

Stream Fusion in HOL

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May 26, 2024

Stream Fusion is a system for removing intermediate list data structures from functional programs, in particular [Haskell](#). This entry adapts stream fusion to Isabelle/HOL and its code generator. We define stream types for finite and possibly infinite lists and stream versions for most of the fusible list functions in the theories *List* and *Coinductive-List*, and prove them correct with respect to the conversion functions between lists and streams. The Stream Fusion transformation itself is implemented as a simproc in the pre-processor of the code generator.

Brian Huffman's AFP entry [3] formalises stream fusion in HOLCF for the domain of lazy lists to prove the GHC compiler rewrite rules correct. In contrast, this work enables Isabelle's code generator to perform stream fusion itself. To that end, it covers both finite and coinductive lists from the HOL library and the Coinductive entry. The fusible list functions require specification and proof principles different from Huffman's.

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1 Stream fusion implementation

```
theory Stream-Fusion
```

```
imports
```

```
  Main
```

```
begin
```

```
ML-file <stream-fusion.ML>
```

```
simproc-setup stream-fusion (f x) = <K Stream-Fusion.fusion-simproc>
```

```
declare [[simproc del: stream-fusion]]
```

Install stream fusion as a simproc in the preprocessor for code equations

```
setup <Code-Preproc.map-pre (fn ss => ss addsimprocs [@{simproc stream-fusion}]>
```

```
end
```

2 Stream fusion for finite lists

```
theory Stream-Fusion-List
```

```
imports Stream-Fusion
```

```
begin
```

```
lemma map-option-mono [partial-function-mono]:
```

```
  mono-option f  $\implies$  mono-option ( $\lambda x. \text{map-option } g (f x)$ )
```

```
apply (rule monotoneI)
```

```
apply (drule (1) monotoneD)
```

```
apply (auto simp add: flat-ord-def split: option.split)
```

```
done
```

2.1 The type of generators for finite lists

```
datatype ('a, 's) step = Done | is-Skip: Skip 's | is-Yield: Yield 'a 's
```

```
type-synonym ('a, 's) raw-generator = 's  $\Rightarrow$  ('a, 's) step
```

Raw generators may not end in *Done*, but may lead to infinitely many *Yields* in a row. Such generators cannot be converted to finite lists, because it corresponds to an infinite list. Therefore, we introduce the type of generators that always end in *Done* after finitely many steps.

```
inductive-set terminates-on :: ('a, 's) raw-generator  $\Rightarrow$  's set
```

```
  for g :: ('a, 's) raw-generator
```

```
where
```

```
  stop: g s = Done  $\implies$  s  $\in$  terminates-on g
```

| *pause*: $\llbracket g \ s = \text{Skip } s'; s' \in \text{terminates-on } g \rrbracket \implies s \in \text{terminates-on } g$
| *unfold*: $\llbracket g \ s = \text{Yield } a \ s'; s' \in \text{terminates-on } g \rrbracket \implies s \in \text{terminates-on } g$

definition *terminates* :: ('a, 's) raw-generator \Rightarrow bool
where *terminates* $g \longleftrightarrow (\text{terminates-on } g = \text{UNIV})$

lemma *terminatesI* [intro?]:
 $(\bigwedge s. s \in \text{terminates-on } g) \implies \text{terminates } g$
by (*auto simp add: terminates-def*)

lemma *terminatesD*:
 $\text{terminates } g \implies s \in \text{terminates-on } g$
by (*auto simp add: terminates-def*)

lemma *terminates-on-stop*:
 $\text{terminates-on } (\lambda-. \text{Done}) = \text{UNIV}$
by (*auto intro: terminates-on.stop*)

