9. A matrix of the form

$$\begin{bmatrix} 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & e & 0 & f \\ g & 0 & h & 0 \end{bmatrix}$$

cannot be invertible.

10. A matrix of the form

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

such that $ae - bd = 0$ cannot be invertible.

4.11 Inner Product Spaces

We now extend the familiar idea of a dot product for geometric vectors to an arbitrary vector space V. This enables us to associate a magnitude with each vector in V and also to define the angle between two vectors in V. The major reason that we want to do this is that, as we will see in the next section, it enables us to construct orthogonal bases in a vector space, and the use of such a basis often simplifies the representation of vectors. We begin with a brief review of the dot product.

Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be two arbitrary vectors in $\mathbb{R}^3$, and consider the corresponding geometric vectors

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \qquad \mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}.$$

The dot product of $\mathbf{x}$ and $\mathbf{y}$ can be defined in terms of the components of these vectors as

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3. \tag{4.11.1}$$

An equivalent geometric definition of the dot product is

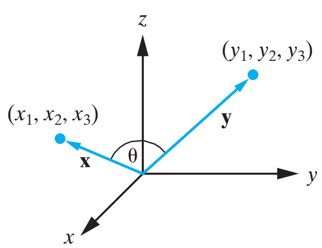

Figure 4.11.1: Defining the dot product in $\mathbb{R}^3$.

$$\mathbf{x} \cdot \mathbf{y} = ||\mathbf{x}||\ ||\mathbf{y}|| \cos\theta, \tag{4.11.2}$$

where $||\mathbf{x}||$, $||\mathbf{y}||$ denote the lengths of $\mathbf{x}$ and $\mathbf{y}$ respectively, and $0 \le \theta \le \pi$ is the angle between them. (See Figure 4.11.1.)

Taking $\mathbf{y} = \mathbf{x}$ in Equations (4.11.1) and (4.11.2) yields

$$||\mathbf{x}||^2 = \mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + x_3^2,$$

so that the length of a geometric vector is given in terms of the dot product by

$$||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Furthermore, from Equation (4.11.2), the angle between any two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||\ ||\mathbf{y}||}, \tag{4.11.3}$$

which implies that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal (perpendicular) if and only if

$$\mathbf{x} \cdot \mathbf{y} = 0.$$

In a general vector space, we do not have a geometrical picture to guide us in defining the dot product, hence our definitions must be purely algebraic. We begin by considering the vector space $\mathbb{R}^n$, since there is a natural way to extend Equation (4.11.1) in this case. Before proceeding, we note that from now on we will use the standard terms *inner product* and *norm* in place of dot product and length, respectively.

DEFINITION 4.11.1

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be vectors in $\mathbb{R}^n$. We define the **standard inner product in** $\mathbb{R}^n$, denoted $\langle \mathbf{x}, \mathbf{y} \rangle$, by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

The **norm** of $\mathbf{x}$ is

$$||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example 4.11.2 If $\mathbf{x} = (1, -1, 0, 2, 4)$ and $\mathbf{y} = (2, 1, 1, 3, 0)$ in $\mathbb{R}^5$, then

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= (1)(2) + (-1)(1) + (0)(1) + (2)(3) + (4)(0) = 7, \\
||\mathbf{x}|| &= \sqrt{1^2 + (-1)^2 + 0^2 + 2^2 + 4^2} = \sqrt{22}, \\
||\mathbf{y}|| &= \sqrt{2^2 + 1^2 + 1^2 + 3^2 + 0^2} = \sqrt{15}.
\end{aligned}$$

□

Basic Properties of the Standard Inner Product in $\mathbb{R}^n$

In the case of $\mathbb{R}^n$, the definition of the standard inner product was a natural extension of the familiar dot product in $\mathbb{R}^3$. To generalize this definition further to an arbitrary vector space, we isolate the most important properties of the standard inner product in $\mathbb{R}^n$ and use them as the defining criteria for a general notion of an inner product. Let us examine the inner product in $\mathbb{R}^n$ more closely. We view it as a mapping that associates with any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ the real number

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

This mapping has the following properties:

For all $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^n$ and all real numbers k,

1. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$. Furthermore, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
2. $\langle \mathbf{y}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.
3. $\langle k\mathbf{x}, \mathbf{y} \rangle = k\langle \mathbf{x}, \mathbf{y} \rangle$.
4. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$.

