Symmetries in Quantum Field Theory and the structure of superselection sectors

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1 Symmetries in quantum theory

"Symmetry" is an extremely broad and versatile concept in physics in general. The realization of symmetries acquires more facets and shadings in quantum theory, than it already exhibits in classical physics. Rather than attempting a full coverage of the topic, I just start by recalling in a schematic way of the richness of the notion in quantum theory, before I turn to a more specific discussion of symmetries in the context of quantum field theory, and finally turn to concentrate on its relation with superselection structure. This relation strongly depends on the number of spacetime dimensions.

Already terminology is quite diverse, and never uniform across contexts. Let us say that, most generally speaking, a "transformation" T is a prescription to associate new object(s) to given objects:

$$X \overset{T}{\mapsto} X_T.$$

It is called a "symmetry" if it preserves some pertinent feature of interest: e.g., the dynamics of a physical system (equations of motion, Hamiltonian), relational structures (algebras), concrete realizations (solutions of equations of motion, states of quantum systems). Noether's groundbreaking work taught us that symmetries of the (classical) Hamiltonian are related to

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dynamical conservation laws, providing a priori knowledge about the physical system under consideration, that may be of eminent importance for both practical applications and structural understanding.

In quantum theory, transformations or symmetries can be given at various levels, indicated with increasing mathematical restrictiveness from left to right and from top to bottom in a schematic diagram for the case of continuous transformations of an algebra:

$$\begin{array}{ccccc} & \delta_T(\cdot) & \stackrel{\text{integrable?}}{\longrightarrow} & \alpha_T(\cdot) & \\ \text{implementable?} & \downarrow & & \downarrow & \text{implementable?} \\ & i[Q_T,\cdot] & \stackrel{\text{self-adjoint?}}{\longrightarrow} & U_T\cdot U_T^* & \end{array}$$

The "$\cdot$" is a placeholder for algebra elements. The first line refers to the purely algebraic perspective: α_T may stand for a one-parameter group of automorphisms of an algebra, and δ_T for the associated infinitesimal derivation. Derivations may not be integrable, and conversely groups of automorphisms may not be differentiable. The situation depends on manifold domain questions in Banach spaces. The second line refers to the "spatial" perspective when the algebra is realized by operators on a Hilbert space. Whether a derivation is implementable by a self-adjoint generator, or whether an automorphism is implemented by a unitary operator, depends on the representations. (The right column pertains also to discrete transformations.)

The questions addressed by the diagram pose nontrivial and possibly hard mathematical problems, which this nontechnical contribution is not the place to discuss. They reach far beyond group theory and representation theory, requiring methods from Functional Analysis. E.g., the issue of self-adjointness of generators to ensure unitary one-parameter groups (Stone-von Neumann) lies at the origin of spectral theory. Suffice it to state that answers will depend on the specific physical system under consideration.

In a "third dimension" to be added to the diagram, one may ask whether several derivations form a Lie algebra, and whether this Lie algebra can be integrated to a possibly non-Abelian group of automorphisms. The corresponding questions in the second line are of cohomological nature and lead to the theory of projective representations and the Bargmann theorem about the conditions under which the latter give rise to true representations of a covering group.

When dynamical quantum systems are addressed, one could add yet another "dimension" to the diagram, concerning distinguished states, and ask whether ground states (or KMS states) are invariant under derivations, automorphisms, or unitaries. If they are not, the symmetry of the algebra is said to be "spontaneously broken" by the state.

2 Quantum field theory

Quantum field theory knows the distinction between spacetime symmetries (usually given by the Poincaré or conformal group), and "inner symmetries", that affect only inner degrees of freedom of quantum fields ("multiplets") but neither the localization of field operators in spacetime nor the momentum transfer on states induced by them. Spacetime symmetries in the first place act on the spacetime, but are expected to lift to unitary operators acting on the Hilbert space or to automorphisms acting on the field algebra, changing the localization in a geometric way.