lemma *wf-terminates*:
assumes *wf R*
and *skip*: $\bigwedge s \ s'. g \ s = \text{Skip } s' \implies (s', s) \in R$
and *yield*: $\bigwedge s \ s' \ a. g \ s = \text{Yield } a \ s' \implies (s', s) \in R$
shows *terminates g*
proof (*rule terminatesI*)
fix *s*
from $\langle \text{wf } R \rangle$ **show** $s \in \text{terminates-on } g$
proof (*induction rule: wf-induct [rule-format, consumes 1, case-names wf]*)
case (*wf s*)
show ?*case*
proof (*cases g s*)
case (*Skip s'*)
hence $(s', s) \in R$ **by** (*rule skip*)
hence $s' \in \text{terminates-on } g$ **by** (*rule wf.IH*)
with *Skip* **show** ?*thesis* **by** (*rule terminates-on.pause*)
next
case (*Yield a s'*)
hence $(s', s) \in R$ **by** (*rule yield*)
hence $s' \in \text{terminates-on } g$ **by** (*rule wf.IH*)
with *Yield* **show** ?*thesis* **by** (*rule terminates-on.unfold*)
qed (*rule terminates-on.stop*)
qed
qed

context *fixes g* :: ('a, 's) raw-generator **begin**

partial-function (*option*) *terminates-within* :: 's ⇒ nat *option* **where**

terminates-within s = (case g s of

Done ⇒ *Some* 0

| *Skip* s' ⇒ *map-option* (λn. n + 1) (*terminates-within* s')

| *Yield* a s' ⇒ *map-option* (λn. n + 1) (*terminates-within* s'))

lemma *terminates-on-conv-dom-terminates-within*:

terminates-on g = *dom* *terminates-within*

proof (*rule set-eqI iffI*)+

fix s

assume s ∈ *terminates-on* g

hence ∃ n. *terminates-within* s = *Some* n

by *induction* (*subst* *terminates-within.simps*, *simp* *add: split-beta*)+

then show s ∈ *dom* *terminates-within* **by** *blast*

next

fix s

assume s ∈ *dom* *terminates-within*

then obtain n **where** *terminates-within* s = *Some* n **by** *blast*

then show s ∈ *terminates-on* g

proof (*induction* *rule: terminates-within.raw-induct*[*rotated* 1, *consumes* 1])

case (1 *terminates-within* s s')

show ?*case*

proof(*cases* g s)

case *Done*

thus ?*thesis* **by** (*simp* *add: terminates-on.stop*)

next

case (*Skip* s')

hence s' ∈ *terminates-on* g **using** 1 **by**(*auto*)

thus ?*thesis* **using** ⟨g s = *Skip* s'⟩ **by** (*simp* *add: terminates-on.pause*)

next

case (*Yield* a s')

hence s' ∈ *terminates-on* g **using** 1 **by**(*auto*)

thus ?*thesis* **using** ⟨g s = *Yield* a s'⟩ **by** (*auto* *intro: terminates-on.unfold*)

qed

qed

qed

end

lemma *terminates-wfE*:

assumes *terminates* g

obtains R

where *wf* R

$\bigwedge s s'. (g\ s = \text{Skip}\ s') \implies (s', s) \in R$
 $\bigwedge s a s'. (g\ s = \text{Yield}\ a\ s') \implies (s', s) \in R$

proof –

let $?R = \text{measure } (\lambda s. \text{the } (\text{terminates-within } g\ s)) :: ('a \times 'a)\ \text{set}$

have $\text{wf } ?R$ **by** simp

moreover {

fix $s\ s'$

assume $g\ s = \text{Skip}\ s'$

moreover from assms **have** $s' \in \text{terminates-on } g$ **by** $(\text{rule } \text{terminatesD})$

then obtain n **where** $\text{terminates-within } g\ s' = \text{Some } n$

unfolding $\text{terminates-on-conv-dom-terminates-within}$ **by** (auto)

ultimately have $\text{the } (\text{terminates-within } g\ s') < \text{the } (\text{terminates-within } g\ s)$

by $(\text{simp add: } \text{terminates-within.simps})$

hence $(s', s) \in ?R$ **by** (auto)

} **moreover** {

fix $s\ s'\ a$

assume $2: g\ s = \text{Yield}\ a\ s'$

moreover from assms **have** $s' \in \text{terminates-on } g$ **by** $(\text{rule } \text{terminatesD})$

then obtain n **where** $\text{terminates-within } g\ s' = \text{Some } n$

unfolding $\text{terminates-on-conv-dom-terminates-within}$ **by** (auto)

ultimately have $(s', s) \in ?R$

by $\text{simp } (\text{subst } \text{terminates-within.simps}, \text{simp add: } \text{split-beta})$