These properties are easily established using Definition 4.11.1. For example, to prove property 1, we proceed as follows. From Definition 4.11.1,

$$\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Since this is a sum of squares of real numbers, it is necessarily nonnegative. Further, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $x_1 = x_2 = \cdots = x_n = 0$—that is, if and only if $\mathbf{x} = \mathbf{0}$. Similarly, for property 2, we have

$$\langle \mathbf{y}, \mathbf{x} \rangle = y_1 x_1 + y_2 x_2 + \cdots + y_n x_n = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \langle \mathbf{x}, \mathbf{y} \rangle.$$

We leave the verification of properties 3 and 4 for the reader.

Definition of a Real Inner Product Space

We now use properties 1–4 as the basic defining properties of an inner product in a real vector space.

DEFINITION 4.11.3

Let V be a real vector space. A mapping that associates with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number, denoted $\langle \mathbf{u}, \mathbf{v} \rangle$, is called an **inner product** in V, provided it satisfies the following properties. For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and all real numbers k,

1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$. Furthermore, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
2. $\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$.
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$.
4. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.

The **norm** of $\mathbf{u}$ is defined in terms of an inner product by

$$||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

A real vector space together with an inner product defined in it is called a **real inner product space**.

Remarks

1. Observe that $||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ takes a well-defined nonnegative real value, since property 1 of an inner product guarantees that the norm evaluates the square root of a nonnegative real number.
2. It follows from the discussion above that $\mathbb{R}^n$ together with the inner product defined in Definition 4.11.1 is an example of a real inner product space.

One of the fundamental inner products arises in the vector space $C^0[a, b]$ of all real-valued functions that are *continuous* on the interval $[a, b]$. In this vector space, we define the mapping $\langle f, g \rangle$ by

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx, \tag{4.11.4}$$

for all f and g in $C^0[a, b]$. We establish that this mapping defines an inner product in $C^0[a, b]$ by verifying properties 1–4 of Definition 4.11.3. If f is in $C^0[a, b]$, then

$$\langle f, f \rangle = \int_a^b [f(x)]^2\,dx.$$

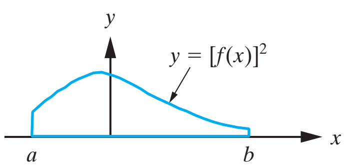

Figure 4.11.2: $\langle f, f \rangle$ gives the area between the graph of $y = [f(x)]^2$ and the x-axis, lying over the interval $[a, b]$.

Since the integrand, $[f(x)]^2$, is a nonnegative continuous function, it follows that $\langle f, f \rangle$ measures the area between the graph $y = [f(x)]^2$ and the x-axis on the interval $[a, b]$. (See Figure 4.11.2.)

Consequently, $\langle f, f \rangle \geq 0$. Furthermore, $\langle f, f \rangle = 0$ if and only if there is zero area between the graph $y = [f(x)]^2$ and the x-axis—that is, if and only if

$$[f(x)]^2 = 0 \qquad \text{for all } x \text{ in } [a, b].$$

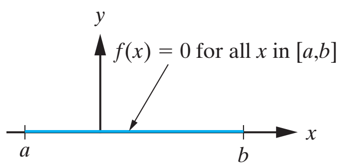

Figure 4.11.3: $\langle f, f \rangle = 0$ if and only if f is the zero function.

Hence, $\langle f, f \rangle = 0$ if and only if $f(x) = 0$, for all x in $[a, b]$, so f must be the zero function. (See Figure 4.11.3.) Consequently, property 1 of Definition 4.11.3 is satisfied. Now let f, g, and h be in $C^0[a, b]$, and let k be an arbitrary real number. Then

$$\langle g, f \rangle = \int_a^b g(x) f(x)\, dx = \int_a^b f(x) g(x)\, dx = \langle f, g \rangle.$$

Hence, property 2 of Definition 4.11.3 is satisfied.
For property 3, we have

$$\langle kf, g \rangle = \int_a^b (kf)(x) g(x)\, dx = \int_a^b k f(x) g(x)\, dx = k \int_a^b f(x) g(x)\, dx = k\langle f, g \rangle,$$

as needed. Finally,

$$\begin{aligned}\langle f + g, h \rangle &= \int_a^b (f + g)(x) h(x)\, dx = \int_a^b [f(x) + g(x)] h(x)\, dx \\ &= \int_a^b f(x) h(x)\, dx + \int_a^b g(x) h(x)\, dx = \langle f, h \rangle + \langle g, h \rangle,\end{aligned}$$

so that property (4) of Definition 4.11.3 is satisfied. We can now conclude that Equation (4.11.4) does define an inner product in the vector space $C^0[a, b]$.

