Chapter 4

Inner-Product Spaces, Euclidean Spaces

As in Chap.2, the term "linear space" will be used as a shorthand for "finite dimensional linear space over $\mathbb{R}$". However, the definitions of an inner-product space and a Euclidean space do not really require finite-dimensionality. Many of the results, for example the Inner-Product Inequality and the Theorem on Subadditivity of Magnitude, remain valid for infinite-dimensional spaces. Other results extend to infinite-dimensional spaces after suitable modification.

41 Inner-Product Spaces

Definition 1: *An* **inner-product space** *is a linear space* $\mathcal{V}$ *endowed with additional structure by the prescription of a non-degenerate quadratic form* $\mathrm{sq} \in \mathrm{Qu}(\mathcal{V})$ (see Sect.27). *The form* sq *is then called the* **inner square** *of* $\mathcal{V}$ *and the corresponding symmetric bilinear form*

$$\mathrm{ip} := \overline{sq} \in \mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R}) \cong \mathrm{Sym}(\mathcal{V}, \mathcal{V}^*)$$

the **inner product** *of* $\mathcal{V}$.

We say that the inner-product space $\mathcal{V}$ *is* **genuine** *if* sq *is strictly positive.*

It is customary to use the following simplified notations:

$$\mathbf{v}^{\cdot 2} := \mathrm{sq}(\mathbf{v}) \quad \text{when} \quad \mathbf{v} \in \mathcal{V}, \tag{41.1}$$

$$\mathbf{u} \cdot \mathbf{v} := \mathrm{ip}\,(\mathbf{u}, \mathbf{v}) = (\mathrm{ip}\,\mathbf{u})\mathbf{v} \quad \text{when} \quad \mathbf{u}, \mathbf{v} \in \mathcal{V}. \tag{41.2}$$

The symmetry and bilinearity of ip is then reflected in the following rules, valid for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathcal{V}$ and $\xi \in \mathbb{R}$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \mathbf{v} \cdot \mathbf{u}, & (41.3)\\ \mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) &= \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}, & (41.4)\\ \mathbf{u} \cdot (\xi \mathbf{v}) &= \xi(\mathbf{u} \cdot \mathbf{v}) = (\xi \mathbf{u}) \cdot \mathbf{v} & (41.5) \end{aligned}$$

We say that $\mathbf{u} \in \mathcal{V}$ is **orthogonal to** $\mathbf{v} \in \mathcal{V}$ if $\mathbf{u} \cdot \mathbf{v} = 0$. The assumption that the inner product is non-degenerate is expressed by the statement that, given $\mathbf{u} \in \mathcal{V}$,

$$(\mathbf{u} \cdot \mathbf{v} = 0 \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}) \implies \mathbf{u} = \mathbf{0}. \tag{41.6}$$

In words, the zero of $\mathcal{V}$ is the only member of $\mathcal{V}$ that is orthogonal to every member of $\mathcal{V}$.

Since ip $\in \mathrm{Lin}(\mathcal{V}, \mathcal{V}^*)$ is injective and since $\dim \mathcal{V} = \dim \mathcal{V}^*$, it follows from the Pigeonhole Principle for Linear Mappings that ip is a linear isomorphism. It induces the linear isomorphism $\mathrm{ip}^\top : \mathcal{V}^{**} \to \mathcal{V}^*$. Since ip is symmetric, we have $\mathrm{ip}^\top = \mathrm{ip}$ when $\mathcal{V}^{**}$ is identified with $\mathcal{V}$ as explained in Sect.22. Thus, this identification is the same as the isomorphism $(\mathrm{ip}^\top)^{-1}\mathrm{ip} : \mathcal{V} \to \mathcal{V}^{**}$ induced by ip . Therefore, there is no conflict if we use ip to identify $\mathcal{V}$ with $\mathcal{V}^*$.

From now on we shall identify $\mathcal{V} \cong \mathcal{V}^*$ by means of ip except that, given $\mathbf{v} \in \mathcal{V}$, we shall write $\mathbf{v}\cdot := \mathrm{ip}\,\mathbf{v}$ for the corresponding element in $\mathcal{V}^*$ so as to be consistent with the notation (41.2).

Every space $\mathbb{R}^I$ of families of numbers, indexed on a given finite set I, carries the natural structure of an inner-product space whose inner square is given by

$$\mathrm{sq}(\lambda) = \lambda^{\cdot 2} := \sum \lambda^2 = \sum_{i \in I} \lambda_i^2 \tag{41.7}$$

for all $\lambda \in \mathbb{R}^I$. The corresponding inner product is given by

$$(\mathrm{ip}\ \lambda)\mu = \lambda \cdot \mu = \sum \lambda\mu = \sum_{i \in I} \lambda_i \mu_i \tag{41.8}$$

for all $\lambda, \mu \in \mathbb{R}^I$. The identification $(\mathbb{R}^I)^* \cong \mathbb{R}^I$ resulting from this inner product is the same as the one described in Sect.23.

Let $\mathcal{V}$ and $\mathcal{W}$ be inner-product spaces. The identifications $\mathcal{V}^* \cong \mathcal{V}$ and $\mathcal{W}^* \cong \mathcal{W}$ give rise to the further identifications such as

$$\mathrm{Lin}(\mathcal{V}, \mathcal{W}) \cong \mathrm{Lin}(\mathcal{V}, \mathcal{W}^*) \cong \mathrm{Lin}_2(\mathcal{V} \times \mathcal{W}, \mathbb{R}),$$

$$\mathrm{Lin}(\mathcal{W},\mathcal{V}) \cong \mathrm{Lin}(\mathcal{W}^*,\mathcal{V}^*).$$

Thus $\mathbf{L} \in \mathrm{Lin}(\mathcal{V},\mathcal{W})$ becomes identified with the bilinear form $\mathbf{L} \in \mathrm{Lin}_2(\mathcal{V}\times\mathcal{W},\mathbb{R})$ whose value at $(\mathbf{v},\mathbf{w}) \in \mathcal{V}\times\mathcal{W}$ satisfies

$$\mathbf{L}(\mathbf{v},\mathbf{w}) = (\mathbf{L}\mathbf{v})\cdot\mathbf{w}, \tag{41.9}$$

and $\mathbf{L}^\top \in \mathrm{Lin}(\mathcal{W}^*,\mathcal{V}^*)$ becomes identified with $\mathbf{L}^\top \in \mathrm{Lin}(\mathcal{W},\mathcal{V}) \cong \mathrm{Lin}_2(\mathcal{W}\times\mathcal{V},\mathbb{R})$ in such a way that

$$(\mathbf{L}^\top\mathbf{w})\cdot\mathbf{v} = \mathbf{L}^\top(\mathbf{w},\mathbf{v}) = \mathbf{L}(\mathbf{v},\mathbf{w}) = (\mathbf{L}\mathbf{v})\cdot\mathbf{w} \tag{41.10}$$

for all $\mathbf{v}\in\mathcal{V}, \mathbf{w}\in\mathcal{W}$.

We identify the supplementary subspaces $\mathrm{Sym}_2(\mathcal{V}^2,\mathbb{R})$ and $\mathrm{Skew}_2(\mathcal{V}^2,\mathbb{R})$ of $\mathrm{Lin}_2(\mathcal{V}^2,\mathbb{R}) \cong \mathrm{Lin}\mathcal{V}$ (see Prop.7 of Sect.24) with the supplementary subspaces

$$\mathrm{Sym}\mathcal{V} := \{\mathbf{S}\in\mathrm{Lin}\mathcal{V} \mid \mathbf{S}=\mathbf{S}^\top\}, \tag{41.11}$$

$$\mathrm{Skew}\mathcal{V} := \{\mathbf{A}\in\mathrm{Lin}\mathcal{V} \mid \mathbf{A}^\top=-\mathbf{A}\} \tag{41.12}$$

of $\mathrm{Lin}\mathcal{V}$. The members of $\mathrm{Sym}\mathcal{V}$ are called **symmetric lineons** and the members of $\mathrm{Skew}\mathcal{V}$ **skew lineons**.

Given $\mathbf{L}\in\mathrm{Lin}(\mathcal{V},\mathcal{W})$ and hence $\mathbf{L}^\top\in\mathrm{Lin}(\mathcal{W},\mathcal{V})$ we have $\mathbf{L}^\top\mathbf{L}\in\mathrm{Sym}\mathcal{V}$ and $\mathbf{L}\mathbf{L}^\top\in\mathrm{Sym}\mathcal{W}$. Clearly, if $\mathbf{L}^\top\mathbf{L}$ is invertible (and hence injective), then $\mathbf{L}$ must also be injective. Also, if $\mathbf{L}$ is invertible, so is $\mathbf{L}^\top\mathbf{L}$. Applying these observations to the linear-combination mapping of a family $\mathbf{f} := (\mathbf{f}_i \mid i\in I)$ in $\mathcal{V}$ and noting the identification

$$G_{\mathbf{f}} := \mathrm{lnc}_{\mathbf{f}}^\top \mathrm{lnc}_{\mathbf{f}} = (\mathbf{f}_i\cdot\mathbf{f}_k \mid (i,k)\in I^2) \in \mathbb{R}^{I^2} \cong \mathrm{Lin}(\mathbb{R}^I) \tag{41.13}$$

we obtain the following results.

Proposition 1: *Let $\mathbf{f} := (\mathbf{f}_i \mid i\in I)$ be a family in $\mathcal{V}$. Then $\mathbf{f}$ is linearly independent if the matrix $G_{\mathbf{f}}$ given by (41.13) is invertible.*

Proposition 2: *A family $\mathbf{b} := (\mathbf{b}_i \mid i\in I)$ in $\mathcal{V}$ is a basis if and only if the matrix $G_{\mathbf{b}}$ is invertible and $\sharp I = \dim\mathcal{V}$.*

Let $\mathbf{b} := (\mathbf{b}_i \mid i\in I)$ be a basis of $\mathcal{V}$. The identification $\mathcal{V}^*\cong\mathcal{V}$ identifies the dual of the basis $\mathbf{b}$ with another basis $\mathbf{b}^*$ of $\mathcal{V}$ in such a way that

$$\mathbf{b}_i^*\cdot\mathbf{b}_k = \delta_{i,k} \quad \text{for all} \quad i,k\in I, \tag{41.14}$$

where $\delta_{i,k}$ is defined by (16.2). Using the notation (41.13) we have

$$\mathbf{b}_k = \sum_{i\in I}(G_{\mathbf{b}})_{k,i}\mathbf{b}_i^* \tag{41.15}$$

and

$$G_{\mathbf{b}^*} = G_{\mathbf{b}}^{-1}. \tag{41.16}$$

Moreover, for each $\mathbf{v} \in \mathcal{V}$, we have

$$\mathrm{lnc}_{\mathbf{b}}^{-1}(\mathbf{v}) = \mathbf{b}^* \cdot \mathbf{v} := (\mathbf{b}_i^* \cdot \mathbf{v} \mid i \in I) \tag{41.17}$$

and hence

$$\mathbf{v} = \mathrm{lnc}_{\mathbf{b}}(\mathbf{b}^* \cdot \mathbf{v}) = \mathrm{lnc}_{\mathbf{b}^*}(\mathbf{b} \cdot \mathbf{v}). \tag{41.18}$$

For all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, we have

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{b} \cdot \mathbf{u}) \cdot (\mathbf{b}^* \cdot \mathbf{v}) = \sum_{i \in I} (\mathbf{b}_i \cdot \mathbf{u})(\mathbf{b}_i^* \cdot \mathbf{v}). \tag{41.19}$$

We say that a family $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ in $\mathcal{V}$ is **orthonormal** if

$$\mathbf{e}_i \cdot \mathbf{e}_k = \left\{ \begin{array}{ccc} 1 \text{ or} -1 & \text{if} & i = k \\ 0 & \text{if} & i \neq k \end{array} \right\} \quad \text{for all} \quad i, k \in I \tag{41.20}$$

We say that $\mathbf{e}$ is **genuinely orthonormal** if, in addition, $\mathbf{e}_i^{\cdot 2} = 1$ for all $i \in I$, which—using the notation (16.2)—means that

$$\mathbf{e}_i \cdot \mathbf{e}_k = \delta_{i,k} \quad \text{for all} \quad i, k \in I. \tag{41.21}$$

To say that $\mathbf{e}$ is orthonormal means that the matrix $G_{\mathbf{e}}$, as defined by (41.13), is diagonal and each of its diagonal terms is 1 or -1. It is clear from Prop.1 that every orthonormal family is linearly independent and from Prop.2 that a given orthonormal family is a basis if and only if $\sharp I = \dim \mathcal{V}$. Comparing (41.21) with (41.14), we see that a basis $\mathbf{b}$ is genuinely orthonormal if and only if it coincides with its dual $\mathbf{b}^*$.