The Coleman-Mandula theorem of QFT [Coleman and Mandula (1967)] states that (under standard assumptions) inner and spacetime symmetries "cannot interfere" with each other: they must commute and form a direct product of groups. An exception is supersymmetry (admitting fermionic generators) [Haag et al. (1975)]. An intriguing feature is that the Hamiltonian arises as the square of a supersymmetry generator, implying that it is automatically positive.

Yet, also without supersymmetry, the interplay of the axioms of Locality and Covariance (including positive energy) provide an indirect relation, when continuous inner symmetries are assumed to be implemented by generators that are associated with covariant and local conserved currents. In this case, the relation (the Goldstone theorem) is of spectral nature: the non-preservation of the vacuum state under the symmetry ("spontaneous symmetry breakdown") requires the presence of massless excitations of the vacuum, known as "Goldstone modes". More differentiated versions of the theorem, according to which the precise situation depends on the decay behavior of correlations, have been elaborated by [Ezawa and Swieca (1967), Buchholz et al. (1992)].

Inner symmetries are often regarded as "global gauge symmetries". "Global" is understood in the sense that they transform fields in the same way irrespective of their localization; "gauge" in the sense that only the invariant quantities are considered as observables. Because the latter commute with the unitary representation U of the gauge group G, its centre $U(G)''$ splits the representation of the fields into inequivalent representations of the observables. These are called superselection sectors, because observables cannot make transitions among them. (Historically, the first recognition of this fact was the superselection rule that local operators cannot interpolate between states of integer and half-inter spin.) The gauge-variant fields are then rather auxiliary mathematical tools whose role is to create from the vacuum "charged states" carrying quantum numbers that cannot be accessed by observables.

"Local gauge transformations" cannot be realized on Hilbert spaces (and consequently many mathematical theorems do not apply; e.g., the Goldstone theorem must fail in order to allow for the so-called Higgs mechanism). They do not relate physical states to each other. They act only on unphysical auxiliary "states" appearing at intermediate levels in the course of the construction of a model. In a famous panel dicussion it was acceded that they should not be regarded as "symmetries" at all [Zichichi (1984)]. Their main role is in fact their seminal power of selecting of renormalizable interactions that lie at the basis of the Standard Model of particles, without being a symmetry of the final quantum theory. Local gauge symmetry falls outside the scope of this contribution and shall not be addressed further. For an alternative method to predict and deal with the interactions of the Standard Model without "quantum field theory on indefinite state spaces", see [E6].

The traditional Wightman axiomatics [Streater and Wightman (1964)] of quantum field theory assumes that (globally) gauge-variant fields obey the same axioms as observable quantum fields, with the only exception that they may be "anti-local". The (analytic) Spin-Statistics theorem of [Streater and Wightman (1964), Jost (1965)] asserts that they cannot be local if they transform in a representation of the Poincaré group with half-integer spin or helicity. The theorem actually implies anti-locality only, when the axioms only offer the two options "local" or "anti-local". Relaxing this axiom, many new possibilities arise, see [E6].

Algebraic quantum field theory, or "Local Quantum Physics" (LQP) [Haag (1992)], avoids to talk (and make assumptions about) non-observable fields altogether. Instead, it axiomatizes the structural properties of observables with the emphasis on their localization properties. Localization and locality in LQP are algebraic properties: An observable is called "localized" in a region if it commutes with all observables localized at spacelike distance from that region. They can therefore by formulated in terms of a C^* algebra without reference to a specific Hilbert space representation. From algebraic axioms imposed on the assignment of localized subalgebras to spacetime regions (the "local net"), LQP allows to draw conclusions about possible states and representations in which these properties possibly can be realized, see Sect. 3.

One may dwell on the "relative" character of algebraic localization, as just defined. This feature opens the way to re-assign different geometric localizations to "the same elements" of the C^* algebra, as long as they are again consistent with locality in the new interpretation. One can thus "transplant" QFT models from one spacetime to another, such as Minkowski, Einstein universe, deSitter and Robertson-Walker [Guido and Longo (2003), Buchholz et al. (2001)]; or re-interpret QFT models on anti-deSitter spacetime as conformal QFTs on the conformal boundary of AdS (which is a completion of Minkowski spacetime) [Rehren (2000)]. The abstract group of spacetime symmetries and its unitary representations are the same, but the geometric interpretation of the group actions on regions are different. In a local setting adapted to curved spacetime, allowing the transplantation of patches of one spacetime into another spacetime in terms of a functor between categories of subregions and categories of subalgebras of C^* algebras, one can give a rigorous meaning to the notion of "the same physics on different spacetimes" [Fewster and Verch (2012)].

