted form [9]

$$\tilde{t} = Q(t),\ \ \tilde{x} = l_1(t)x + \phi(t),\ \ \tilde{y} = l_2(t)y + \psi(t),\ \ \tilde{u} = H_1(t,x,y)u + H_2(t,x,y).$$

Coefficients of $u_{xxx}, u_{xxy}, u_{xyy}$ in $E = 0$ give $H_1 = constant = \beta_3$. Coefficients of $u_{tx}$ and $u_{ty}$ in $E = 0$ give $\tilde{a} = \frac{Q'(t)a(t)}{l_1^2(t)l_2(t)}$ and $\tilde{b} = \frac{Q'(t)b(t)}{l_1^3(t)}$, respectively. Coefficients of $u_{xy}u_x$ give $l_1(t) = constant = \beta_1$ and $\tilde{k} = \frac{k}{\beta_1\beta_3}$. Coefficients of $u_{xx}$, $u_{xy}$, $u_{yy}$, $u_x$, $u_y$ and the terms independent of derivatives give the following six equations:

$$\beta_1 k \frac{\partial H_2}{\partial y} - \beta_3 a(t)\phi'(t) = 0, \tag{3}$$

$$\begin{aligned} &\beta_1 k l_2(t)\frac{\partial H_2}{\partial x} - \beta_1\beta_3 a(t) l_2'(t) y - \beta_3 b(t) l_2(t)\phi'(t) \\ &-\beta_1\beta_3 a(t)\psi'(t) + \beta_1^3\beta_3 l_2(t)\tilde{c}(\tilde{t}) - \beta_1\beta_3 l_2(t)c(t) = 0, \end{aligned} \tag{4}$$

$$b(t)l_2'(t)y + b(t)\psi'(t) - \beta_1^3\tilde{d}(\tilde{t}) + l_2(t)d(t) = 0, \tag{5}$$

$$\frac{\partial^2 H}{\partial x \partial y} = 0, \tag{6}$$

$$k l_2(t)\frac{\partial^2 H_2}{\partial x^2} - \beta_3 l_2'(t)b(t) = 0, \tag{7}$$

$$\begin{aligned} &\beta_1\beta_3 l_2(t)\frac{\partial^4 H_2}{\partial x^3 \partial y} + \beta_1\beta_3 l_2(t)a(t)\frac{\partial^2 H_2}{\partial t \partial x} + \beta_1\beta_3 l_2(t)b(t)\frac{\partial^2 H_2}{\partial t \partial y} \\ &+\left[\beta_1 k l_2(t)\frac{\partial H_2}{\partial y} - \beta_3 l_2(t)a(t)\phi'(t)\right]\frac{\partial^2 H_2}{\partial x^2} \\ &+\left[\beta_1^4\beta_3\tilde{d}(\tilde{t}) - \beta_1\beta_3 b(t)l_2'(t)y - \beta_1\beta_3 b(t)\psi'(t)\right]\frac{\partial^2 H_2}{\partial y^2} \\ &+\left[\beta_1 k l_2(t)\frac{\partial H_2}{\partial x} - \beta_1\beta_3 a(t)l_2'(t)y - \beta_3 l_2(t)b(t)\phi'(t)\right. \\ &\left.+\beta_1^3\beta_3 l_2(t)\tilde{c}(\tilde{t}) - \beta_1\beta_3 a(t)\psi'(t)\right]\frac{\partial^2 H_2}{\partial x \partial y} - \beta_1\beta_3 b(t)l_2'(t)\frac{\partial H_2}{\partial y} = 0. \end{aligned} \tag{8}$$