The standard basis $\delta^I := (\delta_i^I \mid i \in I)$ (see Sect.16) is a genuinely orthonormal basis of $\mathbb{R}^I$.

Let $\mathbf{v} \in \mathcal{V}, \mathbf{w} \in \mathcal{W}$ be given. Then $\mathbf{w} \otimes \mathbf{v} \in \mathrm{Lin}(\mathcal{V}^*, \mathcal{W})$ becomes identified with the element $\mathbf{w} \otimes \mathbf{v}$ of $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$ whose values are given by

$$(\mathbf{w} \otimes \mathbf{v})\mathbf{u} = (\mathbf{v} \cdot \mathbf{u})\mathbf{w} \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}. \tag{41.22}$$

We say that a subspace $\mathcal{U}$ of a given inner-product space $\mathcal{V}$ is **regular** if the restriction $\mathrm{sq}|_{\mathcal{U}}$ of the inner square to $\mathcal{U}$ is non-degenerate, i.e., if for each $\mathbf{u} \in \mathcal{U}$,

$$(\mathbf{u} \cdot \mathbf{v} = 0 \quad \text{for all} \quad \mathbf{v} \in \mathcal{U}) \implies \mathbf{u} = \mathbf{0}. \tag{41.23}$$

If $\mathcal{U}$ is regular, then $\text{sq}|_{\mathcal{U}}$ endows $\mathcal{U}$ with the structure of an inner-product space. If $\mathcal{U}$ is not regular, it does not have a natural structure of an inner-product space.

The identification $\mathcal{V}^* \cong \mathcal{V}$ identifies the annihilator $\mathcal{S}^\perp$ of a given subset $\mathcal{S}$ of $\mathcal{V}$ with the subspace

$$\mathcal{S}^\perp = \{\mathbf{v} \in \mathcal{V} \mid \mathbf{v} \cdot \mathbf{u} = 0 \quad \text{for all} \quad \mathbf{u} \in \mathcal{S}\}$$

of $\mathcal{V}$. Thus, $\mathcal{S}^\perp$ consists of all elements of $\mathcal{V}$ that are orthogonal to every element of $\mathcal{S}$. The following result is very easy to prove with the use of the Formula for Dimension of Annihilators (Sect.21) and of Prop.5 of Sect.17.

Characterization of Regular Subspaces: *Let $\mathcal{U}$ be a subspace of $\mathcal{V}$. Then the following are equivalent:*

(i) $\mathcal{U}$ *is regular,*

(ii) $\mathcal{U} \cap \mathcal{U}^\perp = \{\mathbf{0}\}$,

(iii) $\mathcal{U} + \mathcal{U}^\perp = \mathcal{V}$,

(iv) $\mathcal{U}$ *and* $\mathcal{U}^\perp$ *are supplementary.*

Moreover, if $\mathcal{U}$ is regular, so is $\mathcal{U}^\perp$.

If $\mathcal{U}$ is regular, then its annihilator $\mathcal{U}^\perp$ is also called the **orthogonal supplement** of $\mathcal{U}$. The following two results exhibit a natural one-to-one correspondence between the regular subspaces of $\mathcal{V}$ and the symmetric idempotents in $\text{Lin}\mathcal{V}$, i.e., the lineons $\mathbf{E} \in \text{Lin}\mathcal{V}$ that satisfy $\mathbf{E} = \mathbf{E}^\top$ and $\mathbf{E}^2 = \mathbf{E}$.

Proposition 3: *Let $\mathbf{E} \in \text{Lin}\mathcal{V}$ be an idempotent. Then $\mathbf{E}$ is symmetric if and only if* $\text{Null}\ \mathbf{E} = (\text{Rng}\,\mathbf{E})^\perp$.

Proof: If $\mathbf{E} = \mathbf{E}^\top$, then $\text{Null}\ \mathbf{E} = (\text{Rng}\,\mathbf{E})^\perp$ follows from (21.13).

Assume that $\text{Null}\ \mathbf{E} = (\text{Rng}\,\mathbf{E})^\perp$. It follows from (21.13) that $\text{Null}\ \mathbf{E} = \text{Null}\ \mathbf{E}^\top$ and hence from (22.9) that $\text{Rng}\,\mathbf{E}^\top = \text{Rng}\,\mathbf{E}$. Since, by (21.6), $(\mathbf{E}^\top)^2 = (\mathbf{E}^2)^\top = \mathbf{E}^\top$, it follows that $\mathbf{E}^\top$ is also an idempotent. The assertion of uniqueness in Prop.4 of Sect.19 shows that $\mathbf{E} = \mathbf{E}^\top$. ∎

Proposition 4: *If $\mathbf{E} \in \text{Lin}\mathcal{V}$ is symmetric and idempotent, then $\text{Rng}\,\mathbf{E}$ is a regular subspace of $\mathcal{V}$ and $\text{Null}\ \mathbf{E}$ is its orthogonal supplement. Conversely, if $\mathcal{U}$ is a regular subspace of $\mathcal{V}$, then there is exactly one symmetric idempotent $\mathbf{E} \in \text{Lin}\mathcal{V}$ such that $\mathcal{U} = \text{Rng}\,\mathbf{E}$.*

Proof: Assume that $\mathbf{E} \in \text{Lin}\mathcal{V}$ is symmetric and idempotent. It follows from Prop.3 that $\text{Null}\ \mathbf{E} = (\text{Rng}\,\mathbf{E})^\perp$ and hence by the implication (v) $\Rightarrow$ (i) of Prop.4 of Sect.19 that $\text{Rng}\,\mathbf{E}$ and $(\text{Rng}\,\mathbf{E})^\perp$ are supplementary. Hence, by the implication (iv) $\Rightarrow$ (i) of the Theorem on the Characterization of Regular Subspaces, applied to $\mathcal{U} := \operatorname{Rng} \mathbf{E}$, it follows that $\operatorname{Rng} \mathbf{E}$ is regular.

Assume now that $\mathcal{U}$ is a regular subspace of $\mathcal{V}$. By the implication (i) $\Rightarrow$ (iv) of the Theorem just mentioned, $\mathcal{U}^{\perp}$ is then a supplement of $\mathcal{U}$. By Prop.3, an idempotent $\mathbf{E}$ with $\operatorname{Rng} \mathbf{E} = \mathcal{U}$ is symmetric if and only if $\operatorname{Null} \mathbf{E} = \mathcal{U}^{\perp}$. By Prop.4 of Sect.19 there is exactly one such idempotent. ■

Notes 41

(1) Most textbooks deal only with what we call "genuine" inner-product spaces and use the term "inner-product space" only in this restricted sense. The terms "Euclidean vector space" or even "Euclidean space" are also often used in this sense.

(2) The terms "scalar product" or "dot product" are sometimes used instead of "inner product".

(3) Other notations for the value $\mathbf{u} \cdot \mathbf{v}$ of the inner product are $\langle \mathbf{u}, \mathbf{v} \rangle$, $(\mathbf{u}|\mathbf{v})$, $\langle \mathbf{u}|\mathbf{v} \rangle$, and $(\mathbf{u}, \mathbf{v})$. The last is very common, despite the fact that it clashes with the notation for the pair with terms $\mathbf{u}$ and $\mathbf{v}$.

(4) Some people use simply $\mathbf{v}^2$ instead of $\mathbf{v}^{\cdot 2}$ for an inner square. In fact, I have done this many times myself. We use $\mathbf{v}^{\cdot 2}$ because omitting the dot would lead to confusion of the compositional square with the inner square of a lineon (see Sect.44).

(5) The terms of $\mathbf{b}^* \cdot \mathbf{v}$ are often called the "contravariant components" of $\mathbf{v}$ relative to the basis $\mathbf{b}$ and are denoted by v^i. The terms of $\mathbf{b} \cdot \mathbf{v}$ are called "covariant components" and are denoted by v_i.

(6) In most of the literature, the term "orthonormal" is used for what we call "genuinely orthonormal".

(7) Some people use the term "non-isotropic" or "non-singular" when we speak of a "regular" subspace.

(8) Some authors use the term "perpendicular projection" or the term "orthogonal projection" to mean the same as our "symmetric idempotent". See also Note 1 to Sect.19.

(9) The dual $\mathbf{b}^*$ of a basis $\mathbf{b}$ of an inner-product space $\mathcal{V}$, when regarded as a basis of $\mathcal{V}$ rather than $\mathcal{V}^*$, is often called the "reciprocal" of $\mathbf{b}$.

42 Genuine Inner-Product Spaces

In this section, a genuine inner-product space $\mathcal{V}$ is assumed to be given. We then have

$$\mathbf{v}^{\cdot 2} > 0 \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}^{\times}. \tag{42.1}$$

Of course, if $\mathcal{U}$ is any subspace of $\mathcal{V}$, then the restriction $\operatorname{sq}|_{\mathcal{U}}$ of the inner square of $\mathcal{V}$ to $\mathcal{U}$ is again strictly positive and hence endows $\mathcal{U}$ with the natural structure of a genuine inner-product space. Thus *every* subspace $\mathcal{U}$ of $\mathcal{V}$ is regular and hence has an orthogonal supplement $\mathcal{U}^{\perp}$.

Definition 1: *For every $\mathbf{v} \in \mathcal{V}$, the number $|\mathbf{v}| \in \mathbb{P}$ defined by*

$$|\mathbf{v}| := \sqrt{\mathbf{v}^{\cdot 2}} \tag{42.2}$$

is called the **magnitude** *of* $\mathbf{v}$. *We say that $\mathbf{u} \in \mathcal{V}$ is a* **unit vector** *if* $|\mathbf{u}| = 1$.

In the case when $\mathcal{V}$ is $\mathbb{R}$ the magnitude turns out to be the absolute value; hence the notation (42.2) is consistent with the usual notation for absolute values.

It follows from (42.1) that for all $\mathbf{v} \in \mathcal{V}$,

$$|\mathbf{v}| = 0 \iff \mathbf{v} = \mathbf{0}. \tag{42.3}$$

The following formula follows directly from (42.2) and (41.5):

$$|\xi\mathbf{v}| = |\xi||\mathbf{v}| \quad \text{for all} \quad \mathbf{v} \in \mathcal{V},\ \xi \in \mathbb{R}. \tag{42.4}$$

Inner-Product Inequality: *Every pair $(\mathbf{u}, \mathbf{v})$ in a genuine inner-product space $\mathcal{V}$ satisfies*

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|; \tag{42.5}$$

equality holds if and only if $(\mathbf{u}, \mathbf{v})$ is linearly dependent.

Proof: Assume, first, that $(\mathbf{u}, \mathbf{v})$ is linearly independent. Then, by (42.3), $|\mathbf{u}| \neq 0$, $|\mathbf{v}| \neq 0$. Moreover, if we put $\mathbf{w} := |\mathbf{v}|^2\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{v}$, then $\mathbf{w}$ cannot be zero because it is a linear combination of $(\mathbf{u}, \mathbf{v})$ with at least one coefficient, namely $|\mathbf{v}|^2$, not zero. Hence, by (42.1), we have

$$\begin{aligned} 0 < \mathbf{w}^{\cdot 2} &= (|\mathbf{v}|^2)^2\mathbf{u}^{\cdot 2} - 2|\mathbf{v}|^2(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{v})^2\mathbf{v}^{\cdot 2} \\ &= |\mathbf{v}|^2(|\mathbf{v}|^2|\mathbf{u}|^2 - (\mathbf{u} \cdot \mathbf{v})^2), \end{aligned}$$

which is equivalent to (42.5) with equality excluded.

Assume, now, that $(\mathbf{u}, \mathbf{v})$ is linearly dependent. Then one of $\mathbf{u}$ and $\mathbf{v}$ is a scalar multiple of the other. Without loss we may assume, for example, that $\mathbf{u} = \xi\mathbf{v}$ for some $\xi \in \mathbb{R}$. By (42.4), we then have

$$|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u} \cdot (\xi\mathbf{u})| = |\xi\mathbf{u}^{\cdot 2}| = |\xi||\mathbf{u}||\mathbf{u}| = |\mathbf{v}||\mathbf{u}|,$$

which shows that (42.5) holds with equality. ∎

Subadditivity of Magnitude: *For every $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ we have*

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|. \tag{42.6}$$

Proof: Using (42.5) we find

$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v})^{\cdot 2} = \mathbf{u}^{\cdot 2} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}^{\cdot 2} \\ &\leq |\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}|+|\mathbf{v}|)^2, \end{aligned}$$

which is equivalent to (42.6). ■

Proposition 1: *An inner-product space $\mathcal{V}$ is genuine if and only if it has some genuinely orthonormal basis.*

Proof: Assume that $\mathcal{V}$ has a genuinely orthonormal basis $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ and let $\mathbf{v} \in \mathcal{V}^\times$ be given. Since $\mathbf{e} = \mathbf{e}^*$, we infer from (41.19) that

$$\mathbf{v}^{\cdot 2} = \mathbf{v}\cdot\mathbf{v} = \sum_{i\in I} (\mathbf{e}_i \cdot \mathbf{v})^2,$$

which is strictly positive. Since $\mathbf{v} \in \mathcal{V}^\times$ was arbitrary, it follows that $\mathcal{V}$ is genuine.