A Quantum Noether theorem. As emphasized in [Haag (1992)], a local net of observables together with the unitary representation of the Poincaré group determines "all the physics", including the scattering behaviour (from which phenomenologists are used to "deduce" the interaction Lagrangian). An interesting example how information about the field content is "encoded" in the net (even if the net refers only to algebras and not individual fields), is the "Quantum Noether theorem" of [Doplicher and Longo (1983)], as follows.

A local net is said to satisfy the split property if two von Neumann algebras of observables localized in two spacelike-separated spacetime regions with a finite distance generates an algebra that is isomorphic to the tensor product of the two algebras. This property is related to phase-space properties (localization and momentum transfer), and has been established in several classes of models. It cannot be expected to hold when there is no finite distance because

UV quantum fluctuations create correlations between the two regions that are incompatible with a tensor product.

The tensor product structure allows to show that if a global gauge group is unitarily implemented, then there exist also local implementers, i.e., unitary operators that implement the symmetry on one of the two algebras, and commute with the other algebra. They share this property with exponentiated local charge operators, although it was not assumed that the theory possesses a conserved current. Instead, the local implementers are taken as the LQP counterpart of the latter.

Poincaré symmetry and Modular Theory. Although the focus of this contribution is on the relation between superselection sectors and global "inner symmetries", see Sect. 3 and Sect. 4, we include a paragraph that sheds a new light on spacetime symmetries, and their relations to the Tomita-Takesaki modular theory [E5], see also [E4].

In a nutshell: Modular theory assigns to a von Neumann algebra M and a cyclic and separating vector in a Hilbert space, a unitary one-parameter "modular group" and an anti-unitary "modular conjugation". The former acts by automorphisms on M, and the latter maps M to its commutant M'. The assignment is intrinsic and enjoys many non-trivial "functorial" algebraic features.

A "wedge" W is a Poincaré transform of the spacetime region $W_0 := \{x \in \mathbb{R}^4 : x^1 > |x^0|\}$. Its causal complement is denoted by W'. Let $A(W)$ be the von Neumann algebra associated with quantum fields localized in a wedge, and Ω the vacuum vector. [Bisognano and Wichmann (1976)] showed that in QFT satisfying the Wightman axioms, the modular group of the pair $(A(W), \Omega)$ is the subgroup of the Poincaré group of the Lorentz boosts that preserve the wedge, and the modular conjugation is a PCT transformation mapping $A(W)$ to $A(W')$.

Let now M a von Neumann algebra with a cyclic and separating vector $\Omega \in \mathcal{H}$, and $U(a)$ a unitary one-parameter group on $\mathcal{H}$, leaving Ω fixed. Assume the property $U(a)MU(a)^* \subset M$ for $a > 0$ (which is characteristic for lightlike translations acting on wedge algebras in QFT). [Borchers (1992)] showed: If $U(a)$ has a positive generator, then the modular group $V(s)$ and the modular conjugation of the pair (M, Ω) satisfy the same commutation relations with $U(a)$ as the boosts and the PCT transformation of a two-dimensional QFT with the lightlike translations ("Borchers' theorem"). Also the converse is true: The commutation relations imply positivity of the generator [Wiesbrock (1992)]. [Wiesbrock (1993)] showed a stronger result with two von Neumann algebras $M_1 \subset M$ such that the modular group $V(s)$ of M satisfies $V(s)M_1V(s)^* \subset M_1$ for $s > 0$ ("half-sided modular inclusion"). In this case, it is possible to extract from the two modular groups of M_1 and M another unitary one-parameter group $U(a)$ with positive generator and the commutation relations with $V(s)$ as before.