Coefficient of $y$ in (4) gives $l_2(t) = constant = \beta_2$. From the definition of usual equivalence group, $H_2$ cannot depend on arbitrary elements and, hence, from (3) we have $\phi(t) = constant = \beta_5$ and $H_2(t,x,y) = H_2(t,x)$. From (7) and (9), we deduce that $H_2(t,x) = \beta_4 x + \eta(t)$. We solve for $\tilde{d}(\tilde{t})$ in (5) and for $\tilde{c}(\tilde{t})$ in (4) to find $\tilde{d}(\tilde{t}) = \frac{\beta_2 d(t)}{\beta_1^3} + \frac{\psi'(t)b(t)}{\beta_1^3}$ and $\tilde{c}(\tilde{t}) = \frac{c(t)}{\beta_1^2} + \frac{\psi'(t)a(t)}{\beta_1^2\beta_2} - \frac{\beta_4 k}{\beta_1^2\beta_3}$, respectively. The derived forms of $Q$, $P$, $S$, and $R$ give $\frac{\partial(Q,P,S,R)}{\partial(t,x,y,u)} = \beta_1\beta_2\beta_3 Q'(t)$. Hence, for non-degenerate transformation, we assume that $\beta_1\beta_2\beta_3 Q'(t) \neq 0$. This completes the proof. □

Furthermore, class (1) admits generalized extended equivalence transformations. The results are tabulated in the following theorem.

**Theorem 2.** *The generalized extended equivalence group $\hat{G}$ of class (1) consists of the non-degenerate point transformations*

$$\tilde{t} = Q(t),\ \ \tilde{x} = \beta_1 x + \phi(t),\ \ \tilde{y} = \beta_2 y + \psi(t),\ \ \tilde{u} = \beta_3 u + \theta(t)x + \frac{\beta_3\phi'(t)a(t)}{k\beta_1}y + \eta(t),$$

$$\tilde{a}(\tilde{t}) = \frac{Q'(t)a(t)}{\beta_1^2\beta_2},\ \ \tilde{b}(\tilde{t}) = \frac{Q'(t)b(t)}{\beta_1^3},\ \ \tilde{d}(\tilde{t}) = \frac{\beta_2 d(t)}{\beta_1^3} + \frac{\psi'(t)b(t)}{\beta_1^3},$$
$$\tilde{c}(\tilde{t}) = \frac{c(t)}{\beta_1^2} + \frac{\phi'(t)b(t)}{\beta_1^3} + \frac{\psi'(t)a(t)}{\beta_1^2\beta_2} - \frac{\theta(t)k}{\beta_1^2\beta_3},\ \ \tilde{k} = \frac{k}{\beta_1\beta_3},$$

*where $\beta_1, \beta_2, \beta_3$ are arbitrary constants, $Q(t), \psi(t), \theta(t), \eta(t)$ are arbitrary smooth functions of $t$, and $\phi'(t) = \frac{k\beta_1}{\beta_3 a(t)}\left(\beta_4 - \int \frac{\theta'(t)a(t)}{b(t)}dt\right)$. For non-degenerate transformations, we require that $\beta_1\beta_2\beta_3 Q'(t) \neq 0$.*

**Proof.** We use the initial steps of the proof of Theorem 1 to deduce that the equivalence transformations are of the restricted form

$$\tilde{t} = Q(t),\ \ \tilde{x} = \beta_1 x + \phi(t),\ \ \tilde{y} = l_2(t)y + \psi(t),\ \ \tilde{u} = \beta_3 u + H_2(t,x,y),$$

where the functions $H_2(t,x,y)$, $\phi(t)$, $\psi(t)$, and $l_2(t)$ satisfy the system of Equations (3)–(9). Also, we have the relations

$$\tilde{a} = \frac{Q'(t)a(t)}{\beta_1^2 l_2(t)},\ \ \tilde{b} = \frac{Q'(t)b(t)}{\beta_1^3},\ \ \tilde{k} = \frac{k}{\beta_1\beta_3}.$$