Assume, now, that $\mathcal{V}$ is genuine. Let $\mathfrak{e}$ be an orthonormal set in $\mathcal{V}$. We have seen in the previous section that $\mathfrak{e}$ is linearly independent. Hence it is a basis if and only if $\mathrm{Lsp}\,\mathfrak{e} = \mathcal{V}$. Now, if $\mathrm{Lsp}\,\mathfrak{e}$ is a proper subset of $\mathcal{V}$, then its orthogonal supplement $(\mathrm{Lsp}\,\mathfrak{e})^\perp$ is not the zero-space. Hence we may choose $\mathbf{u} \in ((\mathrm{Lsp}\,\mathfrak{e})^\perp)^\times$ and put $\mathbf{f} := \frac{1}{|\mathbf{u}|}\mathbf{u}$, so that $\mathbf{f}^{\cdot 2} = 1$. It is clear that $\mathfrak{e} \cup \{\mathbf{f}\}$ is an orthonormal set that has $\mathfrak{e}$ as a proper subset. It follows that every maximal orthonormal set in $\mathcal{V}$ is a basis. Such sets exist because the empty set $\emptyset$ is orthonormal and because no orthonormal set in $\mathcal{V}$ can have more than $\dim \mathcal{V}$ members. ■

Definition 2: *The set of all elements of $\mathcal{V}$ with magnitude strictly less than* 1 *is called the* **unit ball** *of $\mathcal{V}$ and is denoted by*

$$\mathrm{Ubl}\mathcal{V} := \{\mathbf{v} \in \mathcal{V} \mid |\mathbf{v}| < 1\} = \mathrm{sq}^{<}([0,1[). \tag{42.7}$$

The set of all element of $\mathcal{V}$ with magnitude less than 1 *is called the* **closed unit ball** *of $\mathcal{V}$ and is denoted by*

$$\overline{\mathrm{Ubl}}\mathcal{V} := \{\mathbf{v} \in \mathcal{V} \mid |\mathbf{v}| \leq 1\} = \mathrm{sq}^{<}([0,1]). \tag{42.8}$$

The set of all unit vectors in $\mathcal{V}$ is called the **unit sphere** *of $\mathcal{V}$ and is denoted by*

$$\mathrm{Usph}\mathcal{V} := \{\mathbf{v} \in \mathcal{V} \mid |\mathbf{v}| = 1\} = \mathrm{sq}^{<}(\{1\}) = \overline{\mathrm{Ubl}}\mathcal{V} \setminus \mathrm{Ubl}\mathcal{V}. \tag{42.9}$$

Notes 42

(1) The magnitude $|\mathbf{v}|$ is often called "Euclidean norm", "norm", or "length".

(2) Many people use $||\mathbf{v}||$ instead of $|\mathbf{v}|$ for the magnitude. I consider this a waste of a good symbol that could be employed for something else. In fact, in this book we reserve the use of double bars for operator-norms only (see Sect.52) and thus have an easy notational distinction between the magnitude $|\mathbf{L}|$ and the operator-norm $||\mathbf{L}||$ of a lineon $\mathbf{L}$ (see Example 3 in Sect.52).

(3) The Inner-Product Inequality is often called "Cauchy's inequality" (particularly in France), "Schwarz's inequality" (particularly in Germany), or "Bunyakovsky's inequality" (particularly in Russia). Various combinations of these three names are also sometimes used.

43 Orthogonal Mappings

Let $\mathcal{V}$ and $\mathcal{V}'$ be inner-product spaces.

Definition 1: *A mapping* $\mathbf{R} : \mathcal{V} \to \mathcal{V}'$ *is called an* **orthogonal mapping** *if it is linear and preserves inner squares, i.e. if* $\mathbf{R} \in \text{Lin}(\mathcal{V}, \mathcal{V}')$ *and*

$$(\mathbf{R}\mathbf{v})^{\cdot 2} = \mathbf{v}^{\cdot 2} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}. \tag{43.1}$$

The set of all orthogonal mappings from $\mathcal{V}$ to $\mathcal{V}'$ will be denoted by $\text{Orth}(\mathcal{V}, \mathcal{V}')$. An orthogonal mapping that has an orthogonal inverse will be called an **orthogonal isomorphism**. We say that $\mathcal{V}$ and $\mathcal{V}'$ are *orthogonally isomorphic* if there exists an orthogonal isomorphism from $\mathcal{V}$ to $\mathcal{V}'$. Of course, if $\mathcal{V}$ and $\mathcal{V}'$ are orthogonally isomorphic, they are also linearly isomorphic; but they may be linearly isomorphic without being orthogonally isomorphic (see Prop.1 of Sect.47 below). The following is evident from Def.1 and from Props.1, 2 of Sect.13.

Proposition 1: *The identity mapping of an inner-product space is orthogonal. The composite of two orthogonal mappings is orthogonal. If an orthogonal mapping is invertible, its inverse is again orthogonal, and hence it is an orthogonal isomorphism.*

The following is a direct consequence of Def.1 and of Prop.3 of Sect.27, applied to $\mathbf{Q} := \text{sq}$.

Proposition 2: $\mathbf{R} \in \text{Lin}(\mathcal{V}, \mathcal{V}')$ *is orthogonal if and only if*

$$\mathbf{R}^\top \mathbf{R} = \mathbf{1}_{\mathcal{V}}, \tag{43.2}$$

or, equivalently, $\mathbf{R}$ *preserves inner products.*

As an immediate consequence of Prop.2 and the Pigeonhole Principle for Linear Mappings we obtain the following result.

Proposition 3: *For every orthogonal mapping* $\mathbf{R} \in \mathrm{Orth}(\mathcal{V}, \mathcal{V}')$ *we have:* $\mathbf{R}$ *is injective,* $\mathrm{Rng}\,\mathbf{R}$ *is a regular subspace of* $\mathcal{V}'$*, and* $\mathbf{R}|^{\mathrm{Rng}}$ *is an orthogonal isomorphism from* $\mathcal{V}$ *to* $\mathrm{Rng}\,\mathbf{R}$*. Moreover,* $\mathbf{R}$ *is an orthogonal isomorphism if and only if* $\mathbf{R}$ *is surjective or, equivalently,*

$$\mathbf{R}\mathbf{R}^\top = \mathbf{1}_{\mathcal{V}'}. \tag{43.3}$$

This is the case if and only if $\dim \mathcal{V} = \dim \mathcal{V}'$*.*

We write $\mathrm{Orth}\mathcal{V} := \mathrm{Orth}(\mathcal{V}, \mathcal{V})$ and call its members **orthogonal lineons**. In view of Prop.1 and Prop.3, $\mathrm{Orth}\mathcal{V}$ is a subgroup of $\mathrm{Lis}\mathcal{V}$ and consists of all orthogonal automorphisms of $\mathcal{V}$. The group $\mathrm{Orth}\mathcal{V}$ is called the **orthogonal group** of the inner product space $\mathcal{V}$.

Proposition 4: *Let* $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ *be a basis of the inner-product space* $\mathcal{V}$*, let* $\mathcal{V}'$ *be an inner-product space and* $\mathbf{R} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}')$*. Then* $\mathbf{R}$ *is orthogonal if and only if*

$$\mathbf{R}\mathbf{b}_i \cdot \mathbf{R}\mathbf{b}_k = \mathbf{b}_i \cdot \mathbf{b}_k \quad \text{for all} \quad i, k \in I. \tag{43.4}$$

In particular, if $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ *is a genuinely orthonormal basis, then* $\mathbf{R}$ *is orthogonal if and only if* $\mathbf{R}\mathbf{e} := (\mathbf{R}\mathbf{e}_i \mid i \in I)$ *is a genuinely orthonormal family.*

Proof: Since $\mathbf{R}\mathbf{b}_i \cdot \mathbf{R}\mathbf{b}_k = (\mathbf{R}^\top\mathbf{R}\mathbf{b}_i) \cdot \mathbf{b}_k$ for all $i, k \in I$, (43.4) states that for each $i \in I$, $(\mathbf{R}^\top\mathbf{R}\mathbf{b}_i)\cdot \in \mathcal{V}^*$ and $\mathbf{b}_i\cdot \in \mathcal{V}^*$ agree on a basis. Hence, by Prop.2 of Sect.16, (43.4) holds if and only if $\mathbf{R}^\top\mathbf{R}\mathbf{b}_i = \mathbf{b}_i$ for all $i \in I$, which, in turn, is the case if and only if $\mathbf{R}^\top\mathbf{R} = \mathbf{1}_{\mathcal{V}}$. The assertion follows then from Prop.2. ■

Proposition 5: *Let* $\mathcal{V}, \mathcal{V}'$ *be genuine inner-product spaces. Then* $\mathcal{V}$ *and* $\mathcal{V}'$ *are orthogonally isomorphic if and only if* $\dim \mathcal{V} = \dim \mathcal{V}'$*.*

Proof: The "only if" part follows from Cor.2 to the Theorem on Characterization of Dimension of Sect.17.

Assume that $\dim \mathcal{V} = \dim \mathcal{V}' =: n$. By Prop.1 of Sect.42 and by Cor.1 to the Theorem on Characterization of Dimension, we may choose genuinely orthonormal bases $\mathbf{e} := (\mathbf{e}_i \mid i \in n^]) $ and $\mathbf{e}' := (\mathbf{e}'_i \mid i \in n^])$ of $\mathcal{V}$ and $\mathcal{V}'$, respectively. By Prop.2 of Sect.16, there is an invertible linear mapping $\mathbf{R} : \mathcal{V} \to \mathcal{V}'$ such that $\mathbf{R}\mathbf{e}_i = \mathbf{e}'_i$ for all $i \in n^]$. By Prop.4, $\mathbf{R}$ is an orthogonal isomorphism. ■

Proposition 6: *Let* $\mathcal{V}, \mathcal{V}'$ *be inner-product spaces. Assume that* $\mathbf{R} : \mathcal{V} \to \mathcal{V}'$ *preserves inner products, i.e., that*

$$\mathbf{R}(\mathbf{u}) \cdot \mathbf{R}(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v} \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathcal{V}, \tag{43.5}$$

and that LspRng $\mathbf{R}$ *is a regular subspace of* $\mathcal{V}'$. *Then* $\mathbf{R}$ *is linear and hence orthogonal.*

Proof: To show that $\mathbf{R}$ preserves sums, let $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ be given. It follows from (43.5) that

$$\begin{aligned} \mathbf{R}(\mathbf{u}+\mathbf{v}) \cdot \mathbf{R}(\mathbf{w}) &= (\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \\ &= \mathbf{R}(\mathbf{u}) \cdot \mathbf{R}(\mathbf{w}) + \mathbf{R}(\mathbf{v}) \cdot \mathbf{R}(\mathbf{w}) \end{aligned}$$

and hence

$$(\mathbf{R}(\mathbf{u}+\mathbf{v}) - \mathbf{R}(\mathbf{u}) - \mathbf{R}(\mathbf{v})) \cdot \mathbf{R}(\mathbf{w}) = 0$$

for all $\mathbf{w} \in \mathcal{V}$. This means that

$$\mathbf{R}(\mathbf{u}+\mathbf{v}) - \mathbf{R}(\mathbf{u}) - \mathbf{R}(\mathbf{v}) \ \in \ (\operatorname{Rng} \mathbf{R})^{\perp}.$$

Since LspRng $\mathbf{R}$ is regular we have $\operatorname{LspRng} \mathbf{R} \cap (\operatorname{Rng} \mathbf{R})^{\perp} = \{\mathbf{0}\}$. It follows that $\mathbf{R}(\mathbf{u}+\mathbf{v}) - \mathbf{R}(\mathbf{u}) - \mathbf{R}(\mathbf{v}) = \mathbf{0}$. Since $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ were arbitrary, we conclude that $\mathbf{R}$ preserves sums. A similar argument shows that $\mathbf{R}$ preserves scalar multiples and hence is linear. ∎

Pitfall: If $\mathbf{R} : \mathcal{V} \to \mathcal{V}'$ preserves inner products but LspRng $\mathbf{R}$ is not regular, then $\mathbf{R}$ need not be linear. For example, if $\mathcal{V} = \{\mathbf{0}\}$ and $\mathbf{n} \in \mathcal{V}'$ is such that $\mathbf{n}^{\cdot 2} = 0$ but $\mathbf{n} \neq \mathbf{0}$, then the mapping $\mathbf{R} : \mathcal{V} \to \mathcal{V}'$ defined by $\mathbf{R}(\mathbf{0}) = \mathbf{n}$ preserves inner products but is not linear. ∎

In view of Prop.3, Prop.6 has the following corollary.