Example 4.11.4

Use Equation (4.11.4) to determine the inner product of the following functions in $C^0[0, 1]$:

$$f(x) = 8x, \qquad g(x) = x^2 - 1.$$

Also find $||f||$ and $||g||$.

Solution: From Equation (4.11.4),

$$\langle f, g \rangle = \int_0^1 8x(x^2 - 1)\, dx = \Big[2x^4 - 4x^2\Big]_0^1 = -2.$$

Moreover, we have

$$||f|| = \sqrt{\int_0^1 64x^2\, dx} = \frac{8}{\sqrt{3}}$$

and

$$||g|| = \sqrt{\int_0^1 (x^2 - 1)^2\, dx} = \sqrt{\int_0^1 (x^4 - 2x^2 + 1)\, dx} = \sqrt{\frac{8}{15}}. \qquad \square$$

We have already seen that the norm concept generalizes the length of a geometric vector. Our next goal is to show how an inner product enables us to define the angle between two vectors in an abstract vector space. The key result is the *Cauchy-Schwarz inequality* established in the next theorem.

Theorem 4.11.5 **(Cauchy-Schwarz Inequality)**

Let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary vectors in a real inner product space V. Then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}||\ ||\mathbf{v}||. \tag{4.11.5}$$

Proof Let k be an arbitrary real number. For the vector $\mathbf{u} + k\mathbf{v}$, we have

$$0 \leq ||\mathbf{u} + k\mathbf{v}||^2 = \langle \mathbf{u} + k\mathbf{v}, \mathbf{u} + k\mathbf{v} \rangle. \tag{4.11.6}$$

But, using the properties of a real inner product,

$$\begin{aligned} \langle \mathbf{u} + k\mathbf{v}, \mathbf{u} + k\mathbf{v} \rangle &= \langle \mathbf{u}, \mathbf{u} + k\mathbf{v} \rangle + \langle k\mathbf{v}, \mathbf{u} + k\mathbf{v} \rangle \\ &= \langle \mathbf{u} + k\mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u} + k\mathbf{v}, k\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle k\mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u}, k\mathbf{v} \rangle + \langle k\mathbf{v}, k\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle k\mathbf{v}, \mathbf{u} \rangle + k\langle \mathbf{v}, k\mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle k\mathbf{v}, \mathbf{u} \rangle + k\langle k\mathbf{v}, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle k\mathbf{v}, \mathbf{u} \rangle + k^2\langle \mathbf{v}, \mathbf{v} \rangle \\ &= ||\mathbf{u}||^2 + 2k\langle \mathbf{v}, \mathbf{u} \rangle + k^2||\mathbf{v}||^2. \end{aligned}$$

Consequently, (4.11.6) implies that

$$||\mathbf{v}||^2 k^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle k + ||\mathbf{u}||^2 \geq 0. \tag{4.11.7}$$

The left-hand side of this inequality defines the quadratic expression

$$P(k) = ||\mathbf{v}||^2 k^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle k + ||\mathbf{u}||^2.$$

The discriminant of this quadratic is

$$\Delta = 4(\langle \mathbf{u}, \mathbf{v} \rangle)^2 - 4||\mathbf{u}||^2||\mathbf{v}||^2.$$

If $\Delta > 0$, then $P(k)$ has two real and distinct roots. This would imply that the graph of P crosses the k-axis and, therefore, P would assume negative values, contrary to (4.11.7). Consequently, we must have $\Delta \leq 0$. That is,

$$4(\langle \mathbf{u}, \mathbf{v} \rangle)^2 - 4||\mathbf{u}||^2||\mathbf{v}||^2 \leq 0,$$

or equivalently,

$$(\langle \mathbf{u}, \mathbf{v} \rangle)^2 \leq ||\mathbf{u}||^2||\mathbf{v}||^2.$$

Hence,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}||\ ||\mathbf{v}||. \qquad \blacksquare$$