Coefficient of $y$ in (4) gives $l_2(t) = \beta_2$. From Equations (3), (6) and (7), we find that $H_2(t,x,y) = \theta(t)x + \frac{\beta_3\phi'(t)a(t)}{k\beta_1}y + \eta(t)$. Equations (4) and (5) give $\tilde{c}(\tilde{t}) = \frac{c(t)}{\beta_1^2} + \frac{\phi'(t)b(t)}{\beta_1^3} + \frac{\psi'(t)a(t)}{\beta_1^2\beta_2} - \frac{\theta(t)k}{\beta_1^2\beta_3}$ and $\tilde{d}(\tilde{t}) = \frac{\beta_2 d(t)}{\beta_1^3} + \frac{\psi'(t)b(t)}{\beta_1^3}$, respectively. As before, $\frac{\partial(Q,P,S,R)}{\partial(t,x,y,u)} = \beta_1\beta_2\beta_3 Q'(t)$. Finally, from (9), $\phi'(t) = \frac{k\beta_1}{\beta_3 a(t)}\left(\beta_4 - \int \frac{\theta'(t)a(t)}{b(t)}dt\right)$, and this completes the proof of Theorem 2. □

From Theorem 1, we deduce that, for the appropriate choice of the function $Q(t)$, we can set $\tilde{b}(\tilde{t}) = 1$ (or $\tilde{a}(\tilde{t}) = 1$). Furthermore, with the suitable choice of $\psi(t)$, we can take $\tilde{c}(\tilde{t}) = 0$ (or $\tilde{d}(\tilde{t}) = 0$). However, from Theorem 2, we have an additional simplification. More precisely, the generalized extended equivalence transformations enable us to fix the functions $\tilde{b}(\tilde{t})$, $\tilde{c}(\tilde{t})$, and $\tilde{d}(\tilde{t})$. That is, the tilded class has a simpler form with $\tilde{b}(\tilde{t}) = 1$, $\tilde{c}(\tilde{t}) = \tilde{d}(\tilde{t}) = 0$, which means that it has only one variable coefficient, the function $\tilde{a}(\tilde{t})$. Therefore, using Theorem 2, we deduce that the transformation

$$\begin{aligned}&\tilde{t} = \int \frac{dt}{b(t)},\ \ \tilde{x} = x + \phi(t),\ \ \tilde{y} = y - \int \frac{d(t)dt}{b(t)},\\ &\tilde{u} = ku + \left(b\phi'(t) + \frac{b(t)c(t) - a(t)d(t)}{b(t)}\right)x + \phi'(t)a(t)y,\ \ \tilde{a} = \frac{a(t)}{b(t)},\end{aligned} \tag{9}$$

where $\phi'(t) = \frac{1}{2\sqrt{|a(t)b(t)|}} \int \sqrt{|\frac{a(t)}{b(t)}|}\left(\frac{a(t)d(t)-b(t)c(t)}{b(t)}\right)' dt$ reduces (1) into

$$\tilde{a}(\tilde{t})\tilde{u}_{\tilde{t}\tilde{x}} + \tilde{u}_{\tilde{t}\tilde{y}} + (\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}})_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{y}} = 0.$$

In the subsequent analysis, we consider this reduced equation without tildes,

$$a(t)u_{tx} + u_{ty} + (u_x u_y)_x + u_{xxxy} = 0. \tag{10}$$

Another observation from Theorem 2 is that the general class (1) can be mapped into a constant coefficient only if $a(t) = \kappa b(t)$, whereas the functions $c(t)$ and $d(t)$ can take any form. This mapping can be obtained from (9) by setting $a(t) = \kappa b(t)$.

In the following theorem, we provide the generalized equivalence group for class (10). We omit the proof, which is similar to those of Theorems 1 and 2.