Proposition 7: *A surjective mapping that preserves inner products is necessarily an orthogonal isomorphism.*

Notes 43

(1) Orthogonal mappings are commonly called "orthogonal transformations", but the definition is often restricted to the case in which the domain and codomain coincide (i.e. when we use "orthogonal lineon") and the spaces involved are genuine inner-product spaces.

(2) The notations $\mathrm{O}_n, \mathrm{O}_n(\mathbb{R})$, or $\mathrm{O}(n, \mathbb{R})$ are often used for the orthogonal group $\operatorname{Orth} \mathbb{R}^n$.

(3) If the inner-product is double-signed and if its index is 1 (see Sect.47), an orthogonal lineon is often called a "Lorentz transformation", especially in the context of the theory of relativity. The orthogonal group is then called the "Lorentz group".

44 Induced Inner Products

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be inner-product spaces. Then $\mathcal{V}_1 \times \mathcal{V}_2$ carries a natural induced structure of a linear space, as explained in Sect.14. We endow $\mathcal{V}_1 \times \mathcal{V}_2$ also with the structure of an inner-product space by prescribing its inner square by the rule

$$(\mathbf{v}_1, \mathbf{v}_2)^{\cdot 2} := \mathbf{v}_1^{\cdot 2} + \mathbf{v}_2^{\cdot 2} \quad \text{for all} \quad \mathbf{v}_1 \in \mathcal{V}_1, \ \mathbf{v}_2 \in \mathcal{V}_2. \tag{44.1}$$

The inner product in $\mathcal{V}_1 \times \mathcal{V}_2$ is given by

$$(\mathbf{u}_1, \mathbf{u}_2) \cdot (\mathbf{v}_1, \mathbf{v}_2) = \mathbf{u}_1 \cdot \mathbf{v}_1 + \mathbf{u}_2 \cdot \mathbf{v}_2$$

for all $(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}_1 \times \mathcal{V}_2$. More generally, if $(\mathcal{V}_i \mid i \in I)$ is a finite family of inner-product spaces, we endow the product-space $\times (\mathcal{V}_i \mid i \in I)$ with the natural structure of an inner-product space by prescribing the inner square by the rule

$$\mathbf{v}^{\cdot 2} := \sum_{i \in I} \mathbf{v}_i^{\cdot 2} \quad \text{for all} \quad \mathbf{v} \in \times_{i \in I} \mathcal{V}_i. \tag{44.2}$$

It is clear that $\times (\mathcal{V}_i \mid i \in I)$ is genuine if all the $\mathcal{V}_i$, $i \in I$, are genuine.

Remark: The inner products on $\mathcal{V}_1$ and $\mathcal{V}_2$ induce on $\mathcal{V}_1 \times \mathcal{V}_2$, in a natural manner, non-degenerate quadratic forms other than the inner square given by (44.1). For example, one such quadratic form $\mathbf{Q} : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{R}$ is given by

$$\mathbf{Q}(\mathbf{v}_1, \mathbf{v}_2) := \mathbf{v}_1^{\cdot 2} - \mathbf{v}_2^{\cdot 2} \tag{44.3}$$

This form is double-signed (see Sect.27) when $\mathcal{V}_1$ and $\mathcal{V}_2$ are genuine inner-product spaces of dimension greater then zero. ■

Theorem on the Induced Inner Product: *Let $\mathcal{V}$ and $\mathcal{W}$ be inner-product spaces. Then $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$ has the natural structure of an inner product space, obtained by prescribing its inner square by the rule*

$$\mathbf{L}^{\cdot 2} := \mathrm{tr}_{\mathcal{V}}(\mathbf{L}^{\top}\mathbf{L}) \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{W}). \tag{44.4}$$

The corresponding inner product is given by

$$\mathbf{L} \cdot \mathbf{M} := \mathrm{tr}_{\mathcal{V}}(\mathbf{L}^{\top}\mathbf{M}) = \mathrm{tr}_{\mathcal{W}}(\mathbf{M}\mathbf{L}^{\top}) \tag{44.5}$$

for all $\mathbf{L}, \mathbf{M} \in \mathrm{Lin}(\mathcal{V}, \mathcal{W})$.

If $\mathcal{V}$ and $\mathcal{W}$ are genuine inner-product spaces, so is $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$.

Proof: The bilinearity of the mapping

$$((\mathbf{L},\mathbf{M}) \mapsto \mathrm{tr}_{\mathcal{V}}(\mathbf{L}^\top\mathbf{M})) \;:\; (\mathrm{Lin}(\mathcal{V},\mathcal{W}))^2 \to \mathbb{R} \tag{44.6}$$

is a consequence of the linearity of transposition, the bilinearity of composition, and the linearity of the trace (see Chap.2). To prove the symmetry of the mapping (44.6) we use (26.7), (21.6), and (22.4) to obtain

$$\begin{aligned} \mathbf{M}\cdot\mathbf{L} &= \mathrm{tr}_{\mathcal{V}}(\mathbf{M}^\top\mathbf{L}) = \mathrm{tr}_{\mathcal{V}}((\mathbf{M}^\top\mathbf{L})^\top) \\ &= \mathrm{tr}_{\mathcal{V}}(\mathbf{L}^\top\mathbf{M}^{\top\top}) = \mathrm{tr}_{\mathcal{V}}(\mathbf{L}^\top\mathbf{M}) = \mathbf{L}\cdot\mathbf{M} \end{aligned}$$

for all $\mathbf{L},\mathbf{M} \in \mathrm{Lin}(\mathcal{V},\mathcal{W})$.

The Representation Theorem for Linear Forms on $\mathrm{Lin}(\mathcal{V},\mathcal{W})$ of Sect.26 states that the mapping that associates with $\mathbf{H} \in \mathrm{Lin}(\mathcal{W},\mathcal{V})$ the linear form $(\mathbf{M} \mapsto \mathrm{tr}_{\mathcal{V}}(\mathbf{HM})) \in (\mathrm{Lin}(\mathcal{V},\mathcal{W}))^*$ is a linear isomorphism. The transposition $(\mathbf{L} \mapsto \mathbf{L}^\top) \;:\; \mathrm{Lin}(\mathcal{V},\mathcal{W}) \to \mathrm{Lin}(\mathcal{W},\mathcal{V})$ is also a linear isomorphism. It follows that their composite, the linear mapping from $\mathrm{Lin}(\mathcal{V},\mathcal{W})$ into $(\mathrm{Lin}(\mathcal{V},\mathcal{W}))^*$ associated with the bilinear mapping (44.6), is an isomorphism and hence injective. This means that (44.4) indeed defines a non-degenerate quadratic form whose associated bilinear form is given by (44.6).

The last equality of (44.5) is an immediate consequence of (26.6).

The last statement of the Theorem is an immediate consequence of Prop.1 of Sect.42 and the following Lemma.

Lemma: *If* $\mathbf{e} := (\mathbf{e}_i \,|i \in I)$ *is a genuinely orthonormal basis of* $\mathcal{V}$ *and* $\mathbf{L} \in \mathrm{Lin}(\mathcal{V},\mathcal{W})$*, then*

$$\mathbf{L}^{\cdot 2} = \mathrm{tr}_{\mathcal{W}}(\mathbf{L}\mathbf{L}^\top) = \sum_{i\in I}(\mathbf{L}\mathbf{e}_i)^{\cdot 2}. \tag{44.7}$$

Proof: Since $\mathbf{e}$ coincides with its dual, we have, by (25.14),

$$\mathbf{1}_{\mathcal{V}} = \sum_{i\in I}\mathbf{e}_i \otimes \mathbf{e}_i.$$

Using Prop.3 of Sect.25, we obtain

$$\mathbf{L}\mathbf{L}^\top = \mathbf{L}\mathbf{1}_{\mathcal{V}}\mathbf{L}^\top = \sum_{i\in I}\mathbf{L}(\mathbf{e}_i\otimes\mathbf{e}_i)\mathbf{L}^\top = \sum_{i\in I}(\mathbf{L}\mathbf{e}_i)\otimes(\mathbf{L}\mathbf{e}_i).$$

Since $\mathrm{tr}_{\mathcal{W}}((\mathbf{L}\mathbf{e}_i)\otimes(\mathbf{L}\mathbf{e}_i)) = (\mathbf{L}\mathbf{e}_i)\cdot(\mathbf{L}\mathbf{e}_i) = (\mathbf{L}\mathbf{e}_i)^{\cdot 2}$ by (26.3), and since the trace is linear, the last of the equalities (44.7) follows. The first is a consequence of (44.5). ∎

Let I and J be finite sets. As we have seen in Sect.41, the spaces $\mathbb{R}^I$, $\mathbb{R}^J$ and $\mathbb{R}^{J\times I}$ can all be regarded as inner-product spaces. Therefore, the Theorem on the Induced Inner Product shows that $\mathrm{Lin}(\mathbb{R}^I, \mathbb{R}^J)$ carries the natural structure of an inner product space. We claim that this structure is compatible with the identification $\mathrm{Lin}(\mathbb{R}^I, \mathbb{R}^J) \cong \mathbb{R}^{J\times I}$ (see Sect.16). Indeed, given $M \in \mathrm{Lin}(\mathbb{R}^I, \mathbb{R}^J) \cong \mathbb{R}^{J\times I}$ we have, by (16.7) and (23.9)

$$(M^\top M)_{i,k} = \sum_{j\in I}(M^\top)_{i,j} M_{j,k} = \sum_{j\in I} M_{j,i} M_{j,k} \text{ for all } i,k \in I$$

and hence, by (26.8),

$$\mathrm{tr}(M^\top M) = \sum_{i\in I}(M^\top M)_{i,i} = \sum_{(j,i)\in J\times I}(M_{j,i})^2. \tag{44.8}$$

We now assume that inner-product spaces $\mathcal{V}$ and $\mathcal{W}$ are given. The following formulas, valid for all $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{W}), \mathbf{v} \in \mathcal{V}$, and $\mathbf{w} \in \mathcal{W}$ follow easily from the definitions (44.4) and (44.5):

$$\mathbf{L}\cdot(\mathbf{w}\otimes\mathbf{v}) = \mathbf{w}\cdot\mathbf{L}\mathbf{v} = \mathbf{L}(\mathbf{v},\mathbf{w}), \tag{44.9}$$
$$(\mathbf{w}\otimes\mathbf{v})^{\cdot 2} = (\mathbf{v}^{\cdot 2})(\mathbf{w}^{\cdot 2}). \tag{44.10}$$

In view of (26.9) we have

$$\mathbf{1}_{\mathcal{V}}^{\cdot 2} = \dim\mathcal{V}. \tag{44.11}$$

Proposition 1: *The transposition* $(\mathbf{L} \mapsto \mathbf{L}^\top) : \mathrm{Lin}(\mathcal{V}, \mathcal{W}) \to \mathrm{Lin}(\mathcal{W}, \mathcal{V})$ *is an orthogonal isomorphism*, i.e. *we have*

$$\mathbf{L}\cdot\mathbf{M} = \mathbf{L}^\top\cdot\mathbf{M}^\top \quad \text{for all} \quad \mathbf{L},\mathbf{M} \in \mathrm{Lin}(\mathcal{V},\mathcal{W}). \tag{44.12}$$

Proof: Given $\mathbf{L},\mathbf{M} \in \mathrm{Lin}(\mathcal{V},\mathcal{W})$, we may use (44.5) to obtain $\mathbf{L}^\top\cdot\mathbf{M}^\top = \mathrm{tr}_{\mathcal{W}}(\mathbf{L}^{\top\top}\mathbf{M}^\top) = \mathrm{tr}_{\mathcal{W}}(\mathbf{L}\mathbf{M}^\top) = \mathbf{M}\cdot\mathbf{L}$ and hence (44.12). ∎

Proposition 2: *The subspaces* $\mathrm{Sym}\mathcal{V}$ *and* $\mathrm{Skew}\mathcal{V}$ *of* $\mathrm{Lin}\mathcal{V}$ *are orthogonal supplements of each other (and hence both regular).*

Proof: By (44.5) we have, for all $\mathbf{S} \in \mathrm{Sym}\mathcal{V}$ and all $\mathbf{A} \in \mathrm{Skew}\mathcal{V}$,

$$\mathbf{S}\cdot\mathbf{A} = \mathrm{tr}(\mathbf{S}^\top\mathbf{A}) = \mathrm{tr}(\mathbf{S}\mathbf{A}) = -\mathrm{tr}(\mathbf{S}\mathbf{A}^\top) = -\mathbf{A}\cdot\mathbf{S}$$

and hence $\mathbf{S}\cdot\mathbf{A} = 0$. It follows that $\mathrm{Sym}\mathcal{V} \subset (\mathrm{Skew}\mathcal{V})^\perp$ and $\mathrm{Skew}\mathcal{V} \subset (\mathrm{Sym}\mathcal{V})^\perp$. We already know (Prop.7 of Sect.24) that $\mathrm{Sym}\mathcal{V}$ and $\mathrm{Skew}\mathcal{V}$ are supplementary. ∎

From now on we assume that both $\mathcal{V}$ and $\mathcal{W}$ are genuine inner-product spaces. By the last assertion of the Theorem on the Induced Inner Product, $\mathrm{Lin}(\mathcal{V},\mathcal{W})$ is then genuine, too. Thus, according to the Definition of Sect.42, every $\mathbf{L} \in \mathrm{Lin}(\mathcal{V},\mathcal{W})$ has a magnitude

$$|\mathbf{L}| := \sqrt{\mathbf{L}^{\cdot 2}} = \sqrt{\mathrm{tr}_{\mathcal{V}}(\mathbf{L}^{\top}\mathbf{L})} = \sqrt{\mathrm{tr}_{\mathcal{W}}(\mathbf{L}\mathbf{L}^{\top})}. \tag{44.13}$$

In addition to the rules concerning magnitudes in general, the magnitude in $\mathrm{Lin}(\mathcal{V},\mathcal{W})$ satifies the following rules.