If $\mathbf{u}$ and $\mathbf{v}$ are arbitrary vectors in a real inner product space V, then $\langle \mathbf{u}, \mathbf{v} \rangle$ is a real number, and so (4.11.5) can be written in the equivalent form

$$-||\mathbf{u}||\ ||\mathbf{v}|| \leq \langle \mathbf{u}, \mathbf{v} \rangle \leq ||\mathbf{u}||\ ||\mathbf{v}||.$$

Consequently, provided that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, we have

$$-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}||\ ||\mathbf{v}||} \leq 1.$$

Thus, each pair of nonzero vectors in a *real* inner product space V determines a unique angle θ by

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}||\ ||\mathbf{v}||}, \qquad 0 \leq \theta \leq \pi. \tag{4.11.8}$$

We call θ the angle between $\mathbf{u}$ and $\mathbf{v}$. In the case when $\mathbf{u}$ and $\mathbf{v}$ are geometric vectors, the formula (4.11.8) coincides with Equation (4.11.3).

Example 4.11.6 Determine the angle between the vectors $\mathbf{u} = (1, -1, 2, 3)$ and $\mathbf{v} = (-2, 1, 2, -2)$ in $\mathbb{R}^4$.

Solution: Using the standard inner product in $\mathbb{R}^4$ yields

$$\langle \mathbf{u}, \mathbf{v} \rangle = -5, \qquad ||\mathbf{u}|| = \sqrt{15}, \qquad ||\mathbf{v}|| = \sqrt{13},$$

so that the angle between $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\cos\theta = -\frac{5}{\sqrt{15}\sqrt{13}} = -\frac{\sqrt{195}}{39}, \qquad 0 \leq \theta \leq \pi.$$

Hence,

$$\theta = \arccos\left(-\frac{\sqrt{195}}{39}\right) \approx 1.937 \text{ radians } \approx 110^\circ\, 58'. \qquad \square$$

Example 4.11.7 Use the inner product (4.11.4) to determine the angle between the functions $f_1(x) = \sin 2x$ and $f_2(x) = \cos 2x$ on the interval $[-\pi, \pi]$.

Solution: Using the inner product (4.11.4), we have

$$\langle f_1, f_2 \rangle = \int_{-\pi}^{\pi} \sin 2x \, \cos 2x \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin 4x \, dx = \frac{1}{8}\left(-\cos 4x\right)\Big|_{-\pi}^{\pi} = 0.$$

Consequently, the angle between the two functions satisfies

$$\cos\theta = 0, \qquad 0 \leq \theta \leq \pi,$$

which implies that $\theta = \pi/2$. We say that the functions are *orthogonal* on the interval $[-\pi, \pi]$, relative to the inner product (4.11.4). In the next section we will have much more to say about orthogonality of vectors. $\square$

Complex Inner Products[9]

The preceding discussion has been concerned with real vector spaces. In order to generalize the definition of an inner product to a complex vector space, we first consider the case of $\mathbb{C}^n$. By analogy with Definition 4.11.1, one might think that the natural inner product in $\mathbb{C}^n$ would be obtained by summing the products of corresponding components of vectors in $\mathbb{C}^n$ in exactly the same manner as in the standard inner product for $\mathbb{R}^n$. However, one reason for introducing an inner product is to obtain a concept of "length" of a vector. In order for a quantity to be considered a reasonable measure of length, we would want it to be a nonnegative real number that vanishes if and only if the vector itself is the zero vector (property 1 of a real inner product). But, if we apply the inner product in $\mathbb{R}^n$ given in Definition 4.11.1 to vectors in $\mathbb{C}^n$, then, since the components of vectors in $\mathbb{C}^n$ are complex numbers, it follows that the resulting norm of a vector in

[9] In the remainder of the text, the only complex inner product that we will require is the standard inner product in $\mathbb{C}^n$, and this is needed only in Section 5.10.

$\mathbb{C}^n$ would be a complex number also. Furthermore, applying the $\mathbb{R}^2$ inner product to, for example, the vector $\mathbf{u} = (1-i, 1+i)$, we obtain

$$||\mathbf{u}||^2 = (1-i)^2 + (1+i)^2 = 0,$$

which means that a nonzero vector would have zero "length." To rectify this situation, we must define an inner product in $\mathbb{C}^n$ more carefully. We take advantage of complex conjugation to do this, as the definition shows.

