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Location of Triangle Centers Relative to the Incircle and Circumcircle

Stanley Rabinowitz
545 Elm St Unit 1, Milford, New Hampshire 03055, USA
e-mail: stan.rabinowitz@comcast.net
web: http://www.StanleyRabinowitz.com/

Abstract. We use a computer to determine the relative location of Kimberling centers X_1 through X_{100} with respect to the incircle of a triangle. For example, for all triangles, we determine which centers must lie inside the incircle and which must lie outside the incircle. We also present inequalities representing how far away these centers can get from the incenter and circumcenter.

Keywords. triangle geometry, triangle centers, incircle, circumcircle, inequalities, computer-discovered mathematics, Blundon's Fundamental Inequality.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

Let X_n denote the nth named triangle center as cataloged in the Encyclopedia of Triangle Centers [5].

It is obvious that the incenter, X_1, lies inside the incircle. It is easy to see geometrically that the Gergonne point, X_7, must lie inside the incircle (Figure 1).

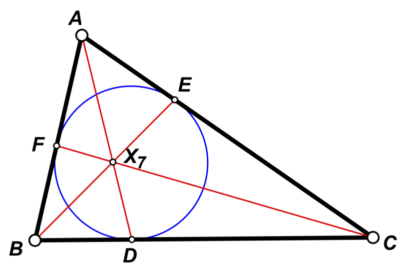

Figure 1. Gergonne point

It is obvious that the Feuerbach point, X_{11}, must lie on the incircle, since it is the point of contact of the incircle with the nine-point circle.

In this paper, we use a computer to study triangle centers X_1 through X_{100} to determine whether they must lie inside, on, or outside the incircle of the triangle. We also investigate the relationship between the distances from these centers to X_1 and r, the inradius of the triangle. Similar analyses are made for the circumcircle and R, the circumradius of the triangle.

Many of the inequalities we found could be proven by hand computation. For example, here is a proof that the X_{55} point of a triangle must lie inside the incircle.

Proof. We have $X_{55} = (a^2(a-b-c) : \cdots : \cdots)$. A straightforward algebra computation shows that

$$(IX_{55})^2 - r^2 = \frac{(a-b-c)^2(a+b-c)^2(a-b+c)^2\,(a^2-2ab-2ac+b^2-2bc+c^2)}{4\left(-a^2b-a^2c+a^3-ab^2-ac^2-b^2c+b^3-bc^2+c^3\right)^2}.$$

From the triangle inequalities, it follows that

$$\begin{cases} a^2 < a(b+c) \\ b^2 < b(b+c) \\ c^2 < c(a+b) \end{cases} \quad \Rightarrow \quad a^2+b^2+c^2 < 2ab+2bc+2ca$$

which implies that the last factor in the numerator must be negative. Since all the other factors are perfect squares, they must be positive and the entire fraction is negative.

Thus, $(IX_{55})^2 - r^2 < 0$ or $IX_{55} < r$. □

Although the proof is elementary, the computation is tedious and required a bit of ingenuity. Furthermore, when we have hundreds of inequalities to prove, the work is best left to a computer.

2. Methodology

We used Mathematica for our investigation.

The following Mathematica code computes the distance between two points `p1` and `p2` in terms of their barycentric coordinates. The formula used comes from [4].

```
normed[{x_, y_, z_}] := {x, y, z}/(x + y + z);
squaredDistance[p1_, p2_] := Block[{x, y, z},
   If[Total[p2] == 0, Return[Infinity]];
   {x, y, z} = normed[p1] - normed[p2];
   Return[-a^2*y*z - b^2*z*x - c^2*x*y];
   ];
```

The barycentric coordinates for triangle centers are given in [5] in terms of the sides of the triangle, a, b, and c, and trigonometric functions of the angles of the triangle A, B, and C.

To avoid radicals, we replaced inequalities of the form

$$\texttt{dist}[X_n, X_m] \le kr$$

by the equivalent inequality

$$\texttt{dist}[X_n, X_m]^2 \le k^2 r^2$$

where `dist`$[X_n, X_m]$ represents the distance between the two specified centers.

We then used *Blundon's Fundamental Inequality* [1] to find the best value for k that makes the inequality true. Examples of how this procedure works can be found in section I.1 of [6], the proof of Theorem 1 in [8], and [7]. Details of the specific algorithm used can be found in [9].

3. Results

The following result is well-known.