Proposition 3: *For all* $\mathbf{v} \in \mathcal{V}$, $\mathbf{w} \in \mathcal{W}$, *and* $\mathbf{L} \in \mathrm{Lin}(\mathcal{V},\mathcal{W})$ *we have*

$$|\mathbf{w} \otimes \mathbf{v}| = |\mathbf{w}||\mathbf{v}|, \tag{44.14}$$

$$|\mathbf{L}\mathbf{v}| \leq |\mathbf{L}||\mathbf{v}|. \tag{44.15}$$

Proof: (44.14) is an immediate consequence of (44.10). To prove (44.15), we use (44.9), the Inner-Product Inequality of Sect.42 as applied to $\mathrm{Lin}(\mathcal{V},\mathcal{W})$, and (44.14) to show that

$$|\mathbf{w} \cdot \mathbf{L}\mathbf{v}| = |\mathbf{L} \cdot (\mathbf{w} \otimes \mathbf{v})| \leq |\mathbf{L}||\mathbf{w} \otimes \mathbf{v}| = |\mathbf{L}||\mathbf{w}||\mathbf{v}|.$$

Putting $\mathbf{w} := \mathbf{L}\mathbf{v}$ we get $|\mathbf{L}\mathbf{v}|^2 \leq |\mathbf{L}\mathbf{v}||\mathbf{L}||\mathbf{v}|$, which is equivalent to (44.15). ∎

In view of (44.11), we have

$$|\mathbf{1}_{\mathcal{V}}| = \sqrt{\dim(\mathcal{V})}. \tag{44.16}$$

Proposition 4: *Let* $\mathcal{V}$, $\mathcal{V}'$ *and* $\mathcal{V}''$ *be genuine inner-product spaces and let* $\mathbf{L} \in \mathrm{Lin}(\mathcal{V},\mathcal{V}')$, $\mathbf{M} \in \mathrm{Lin}(\mathcal{V}',\mathcal{V}'')$. *Then*

$$|\mathbf{M}\mathbf{L}| \leq |\mathbf{M}||\mathbf{L}|. \tag{44.17}$$

Proof: Choose a genuinely orthonormal basis $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ of $\mathcal{V}$. By the Lemma, (44.7), we have

$$|\mathbf{M}\mathbf{L}|^2 = (\mathbf{M}\mathbf{L})^{\cdot 2} = \sum_{i \in I} (\mathbf{M}\mathbf{L}\mathbf{e}_i)^{\cdot 2} = \sum_{i \in I} |\mathbf{M}(\mathbf{L}\mathbf{e}_i)|^2.$$

Applying (44.15) to each term, we get

$$|\mathbf{M}\mathbf{L}|^2 \leq \sum_{i \in I} |\mathbf{M}|^2 |\mathbf{L}\mathbf{e}_i|^2 = |\mathbf{M}|^2 \sum_{i \in I} (\mathbf{L}\mathbf{e}_i)^{\cdot 2}.$$

Applying the Lemma, (44.7), again, we obtain $|\mathbf{M}\mathbf{L}|^2 \leq |\mathbf{M}|^2 \mathbf{L}^{\cdot 2}$, which is equivalent to (44.17). ∎

45 Euclidean Spaces

Definition 1: *A* **Euclidean space** *is a finite-dimensional flat space $\mathcal{E}$ endowed with additional structure by the prescription of a* **separation** *function*

$$\text{sep} \; : \; \mathcal{E} \times \mathcal{E} \longrightarrow \mathbb{R}$$

such that

(a) sep *is translation invariant, i.e.*

$$\text{sep} \circ (\mathbf{v} \times \mathbf{v}) = \text{sep} \tag{45.1}$$

for all $\mathbf{v}$ in the translation space $\mathcal{V}$ and $\mathcal{E}$,

(b) *For some $x \in \mathcal{E}$, the mapping*

$$(\mathbf{v} \mapsto \text{sep}(x, x + \mathbf{v})) \; : \; \mathcal{V} \longrightarrow \mathbb{R} \tag{45.2}$$

is a non-degenerate quadratic form on $\mathcal{V}$.

It follows from (a) that the mapping (45.2) from $\mathcal{V}$ to $\mathbb{R}$ does not depend on the choice of x. Hence the translation space $\mathcal{V}$ of a Euclidean space $\mathcal{E}$ carries the natural structure of an inner-product space whose inner square $\mathbf{v} \mapsto \mathbf{v}^{\cdot 2}$ satisfies

$$\text{sep}(x, y) = (y - x)^{\cdot 2} \quad \text{for all} \quad x, y \in \mathcal{E}. \tag{45.3}$$

Conversely, if $\mathcal{E}$ is a finite-dimensional flat space and if its translation space $\mathcal{V}$ is endowed with additional structure as in Def.1 of Sect.41 so as to make it an inner-product space, then $\mathcal{E}$ acquires the natural structure of a Euclidean space when we use (45.3) to *define* the separation function.

Every inner-product space $\mathcal{V}$ has the natural structure of a Euclidean space that is its own (external) translation space. Indeed, the natural structure of $\mathcal{V}$ as a flat space is the one described by Prop.3 of Sect.32, and the inner-product of $\mathcal{V}$ then gives $\mathcal{V}$ the natural structure of a Euclidean space as remarked above.

We say that a flat $\mathcal{F}$ in a Euclidean space $\mathcal{E}$ is **regular** if its direction space is a regular subspace of $\mathcal{V}$. If $\mathcal{F}$ is regular, then $\text{sep}|_{\mathcal{F} \times \mathcal{F}}$ endows $\mathcal{F}$ with the structure of a Euclidean space; but if $\mathcal{F}$ is not regular, it does not have a natural structure of a Euclidean space.

Definition 2: *Let $\mathcal{E}, \mathcal{E}'$ be Euclidean spaces. We say that the mapping $\alpha \; : \; \mathcal{E} \to \mathcal{E}'$ is* **Euclidean** *if it is flat and preserves separation, so that*

$$\text{sep}' \circ (\alpha \times \alpha) = \text{sep}, \tag{45.4}$$

where sep *and* sep' *are the separation functions of* $\mathcal{E}$ *and* $\mathcal{E}'$*, respectively.*

A Euclidean mapping that has a Euclidean inverse is called a **Euclidean isomorphism**.

The following is evident from Def.2, (45.3), (33.4), and Def.1 in Sect.43.

Proposition 1: *The mapping* $\alpha : \mathcal{E} \to \mathcal{E}'$ *is Euclidean if and only if it is flat and has an orthogonal gradient.*

Using this Proposition and Prop.1 of Sect.43, we obtain:

Proposition 2: *The identity mapping of a Euclidean space is a Euclidean mapping. The composite of two Euclidean mappings is Euclidean. If a Euclidean mapping is invertible, then its inverse is again Euclidean, and hence it is a Euclidean isomorphism.*

In view of Prop.3 of Sect.43, we have:

Proposition 3: *Every Euclidean mapping* $\alpha : \mathcal{E} \to \mathcal{E}'$ *is injective, its range* $\text{Rng}\,\alpha$ *is a regular subspace of* $\mathcal{E}'$*, and* $\alpha|^{\text{Rng}}$ *is a Euclidean isomorphorism from* $\mathcal{E}$ *onto* $\text{Rng}\,\alpha$*. Moreover,* α *is a Euclidean isomorphism if and only if* α *is surjective, and this is the case if and only if* $\dim \mathcal{E} = \dim \mathcal{E}'$*.*

The set of all Euclidean automorphisms of a Euclidean space $\mathcal{E}$ will be denoted by $\text{Eis}\mathcal{E}$. It is a subgroup of the group $\text{Fis}\,\mathcal{E}$ of all flat automorphisms of $\mathcal{E}$. The mapping $g : \text{Eis}\mathcal{E} \to \text{Orth}\mathcal{V}$ defined by $g(\alpha) := \nabla\alpha$ is a surjective group-homomorphism whose kernel is the translation group $\mathcal{V}$. In fact, g is obtained from the homomorphism $h : \text{Fis}\,\mathcal{E} \to \text{Lis}\mathcal{V}$ described at the end of Sect.33 by the adjustment $g := h|_{\text{Eis}\mathcal{E}}^{\text{Orth}\mathcal{V}}$.

Proposition 4: *Let* $\mathcal{E}$*,* $\mathcal{E}'$ *be Euclidean spaces, let* $\alpha : \mathcal{E} \to \mathcal{E}'$ *be a flat mapping, and let* $p := (p_i \mid i \in I)$ *be a flat basis of* $\mathcal{E}$ *such that*

$$\text{sep}(p_i, p_j) = \text{sep}'(\alpha(p_i), \alpha(p_j)) \quad \text{for all} \quad i, j \in I. \tag{45.5}$$

Then α *is a Euclidean mapping.*

Proof: Since α is flat we have, by (33.4),

$$\alpha(p_i) - \alpha(p_j) = \nabla\alpha(p_i - p_j) \quad \text{for all} \quad i, j \in I. \tag{45.6}$$

We now choose $k \in I$ and put $I' := I \setminus \{k\}$ and $\mathbf{u}_i := p_i - p_k$ for all $i \in I'$. Then (45.6) gives

$$\alpha(p_i) - \alpha(p_k) = (\nabla\alpha)\mathbf{u}_i \quad \text{for all} \quad i \in I'$$

and

$$\alpha(p_i) - \alpha(p_j) = (\nabla\alpha)(\mathbf{u}_i - \mathbf{u}_j) \quad \text{for all} \quad i, j \in I'.$$

In view of (45.3), it follows from (45.5) that

$$((\nabla\alpha)\mathbf{u}_i)^{\cdot 2} = \mathbf{u}_i^{\cdot 2} \quad \text{for all} \quad i \in I' \tag{45.7}$$

and

$$((\nabla\alpha)(\mathbf{u}_i - \mathbf{u}_j))^{\cdot 2} = (\mathbf{u}_i - \mathbf{u}_j)^{\cdot 2} \quad \text{for all} \quad i, j \in I',$$

which is equivalent to

$$((\nabla\alpha)\mathbf{u}_i)^{\cdot 2} - 2((\nabla\alpha)\mathbf{u}_i) \cdot ((\nabla\alpha)\mathbf{u}_j) + ((\nabla\alpha)\mathbf{u}_j)^{\cdot 2} = \mathbf{u}_i^{\cdot 2} - 2\mathbf{u}_i \cdot \mathbf{u}_j + \mathbf{u}_j^{\cdot 2}.$$

Using (45.7) this gives

$$(\nabla\alpha)\mathbf{u}_i \cdot (\nabla\alpha)\mathbf{u}_j = \mathbf{u}_i \cdot \mathbf{u}_j \quad \text{for all} \quad i, j \in I'.$$

Since $(\mathbf{u}_i \mid i \in I')$ is a basis of $\mathcal{V}$ by Prop.2 of Sect.35, it follows from Prop.4 of Sect.43 that $\nabla\alpha$ is orthogonal. Hence α is Euclidean by Prop.1. ■

The following result shows that, in Def.2, the requirement that α be flat can in many cases be omitted.