**Theorem 3.** *The generalized extended equivalence group $\hat{G}_1$ of class (10) consists of the transformations*

$$\tilde{t} = \beta_1^3 t + \beta_3, \;\; \tilde{x} = \beta_1 x + \beta_5 \int \frac{dt}{\sqrt{|a(t)|}} + \beta_6, \;\; \tilde{y} = \beta_2 y + \beta_4,$$

$$\tilde{u} = \frac{1}{\beta_1} u + \frac{\beta_5}{\beta_1^2 \sqrt{|a(t)|}} x + \frac{\beta_5 \sqrt{|a(t)|}}{\beta_1^2} y + \eta(t), \;\; \tilde{a}(\tilde{t}) = \frac{\beta_1 a(t)}{\beta_2}$$

*where $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ are arbitrary constants with $\beta_1\beta_2 \neq 0$ and $\eta(t)$ is an arbitrary smooth function.*

The usual equivalence group $G_1$ of class (10) can be obtained from Theorem 3 by setting $\beta_5 = 0$. Equivalence transformations can be used to simplify the form of $a(t)$ in the analysis of the group classification of (10). For example, if $a(t) = (\mu_1 t + \mu_2)^n$, then it can be taken as $a(t) = t^n$, without loss of generality.

## 3. Lie Group Classification

We perform the Lie group classification for the simplified class (10). Using these results and the equivalence transformations in Theorem 2, we derive the corresponding classification for the general class (1), which is presented in the Appendix A. The Lie symmetry method is well known and regularly used in recent decades [1,10–14]. For this reason, we omit the analysis for obtaining the desired Lie symmetries. We search for Lie operators of the form

$$\Gamma = T(t,x,y,u)\partial_t + X(t,x,y,u)\partial_x + Y(t,x,y,u)\partial_y + U(t,x,y,u)\partial_u \tag{11}$$

corresponding to the infinitesimal transformations: $\tilde{t} = t + \epsilon T$, $\tilde{x} = x + \epsilon X$, $\tilde{y} = y + \epsilon Y$, $\tilde{u} = u + \epsilon U$, to the first order of $\epsilon$. The analysis leads to four cases:

1. If $a(t)$ is arbitrary, the Lie algebra is spanned by the operators

$$X_1 = \partial_x, \;\; X_2 = \partial_y, \;\; X_3 = \left(\int \frac{dt}{\sqrt{|a(t)|}}\right)\partial_x + \left(\frac{x + a(t)y}{\sqrt{|a(t)|}}\right)\partial_u, \;\; X_\eta = \eta(t)\partial_u.$$

Additional Lie symmetries exist in the cases where $a(t) = t^n$, $\mathrm{e}^{nt}$, 1.

2. If $a(t) = t^n$, the additional Lie operator is of the form

$$X_4 = 3t\partial_t + x\partial_x + (1 - 3n)y\partial_y - u\partial_u.$$

3. If $a(t) = \mathrm{e}^{nt}$, the additional Lie operator is of the form

$$Y_4 = \partial_t - ny\partial_y.$$

4. If $a(t) = 1$, we have two additional Lie operators: $X_4(n = 0)$ and $Y_4(n = 0)$,

$$X_4 = 3t\partial_t + x\partial_x + y\partial_y - u\partial_u, \;\; Y_4 = \partial_t.$$

We use the above Lie symmetries in the next section to derive few similarity reductions.

## 4. Examples of Similarity Reductions

The main application of Lie symmetries is the construction of similarity mappings that have the property of reducing the number of independent variables of the equation under study. The complete list of reductions can be obtained using the one- and two-dimensional subalgebras of the Lie invariance algebra admitted by the equation. The mappings obtained from one-dimensional subalgebras reduce the number of independent variables by one, whereas the two-dimensional algebras reduce the number by two. Detailed analysis of how to construct these subalgebras can be found in recent work [15]. Here, we present certain examples of similarity reductions using one-dimensional subalgebras. These mappings are derived solving the invariance surface condition

$$T(t,x,y,u)u_t + X(t,x,y,u)u_x + Y(t,x,y,u)u_y = U(t,x,y,u). \tag{12}$$

For the case of $a(t)$ being arbitrary, we give two examples. The Lie symmetry $X_2 + cX_1$ produces the similarity mapping $u(t,x,y) = w(t, x-cy)$, which reduces (10) to