} **ultimately**

show thesis **by** $(\text{rule } \text{that})$

qed

typedef $('a, 's)\ \text{generator} = \{g :: ('a, 's)\ \text{raw-generator. } \text{terminates } g\}$

morphisms generator Generator

proof

show $(\lambda-. \text{Done}) \in ?\text{generator}$

by $(\text{simp add: } \text{terminates-on-stop } \text{terminates-def})$

qed

setup-lifting $\text{type-definition-generator}$

2.2 Conversion to 'a list

context $\text{fixes } g :: ('a, 's)\ \text{generator}$ **begin**

function $\text{unstream} :: 's \Rightarrow 'a\ \text{list}$

where

$\text{unstream } s = (\text{case } \text{generator } g\ s\ \text{of}$

$\text{Done} \Rightarrow []$

$| \text{Skip } s' \Rightarrow \text{unstream } s'$

```

  | Yield x s'  $\Rightarrow$  x # unstream s')
by pat-completeness auto
termination
proof –
  have terminates (generator g) using generator[of g] by simp
  thus ?thesis by(rule terminates-wfE)(erule termination)
qed

```

```

lemma unstream-simps [simp]:
  generator g s = Done  $\Longrightarrow$  unstream s = []
  generator g s = Skip s'  $\Longrightarrow$  unstream s = unstream s'
  generator g s = Yield x s'  $\Longrightarrow$  unstream s = x # unstream s'
by(simp-all)

```

```

declare unstream.simps[simp del]

```

```

function force :: 's  $\Rightarrow$  ('a  $\times$  's) option
where
  force s = (case generator g s of Done  $\Rightarrow$  None
    | Skip s'  $\Rightarrow$  force s'
    | Yield x s'  $\Rightarrow$  Some (x, s'))
by pat-completeness auto
termination
proof –
  have terminates (generator g) using generator[of g] by simp
  thus ?thesis by(rule terminates-wfE)(rule termination)
qed

```

```

lemma force-simps [simp]:
  generator g s = Done  $\Longrightarrow$  force s = None
  generator g s = Skip s'  $\Longrightarrow$  force s = force s'
  generator g s = Yield x s'  $\Longrightarrow$  force s = Some (x, s')
by(simp-all)

```

```

declare force.simps[simp del]

```

```

lemma unstream-force-None [simp]: force s = None  $\Longrightarrow$  unstream s = []
proof(induction s rule: force.induct)
  case (1 s)
  thus ?case by(cases generator g s) simp-all
qed

```

```

lemma unstream-force-Some [simp]: force s = Some (x, s')  $\Longrightarrow$  unstream s = x #
unstream s'

```



```

proof(induction s rule: force.induct)
  case (1 s)
  thus ?case by(cases generator g s simp-all)
qed

```

```

end

```

```

setup ‹Context.theory-map (Stream-Fusion.add-unstream @{const-name unstream})›

```

2.3 Producers

2.3.1 Conversion to streams

```

fun stream-raw :: 'a list ⇒ ('a, 'a list) step
where
  stream-raw [] = Done
| stream-raw (x # xs) = Yield x xs

```

```

lemma terminates-stream-raw: terminates stream-raw

```

```

proof (rule terminatesI)
  fix s :: 'a list
  show s ∈ terminates-on stream-raw
  by(induction s)(auto intro: terminates-on.intros)
qed

```

```

lift-definition stream :: ('a, 'a list) generator is stream-raw by(rule terminates-stream-raw)

```

```

lemma unstream-stream: unstream stream xs = xs
by(induction xs)(auto simp add: stream.rep-eq)

```

2.3.2 replicate

```

fun replicate-raw :: 'a ⇒ ('a, nat) raw-generator
where
  replicate-raw a 0 = Done
| replicate-raw a (Suc n) = Yield a n

```

```

lemma terminates-replicate-raw: terminates (replicate-raw a)