Theorem 1 (Points on the Incircle)**.** *The Feuerbach point, X_{11}, lies on the incircle and no other Kimberling center, X_1 through X_{100} must lie on the incircle.*

A list of the Kimberling centers up to X_{3025} that must lie on the incircle can be found in [10].

Theorem 2 (Points Inside the Incircle)**.** *The Kimberling centers X_n must lie inside the incircle for the following values of n.*

$$1, 7, 12, 55, 56, 57, 65$$

Theorem 3 (Points Outside the Incircle)**.** *The Kimberling centers X_n must lie outside the incircle for the following values of n.*

$$14, 16, 23, 30, 36, 44, 67, 74, 80, 88, 98, 99, 100$$

The following result is well-known.

Theorem 4 (Points on the Circumcircle)**.** *The Kimberling centers X_n must lie on the circumcircle for the following values of n.*

$$74, 98{-}100$$

A list of the Kimberling centers up to X_{2868} that must lie on the circumcircle can be found in [11].

Theorem 5 (Points Inside the Circumcircle)**.** *The Kimberling centers X_n must lie inside the circumcircle for the following values of n.*

$$1{-}3, 6{-}12, 15, 21, 24, 31, 32, 35, 37{-}42, 45, 55{-}61,$$
$$63, 65, 69, 71, 72, 75, 76, 78, 81{-}83, 85, 86, 89$$

Theorem 6 (Points Outside the Circumcircle)**.** *The Kimberling centers X_n must lie outside the circumcircle for the following values of n.*

$$16, 23, 30, 36, 44$$

Corollary 7 (Points Between the Circles)**.** *There are no Kimberling centers from X_1 to X_{100} that must lie outside the incircle but inside the circumcircle of the triangle.*

Such centers do exist. The one with the smallest index is X_{115}.

4. Refinements For the Incenter

We have determined when a point must lie inside or outside the incircle. But how close can they get to the incircle? There are many centers, P, that do not lie inside the incircle. This means that $\texttt{dist}[P, X_1] > r$, where $\texttt{dist}[P, Q]$ denotes the distance between points P and Q. Perhaps the point P cannot get too far away from the incenter and there is some constant k such that $\texttt{dist}[P, X_1] \leq kr$. We investigated these questions and found the following results.

Theorem 8. *The following Kimberling centers X_n satisfy the associated inequality indicating that they cannot get too far away from the incenter. The listed coefficients of r are best possible.*

$$\begin{gathered}
\texttt{dist}[X_1, X_7] < r \\
\texttt{dist}[X_1, X_{11}] = r \\
\texttt{dist}[X_1, X_{12}] < r \\
\texttt{dist}[X_1, X_{35}] < 2r \\
\texttt{dist}[X_1, X_{46}] < 2r \\
\texttt{dist}[X_1, X_{55}] < r \\
\texttt{dist}[X_1, X_{56}] < r \\
\texttt{dist}[X_1, X_{57}] < r \\
\texttt{dist}[X_1, X_{65}] < r \\
\texttt{dist}[X_1, X_{79}] < 2r \\
\texttt{dist}[X_1, X_{80}] = 2r
\end{gathered}$$

All other centers X_n with $1 < n \leq 100$ not in the above list can get arbitrarily far away from the incenter in the sense that there is no positive constant k such that the inequality $\texttt{dist}[X_1, X_n] < kr$ is true for all triangles.

Theorem 9. *The following Kimberling centers X_n satisfy the associated inequality indicating that they cannot get too close to the incenter. The listed coefficients of r are best possible.*

$$\begin{gathered}
\texttt{dist}[X_1, X_{11}] = 1 \\
\texttt{dist}[X_1, X_{14}] > \frac{1}{3}(3 + \sqrt{3})r \\
\texttt{dist}[X_1, X_{16}] > (1 + \sqrt{3})r \\
\texttt{dist}[X_1, X_{23}] > \frac{7}{3}r \\
\texttt{dist}[X_1, X_{30}] = \infty \\
\texttt{dist}[X_1, X_{36}] > 2r \\
\texttt{dist}[X_1, X_{44}] > 3r
\end{gathered}$$

$$\begin{gathered}
\texttt{dist}[X_1, X_{59}] > \frac{\sqrt{7}}{4}r \\
\texttt{dist}[X_1, X_{67}] > r \\
\texttt{dist}[X_1, X_{74}] > r \\
\texttt{dist}[X_1, X_{80}] = 2r \\
\texttt{dist}[X_1, X_{88}] > \frac{3}{2}r \\
\texttt{dist}[X_1, X_{98}] > r \\
\texttt{dist}[X_1, X_{99}] > r \\
\texttt{dist}[X_1, X_{100}] > r
\end{gathered}$$

All other Kimberling centers X_n with $1 \leq n \leq 100$ can get arbitrarily close to the incenter in the sense that there is no positive constant k such that the inequality $\texttt{dist}[X_1, X_n] > kr$ is true for all triangles.