$$a(t)w_{t\xi} - cw_{t\xi} - cw_{\xi\xi\xi\xi} - 2cw_\xi w_{\xi\xi} = 0, \;\; \xi = x - cy$$

and integrating with respect to $\xi$, taking the integrating function equal to zero, we find the variable coefficient potential KdV equation

$$[a(t) - c]w_t - cw_{\xi\xi\xi} - cw_\xi^2 = 0.$$

Lie symmetries of a similar equation are presented in [16]. The Lie symmetry $X_3 + X_2$ leads to the reduction

$$u(t,x,y) = \frac{1}{2}\left(\frac{1}{\phi'(t)} - \phi(t)\phi'(t)\right)y^2 + \phi'(t)xy + w(t,\xi), \;\; \xi = x - \phi(t)y, \; a(t) = \frac{1}{\phi'(t)^2}$$

that maps (10) into

$$\left(1 - \phi\phi'^2\right)w_{t\xi} - \phi\phi'^2 w_{\xi\xi\xi\xi} - 2\phi\phi'^2 w_\xi w_{\xi\xi} + \phi'^3\xi w_{\xi\xi} + \phi'^2\phi''\xi = 0,$$

which can be integrated once with respect to $\xi$.

In the case $a(t) = 1$, Lie symmetry $Y_4 + c_1X_1 + c_2X_2$ produces the similarity mapping $u = w(\xi,\eta), \; \xi = x - c_1t, \; \eta = y - c_2t$, which transforms (10) ($a(t) = 1$) into

$$w_{\eta\xi\xi\xi} + (w_\eta w_\xi)_\xi - (c_1 + c_2)w_{\eta\xi} - c_2 w_{\eta\eta} - c_1 w_{\xi\xi} = 0.$$

This last equation admits four Lie symmetries, $\partial_\eta$, $\partial_\xi$, $\partial_w$, and $3\eta\partial_\eta + \xi\partial_\xi + [4c_1\eta + 2(c_1 + c_2)\xi - w]\partial_w$, which can be employed for further reductions. For example, using the Lie symmetry $\partial_\eta + c_3\partial_\xi$ and the above mapping, we construct the double reduction $u = F(\theta), \; \theta = x - c_1y - c_2t$, which reduces (10) ($a(t) = 1$) into

$$c_1F^{(iv)}(\theta) + 2c_1F'(\theta)F''(\theta) + c_2(1 - c_1)F''(\theta) = 0.$$

We integrate and rename the constants to find

$$G''(\theta) + G^2(\theta) + k_1G(\theta) + k_2 = 0, \quad G(\theta) = F'(\theta)$$

and we set $G(\theta) = H(\theta) - \frac{k_1}{2}$ to get

$$H''(\theta) + H^2(\theta) + k_2 - \tfrac{1}{4}k_1^2 = 0.$$

We integrate again to give

$$\tfrac{1}{2}[H'(\theta)]^2 + \tfrac{1}{3}H^3(\theta) + (k_2 - \tfrac{1}{4}k_1^2)H(\theta) = k_3.$$

## 5. A Note on Non-Lie Reductions

Bluman and Cole [17,18] introduced a method to derive non-Lie reductions. In this approach, we require invariance of Equation (10) in conjunction with the invariant surface condition (12) under the infinitesimal transformations generated by the Lie operator (11). Motivated by the work in reference [16], we state that, for Equation (10), we can search for three forms of reduction operators:

1. $\Gamma = \partial_t + X(t,x,y,u)\partial_x + Y(t,x,y,u)\partial_y + U(t,x,y,u)\partial_u$
2. $\Gamma = X(t,x,y,u)\partial_x + \partial_y + U(t,x,y,u)\partial_u$
3. $\Gamma = \partial_x + U(t,x,y,u)\partial_u$

Unlike the Lie symmetries, the determining system for the coefficient functions of the operator is nonlinear. This makes the task for deriving such non-Lie operators very difficult. Here, we do not carry out the complete analysis for deriving these forms of operators, but we only present some examples of the second and third forms. The search of operators of the first form leads to equivalent Lie operators.