```

```

proof (rule terminatesI)
  fix s :: nat
  show s ∈ terminates-on (replicate-raw a)
  by(induction s)(auto intro: terminates-on.intros)
qed

```

```

lift-definition replicate-prod :: 'a ⇒ ('a, nat) generator is replicate-raw

```

by(rule terminates-replicate-raw)

lemma *unstream-replicate-prod* [stream-fusion]: *unstream* (replicate-prod x) n = replicate n x

by(induction n)(simp-all add: replicate-prod.rep-eq)

2.3.3 *upt*

definition *upt-raw* :: $\text{nat} \Rightarrow (\text{nat}, \text{nat})$ raw-generator

where *upt-raw* n m = (if $m \geq n$ then Done else Yield m (Suc m))

lemma *terminates-upt-raw*: terminates (*upt-raw* n)

proof (rule terminatesI)

fix s :: nat

show $s \in$ terminates-on (*upt-raw* n)

by(induction $n-s$ arbitrary: s rule: nat.induct)(auto 4 3 simp add: upt-raw-def intro: terminates-on.intros)

qed

lift-definition *upt-prod* :: $\text{nat} \Rightarrow (\text{nat}, \text{nat})$ generator **is** *upt-raw* **by**(rule terminates-upt-raw)

lemma *unstream-upt-prod* [stream-fusion]: *unstream* (*upt-prod* n) m = *upt* m n

by(induction $n-m$ arbitrary: n m)(simp-all add: upt-prod.rep-eq upt-conv-Cons upt-raw-def unstream.simps)

2.3.4 *upto*

definition *upto-raw* :: $\text{int} \Rightarrow (\text{int}, \text{int})$ raw-generator

where *upto-raw* n m = (if $m \leq n$ then Yield m ($m + 1$) else Done)

lemma *terminates-upto-raw*: terminates (*upto-raw* n)

proof (rule terminatesI)

fix s :: int

show $s \in$ terminates-on (*upto-raw* n)

by(induction $\text{nat}(n-s+1)$ arbitrary: s)(auto 4 3 simp add: upto-raw-def intro: terminates-on.intros)

qed

lift-definition *upto-prod* :: $\text{int} \Rightarrow (\text{int}, \text{int})$ generator **is** *upto-raw* **by** (rule terminates-upto-raw)

lemma *unstream-upto-prod* [stream-fusion]: *unstream* (*upto-prod* n) m = *upto* m n

by(induction $\text{nat}(n-m+1)$ arbitrary: m)(simp-all add: upto-prod.rep-eq upto.simps upto-raw-def)

2.3.5 []

lift-definition *Nil-prod* :: ('a, unit) generator is λ -. *Done*
by(*auto simp add: terminates-def intro: terminates-on.intros*)

lemma *generator-Nil-prod*: *generator Nil-prod* = (λ -. *Done*)
by(*fact Nil-prod.rep-eq*)

lemma *unstream-Nil-prod* [*stream-fusion*]: *unstream Nil-prod* () = []
by(*simp add: generator-Nil-prod*)

2.4 Consumers

2.4.1 (!)

context fixes *g* :: ('a, 's) generator **begin**

definition *nth-cons* :: 's \Rightarrow nat \Rightarrow 'a
where [*stream-fusion*]: *nth-cons s n* = *unstream g s ! n*

lemma *nth-cons-code* [*code*]:
 nth-cons s n =
 (*case generator g s of Done* => *undefined n*
 | *Skip s'* => *nth-cons s' n*
 | *Yield x s'* => (*case n of 0* => *x* | *Suc n'* => *nth-cons s' n'*))
by(*cases generator g s*)(*simp-all add: nth-cons-def nth-def split: nat.split*)

end

2.4.2 *length*

context fixes *g* :: ('a, 's) generator **begin**

definition *length-cons* :: 's \Rightarrow nat
where *length-cons s* = *length (unstream g s)*

lemma *length-cons-code* [*code*]:
 length-cons s =
 (*case generator g s of*
 Done \Rightarrow 0
 | *Skip s'* \Rightarrow *length-cons s'*
 | *Yield a s'* \Rightarrow 1 + *length-cons s'*)
by(*cases generator g s*)(*simp-all add: length-cons-def*)

definition *gen-length-cons* :: nat \Rightarrow 's \Rightarrow nat

where $gen\text{-}length\text{-}cons\ n\ s = n + length\ (unstream\ g\ s)$

lemma $gen\text{-}length\text{-}cons\text{-}code$ [code]:

$gen\text{-}length\text{-}cons\ n\ s = (case\ generator\ g\ s\ of$
 $Done \Rightarrow n \mid Skip\ s' \Rightarrow gen\text{-}length\text{-}cons\ n\ s' \mid Yield\ a\ s' \Rightarrow gen\text{-}length\text{-}cons\ (Suc\ n)$
 $s')$

by($simp\ add: gen\text{-}length\text{-}cons\text{-}def\ split: step.split$)

lemma $unstream\text{-}gen\text{-}length$ [stream-fusion]: $gen\text{-}length\text{-}cons\ 0\ s = length\ (unstream\ g\ s)$

by($simp\ add: gen\text{-}length\text{-}cons\text{-}def$)

lemma $unstream\text{-}gen\text{-}length2$ [stream-fusion]: $gen\text{-}length\text{-}cons\ n\ s = List.gen\text{-}length\ n\ (unstream\ g\ s)$

by($simp\ add: List.gen\text{-}length\text{-}def\ gen\text{-}length\text{-}cons\text{-}def$)

end

2.4.3 $foldr$

context

fixes $g :: ('a, 's)\ generator$

and $f :: 'a \Rightarrow 'b \Rightarrow 'b$

and $z :: 'b$

begin

definition $foldr\text{-}cons :: 's \Rightarrow 'b$

where [stream-fusion]: $foldr\text{-}cons\ s = foldr\ f\ (unstream\ g\ s)\ z$

lemma $foldr\text{-}cons\text{-}code$ [code]:

$foldr\text{-}cons\ s =$
 $(case\ generator\ g\ s\ of$
 $Done \Rightarrow z$
 $\mid Skip\ s' \Rightarrow foldr\text{-}cons\ s'$
 $\mid Yield\ a\ s' \Rightarrow f\ a\ (foldr\text{-}cons\ s'))$

by($cases\ generator\ g\ s$)($simp\text{-}all\ add: foldr\text{-}cons\text{-}def$)

end

2.4.4 $foldl$

context

fixes $g :: ('b, 's)\ generator$

and $f :: 'a \Rightarrow 'b \Rightarrow 'a$

begin

definition *foldl-cons* :: 'a ⇒ 's ⇒ 'a

where [*stream-fusion*]: *foldl-cons* z s = *foldl* f z (*unstream* g s)

lemma *foldl-cons-code* [*code*]:

foldl-cons z s =
 (case generator g s of
 Done ⇒ z
 | Skip s' ⇒ *foldl-cons* z s'
 | Yield a s' ⇒ *foldl-cons* (f z a) s')

by (cases generator g s)(*simp-all add: foldl-cons-def*)

end

2.4.5 *fold*

context

fixes g :: ('a, 's) generator

and f :: 'a ⇒ 'b ⇒ 'b

begin

definition *fold-cons* :: 'b ⇒ 's ⇒ 'b

where [*stream-fusion*]: *fold-cons* z s = *fold* f (*unstream* g s) z

lemma *fold-cons-code* [*code*]:

fold-cons z s =
 (case generator g s of
 Done ⇒ z
 | Skip s' ⇒ *fold-cons* z s'
 | Yield a s' ⇒ *fold-cons* (f a z) s')

by (cases generator g s)(*simp-all add: fold-cons-def*)

end

2.4.6 *List.null*

definition *null-cons* :: ('a, 's) generator ⇒ 's ⇒ bool

where [*stream-fusion*]: *null-cons* g s = *List.null* (*unstream* g s)

lemma *null-cons-code* [*code*]:

null-cons g s = (case generator g s of Done ⇒ True | Skip s' ⇒ *null-cons* g s' | Yield
- - ⇒ False)

by(cases generator g s)(*simp-all add: null-cons-def null-def*)