These results are most remarkable because they provide the entrance gate to extend the "modular nature" of Lorentz boosts to the Poincaré group. They can be turned around in various ways, and extended to the Poincaré group in four-dimensional spacetime (4D):

(i) A "Borchers triple" (see, e.g., [Buchholz et al. (2011)]) is a von Neumann algebra M

with a cyclic and separating vector $\Omega \in \mathcal{H}$, a unitary positive-energy representation U of the translation group on $\mathcal{H}$ that leaves Ω invariant and acts on M such that the semigroup of translations inside W_0 maps M into itself. In two spacetime dimensions, these data suffice to construct a full-fledged Poincaré covariant net of local algebras $A(W)$ for all wedge regions. Namely, for the reference wedge, $A(W_0) := M$ and all other wedge algebras are defined by acting with U. The conditions of the Borchers triple ensure that the resulting net of wedges is a local net. Algebras of observables localized in doublecones (= intersections of a left and a right wedge) may be defined by intersections of the wedge algebras. Since the local net determines all the physics (including, e.g., the S-matrix, [Haag (1992)]), one has, in principle, a completely "non-Lagrangian" way to construct dynamical models "out of symmetries" in two spacetime dimensions. The difficulty is to ensure that Ω is cyclic and separating also for algebras of compactly localized observables. These methods have been pivotal for the construction of models with factorizing S-matrices [Lechner (2008)]. See [E2].

In four dimensions, Borchers' theorem does not suffice to construct the Lorentz boosts in all directions. One might add further assumptions concerning the rotations. A stronger result, not even assuming the translations, is:

(ii) The modular groups of a small number of von Neumann algebras with a common cyclic and invariant vector and in a "suitable modular position" relative to each other [Kähler and Wiesbrock (2001)] suffice to construct a unitary positive-energy representation U of the Poincaré group. As before, by identifying the algebra of a reference wedge with one of the given von Neumann algebras and acting with the Poincaré group, one obtains a net of wedge algebras, and by intersections, one obtains a net of doublecone algebras. This is, in principle, another non-Lagrangian way to construct dynamical models including their spacetime symmetry by Modular Theory. A particularly elegant version of this result applying to chiral conformal QFT (see Sect. 4) was given by [Guido et al. (1998)], see [E2].

3 Superselection sectors in 4D

The highlight of the analysis of representations of local nets of C^* algebras is the theory of superselection sectors developped by Doplicher, Haag and Roberts [Doplicher et al. (1971, 1974)] (DHR theory). There is a detailed account in [E1]. It is presented here rather briefly as a "benchmark" to which the theory of superselection sectors in conformal QFT in two dimensions (Sect. 4) should be compared.

The authors concentrate on states (considered as "relevant for scattering theory") that belong to positive-energy representations of the Poincaré group, and that are indistinguishable from states in the vacuum sector by measurements in the causal complement of any open spacetime region. Thus, the inequivalence from the vacuum representation is a global feature, generically called "charge", to label superselection sectors = inequivalent representations of the local net. The selection criterium excludes, say, thermal states in which only number or charge densities are defined but no total particle number or charge operators.

The authors assume that the local observables satisfy a strengthened version of Locality: Haag duality. It asserts that the von Neumann algebras generated by observables localized in doublecone regions O and by observables localized in the causal complement O', are exactly each others' commutants. Under this maximality assumption, they discovered that one can define a "tensor product" among the charged representations. This then turns the latter into the objects of a C^* tensor category, equipped with an intrinsic unitary symmetry in terms of "statistics operators" which relate the tensor product (in the sense of the category) of representations with the opposite tensor product. The statistics operators in their turn define a representation of the infinite permutation group. From these data, one can then extract two intrinsic quantum numbers which are invariants of the superselection sectors: the "statistical dimension" and a sign. The statistical dimension is necessarily a positive integer (possibly infinite), and the sign is -1 if and only if the representation of the Poincaré group in the sector has half-integer spin (an algebraic Spin-Statistics theorem).