Proposition 5: *Let $\mathcal{E}, \mathcal{E}'$ be Euclidean spaces and let $\alpha : \mathcal{E} \to \mathcal{E}'$ be a mapping that preserves separation, i.e. satisfies (45.4). Then α is flat and hence Euclidean provided that the flat span of* $\operatorname{Rng}\alpha$ *is regular.*

Proof: Let $\mathcal{V}$ and $\mathcal{V}'$ be the translation spaces of $\mathcal{E}$ and $\mathcal{E}'$, respectively. We choose $q \in \mathcal{E}$ and define $\mathbf{R} : \mathcal{V} \to \mathcal{V}'$ by

$$\mathbf{R}(\mathbf{v}) := \alpha(q + \mathbf{v}) - \alpha(q) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}. \tag{45.8}$$

We have, for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$

$$\mathbf{R}(\mathbf{u}) - \mathbf{R}(\mathbf{v}) = \alpha(q + \mathbf{u}) - \alpha(q + \mathbf{v}).$$

Since α preserves separation, it follows that

$$(\mathbf{R}(\mathbf{u}) - \mathbf{R}(\mathbf{v}))^{\cdot 2} = (\mathbf{u} - \mathbf{v})^{\cdot 2} \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathcal{V}.$$

By an argument similar to one given in the proof of Prop.4, we conclude that

$$\mathbf{R}(\mathbf{u}) \cdot \mathbf{R}(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v} \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathcal{V}.$$

Now, (45.8) shows that $\operatorname{Rng}\mathbf{R} = (\operatorname{Rng}\alpha) - \alpha(q)$. Hence, by (32.6), $\operatorname{Lsp}\operatorname{Rng}\mathbf{R}$ is the direction space of $\operatorname{Fsp}\operatorname{Rng}\alpha$. If $\operatorname{Fsp}\operatorname{Rng}\alpha$ is regular, so is $\operatorname{Lsp}\operatorname{Rng}\mathbf{R}$, and we can apply Prop.6 of Sect.43 to conclude that $\mathbf{R}$ is orthogonal. It follows from (45.8) that α is flat and that $\mathbf{R} = \nabla\alpha$. ■

The following corollary to Prop.5 shows that the prescription of the separation function alone is sufficient to determine the structure of a Euclidean space on a set.

Proposition 6: *Let a set $\mathcal{E}$ and a function* $\operatorname{sep} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *be given. Then $\mathcal{E}$ can be endowed with at most one structure of a Euclidean space whose separation function is* sep.

Proof: Let two such structures be given with $\mathcal{V}$ and $\mathcal{V}'$ as the corresponding translation spaces. The identity $1_{\mathcal{E}}$ preserves separation and hence Prop.5 may be applied to it when $\mathcal{E}$ as domain of $1_{\mathcal{E}}$ is considered as endowed with the first structure and as codomain of $1_{\mathcal{E}}$ with the second. Let $\mathbf{R} \in \operatorname{Orth}(\mathcal{V}, \mathcal{V}')$ be the gradient of $1_{\mathcal{E}}$ when interpreted in this manner. In view of (33.1), it follows that $1_{\mathcal{E}} \circ \mathbf{v} = (\mathbf{R}\mathbf{v}) \circ 1_{\mathcal{E}}$ and hence $\mathbf{v} = \mathbf{R}\mathbf{v}$ for all $\mathbf{v} \in \mathcal{V}$, which means that $\mathcal{V} \subset \mathcal{V}'$ and $\mathbf{R} = \mathbf{1}_{\mathcal{V} \subset \mathcal{V}'}$. Reversing the roles of $\mathcal{V}$ and $\mathcal{V}'$ we get $\mathcal{V}' \subset \mathcal{V}$ and hence $\mathcal{V} = \mathcal{V}'$ and $\mathbf{R} = \mathbf{1}_{\mathcal{V}}$. Since $\mathbf{R}$ is an orthogonal isomorphism, it follows that $\mathcal{V}$ and $\mathcal{V}'$ coincide not only as subgroups of $\operatorname{Perm}\mathcal{E}$ but also as inner-product spaces. ■

Proposition 7: *Let $\mathcal{E}$ be a Euclidean space and $p := (p_i \mid i \in I)$ a flatly spanning family of points in $\mathcal{E}$. Then for all $x, y \in \mathcal{E}$*

$$(\operatorname{sep}(x, p_i) = \operatorname{sep}(y, p_i) \quad \text{for all} \quad i \in I) \quad \Longrightarrow \quad x = y.$$

Proof: Let $x, y \in \mathcal{E}$ be given and let z be the midpoint of (x, y) (see Example 1 of Sect.34). This means that $x = z + \mathbf{u}$ and $y = z - \mathbf{u}$ for a suitable $\mathbf{u} \in \mathcal{V}$. The family $\mathbf{b} := ((p_i - z) \mid i \in I)$ spans $\mathcal{V}$ and we have

$$\begin{aligned} \operatorname{sep}(x, p_i) &= (p_i - x)^{\cdot 2} = (\mathbf{b}_i - \mathbf{u})^{\cdot 2} \\ \operatorname{sep}(y, p_i) &= (p_i - y)^{\cdot 2} = (\mathbf{b}_i + \mathbf{u})^{\cdot 2} \end{aligned}$$

for all $i \in I$. Hence, if $\operatorname{sep}(x, p_i) = \operatorname{sep}(y, p_i)$ for all $i \in I$ we have

$$\mathbf{b}_i \cdot \mathbf{u} = \frac{1}{4}((\mathbf{b}_i + \mathbf{u})^{\cdot 2} - (\mathbf{b}_i - \mathbf{u})^{\cdot 2}) = 0$$

for all $i \in I$. Since $\mathbf{b}$ is spanning, it follows that $\mathbf{u} \in (\operatorname{Rng}\mathbf{b})^{\perp} = \mathcal{V}^{\perp} = \{\mathbf{0}\}$ and hence $\mathbf{u} = \mathbf{0}$, which means that $x = z = y$. ■

Notes 45

(1) What we call a "genuine Euclidean space" (see Def.1 of Sect.46) is usually just called a "Euclidean space", and non-genuine Euclidean spaces are often referred to as "pseudo-Euclidean spaces". I believe it is more convenient to have a term that covers both cases.

The term "Euclidean space" is often misused to mean a genuine inner-product space or even $\mathbb{R}^n$.

(2) In the past, the term "separation" has been used only in the special case when the space $\mathcal{E}$ is a model of the event-world in the theory of relativity.

46 Genuine Euclidean Spaces, Congruences

Definition 1: *We say that a Euclidean space* $\mathcal{E}$ *is* **genuine** *if its separation function* sep *has only positive values, i.e. if* $\mathrm{Rng}\,\mathrm{sep} \subset \mathbb{P}$. *The* **distance** *function*

$$\mathrm{dst} \ : \ \mathcal{E} \times \mathcal{E} \to \mathbb{P}$$

is then defined by

$$\mathrm{dst}(x,y) := \sqrt{\mathrm{sep}(x,y)} \quad \text{for all} \quad x,y \in \mathcal{E}. \tag{46.1}$$

It is clear from (45.3) that a Euclidean space $\mathcal{E}$ is genuine if and only if its translation space $\mathcal{V}$ is a genuine inner-product space. Hence all flats in a genuine Euclidean space are regular and have natural structures of genuine Eucliden spaces. Comparing (46.1) with (42.2), we see that (45.3) is equivalent to

$$\mathrm{dst}(x,y) = |y-x| \quad \text{for all} \quad x,y \in \mathcal{E}. \tag{46.2}$$

Using (42.4) with $\xi := -1$ and (42.6) and (42.3), we obtain:

Proposition 1: *The distance function* dst *of a genuine Euclidean space* $\mathcal{E}$ *has the following properties, valid for all* $x,y,z \in \mathcal{E}$ *:*

$$\mathrm{dst}(x,y) = \mathrm{dst}(y,x), \tag{46.3}$$

$$\mathrm{dst}(x,y) + \mathrm{dst}(y,z) \geq \mathrm{dst}(x,z), \tag{46.4}$$

$$\mathrm{dst}(x,y) = 0 \iff x = y. \tag{46.5}$$

Let $q \in \mathcal{E}$ and $\rho \in \mathbb{P}^\times$ be given. Then

$$\mathrm{Ball}_{q,\rho}(\mathcal{E}) := \{x \in \mathcal{E} \mid \mathrm{dst}(x,q) < \rho\} \tag{46.6}$$

is called the **ball of radius** ρ **centered at** q,

$$\overline{\mathrm{Ball}}_{q,\rho}(\mathcal{E}) := \{x \in \mathcal{E} \mid \mathrm{dst}(x,q) \leq \rho\} \tag{46.7}$$

is called the **closed ball** of radius ρ centered at q, and

$$\begin{aligned}\mathrm{Sph}_{q,\rho}(\mathcal{E}) &:= \{x \in \mathcal{E} \mid \mathrm{dst}(x,q) = \rho\} \\ &= \overline{\mathrm{Ball}}_{q,\rho}(\mathcal{E}) \setminus \mathrm{Ball}_{q,\rho}(\mathcal{E})\end{aligned} \tag{46.8}$$

is called the **sphere** of radius ρ centered at q. If $\dim \mathcal{E} = 2$, the term “disc” is often used instead of “ball”, and the term “circle” instead of “sphere”.

In view of Def.2 of Sect.42, we have

$$\begin{aligned} \mathrm{Ball}_{q,\rho}(\mathcal{E}) &= q + \rho\mathrm{Ubl}\mathcal{V}, & (46.9) \\ \overline{\mathrm{Ball}}_{q,\rho}(\mathcal{E}) &= q + \rho\overline{\mathrm{Ubl}}\mathcal{V}, & (46.10) \\ \mathrm{Sph}_{q,\rho}(\mathcal{E}) &= q + \rho\mathrm{Usph}\mathcal{V}. & (46.11) \end{aligned}$$

We now assume that a genuine Euclidean space $\mathcal{E}$ is given.

Definition 2: *We say that the families $p := (p_i \mid i \in I)$ and $p' := (p'_i \mid i \in I)$ of points in $\mathcal{E}$ are* **congruent** *in $\mathcal{E}$ if there is a Euclidean automorphism α of $\mathcal{E}$ such that $\alpha(p_i) = p'_i$ for all $i \in I$.*

Congruence Theorem: *The families $p := (p_i \mid i \in I)$ and $p' := (p'_i \mid i \in I)$ are congruent if and only if*

$$\mathrm{dst}(p_i, p_j) = \mathrm{dst}(p'_i, p'_j) \quad \text{for all} \quad i, j \in I. \tag{46.12}$$

Proof: The "only if" part follows from the definitions.

Assume that (46.12) holds. Put $\mathcal{F} := \mathrm{Fsp}\,\mathrm{Rng}\,p$ and choose a subset K of I such that $p|_K = (p_k \mid k \in K)$ is a flat basis of $\mathcal{F}$. Put $\mathcal{F}' := \mathrm{Fsp}\,\mathrm{Rng}\,p'$. By Prop.5 of Sect.35, there is a unique flat mapping $\beta : \mathcal{F} \to \mathcal{F}'$ such that $\beta(p_k) = p'_k$ for all $k \in K$. By Prop.4 of Sect.45, we can conclude from (46.12) that β must be Euclidean and hence injective (see Prop.3 of Sect.45). It follows that $\dim \mathcal{F} \leq \dim \mathcal{F}'$. Reversing the roles of p and p', we also conclude that $\dim \mathcal{F}' \leq \dim \mathcal{F}$. Therefore, we have $\dim \mathcal{F} = \dim \mathcal{F}'$ and β is a Euclidean isomorphism. Hence $p'|_K = \beta_>(p|_K)$ is a flat basis of $\mathcal{F}'$. Since β preserves distance, we have for all $x \in \mathcal{F}$

$$\mathrm{dst}(x, p_k) = \mathrm{dst}(\beta(x), \beta(p_k)) = \mathrm{dst}(\beta(x), p'_k)$$

for all $k \in K$. Thus, using (46.12) we get

$$\mathrm{dst}(\beta(p_i), p'_k) = \mathrm{dst}(p_i, p_k) = \mathrm{dst}(p'_i, p'_k)$$

for all $k \in K$, $i \in I$. Using Prop.7 of Sect.45, we conclude that

$$\beta(p_i) = p'_i \quad \text{for all} \quad i \in I. \tag{46.13}$$

Let $\mathcal{U}$ and $\mathcal{U}'$ be the direction spaces of $\mathcal{F}$ and $\mathcal{F}'$. Since

$$\dim \mathcal{U} = \dim \mathcal{F} = \dim \mathcal{F}' = \dim \mathcal{U}',$$

it follows that $\dim \mathcal{U}^\perp = \dim \mathcal{U}'^\perp$. Therefore, in view of Prop.5 of Sect.43, we may choose an orthogonal isomorphism $\mathbf{R} : \mathcal{U}^\perp \to \mathcal{U}'^\perp$. Since $\mathcal{E} =$

$\mathcal{F} + \mathcal{U}^{\perp} = \mathcal{F}' + \mathcal{U}'^{\perp}$, there is a unique Euclidean isomorphism $\alpha : \mathcal{E} \to \mathcal{E}$ such that

$$\alpha(x + \mathbf{w}) = \beta(x) + \mathbf{R}\mathbf{w} \text{ for all } x \in \mathcal{F}, \ \mathbf{w} \in \mathcal{U}^{\perp}.$$

By (46.13) we have $\alpha(p_i) = p'_i$ for all $i \in I$, showing that p and p' are congruent. ∎

The Congruence Theorem shows that two families of points in $\mathcal{E}$ are congruent in $\mathcal{E}$ if and only if they are congruent in any flat $\mathcal{F}$ that includes the ranges of both families. It is for this reason that we may simply say "congruent" rather than "congruent in $\mathcal{E}$".