2.4.7 *hd*

context fixes $g :: ('a, 's)$ generator **begin**

definition $hd-cons :: 's \Rightarrow 'a$

where [*stream-fusion*]: $hd-cons\ s = hd\ (unstream\ g\ s)$

lemma $hd-cons-code$ [*code*]:

$hd-cons\ s =$
 (*case generator g s of*
 Done \Rightarrow *undefined*
 | *Skip s'* \Rightarrow $hd-cons\ s'$
 | *Yield a s'* \Rightarrow a)

by (*cases generator g s*)(*simp-all add: hd-cons-def hd-def*)

end

2.4.8 *last*

context fixes $g :: ('a, 's)$ generator **begin**

definition $last-cons :: 'a\ option \Rightarrow 's \Rightarrow 'a$

where $last-cons\ x\ s =$ (*if unstream g s = [] then the x else last (unstream g s)*)

lemma $last-cons-code$ [*code*]:

$last-cons\ x\ s =$
 (*case generator g s of Done* \Rightarrow *the x*
 | *Skip s'* \Rightarrow $last-cons\ x\ s'$
 | *Yield a s'* \Rightarrow $last-cons\ (Some\ a)\ s'$)

by (*cases generator g s*)(*simp-all add: last-cons-def*)

lemma $unstream-last-cons$ [*stream-fusion*]: $last-cons\ None\ s = last\ (unstream\ g\ s)$

by (*simp add: last-cons-def last-def option.the-def*)

end

2.4.9 *sum-list*

context fixes $g :: ('a :: monoid-add, 's)$ generator **begin**

definition $sum-list-cons :: 's \Rightarrow 'a$

where [*stream-fusion*]: $sum-list-cons\ s = sum-list\ (unstream\ g\ s)$

lemma $sum-list-cons-code$ [*code*]:

$sum-list-cons\ s =$

```

    (case generator g s of
      Done  $\Rightarrow$  0
    | Skip s'  $\Rightarrow$  sum-list-cons s'
    | Yield a s'  $\Rightarrow$  a + sum-list-cons s')
  by (cases generator g s)(simp-all add: sum-list-cons-def)

```

end

2.4.10 list-all2

context

```

  fixes g :: ('a, 's1) generator
  and h :: ('b, 's2) generator
  and P :: 'a  $\Rightarrow$  'b  $\Rightarrow$  bool

```

begin

definition list-all2-cons :: 's1 \Rightarrow 's2 \Rightarrow bool

where [stream-fusion]: list-all2-cons sg sh = list-all2 P (unstream g sg) (unstream h sh)

definition list-all2-cons1 :: 'a \Rightarrow 's1 \Rightarrow 's2 \Rightarrow bool

where list-all2-cons1 x sg' sh = list-all2 P (x # unstream g sg') (unstream h sh)

lemma list-all2-cons-code [code]:

```

  list-all2-cons sg sh =
    (case generator g sg of
      Done  $\Rightarrow$  null-cons h sh
    | Skip sg'  $\Rightarrow$  list-all2-cons sg' sh
    | Yield a sg'  $\Rightarrow$  list-all2-cons1 a sg' sh)

```

by(simp split: step.split add: list-all2-cons-def null-cons-def List.null-def list-all2-cons1-def)

lemma list-all2-cons1-code [code]:

```

  list-all2-cons1 x sg' sh =
    (case generator h sh of
      Done  $\Rightarrow$  False
    | Skip sh'  $\Rightarrow$  list-all2-cons1 x sg' sh'
    | Yield y sh'  $\Rightarrow$  P x y  $\wedge$  list-all2-cons sg' sh')