5. Refinements For the Circumcenter

Theorem 10. *The following Kimberling centers X_n satisfy the associated inequality indicating that they cannot get too far away from the circumcenter. The listed coefficients of R are best possible.*

$$\mathtt{dist}[X_3, X_4] < 3R$$
$$\mathtt{dist}[X_3, X_5] < \frac{3}{2}R$$
$$\mathtt{dist}[X_3, X_{13}] < \frac{2}{\sqrt{3}}R$$
$$\mathtt{dist}[X_3, X_{14}] < \frac{2}{\sqrt{3}}R$$
$$\mathtt{dist}[X_3, X_{17}] \le k_1R$$
$$\mathtt{dist}[X_3, X_{19}] < k_2R$$
$$\mathtt{dist}[X_3, X_{20}] < 3R$$
$$\mathtt{dist}[X_3, X_{25}] < \frac{2}{\sqrt{3}}R$$
$$\mathtt{dist}[X_3, X_{27}] < k_3R$$
$$\mathtt{dist}[X_3, X_{28}] < k_4R$$
$$\mathtt{dist}[X_3, X_{34}] < k_5R$$
$$\mathtt{dist}[X_3, X_{43}] < \frac{8\sqrt{6}-3}{15}R$$
$$\mathtt{dist}[X_3, X_{46}] < R\sqrt{6\sqrt{3}-9}$$
$$\mathtt{dist}[X_3, X_{51}] < \frac{\sqrt{10}}{3}R$$
$$\mathtt{dist}[X_3, X_{52}] < 2R$$
$$\mathtt{dist}[X_3, X_{53}] < k_6R$$
$$\mathtt{dist}[X_3, X_{67}] < (6-2\sqrt{6})R$$
$$\mathtt{dist}[X_3, X_{79}] < k_7R$$
$$\mathtt{dist}[X_3, X_{80}] < \frac{5}{4}R$$
$$\mathtt{dist}[X_3, X_{88}] < (15-8\sqrt{3})R$$

where
$k_1 \approx 1.025582719$ *is the positive root of* $49x^3 + 22x^2 - 39x - 36$,
$k_2 \approx 1.315481999$ *is the largest positive root of* $19x^5 + 111x^4 + 334x^3 - 730x^2 - 337x + 539$,
$k_3 \approx 1.477671258$ *is the largest positive root of* $27x^5 + 729x^4 + 4734x^3 - 9566x^2 - 169x + 2197$,
$k_4 \approx 1.205917120$ *is the positive root of* $59x^3 + 77x^2 - 75x - 125$,
$k_5 \approx 1.054067061$ *is the positive root of* $11x^4 + 126x^3 + 136x^2 - 134x - 171$,
$k_6 \approx 1.089001927$ *is the largest positive root of* $32x^4 + 16x^3 + 8x^2 - 204x + 147$,
$k_7 \approx 1.057298324$ *is the smallest positive root of* $4x^3 - 107x^2 - 2x + 117$.

For each center X known to lie inside the circumcircle as listed in Theorem 5, we have the inequality $\mathtt{dist}[X_3, X] < R$ *where the coefficient, 1, of R is the best possible. These points can get arbitrarily close to the circumcircle.*

All other centers X_n with $1 \le n \le 100$ not on the circumcircle can get arbitrarily far away from the circumcenter in the sense that there is no positive constant k such that the inequality $\mathtt{dist}[X_3, X_n] \le kR$ *is true for all triangles.*

Theorem 11. *The following Kimberling centers X_n satisfy the associated inequality indicating that they cannot get too close to the circumcenter.*

$$\mathtt{dist}[X_3, X_{30}] = \infty$$
$$\mathtt{dist}[X_3, X_{62}] \ge \frac{1}{3}R$$
$$\mathtt{dist}[X_3, X_{94}] \ge \frac{\sqrt{3}}{2}R$$

For each center X known to lie outside the circumcircle as listed in Theorem 6, we have the inequality $\mathtt{dist}[X_3, X] > R$ *where the coefficient, 1, of R is the best possible. These points can get arbitrarily close to the circumcircle.*

All other Kimberling centers X_n with $1 < n \leq 100$ not on the circumcircle, can get arbitrarily close to the circumcenter in the sense that there is no positive constant k such that the inequality $\mathtt{dist}[X_3, X_n] \geq kR$ is true for all triangles.