DEFINITION 4.11.8

If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in $\mathbb{C}^n$, we define the **standard inner product in** $\mathbb{C}^n$ by[10]

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1\overline{v}_1 + u_2\overline{v}_2 + \cdots + u_n\overline{v}_n.$$

The **norm** of $\mathbf{u}$ is defined to be the *real number*

$$||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2}.$$

The preceding inner product is a mapping that associates with the two vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in $\mathbb{C}^n$ the *scalar*

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1\overline{v}_1 + u_2\overline{v}_2 + \cdots + u_n\overline{v}_n.$$

In general, $\langle \mathbf{u}, \mathbf{v} \rangle$ will be nonreal (i.e., it will have a nonzero imaginary part). The key point to notice is that the norm of $\mathbf{u}$ is always a *real* number, even though the separate components of $\mathbf{u}$ are complex numbers.

Example 4.11.9 If $\mathbf{u} = (1+2i, 2-3i)$ and $\mathbf{v} = (2-i, 3+4i)$, find $\langle \mathbf{u}, \mathbf{v} \rangle$ and $||\mathbf{u}||$.

Solution: Using Definition 4.11.8,

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= (1+2i)(2+i) + (2-3i)(3-4i) = 5i - 6 - 17i = -6 - 12i,\\ ||\mathbf{u}|| &= \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{(1+2i)(1-2i) + (2-3i)(2+3i)} = \sqrt{5+13} = 3\sqrt{2}. \quad \square\end{aligned}$$

The standard inner product in $\mathbb{C}^n$ satisfies properties (1), (3), and (4), but not property (2). We now derive the appropriate generalization of property (2) when using the standard inner product in $\mathbb{C}^n$. Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ be vectors in $\mathbb{C}^n$. Then, from Definition 4.11.8,

$$\langle \mathbf{v}, \mathbf{u} \rangle = v_1\overline{u}_1 + v_2\overline{u}_2 + \cdots + v_n\overline{u}_n = \overline{u_1\overline{v}_1 + u_2\overline{v}_2 + \cdots + u_n\overline{v}_n} = \overline{\langle \mathbf{u}, \mathbf{v} \rangle}.$$

Thus,

$$\langle \mathbf{v}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{v} \rangle}.$$

We now use the properties satisfied by the standard inner product in $\mathbb{C}^n$ to define an inner product in an arbitrary (that is, real or complex) vector space.

[10] Recall that if $z = a + ib$, then $\overline{z} = a - ib$ and $|z|^2 = z\overline{z} = (a+ib)(a-ib) = a^2 + b^2$.

DEFINITION 4.11.10

Let V be a (real or complex) vector space. A mapping that associates with each pair of vectors $\mathbf{u}$, $\mathbf{v}$ in V a scalar, denoted $\langle \mathbf{u}, \mathbf{v} \rangle$, is called an **inner product** in V, provided it satisfies the following properties. For all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V and all (real or complex) scalars k,

1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$. Furthermore, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
2. $\langle \mathbf{v}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{v} \rangle}$.
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$.
4. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.

The **norm** of $\mathbf{u}$ is defined in terms of the inner product by

$$||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

Remark Notice that the properties in the preceding definition reduce to those in Definition 4.11.3 in the case that V is a *real* vector space, since in such a case the complex conjugates are unnecessary. Thus, this definition is a consistent extension of Definition 4.11.3.

Example 4.11.11 Use properties 2 and 3 of Definition 4.11.10 to prove that in an inner product space

$$\langle \mathbf{u}, k\mathbf{v} \rangle = \overline{k}\langle \mathbf{u}, \mathbf{v} \rangle$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ and all scalars k.

Solution: From properties 2 and 3, we have

$$\langle \mathbf{u}, k\mathbf{v} \rangle = \overline{\langle k\mathbf{v}, \mathbf{u} \rangle} = \overline{k\langle \mathbf{v}, \mathbf{u} \rangle} = \overline{k}\ \overline{\langle \mathbf{v}, \mathbf{u} \rangle} = \overline{k}\ \langle \mathbf{u}, \mathbf{v} \rangle.$$

Notice that in the particular case of a real vector space, the foregoing result reduces to

$$\langle \mathbf{u}, k\mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle,$$

since in such a case the scalars are real numbers. □

Exercises for 4.11

Key Terms

Inner product, Axioms of an inner product, Real (complex) inner product space, Norm, Angle, Cauchy-Schwarz inequality.