There is no assumption that the observables arise as the invariants of a larger "field algebra" of unobservable operators, under the action of an inner symmetry. To the contrary, the analysis was crowned by the "duality result" of [Doplicher and Roberts (1990)]: The DHR symmetric tensor category of representations of the local net is isomorphic to the category of representations of a compact group G, such that the statistical dimension is identified with the natural dimension of the associated representation of G. Then, one can construct a field algebra as a graded-local net with the grading given by the Spin-Statistic theorem, such that the observables are the invariants under an action of G by inner symmetry automorphisms (global gauge group). Thus, the above scenario with a global symmetry being "responsible" for the existence of superselection sectors can be deduced, rather than assumed. It is a subtle consequence of locality and covariance of the observables in states "sufficiently close" to the vacuum state.

The axioms of the DHR approach have to be relaxed in various cases of physical relevance. Roberts [Roberts (1976)] pointed out that one of the main assumptions, Haag duality in the vacuum sector, does not hold in theories with spontaneously broken global symmetries. As a consequence, spontaneously broken symmetries do not give rise to superselection rules. [Buchholz and Roberts (2014)] addressed the fact that there are "too many" charged sectors in QED, because of long-range electromagnetic fields accompanying the charges which would produce continuously many inequivalent sectors for each total electric charge. But these sectors cannot be distinguished by measurements within future lightcones (which are the only options experimenters have). By relaxing the notion of sector accordingly, the authors could define more appropriate, coarser equivalence classes and establish (for sectors of statistical dimension 1) the same sector structure as in DHR theory.

QFTs with long-range interactions or topological charges admit superselection sectors that do not satisfy the DHR selection criterium (localization in doublecones). For charged states of QED, "photon clouds" extending to infinity define inequivalent sectors [Fröhlich et al. (1979)]. Their electric flux cannot be compactly localized but possibly along infinite spacelike cones, while the restriction of sectors to lightcones can distinguish only their total electric charge [Buchholz (1982)], see above. Motivated by the search for confinement criteria, [Buchholz and Fredenhagen (1982)] showed for theories with a mass gap that the intrinsic localization of sectors cannot be worse than arbitrary spacelike cones. It is rather clear from the DHR analysis, that in such cases, the reconstruction as in [Doplicher and Roberts (1990)] cannot give rise to a graded-local field net. Instead, the charged fields can only be localized, relative to the local observables, along spacelike cones.

The Wightman axiomatics is therefore too restrictive for realistic quantum field theories. A perturbative constructive scheme, in which string-localized charged quantum fields naturally emerge, is the topic of [E6].

4 Superselection sectors in conformal QFT in 2D

In two spacetime dimensions (2D), the causal complement of a doublecone is not connected. This circumstance is responsible for the fact that in the DHR theory of superselection sectors, the tensor category of representations of the (chiral or 2D) observables is *braided*, i.e., it is not equipped with a unitary representation of the infinite permutation group but of the infinite braid group [Fröhlich and Gabbiani (1990), Fredenhagen et al. (1989)]. Namely, the cohomological argument to show that the statistics operators (the representers of transpositions) are their own inverses, fails.

Braided tensor categories are much richer than symmetric ones. In particular, a general duality theorem as in [Doplicher and Roberts (1990)] does not exist, and there is no obvious analogue of a global gauge group that can a posteriori be made "responsible" for the presence of superselection sectors, as is the case in 4D. The statistical dimensions of sectors are in general positive but not integer numbers; they can therefore not be identified with the natural dimension of a representation of some compact group (or Hopf algebra) – it is rather the square root of the Jones index of a subfactor that characterizes the failure of Haag duality in the sector under consideration [Longo (1989)]. Also, the sign of the statistics (distinguishing fermions from bosons in 4D) may be rather a complex phase.

A natural question arises: are there other "reconstructions" of an algebra of fields that create the superselection sectors of the given observables, and what are the algebraic properties of the "charged fields"? The question is void, though, for massive quantum field theories in which the split property (see Sect. 2) holds for wedge regions: [Müger (1998)] has shown that such theories do not have any DHR sectors besides the vacuum sector.