6. Miscellaneous Inequalities

We present a selection of other inequalities found.

Theorem 12. *The following inequalities are best possible.*

$$\mathtt{dist}[X_1, X_2] < \frac{2}{3}R$$
$$\mathtt{dist}[X_1, X_3] < R$$
$$\mathtt{dist}[X_1, X_4] < 2R$$
$$\mathtt{dist}[X_1, X_5] < \frac{1}{2}R$$
$$\mathtt{dist}[X_1, X_8] < 2R$$
$$\mathtt{dist}[X_1, X_9] < R$$
$$\mathtt{dist}[X_1, X_{10}] < R$$
$$\mathtt{dist}[X_1, X_{11}] < \frac{1}{2}R$$
$$\mathtt{dist}[X_1, X_{12}] < \left(\frac{3}{2} - \sqrt{2}\right) R$$
$$\mathtt{dist}[X_1, X_{15}] < kR$$
$$\mathtt{dist}[X_1, X_{19}] < \frac{1}{\sqrt{2}}R$$
$$\mathtt{dist}[X_1, X_{20}] < 4R$$
$$\mathtt{dist}[X_1, X_{21}] < \frac{2}{3}R$$
$$\mathtt{dist}[X_1, X_{34}] < \frac{\left(3\sqrt{3} - 4 + \sqrt{42\sqrt{3} - 45}\right)}{11}R$$
$$\mathtt{dist}[X_1, X_{46}] < R\sqrt{10\sqrt{5} - 22}$$
$$\mathtt{dist}[X_1, X_{51}] < \frac{1}{3}R$$
$$\mathtt{dist}[X_1, X_{57}] < \frac{2}{\sqrt{37 + 14\sqrt{7}}}R$$
$$\mathtt{dist}[X_1, X_{61}] < \frac{2}{3}R$$
$$\mathtt{dist}[X_1, X_{63}] < 2R$$
$$\mathtt{dist}[X_1, X_{65}] < \frac{1}{3\sqrt{3}}R$$
$$\mathtt{dist}[X_1, X_{69}] < 2R$$
$$\mathtt{dist}[X_1, X_{71}] < \frac{4}{3}R$$
$$\mathtt{dist}[X_1, X_{88}] < \left(18 - 12\sqrt{2}\right) R$$

where $k \approx 0.2370406267$ *is the positive root of*
$4x^6 + 36x^5 + 120x^4 + 288x^3 + 513x^2 - 72x - 16$.

Theorem 13. *The following inequalities are best possible.*

$$\mathtt{dist}[X_1, X_2] < \frac{1}{3}s$$
$$\mathtt{dist}[X_1, X_6] < \frac{s}{54}\sqrt{14\sqrt{7} - 20}$$
$$\mathtt{dist}[X_1, X_8] < s$$
$$\mathtt{dist}[X_1, X_9] < \frac{1}{2}s$$
$$\mathtt{dist}[X_1, X_{10}] < \frac{1}{2}s$$
$$\mathtt{dist}[X_1, X_{11}] < \frac{1}{3\sqrt{3}}s$$
$$\mathtt{dist}[X_1, X_{13}] < \frac{1}{2\sqrt{3}}s$$
$$\mathtt{dist}[X_1, X_{14}] < \frac{2s}{243}\sqrt{1257 + 390\sqrt{10}}$$
$$\mathtt{dist}[X_1, X_{15}] < \frac{1}{2\sqrt{3}}s$$
$$\mathtt{dist}[X_1, X_{19}] < \frac{1}{2}s$$
$$\mathtt{dist}[X_1, X_{21}] < \frac{1}{3}s$$
$$\mathtt{dist}[X_1, X_{25}] < \frac{1}{2\sqrt{2}}s$$
$$\mathtt{dist}[X_1, X_{27}] < \frac{1}{\sqrt{5}}s$$
$$\mathtt{dist}[X_1, X_{28}] < \frac{1}{2\sqrt{2}}s$$
$$\mathtt{dist}[X_1, X_{39}] < \frac{1}{2\sqrt{37 + 14\sqrt{7}}}s$$
$$\mathtt{dist}[X_1, X_{41}] < \frac{s}{4}\sqrt{10\sqrt{5} - 22}$$