Definition 3: *Let $\mathcal{S}$ and $\mathcal{S}'$ be two subsets of $\mathcal{E}$. We say that a mapping $\gamma : \mathcal{S} \to \mathcal{S}'$ is a* **congruence** *if $(x \mid x \in \mathcal{S})$ and $(\gamma(x) \mid x \in \mathcal{S})$ are congruent families of points. The set of all congruences from a given set $\mathcal{S}$ to itself is called the* **symmetry-group** *of $\mathcal{S}$ and is denoted by* $\mathrm{Cong}\mathcal{S}$.

The Congruence Theorem has the following immediate consequence:

Corollary: *The mapping $\gamma : \mathcal{S} \to \mathcal{S}'$ is a congruence if and only if it is surjective and preserves distance.*

All congruences are invertible mappings and $\mathrm{Cong}\mathcal{S}$ is a subgroup of the permutation group $\mathrm{Perm}\,\mathcal{S}$ of $\mathcal{S}$. The group $\mathrm{Cong}\mathcal{S}$ describes the internal symmetry of $\mathcal{S}$. For example, if $\sharp\mathcal{S} = 3$, then $\mathcal{S}$ can be viewed as the set of vertices of a triangle. If the triangle is equilateral, then $\mathrm{Cong}\mathcal{S} = \mathrm{Perm}\,\mathcal{S}$. If the triangle is isoceles but not equilateral, then $\mathrm{Cong}\mathcal{S}$ is a two-element subgroup of $\mathrm{Perm}\,\mathcal{S}$ that contains $1_{\mathcal{S}}$ and one switch. If the triangle is scalene, then $\mathrm{Cong}\mathcal{S}$ is the identity-subgroup $\{1_{\mathcal{S}}\}$ of $\mathrm{Perm}\,\mathcal{S}$.

Notes 46

(1) In view of Prop.5 of Sect.45, it turns out that a mapping between genuine Euclidean spaces is a Euclidean isomorphism if and only if it is an isometry, i.e. an invertible mapping that preserves distances. It is for this reason that many people say "isometry" in place of "Euclidean isomorphism".

47 Double-Signed Inner-Product Spaces

We assume that an inner-product space $\mathcal{V}$ with inner square sq is given. If sq is strictly positive, i.e. if the space is a genuine inner-product space, then the results of Sect.42 apply. If sq is strictly negative, then a change from sq to $-$sq converts the space into a genuine inner-product space, and the results of Sect.42, with suitable adjustments of sign, still apply. In particular, two inner-product spaces with strictly negative inner squares are isomorphic if they have the same dimension.

The results of this section are of interest only if the inner product is *double-signed*, i.e. if sq is neither strictly positive nor strictly negative. Recall that a subspace $\mathcal{U}$ of $\mathcal{V}$ is called *regular* if $\mathrm{sq}|_{\mathcal{U}}$ is non-degenerate (see Sect.41). We say that $\mathcal{U}$ is

singular if it is not regular,

totally singular if $\mathrm{sq}|_{\mathcal{U}} = 0$,

positive-regular if $\mathrm{sq}|_{\mathcal{U}}$ is strictly positive,

negative-regular if $\mathrm{sq}|_{\mathcal{U}}$ is strictly negative.

Definition 1: *Let $\mathcal{V}$ be an inner product space. The greatest among the dimensions of all positive-regular [negative-regular] subspaces of $\mathcal{V}$ will be denoted by* $\mathrm{sig}^{+}\mathcal{V}$ $[\mathrm{sig}^{-}\mathcal{V}]$. *The pair* $(\mathrm{sig}^{+}\mathcal{V}, \mathrm{sig}^{-}\mathcal{V})$ *is called the* **signature** *of $\mathcal{V}$.*

The structure of double-signed inner-product spaces is described by the following result.

Inner-Product Signature Theorem: *Let $\mathcal{V}$ be an inner-product space. If a positive-regular [negative-regular] subspace $\mathcal{U}$ of $\mathcal{V}$ satisfies one of the following conditions, then it satisfies all three:*

(a) $\dim\mathcal{U} = \mathrm{sig}^{+}\mathcal{V} \quad [\dim\mathcal{U} = \mathrm{sig}^{-}\mathcal{V}]$,

(b) *$\mathcal{U}$ is maximal among the positive-regular [negative-regular] subspaces of $\mathcal{V}$,*

(c) *$\mathcal{U}^{\perp}$ is negative-regular [positive-regular].*

Proof: It is sufficient to consider the case when $\mathcal{U}$ is positive-regular, because a change from sq to $-$sq will then take care of the other case. We choose a positive-regular subspace $\mathcal{W}$ such that $\dim\mathcal{W} = \mathrm{sig}^{+}\mathcal{V}$; then $\dim\mathcal{U} \le \dim\mathcal{W}$.

(a) $\Rightarrow$ **(b):** This is trivial.

(b) $\Rightarrow$ **(c):** Suppose that $\mathcal{U}^{\perp}$ is not negative-regular. Since $\mathcal{U}^{\perp}$ is regular by the last assertion of the Theorem on the Characterization of Regular Subspaces of Sect.41, $\mathrm{sq}|_{\mathcal{U}^{\perp}}$ cannot be negative by Prop.4 of Sect.27. Hence we may choose $\mathbf{w} \in \mathcal{U}^{\perp}$ such that $\mathbf{w}^{\cdot 2} > 0$. Then $\mathcal{U} + \mathbb{R}\mathbf{w}$ strictly includes $\mathcal{U}$ and is still positive-regular. Hence $\mathcal{U}$ is not maximal.

(c) $\Rightarrow$ **(a):** Since $\mathcal{U}$ is regular, $\mathcal{U}^{\perp}$ is a supplement of $\mathcal{U}$ by the Theorem on the Characterization of Regular Subspaces. Hence, by Prop.4 of Sect.19, there is a projection $\mathbf{P} : \mathcal{V} \to \mathcal{U}$ such that $\mathcal{U}^{\perp} = \mathrm{Null}\ \mathbf{P}$. Let $\mathbf{w} \in \mathcal{W} \cap \mathcal{U}^{\perp}$ be given. Since $\mathcal{W}$ is positive-regular and $\mathcal{U}^{\perp}$ is negative regular, we would have $\mathbf{w}^{\cdot 2} > 0$ and $\mathbf{w}^{\cdot 2} < 0$ if $\mathbf{w}$ were not zero. Since this cannot be, it follows that

$$\text{Null } \mathbf{P}|_{\mathcal{W}} = \mathcal{W} \cap \mathcal{U}^{\perp} = \{\mathbf{0}\}$$

and hence that $\mathbf{P}|_{\mathcal{W}}$ is injective. By the Pigeonhole Principle for Linear Mappings, it follows that $\dim \mathcal{W} \leq \dim \mathcal{U}$ and hence $\dim \mathcal{U} = \dim \mathcal{W} = \text{sig}^{+}\mathcal{V}$, ∎

Using the eqivalence (a) $\Leftrightarrow$ (c) twice, once for $\mathcal{U}$ and once when $\mathcal{U}$ is replaced by $\mathcal{U}^{\perp}$, we obtain the following:

Corollary: *There exist positive-regular subspaces $\mathcal{U}$ such that $\mathcal{U}^{\perp}$ is negative-regular. If $\mathcal{U}$ is such a subspace, then*

$$\dim \mathcal{U} = \text{sig}^{+}\mathcal{V}, \quad \dim \mathcal{U}^{\perp} = \text{sig}^{-}\mathcal{V}. \tag{47.1}$$

We have

$$\text{sig}^{+}\mathcal{V} + \text{sig}^{-}\mathcal{V} = \dim \mathcal{V}. \tag{47.2}$$

Of course, spaces that are orthogonally isomorphic have the same signature. The converse is also true:

Proposition 1: *Two inner-product spaces are orthogonally isomorphic if (and only if) they have the same signature.*

Proof: Assume that $\mathcal{V}$ and $\mathcal{V}'$ are inner-product spaces with the same signature. In view of the Corollary above, we may choose positive-regular subspaces $\mathcal{U}$ and $\mathcal{U}'$ of $\mathcal{V}$ and $\mathcal{V}'$, respectively, such that $\mathcal{U}^{\perp}$ and $\mathcal{U}'^{\perp}$ are negative regular. Since $\mathcal{V}$ and $\mathcal{V}'$ have the same signature, we have, by (47.1),

$$\dim \mathcal{U} = \dim \mathcal{U}', \quad \dim \mathcal{U}^{\perp} = \dim \mathcal{U}'^{\perp}.$$

Therefore, we may apply Prop.5 of Sect.43 and choose orthogonal isomorphisms $\mathbf{R}_1 : \mathcal{U} \to \mathcal{U}'$ and $\mathbf{R}_2 : \mathcal{U}^{\perp} \to \mathcal{U}'^{\perp}$. Since $\mathcal{U}$ and $\mathcal{U}^{\perp}$ are supplementary, there is a unique linear mapping $\mathbf{R} : \mathcal{V} \to \mathcal{V}'$ such that $\mathbf{R}|_{\mathcal{U}} = \mathbf{R}_1|^{\mathcal{V}'}$ and $\mathbf{R}|_{\mathcal{U}^{\perp}} = \mathbf{R}_2|^{\mathcal{V}'}$ (See Prop.5 of Sect.19). It is easily seen that $\mathbf{R}$ is an orthogonal isomoprhism. ∎

Proposition 2: *Every inner-product space $\mathcal{V}$ has orthonormal bases.*

Moreover, if $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ is an orthonormal basis of $\mathcal{V}$, then the signature of $\mathcal{V}$ is given by

$$\text{sig}^{+}\mathcal{V} = \sharp\{i \in I \mid \mathbf{e}_i^{\cdot 2} = 1\}, \tag{47.3}$$

$$\text{sig}^{-}\mathcal{V} = \sharp\{i \in I \mid \mathbf{e}_i^{\cdot 2} = -1\}. \tag{47.4}$$

Proof: In view of the Corollary above, we may choose a positive-regular subspace $\mathcal{U}$ of $\mathcal{V}$ such that $\mathcal{U}^{\perp}$ is negative regular. By Prop.1 of Sect.42, we may choose a genuinely orthonormal set basis $\mathfrak{b}$ of $\mathcal{U}$ and an orthonormal set basis $\mathfrak{c}$ of $\mathcal{U}^\perp$ such that $\mathbf{e}^{\cdot 2} = -1$ for all $\mathbf{e} \in \mathfrak{c}$. Then $\mathfrak{b} \cup \mathfrak{c}$ is an orthonormal set basis of $\mathcal{V}$.

Let an orthonormal basis $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ of $\mathcal{V}$ be given. We put

$$\mathcal{U}_1 := \mathrm{Lsp}\{\mathbf{e}_i \mid i \in I, \mathbf{e}_i^{\cdot 2} = 1\}, \quad \mathcal{U}_2 := \mathrm{Lsp}\{\mathbf{e}_i \mid i \in I, \mathbf{e}_i^{\cdot 2} = -1\}.$$

It is clear that $\mathcal{U}_1$ is positive-regular and that $\mathcal{U}_2$ is negative regular and that the right sides of (47.3) and (47.4) are the dimensions of $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. Hence, by Def.1, $\dim\mathcal{U}_1 \leq \mathrm{sig}^+\mathcal{V}$, $\dim\mathcal{U}_2 \leq \mathrm{sig}^-\mathcal{V}$. On the other hand, since $\mathbf{e}$ is a basis, $\mathcal{U}_1$ and $\mathcal{U}_2$ are supplementary, in $\mathcal{V}$ and hence by Prop.5 of Sect.17, $\dim\mathcal{U}_1 + \dim\mathcal{U}_2 = \dim\mathcal{V}$. This is compatible with (47.2) only if $\dim\mathcal{U}_1 = \mathrm{sig}^+\mathcal{V}$ and $\dim\mathcal{U}_2 = \mathrm{sig}^-\mathcal{V}$. ■

The greatest among the dimensions of all totally singular subspaces of a given inner product space $\mathcal{V}$ is called the **index** of $\mathcal{V}$ and is denoted by $\mathrm{ind}\mathcal{V}$.

Inner-Product Index Theorem: *The index of an inner-product space $\mathcal{V}$ is given by*

$$\mathrm{ind}\mathcal{V} = \min\{\mathrm{sig}^+\mathcal{V},\ \mathrm{sig}^-\mathcal{V}\} \tag{47.5}$$

and all maximal totally singular subspaces of $\mathcal{V}$ have dimension $\mathrm{ind}\mathcal{V}$.