As a first example, we have the operator of the above form 2,

$$\Gamma = \partial_y + \psi(t,x)\partial_u, \;\; a(t)\psi_{tx} + (\psi\psi_x)_x = 0.$$

This operator produces the ansatz $u(t,x,y) = \psi(t,x)y + \phi(t,x)$ that reduces Equation (10) to the system

$$a(t)\phi_{tx} + (\psi\phi_x)_x + \psi_t + \psi_{xxx} = 0, \;\; a(t)\psi_{tx} + (\psi\psi_x)_x = 0.$$

Both above equations can be integrated once with respect to $x$. The second example is the non-Lie operator $\Gamma = \partial_y + a(t)\partial_x$, which produces the ansatz $u(t,x,y) = \phi(t,\xi), \; \xi = x - a(t)y$ that maps (10) into

$$a'(t)\phi_\xi + a(t)\phi_{\xi\xi\xi\xi} + 2a(t)\phi_\xi\phi_{\xi\xi} = 0.$$

This differential equation can be integrated once with respect to $\xi$.

A trivial example of the third form is the operator $\partial_x + [f(t) + g(y)]\partial_u$, which leads to the trivial ansatz $u(t,x,y) = [f(t) + g(y)]x + \phi(t,y)$, and the reduced equation is

$$\phi_{ty} + [f(t) + g(y)]g'(y) + a(t)f'(t) = 0,$$

which can be integrated and the solution is expressed in terms of the arbitrary functions $a(t), \; f(t)$, and $g(y)$.

## 6. Final Remarks

The difficult task of the Lie group classification of the variable coefficient Boiti–Leon–Manna–Pempinelli Equation (1) has been achieved with the aid of the equivalence group. Equivalence transformations provides the tools to fix a number of variable coefficients, which makes the symmetry analysis simpler. As another example, in order to show the significance of the equivalence group, we consider (1) with $a(t) = 0$,

$$b(t)u_{ty} + c(t)u_{xy} + d(t)u_{yy} + k(u_x u_y)_x + u_{xxxy} = 0. \tag{13}$$

The variable coefficient Equation (13) can be mapped into the standard Boiti–Leon–Manna–Pempinelli equation

$$\tilde{u}_{\tilde{t}\tilde{y}} + (\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}})_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{y}} = 0 \tag{14}$$

under the mapping

$$\tilde{t} = t, \;\; \tilde{x} = [b(t)]^{\frac{1}{3}}x, \;\; \tilde{y} = y - \int \frac{d(t)}{b(t)}dt, \;\; \tilde{u} = \frac{1}{6}[b(t)]^{-\frac{1}{3}}\Big[6ku + b'(t)x^2 + 6c(t)x\Big].$$

Lie symmetries admitted by Boiti–Leon–Manna–Pempinelli Equation (14) can be found in [19]. The Lie group classification of class (13) can be achieved using the above mapping and the Lie symmetries of (14). To our knowledge, non-Lie reduction operators do not exist in the literature. We list four examples of such operators that have been derived by the method of Bluman and Cole:

$$\partial_y + \phi(t)\psi(y)\partial_u,$$
$$\partial_y + \sqrt{\phi_1(t)x + \phi_2(t)}\partial_u,$$
$$\partial_x + [\phi(t) + \psi(y)]\partial_u,$$
$$\partial_x + \frac{2x}{2\psi(y) + 3t}\partial_u,$$

where all the functions that appear in the reduction operators are arbitrary. An open problem, although not an easy one, is the classification of all such reduction operators for Boiti–Leon–Manna–Pempinelli Equation (14).