On the other hand, a conformal quantum field theory (CQFT) neither has a mass gap, nor does the split property hold for wedge regions. The latter is because two wedges at a finite distance can be mapped by a conformal transformation to a pair of doublecones touching each other in a point. Indeed, large classes of models of CQFT in 2D have a rich superselection structure.

We therefore limit the subsequent discussion to the case of CQFT in 2D.

Geometric preliminaries. Conformal quantum field theory is QFT in which the Poincaré group is extended to the conformal group. In two spacetime dimensions, due to the factorization of the metric $ds^2 = dt^2 - dx^2 = d(t-x)d(t+x)$, the conformal group is infinite-dimensional. It acts geometrically on a "conformal completion" of the spacetime. The latter can be visualized with the help of the Cayley map (the inverse of the stereographic projection)

$$u_\pm = t \pm x \mapsto z_\pm = \frac{1 + iu_\pm}{1 - iu_\pm} = e^{i\theta_\pm}$$

which maps the left and right chiral lightrays to the sphere $S^1 \subset \mathbb{C}$. The completion is the addition of the points $z_\pm = -1$. Thus, the conformal spacetime is $S^1 \times S^1$. The 2D Einstein universe is a covering space of $S^1 \times S^1$ respecting the spacelike periodicity $(\theta_+, \theta_-) = (\theta_+ + 2\pi, \theta_- - 2\pi)$. The conformal group is then (a covering of) $\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$, where $\mathrm{Diff}_+(S^1)$ are the orientation-preserving diffeomorphisms of the circle.

The conformal completion "at infinity" (where opposite wedges touch each other) is the reason why the split property for wedges fails, as stated above.

Conformal quantum fields. Before we turn to the DHR superselection theory of CQFT in two dimensions, we explain that conformal symmetry is a very restrictive property for the structure of conformal fields.

The challenge for conformal quantum field theory (CQFT) in 2D is to understand how conformal symmetry can be implemented as an action on algebras of local quantum fields on a Hilbert space. A first, and most prominent in the sequel, indicator for the individual fields is the "scaling dimension", that characterizes their transformation law under the scale transformations of the (embedded) Minkowski spacetime.

The remarkable fact about CQFT is the existence of "chiral fields", depending on only one of the lightcone coordinates $t \pm x$. The name comes from the left- or right-handed free *massless* Dirac fields in 2D which are chiral in the present sense. Beyond this example, every conserved symmetric and traceless tensor field of rank r (e.g., conserved currents of rank 1 or the stress-energy tensor of rank 2) has only two linearly independent components, and these provide a pair of chiral fields if and only its scaling dimension equals its rank r. Thus, chiral fields abound in conformal field theories whenever there are symmetries associated with conserved tensor fields of canonical dimension.

When the Hamiltonian is positive, correlation functions are boundary values of analytic functions with ordered imaginary parts of the time variables. Under the Cayley map, this ordering becomes "radial ordering" in the complex variables $z_\pm$. In the complex domain, z_+ and z_- are independent complex variables, and chiral fields are holomorphic functions of either z_+ or z_-. At the "Euclidean points" with real x and imaginary time (where the metric becomes Euclidean), one has $u_- = -\overline{u}_+$ and (after the Cayley map) $z_- = \overline{z}_+$. This justifies the popular terminology "holomorphic" and "anti-holomorphic" [di Francesco et al. (1997)].

In the "Euclidean" formulation, conformal field theory has a non-relativistic interpretation as critical equilibrium points of systems of Statistical Mechanics with infinitely many degrees of freedom in two spatial dimensions. Namely, conformal invariance emerges at critical points by the loss of a finite correlation length to set a scale.

Most of the structures of CQFT "in real time" (Lorentzian metric) can be "translated" to Euclidean CFT. However, Euclidean CFT is much less restrictive than CQFT which needs a Hilbert space, and not every Euclidean model will have correlation functions obeying the Osterwalder-Schrader positivity conditions, so as to be the "Wick rotation" of a relativistic QFT. In particular, many of the classification results for symmetries and representations of CQFT will not apply, or have to be extended to Euclidean CFT.