Proof: By the Corollary above, we may choose a positive-regular subspace $\mathcal{U}$ of $\mathcal{V}$ such that $\mathcal{U}^\perp$ is negative-regular. Let $\mathbf{P}_1 : \mathcal{V} \to \mathcal{U}$ and $\mathbf{P}_2 : \mathcal{V} \to \mathcal{U}^\perp$ be the projections for which $\mathrm{Null}\ \mathbf{P}_1 = \mathcal{U}^\perp$ and $\mathrm{Null}\ \mathbf{P}_2 = \mathcal{U}$ (see Prop.4 of Sect.19).

Let a totally singular subspace $\mathcal{N}$ of $\mathcal{V}$ be given. Since $\mathcal{U}^\perp$ is negative-regular, every non-zero element of $\mathcal{U}^\perp$ has a strictly negative inner square and hence cannot belong to $\mathcal{N}$. It follows that

$$\mathrm{Null}\ \mathbf{P}_1|_{\mathcal{N}} = \mathcal{N} \cap \mathcal{U}^\perp = \{\mathbf{0}\}.$$

and hence that $\mathbf{P}_1|_{\mathcal{N}}$ is injective. By the Pigeonhole Principle for Linear Mappings, it follows that

$$\dim\mathcal{N} = \dim(\mathrm{Rng}\,\mathbf{P}_1|_{\mathcal{N}}) \leq \dim\mathcal{U} = \mathrm{sig}^+\mathcal{V}. \tag{47.6}$$

A similar argument shows that

$$\dim\mathcal{N} = \dim(\mathrm{Rng}\,\mathbf{P}_2|_{\mathcal{N}}) \leq \dim\mathcal{U}^\perp = \mathrm{sig}^-\mathcal{V}. \tag{47.7}$$

If the inequalities in (47.6) and (47.7) are both strict, then the orthogonal supplements of $\mathrm{Rng}\,\mathbf{P}_1|_{\mathcal{N}}$ and $\mathrm{Rng}\,\mathbf{P}_2|_{\mathcal{N}}$ relative to $\mathcal{U}$ and $\mathcal{U}^\perp$, respectively, are both non-zero. Hence we may choose $\mathbf{u} \in \mathcal{U}$ and $\mathbf{w} \in \mathcal{U}^{\perp}$ such that $\mathbf{u}^{\cdot 2} = 1$, $\mathbf{w}^{\cdot 2} = -1$, and $\mathbf{u} \cdot \mathbf{P}_1\mathbf{n} = \mathbf{w} \cdot \mathbf{P}_2\mathbf{n} = 0$ for all $\mathbf{n} \in \mathcal{N}$. It is easily verified that $\mathcal{N} + \mathbb{R}(\mathbf{u} + \mathbf{w})$ is a totally singular subspace of $\mathcal{V}$ that includes $\mathcal{N}$ as a proper subspace. We conclude that $\mathcal{N}$ is maximal among the totally singular subspaces of $\mathcal{V}$ if and only if at least one of the inequalities (47.6) and (47.7) reduces to an equality, i.e., if and only if $\dim \mathcal{N} = \min \{\text{sig}^{+}\mathcal{V},\ \text{sig}^{-}\mathcal{V}\}$. ■

Remark: The mathematical model for space-time in the Theory of Special Relativity has the structure of a (non-genuine) Euclidean space $\mathcal{E}$ whose translation space $\mathcal{V}$ has index 1. It is customary, by convention, to assume

$$1 = \text{ind}\mathcal{V} = \text{sig}^{-}\mathcal{V} \leq \text{sig}^{+}\mathcal{V}.$$

(However, many physicists now use the only other possible convention: $1 = \text{ind}\mathcal{V} = \text{sig}^{+}\mathcal{V} \leq \text{sig}^{-}\mathcal{V}$). The elements of $\mathcal{V}$ are called *world-vectors*. A non-zero world-vector $\mathbf{v}$ is said to be *time-like*, *space-like*, or *signal-like*, depending on whether $\mathbf{v}^{\cdot 2} < 0$, $\mathbf{v}^{\cdot 2} > 0$, or $\mathbf{v}^{\cdot 2} = 0$, respectively. ■

Notes 47

(1) The terms "isotropic" and "totally isotropic" are sometimes used for our "singular" and "totally singular".

(2) Some people use the term "signature" to mean an appropriate list of + signs and − signs rather than just a pair of numbers in $\mathbb{N}$.

(3) The Inner-Product Signature Theorem, or the closely related Prop.2, is often referred to as "the Law of Inertia" or "Sylvester's Law of Inertia".

48 Problems for Chapter 4

(1) Let $\mathcal{V}$ be an inner-product space.

(a) Prove: If $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ is an orthonormal family in $\mathcal{V}$ and if $\mathbf{v} \in \mathcal{V}$, then

$$\mathbf{w} := \mathbf{v} - \textstyle\sum_{i \in I} \text{sgn}\,(\mathbf{e}_i^{\cdot 2})(\mathbf{e}_i \cdot \mathbf{v})\mathbf{e}_i \quad \text{(P4.1)}$$

satisfies $\mathbf{w} \cdot \mathbf{e}_j = 0$ for all $j \in I$. (For the definition of sgn see (08.13)).

(b) Assume that $\mathcal{V}$ is genuine. Prove: If $\mathbf{b} := (\mathbf{b}_i \mid i \in n^{]})$ is a list-basis of $\mathcal{V}$ then there is exactly one orthonormal list basis $\mathbf{e} := (\mathbf{e}_i \mid i \in n^{]})$ of $\mathcal{V}$ such that $\mathbf{b}_k \cdot \mathbf{e}_k > 0$ and

$$\mathrm{Lsp}\{\mathbf{e}_i \mid i \in k^]\} = \mathrm{Lsp}\{\mathbf{b}_i \mid i \in k^]\} \quad \text{(P4.2)}$$

for all $k \in n^]$. (Hint: Use induction and Part (a).)

Note: The procedure for obtaining the orthonormal basis $\mathbf{e}$ of (b) is often called "Gram-Schmidt orthogonalization". I consider it of far less importance than some mathematicians do.

(2) Let $\mathcal{V}$ be a genuine inner-product space of dimension 2.

(a) Show that the set $\mathrm{Skew}\mathcal{V} \cap \mathrm{Orth}\mathcal{V}$ has exactly two members; moreover, if $\mathbf{J}$ is one of them then $-\mathbf{J}$ is the other, and we have $\mathbf{J}^2 = -\mathbf{1}_\mathcal{V}$.

(b) Show that $\mathbf{L} - \mathbf{L}^\top = -\mathrm{tr}(\mathbf{LJ})\mathbf{J}$ for all $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$.

(3) Let $\mathcal{V}$ be a genuine inner-product space.

(a) Prove: If $\mathbf{A}$ is a skew lineon on $\mathcal{V}$, then $\mathbf{1}_\mathcal{V} - \mathbf{A}$ is invertible, so that we may define $\Phi : \mathrm{Skew}\mathcal{V} \to \mathrm{Lin}\mathcal{V}$ by

$$\Phi(\mathbf{A}) := (\mathbf{1}_\mathcal{V} + \mathbf{A})(\mathbf{1}_\mathcal{V} - \mathbf{A})^{-1} \quad \text{for all} \quad \mathbf{A} \in \mathrm{Skew}\mathcal{V}.$$
(P4.3)

(b) Show that $\Phi(\mathbf{A})$ is orthogonal when $\mathbf{A} \in \mathrm{Skew}\mathcal{V}$.

(c) Show that Φ is injective and that

$$\mathrm{Rng}\,\Phi = \{\mathbf{R} \in \mathrm{Orth}\mathcal{V} \mid \ \mathbf{R} + \mathbf{1}_\mathcal{V} \quad \text{is invertible}\} \quad \text{(P4.4)}$$

(4) Let $\mathcal{V}$ be a genuine inner-product space.

(a) Show that, for each $\mathbf{a} \in \mathcal{V}^\times$, there is exactly one orthogonal lineon $\mathbf{R}$ on $\mathcal{V}$ such that $\mathbf{Ra} = -\mathbf{a}$ and $\mathbf{Ru} = \mathbf{u}$ for all $\mathbf{u} \in \{\mathbf{a}\}^\perp$. Show that, for all $\mathbf{v} \in \mathcal{V}$,

$$\mathbf{Rv} = -\mathbf{v} \iff \mathbf{v} \in \mathbb{R}\mathbf{a}. \quad \text{(P4.5)}$$

(b) Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ be given. Show that $\mathbf{L}$ commutes with every orthogonal lineon on $\mathcal{V}$ if and only if $\mathbf{L} \in \mathbb{R}\mathbf{1}_\mathcal{V}$. (Hint: use Part (a).)

(5) Let $\mathcal{E}$ be a genuine Euclidean space. Prove that the sum of the squares of the lengths of the sides of a parallelogram (see Problem 5 of Chapt. 3) in $\mathcal{E}$ equals the sum of the squares of the lengths of its diagonals. (Hint: Reduce the problem to an identity involving magnitudes of vectors in $\mathcal{V} := \mathcal{E} - \mathcal{E}$.)

Note: This result is often called the "parallelogram law".

(6) Let $\mathcal{E}$ be a genuine Euclidean space with $\dim\mathcal{E} > 0$ and let p be a flat basis of $\mathcal{E}$. Show that there is exactly one sphere that includes $\operatorname{Rng} p$, i.e. there is exactly one $q \in \mathcal{E}$ and one $\rho \in \mathbb{P}^\times$ such that $\operatorname{Rng} p \subset \operatorname{Sph}_{q,\rho}\mathcal{E}$ (see (46.8)). (Hint: Use induction over $\dim\mathcal{E}$.)

(7) Let $p := (p_i \,|\, i \in I)$ and $p' := (p'_i \,|\, i \in I)$ be families in a genuine Euclidean space $\mathcal{E}$ and assume that there is a $k \in I$ such that

$$(p_i - p_k) \cdot (p_j - p_k) = (p'_i - p'_k) \cdot (p'_j - p'_k) \quad \text{for all} \quad i, j \in I. \tag{P4.6}$$

Show that p and p' are congruent.

(8) Let $\mathcal{V}$ and $\mathcal{V}'$ be inner-product spaces and consider $\operatorname{Lin}(\mathcal{V}, \mathcal{V}')$ with its natural inner-product space structure characterized by (44.4) and (44.5). Let (p, n) and (p', n') be the signatures of $\mathcal{V}$ and $\mathcal{V}'$, respectively (see Sect.47). Show that the signature of $\operatorname{Lin}(\mathcal{V}, \mathcal{V}')$ is given by $(pp' + nn', pn' + np')$. (Hint: Use Prop.4 of Sect.25 and Prop.2 of Sect.47.)

(9) Let $\mathcal{V}$ be an inner-product space with $\operatorname{sig}^+\mathcal{V} = 1$ and $\dim\mathcal{V} > 1$. Put

$$\mathcal{V}_+ := \{\mathbf{v} \in \mathcal{V} \,|\; \mathbf{v}^{\cdot 2} > 0\}. \tag{P4.7}$$

(a) Prove: For all $\mathbf{u}, \mathbf{v} \in \mathcal{V}_+$ we have

$$(\mathbf{u} \cdot \mathbf{v})^2 \geq \mathbf{u}^{\cdot 2}\mathbf{v}^{\cdot 2}; \tag{P4.8}$$

equality holds if and only if $(\mathbf{u}, \mathbf{v})$ is linearly dependent. (Hint: Use a trick similar to the one used in the proof of the Inner-Product Inequality, Sect.42, and use the Inner-Product Signature Theorem.)

(b) Prove: For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}_+$ we have

$$(\mathbf{u} \cdot \mathbf{v})(\mathbf{v} \cdot \mathbf{w})(\mathbf{w} \cdot \mathbf{u}) > 0. \tag{P4.9}$$

(Hint: Consider $\mathbf{z} := (\mathbf{v}\cdot\mathbf{w})\mathbf{u} - (\mathbf{u}\cdot\mathbf{w})\mathbf{v}$ and use the Inner-Product Signature Theorem).

(c) Define the relation $\sim$ on $\mathcal{V}_+$ by

$$\mathbf{u} \sim \mathbf{v} \;\; :\Longleftrightarrow \;\; \mathbf{u}\cdot\mathbf{v} > 0. \tag{P4.10}$$

Show that $\sim$ is an equivalence relation on $\mathcal{V}_+$ and that the corresponding partition of $\mathcal{V}_+$ (see Sect.01) has exactly two members; moreover, if $\mathcal{C}$ is one of them then $-\mathcal{C}$ is the other.

Note: In the application to the theory of relativity, one of the two equivalence classes of Part (b), call it $\mathcal{C}$, is singled out. The elements of $\mathcal{C}$ are then called "future-directed" world-vectors while the members of $-\mathcal{C}$ are called "past-directed" world vectors. ■