## 7. Conclusions

Recently, a generalization of the Boiti–Leon–Manna–Pempinelli equation has appeared in the literature [8]. This general equation has four coefficients as functions of time, which makes it difficult to be studied. However, before studying such equations, it is very useful to attempt to simplify it. Equivalence transformations are good tools that enable us, in most cases, to simplify variable coefficient equations. Here, we have derived the equivalence transformations for class (1) and used them to transform it to the simplified class (10) with only one variable coefficient. We present the group classification of both classes (1) and (10) with the employment of the equivalence transformations. This completes the existing results for the group classification of the general class (1) that appear in the literature.

The present work can be extended to find exact or numerical solutions for the generalized Boiti–Leon–Manna–Pempinelli equation. Lie reductions in Section 4 and non-Lie reductions in Section 5 can be used to derive solutions for (10). It is pointed out that in both sections we have only presented partial results. Another future task is to complete the analysis in these two sections. However, finding the complete list of non-Lie reduction operators (nonclassical symmetries) is an extremely difficult task because most of the determining equations are of a nonlinear form.

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## Appendix A

We apply transformations (9) that are derived from Theorem 2 and the Lie symmetry classification of (10) to obtain the corresponding results of the general class of variable coefficient Boiti–Leon–Manna–Pempinelli Equations (1).

1. If $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $k$ are arbitrary, the Lie algebras admitted by (1) is spanned by the operators

$$X_1 = \partial_x, \;\; X_2 = \partial_y, \;\; X_3 = \left(\int \frac{dt}{\sqrt{|a(t)b(t)|}}\right)\partial_x + \frac{1}{k}\left(\frac{b(t)x + a(t)y}{\sqrt{|a(t)b(t)|}}\right)\partial_u, \;\; X_\eta = \eta(t)\partial_u.$$

The above Lie symmetries agree with the results in [5].

Additional Lie symmetries exist in the cases where $a(t) = b(t)(\int b)^n$, $b(t)e^{n(\int b)}, and\ b(t)$; and $c(t) and\ d(t)$ are arbitrary. The following cases do not appear in the literature.

2. If $a(t) = b(t)(\int b)^n$, the additional Lie operator is of the form

$$X_4 = 3(2-n)b(t)\left(\int \frac{1}{b}dt\right)\partial_t + (2-n)\left[(1-3n)y + 3d(t)\int \frac{1}{b}dt + (3n-1)\int \frac{d}{b}dt\right]\partial_y$$
$$+\left\{(2-n)x + \frac{1}{4}\left[6(n-2)d(t)\left(\int \frac{1}{b}dt\right)^{n+1} + 4\int \frac{d}{b}\left(\int \frac{1}{b}dt\right)^n dt - 4\int \frac{c}{b}dt\right.\right.$$
$$-6(n-2)c(t)\left(\int \frac{1}{b}dt\right)$$
$$\left.\left.+(3n^2-4n)\left(\int \frac{1}{b}dt\right)^{1-\frac{n}{2}}\int\left(\frac{c}{b}\left(\int \frac{1}{b}dt\right)^{\frac{n}{2}-1} - \frac{d}{b}\left(\int \frac{1}{b}dt\right)^{\frac{3n}{2}-1}\right)dt\right]\right\}\partial_x$$
$$+(n-2)\left\{u + \frac{1}{8k}\left[12b(t)c'(t)\left(\int \frac{1}{b}dt\right) - 12b(t)d'(t)\left(\int \frac{1}{b}dt\right)^{n+1} - (18n+8)d(t)\left(\int \frac{1}{b}dt\right)^n\right.\right.$$
$$\left.+(6n+8)c(t) + (3n^2-4n)\left(\int \frac{1}{b}dt\right)^{-\frac{n}{2}}\int\left(\frac{d}{b}\left(\int \frac{1}{b}dt\right)^{\frac{3n}{2}-1} - \frac{c}{b}\left(\int \frac{1}{b}dt\right)^{\frac{n}{2}-1}\right)dt\right]x$$
$$-\frac{1}{8k}\left[12b(t)c'(t)\left(\int \frac{1}{b}dt\right)^{n+1} - 12b(t)d'(t)\left(\int \frac{1}{b}dt\right)^{2n+1} - (6n+8)d(t)\left(\int \frac{1}{b}dt\right)^{2n}\right.$$
$$-(6n-8)c(t)\left(\int \frac{1}{b}dt\right)^n$$
$$\left.\left.-(3n^2-4n)\left(\int \frac{1}{b}dt\right)^{\frac{n}{2}}\int\left(\frac{d}{b}\left(\int \frac{1}{b}dt\right)^{\frac{3n}{2}-1} - \frac{c}{b}\left(\int \frac{1}{b}dt\right)^{\frac{n}{2}-1}\right)dt\right]y\right\}\partial_u.$$