Models of Euclidean CFT exhibit a mathematically far richer structure, including rational models with "forbidden" central charges (see below) like the Yang-Lee model with $c = -\frac{22}{5}$, or logarithmic CFT models with not completely decomposable fusion structure (appearing, e.g., in percolating systems), that have no counterpart in CQFT.

Classifications and superselection structure. It was known by [Lüscher and Mack (1976)] that a conserved symmetric and traceless stress-energy tensor in two spacetime dimensions, without any further model input, splits into two chiral components, whose commutation relations are a position-space version of (two copies of) the Virasoro algebra. Conversely, the Virasoro algebra is a mode expansion of the chiral stress-energy tensor. The Virasoro algebra is the unique central extension of the Lie algebra of $\mathrm{Diff}_+(S^1)$, where the eigenvalue of the central extension is called the "central charge" c. This parameter is the only model-dependent quantum number of conformal stress-energy tensors.

It is worth noting here, that the Virasoro generators L_n with $|n| \leq 1$ including the "conformal Hamiltonian" L_0 serve a double role: as the generators of the "unbroken" Möbius subgroup of $\mathrm{Diff}_+(S^1)$, under which the vacuum vector is invariant, and as field observables (integrals over the chiral stress-energy tensor). This is the reason why the vacuum state cannot be invariant under the infinite-dimensional conformal group: the stress-energy tensor would have to be zero.

A breakthrough in CQFT in two spacetime dimensions was made by [Friedan et al. (1984)]: There are many inequivalent positive-energy representations of the Virasoro algebra, characterized by their "lowest weight" h (= lowest eigenvalue of L_0) which is interpreted as the scaling dimension of an associated quantum field. The surprising discovery (already anticipated in [Lüscher and Mack (1976)]) is that in the range $c < 1$, the possible values of c and h are discretely quantized by Hilbert space positivity. (Later, many other nontrivial classification results could be achieved along similar lines.)

The kinematics of chiral fields is particularly simple and often allows to determine their correlation functions and commutation relations. The "more interesting" fields of CQFT are non-chiral local fields that can coexist with the chiral ones. Among the non-chiral fields there must be "primary" ones with a pair (h_+, h_-) of chiral scaling dimensions so that $h_+ - h_+$ is the spin. They create states from the vacuum which belong to representations of the chiral Virasoro algebras with lowest weights h_+ and h_-, respectively. Conversely, the a priori classification of admissible values of $h_\pm$ is taken as signal for the existence of fields with these quantum numbers.

A second breakthrough discovery was made by [Belavin et al. (1984)]: the fulfilment of the quantization conditions in [Friedan et al. (1984)] leads to Ward identities for the correlation functions of putative primary fields. These are linear differential equations which can be used to actually compute the correlation functions (up to a finite number of undetermined coefficients), and to derive "fusion rules" for the operator product expansion of these fields. Although obtained in a completely different way, these result are in a certain sense comparable to the reconstruction of a field algebra [Doplicher and Roberts (1990)] out of the representation theory of the observables (see Sect. 3), with the fusion rules corresponding to the tensor product of the category of representations. The picture was completed by constructive proofs [Goddard et al. (1986)] that all admissible values of $(c < 1, h)$ can indeed be realized.

For a model-independent treatment, we turn to the DHR theory of superselection sectors. As said, the theory of Sect. 3 can be adapted to CQFT, with the chiral stress-energy tensors (the local version of the Virasoro algebra) or other chiral fields as algebras of observables. The main difference as compared to 4D is that the tensor category of DHR representations is *braided*. As a consequence, a duality theorem as in [Doplicher and Roberts (1990)] does not exist.

The absence of a group underlying the superselection structure of CQFT in 2D does not exclude that CQFT models B can have actions of compact inner symmetry groups G. In this case, one may descend to the invariant subalgebra $A = B^G$ ("orbifold model"). Then, as in 4D, the centre of $U(G)$ splits the vacuum representation of B into superselection sectors of A in correspondence with the representations of the group. But due to the geometry of the 2D conformal spacetime, there arise on top so-called "twisted sectors". These are restrictions of solitonic sectors of B, differing at the "left" and "right" infinity of spacetime by the action of group elements $g \in G$. Because of the twisted representations, the resulting tensor category of sectors of A is no longer isomorphic to the tensor category of representations of G, but to the tensor category of the "Drinfel'd double" of G, which is a Hopf algebra.