Skills

- Know the four inner product space axioms.
- Be able to check whether or not a proposed inner product on a vector space V satisfies the inner product space axioms.
- Be able to compute the inner product of two vectors in an inner product space.
- Be able to find the norm of a vector in an inner product space.
- Be able to find the angle between two vectors in an inner product space.

True-False Review

For Questions 1–7, decide if the given statement is **true** or **false**, and give a brief justification for your answer. If true, you can quote a relevant definition or theorem from the text. If false, provide an example, illustration, or brief explanation of why the statement is false.

1. If $\mathbf{v}$ and $\mathbf{w}$ are linearly independent vectors in an inner product space V, then $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

2. In any inner product space V, we have
$$\langle k\mathbf{v}, k\mathbf{w} \rangle = k\langle \mathbf{v}, \mathbf{w} \rangle.$$

3. If $\langle \mathbf{v}_1, \mathbf{w} \rangle = \langle \mathbf{v}_2, \mathbf{w} \rangle = 0$ in an inner product space V, then
$$\langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \mathbf{w} \rangle = 0.$$

4. In any inner product space V, $\langle \mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle < 0$ if and only if $||\mathbf{x}|| < ||\mathbf{y}||$.

5. In any vector space V, there is at most one valid inner product $\langle\ ,\ \rangle$ that can be defined on V.

6. The angle between the vectors $\mathbf{v}$ and $\mathbf{w}$ in an inner product space V is the same as the angle between the vectors $-2\mathbf{v}$ and $-2\mathbf{w}$.

7. If $p(x) = a_0+a_1x+a_2x^2$ and $q(x) = b_0+b_1x+b_2x^2$, then we can define an inner product on P_2 via $\langle p, q \rangle = a_0b_0$.

Problems

1. Use the standard inner product in $\mathbb{R}^4$ to determine the angle between the vectors $\mathbf{v} = (1, 3, -1, 4)$ and $\mathbf{w} = (-1, 1, -2, 1)$.

2. If $f(x) = \sin x$ and $g(x) = x$ on $[0, \pi]$, use the function inner product defined in the text to determine the angle between f and g.

3. If $\mathbf{v} = (2+i, 3-2i, 4+i)$ and $\mathbf{w} = (-1+i, 1-3i, 3-i)$, use the standard inner product in $\mathbb{C}^3$ to determine, $\langle \mathbf{v}, \mathbf{w} \rangle$, $||\mathbf{v}||$, and $||\mathbf{w}||$.

4. Let
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$
be vectors in $M_2(\mathbb{R})$. Show that the mapping
$$\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} \tag{4.11.9}$$
defines an inner product in $M_2(\mathbb{R})$.

5. Referring to A and B in the previous problem, show that the mapping
$$\langle A, B \rangle = a_{11}b_{22} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{11}$$
does *not* define a valid inner product on $M_2(\mathbb{R})$.

For Problems 6–7, use the inner product (4.11.9) to determine $\langle A, B \rangle$, $||A||$, and $||B||$.

6. $A = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$.

7. $A = \begin{bmatrix} 3 & 2 \\ -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$.

8. Let $p_1(x) = a + bx$ and $p_2(x) = c + dx$ be vectors in P_1. Determine a mapping $\langle p_1, p_2 \rangle$ that defines an inner product on P_1.

Consider the vector space $\mathbb{R}^2$. Define the mapping $\langle\ ,\ \rangle$ by
$$\langle \mathbf{v}, \mathbf{w} \rangle = 2v_1w_1 + v_1w_2 + v_2w_1 + 2v_2w_2 \tag{4.11.10}$$
for all vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ in $\mathbb{R}^2$. This mapping is required for Problems 9–12.

9. Verify that Equation (4.11.10) defines an inner product on $\mathbb{R}^2$.

For Problems 10–12, determine the inner product of the given vectors using **(a)** the inner product (4.11.10), **(b)** the standard inner product in $\mathbb{R}^2$.

10. $\mathbf{v} = (1, 0)$, $\mathbf{w} = (-1, 2)$.

11. $\mathbf{v} = (2, -1)$, $\mathbf{w} = (3, 6)$.