Theorem 14. *The following inequalities are best possible.*

$$\texttt{dist}[X_1, X_{80}] < \frac{2}{3\sqrt{3}}s$$

$$\texttt{dist}[X_1, X_{83}] < \frac{1}{5}s$$

$$\texttt{dist}[X_1, X_{85}] < \frac{1}{5}s$$

$$\texttt{dist}[X_1, X_{86}] < \frac{1}{5}s$$

Theorem 15. *The following inequality is best possible.*

$$\texttt{dist}[X_3, X_{52}] < s$$

Theorem 16. *The following inequalities are best possible.*

$$\texttt{dist}[X_3, X_{36}] > \frac{1}{2}s$$

$$\texttt{dist}[X_3, X_{44}] > \frac{1}{2}s$$

$$\texttt{dist}[X_3, X_{74}] > \frac{2}{3\sqrt{3}}s$$

Theorem 17. *The following inequalities are best possible.*

$$\frac{\texttt{dist}[X_1, X_2]}{\texttt{dist}[X_3, X_2]} > \frac{1}{3}\frac{r}{R}$$

$$\frac{\texttt{dist}[X_1, X_4]}{\texttt{dist}[X_3, X_4]} > \frac{8}{9}\frac{r}{R}$$

$$\frac{\texttt{dist}[X_1, X_6]}{\texttt{dist}[X_3, X_6]} > \frac{1}{3}\frac{r}{R}$$

$$\frac{8}{17}\frac{r}{R} < \frac{\texttt{dist}[X_1, X_7]}{\texttt{dist}[X_3, X_7]} < \frac{r}{R}$$

$$\frac{\texttt{dist}[X_1, X_8]}{\texttt{dist}[X_3, X_8]} > \frac{r}{R}$$

$$\frac{\texttt{dist}[X_1, X_9]}{\texttt{dist}[X_3, X_9]} > \frac{r}{R}$$

$$\frac{\texttt{dist}[X_1, X_{10}]}{\texttt{dist}[X_3, X_{10}]} > \frac{1}{2}\frac{r}{R}$$

$$\frac{\texttt{dist}[X_1, X_{12}]}{\texttt{dist}[X_3, X_{12}]} < \frac{r}{R} < \frac{\texttt{dist}[X_1, X_{11}]}{\texttt{dist}[X_3, X_{11}]}$$

7. Conditions for a Center to Lie on the Incircle

Typically, the condition for a center to lie on the incircle is messy. We present a few cases where the condition is simple.

Theorem 18 (Conditions for a Point to lie on the Incircle)**.**
(a) X_2 *lies on the incircle if and only if* $s^2 = 4r(4R + r)$.
(b) X_3 *lies on the incircle if and only if* $r = (\sqrt{2} - 1)R$.
(c) X_4 *lies on the incircle if and only if* $s^2 = 2r^2 + 4rR + 4R^2$.
(d) X_5 *lies on the incircle if and only if* $R = 4r$.
(e) X_8 *lies on the incircle if and only if* $s^2 = 4r(4R - r)$.
(f) X_{10} *lies on the incircle if and only if* $s^2 = r(16R - r)$.
(g) X_{20} *lies on the incircle if and only if* $3s^2 = 8R(2R + r)$.
(h) X_{35} *lies on the incircle if and only if* $2r = (2\sqrt{3} - 3)R$.
(i) X_{40} *lies on the incircle if and only if* $r = 2(\sqrt{5} - 2)R$.
(j) X_{46} *lies on the incircle if and only if* $r = (2\sqrt{3} - 3)R$.
(k) X_{77} *lies on the incircle if and only if* $s^2 = R(5R + 2r)$.
(l) X_{79} *lies on the incircle if and only if* $8s^2 = 4r^2 + 20rR + 27R^2$.

Theorem 19 (Conditions for a Point to lie on the Circumcircle)**.**
(a) X_4, X_{19}, X_{20}, *and* X_{34} *lie on the circumcircle if and only if* $s = r + 2R$.
(a) X_{46} *lies on the circumcircle if and only if* $r = (\sqrt{2} - 1)R$.
(b) X_{48} *lies on the circumcircle if and only if* $s^2 = (2R + r)(2R - r)$.

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