However, compact Hopf algebras still fall short as candidates for a symmetry notion, that could be made responsible for the existence of superselection sectors in general. In particular, they cannot account for non-integer statistical dimensions, that abound in the multitude of known CQFT models. Instead, the fusion rules of minimal models suggest that quantum groups with the deformation parameter q a root of unity may play a role. There is, however, for complex values of q a conflict between their $*$-structure and their actions on C^* algebras.

The notion of Q-system [Longo and Rehren (1995)] (also known as Frobenius algebra) allows to characterize extensions of a local net A by local (or non-local) nets B which contain A as fixed points of a global conditional expectation – an abstract generalization of averages over compact group actions. A Q-system is a set of data within the DHR tensor category of A, which determines the vacuum Hilbert space of the extension B, its "field content" in an algebraic sense, as well as the multiplication law of the new generators. Q-systems can be classified in terms of the (given) tensor category, but there is no unique or maximal extension like the field algebra reconstructed in [Doplicher and Roberts (1990)]. The method can be used both for "chiral extensions" where A and B are chiral theories, or for two-dimensional extensions B of subtheories $A = A_+ \otimes A_-$ which are tensor products of left and right chiral subtheories. This result "in principle" answers the quest for "generalized symmetries" that can be made responsible for the existence of superselection sectors. However, the resulting notion of "symmetry" is very far from Hopf algebras or quantum groups, and essentially resists a classification.

The fact that the tensor product of sectors is only braided-symmetric means that putative fields of a field algebra that would create these sectors from the vacuum, are neither local nor anti-local, but "anyonic" (with commutation relations controlled by complex phases or matrices, rather than signs). A satisfactory general axiomatization of such fields is hardly possible beyond the mere description of the structure of known models, see [Moore and Seiberg (1989)] or [Fredenhagen et al. (1992)]. This is another instance where the traditional Wightman axiomatization falls short.

Modular tensor categories are braided tensor categories with a maximal non-degeneracy of the braiding, that allows to distinguish sectors only in terms of their braiding with other sectors. [Kawahigashi et al. (2001)] discovered that the superselection category of a chiral CQFT is in fact modular, if the failure of Haag duality for disconnected chiral intervals gives rise to a subfactor of finite Jones index (and a strong additivity property holds). Modular tensor categories share a new kind of symmetry, known before from models [Cardy (1986)], namely a unitary representation of the noncompact discrete group $SL(2,\mathbb{Z})/\mathbb{Z}_2$ on the fusion algebra, which in some cases can be realized as an action on the (complex) temperature parameter of thermal partition functions associated with the sectors [Cappelli et al. (1987)]. This new symmetry is also a necessary condition for the validity of certain glueing prescriptions in the conformal wordsheet approach to (super-) String theory. By [Kawahigashi et al. (2001)], it is automatic if the mentioned structural conditions are satisfied.

The study of modular tensor categories revealed many highly non-trivial mathematical structures [Fuchs et al. (2003), Fuchs et al. (2007)]. These structures could (not least) be identified [Bischof et al. (2016)] with constraints on the possibilities of "merging" one CQFT algebra with another, i.e., the question whether and how the two algebras of their local fields can be defined on a common Hilbert space where they are local relative to each other, and on imposing boundary conditions on a given CQFT.

This is not the place to go into further details, or to relate all the many other branches of CQFT. Suffice it to state that there exist not only large classes of "elementary" models like the minimal models of [Belavin et al. (1984), Goddard et al. (1986)] and the non-Abelian current algebra models of [Knizhnik and Zamolodchikov (1984)], but there is also a large tool box of methods to construct new theories from given ones, including the coset construction [Goddard et al. (1986)], the lattice construction [Buchholz et al. (1988), Dong and Xu (2006)], and "braided products" of models [Bischof et al. (2016)]. These models can be used to explore further general features of CQFT.

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