12. $\mathbf{v} = (1, -2)$, $\mathbf{w} = (2, 1)$.

13. Consider the vector space $\mathbb{R}^2$. Define the mapping $\langle\ ,\ \rangle$ by
$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1w_1 - v_2w_2, \tag{4.11.11}$$
for all vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$. Verify that all of the properties in Definition 4.11.3 except (1) are satisfied by (4.11.11).

The mapping (4.11.11) is called a **pseudo-inner product** in $\mathbb{R}^2$ and, when generalized to $\mathbb{R}^4$, is of fundamental importance in Einstein's special relativity theory.

14. Using Equation (4.11.11), determine all nonzero vectors satisfying $\langle \mathbf{v}, \mathbf{v} \rangle = 0$. Such vectors are called **null** vectors.

15. Using Equation (4.11.11), determine all vectors satisfying $\langle \mathbf{v}, \mathbf{v} \rangle < 0$. Such vectors are called **timelike** vectors.

16. Using Equation (4.11.11), determine all vectors satisfying $\langle \mathbf{v}, \mathbf{v} \rangle > 0$. Such vectors are called **spacelike** vectors.

17. Make a sketch of $\mathbb{R}^2$ and indicate the position of the null, timelike, and spacelike vectors.

18. Consider the vector space $\mathbb{R}^n$, and let $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ be vectors in $\mathbb{R}^n$. Show that the mapping $\langle , \rangle$ defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = k_1 v_1 w_1 + k_2 v_2 w_2 + \cdots + k_n v_n w_n$$

is a valid inner product on $\mathbb{R}^n$ if and only if the constants $k_1, k_2, \dots, k_n$ are all positive.

19. Prove from the inner product axioms that, in any inner product space V, $\langle \mathbf{v}, \mathbf{0} \rangle = 0$ for all $\mathbf{v}$ in V.

20. Let V be a real inner product space.

(a) Prove that for all $\mathbf{v}, \mathbf{w} \in V$,

$$||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + ||\mathbf{w}||^2.$$

[**Hint**: $||\mathbf{v} + \mathbf{w}||^2 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle$.]

(b) Two vectors $\mathbf{v}$ and $\mathbf{w}$ in an inner product space V are called *orthogonal* if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. Use (a) to prove the general **Pythagorean theorem:** If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal in an inner product space V, then

$$||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2.$$

(c) Prove that for all $\mathbf{v}, \mathbf{w}$ in V,

(i) $||\mathbf{v} + \mathbf{w}||^2 - ||\mathbf{v} - \mathbf{w}||^2 = 4\langle \mathbf{v}, \mathbf{w} \rangle$.

(ii) $||\mathbf{v} + \mathbf{w}||^2 + ||\mathbf{v} - \mathbf{w}||^2 = 2(||\mathbf{v}||^2 + ||\mathbf{w}||^2)$.

21. Let V be a complex inner product space. Prove that for all $\mathbf{v}, \mathbf{w}$ in V,

$$||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + 2\mathrm{Re}(\langle \mathbf{v}, \mathbf{w} \rangle) + ||\mathbf{v}||^2,$$

where Re denotes the real part of a complex number.

4.12 Orthogonal Sets of Vectors and the Gram-Schmidt Process

The discussion in the previous section has shown how an inner product can be used to define the angle between two nonzero vectors. In particular, if the inner product of two nonzero vectors is zero, then the angle between those two vectors is $\pi/2$ radians, and therefore it is natural to call such vectors orthogonal (perpendicular). The following definition extends the idea of orthogonality into an arbitrary inner product space.

DEFINITION 4.12.1

Let V be an inner product space.

1. Two vectors $\mathbf{u}$ and $\mathbf{v}$ in V are said to be **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.
2. A set of nonzero vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in V is called an **orthogonal set** of vectors if
$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0, \qquad \text{whenever } i \neq j.$$
(That is, every vector is orthogonal to every other vector in the set.)
3. A vector $\mathbf{v}$ in V is called a **unit vector** if $||\mathbf{v}|| = 1$.
4. An *orthogonal* set of *unit* vectors is called an **orthonormal set** of vectors. Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in V is an orthonormal set if and only if

   (a) $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ whenever $i \neq j$.

   (b) $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$ for all $i = 1, 2, \dots, k$.