3. If $a(t) = b(t)\mathrm{e}^{n(\int b)}$, the additional Lie operator is of the form

$$Y_4 = 6b(t)\partial_t + 6\left[n\int \frac{d}{b}dt + d - ny\right]\partial_y$$
$$+\frac{3}{2}e^{-\frac{n}{2}\int \frac{1}{b}dt}\left[2c(t)e^{\frac{n}{2}\int \frac{1}{b}dt} - 2d(t)e^{\frac{3n}{2}\int \frac{1}{b}dt} + n\int \frac{d}{b}e^{\frac{3n}{2}\int \frac{1}{b}dt}dt - n\int \frac{c}{b}e^{\frac{n}{2}\int \frac{1}{b}dt}dt\right]\partial_x$$
$$+\frac{3}{4k}e^{-\frac{n}{2}\int \frac{1}{b}dt}\left\{\left[(4b(t)d'(t) + 6nd'(t))e^{\frac{n}{2}\int \frac{1}{b}dt} - (4b(t)c'(t) + 2nc(t))e^{\frac{3n}{2}\int \frac{1}{b}dt}\right.\right.$$
$$\left.+n^2\int \frac{c}{b}e^{\frac{n}{2}\int \frac{1}{b}dt}dt - n^2\int \frac{d}{b}e^{\frac{3n}{2}\int \frac{1}{b}dt}dt\right]x$$
$$+e^{n\int \frac{1}{b}dt}\left[(4b(t)c'(t) - 2nc(t))e^{\frac{n}{2}\int \frac{1}{b}dt} - (4b(t)d'(t) + 2nd(t))e^{\frac{3n}{2}\int \frac{1}{b}dt}\right.$$
$$\left.\left.+n^2\int \frac{c}{b}e^{\frac{n}{2}\int \frac{1}{b}dt}dt - n^2\int \frac{d}{b}e^{\frac{3n}{2}\int \frac{1}{b}dt}dt\right]y\right\}\partial_u.$$

4. If $a(t) = b(t)$, we have two additional Lie operators: $X_4(n=0)$ and $Y_4(n=0)$,

$$X_4 = 6b(t)\left(\int \frac{1}{b}dt\right)\partial_t + \left[2x + 3(c(t)-d(t))\left(\int \frac{1}{b}dt\right) + \left(\int \frac{d-c}{b}dt\right)\right]\partial_x$$
$$+2\left[y + 3d(t)\left(\int \frac{1}{b}dt\right) - \left(\int \frac{d}{b}dt\right)\right]\partial_y$$
$$+\frac{1}{k}\left\{-2ku + \left[3b(t)(c'(t)-d'(t))\left(\int \frac{1}{b}dt\right) + 2(c(t)-d(t))\right][y-x]\right\}\partial_u,$$
$$Y_4 = 6b(t)\partial_t + 3[c(t)-d(t)]\partial_x + 6d(t)\partial_y + \frac{3}{k}b(t)[c'(t)-d'(t)][y-x]\partial_u.$$

As we pointed out in Section 2, case 4 is equivalent to the constant coefficient equation.

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