```

by(simp split: step.split add: list-all2-cons-def null-cons-def List.null-def list-all2-cons1-def)

end

2.4.11 list-all

context

```

  fixes g :: ('a, 's) generator

```

and $P :: 'a \Rightarrow \text{bool}$
begin

definition $\text{list-all-cons} :: 's \Rightarrow \text{bool}$
where [*stream-fusion*]: $\text{list-all-cons } s = \text{list-all } P (\text{unstream } g \ s)$

lemma $\text{list-all-cons-code}$ [*code*]:
 $\text{list-all-cons } s \longleftrightarrow$
 (case generator $g \ s$ of
 $\text{Done} \Rightarrow \text{True} \mid \text{Skip } s' \Rightarrow \text{list-all-cons } s' \mid \text{Yield } x \ s' \Rightarrow P \ x \wedge \text{list-all-cons } s')$
by(*simp add: list-all-cons-def split: step.split*)

end

2.4.12 *ord.lexordp*

context *ord* **begin**

definition $\text{lexord-fusion} :: ('a, 's1) \text{ generator} \Rightarrow ('a, 's2) \text{ generator} \Rightarrow 's1 \Rightarrow 's2 \Rightarrow \text{bool}$
where [*code del*]: $\text{lexord-fusion } g1 \ g2 \ s1 \ s2 = \text{ord-class.lexordp} (\text{unstream } g1 \ s1) (\text{unstream } g2 \ s2)$

definition $\text{lexord-eq-fusion} :: ('a, 's1) \text{ generator} \Rightarrow ('a, 's2) \text{ generator} \Rightarrow 's1 \Rightarrow 's2 \Rightarrow \text{bool}$
where [*code del*]: $\text{lexord-eq-fusion } g1 \ g2 \ s1 \ s2 = \text{lexordp-eq} (\text{unstream } g1 \ s1) (\text{unstream } g2 \ s2)$

lemma $\text{lexord-fusion-code}$:
 $\text{lexord-fusion } g1 \ g2 \ s1 \ s2 \longleftrightarrow$
 (case generator $g1 \ s1$ of
 $\text{Done} \Rightarrow \neg \text{null-cons } g2 \ s2$
 $\mid \text{Skip } s1' \Rightarrow \text{lexord-fusion } g1 \ g2 \ s1' \ s2$
 $\mid \text{Yield } x \ s1' \Rightarrow$
 (case force $g2 \ s2$ of
 $\text{None} \Rightarrow \text{False}$
 $\mid \text{Some } (y, s2') \Rightarrow x < y \vee \neg y < x \wedge \text{lexord-fusion } g1 \ g2 \ s1' \ s2')$)

unfolding lexord-fusion-def
by(*cases generator g1 s1 force g2 s2 rule: step.exhaust[case-product option.exhaust]*)(*auto simp add: null-cons-def null-def*)

lemma $\text{lexord-eq-fusion-code}$:
 $\text{lexord-eq-fusion } g1 \ g2 \ s1 \ s2 \longleftrightarrow$
 (case generator $g1 \ s1$ of


```

    Done  $\Rightarrow$  True
  | Skip s1'  $\Rightarrow$  lexord-eq-fusion g1 g2 s1' s2
  | Yield x s1'  $\Rightarrow$ 
    (case force g2 s2 of
      None  $\Rightarrow$  False
    | Some (y, s2')  $\Rightarrow$  x < y  $\vee$   $\neg$  y < x  $\wedge$  lexord-eq-fusion g1 g2 s1' s2')
unfolding lexord-eq-fusion-def
by(cases generator g1 s1 force g2 s2 rule: step.exhaust[case-product option.exhaust]) auto
end

```

```

lemmas [code] =
  lexord-fusion-code ord.lexord-fusion-code
  lexord-eq-fusion-code ord.lexord-eq-fusion-code

```

```

lemmas [stream-fusion] =
  lexord-fusion-def ord.lexord-fusion-def
  lexord-eq-fusion-def ord.lexord-eq-fusion-def

```

2.5 Transformers

2.5.1 map

definition *map-raw* :: ('a \Rightarrow 'b) \Rightarrow ('a, 's) raw-generator \Rightarrow ('b, 's) raw-generator
where

```

  map-raw f g s = (case g s of
    Done  $\Rightarrow$  Done
  | Skip s'  $\Rightarrow$  Skip s'
  | Yield a s'  $\Rightarrow$  Yield (f a) s')

```

lemma *terminates-map-raw*:

```

  assumes terminates g
  shows terminates (map-raw f g)

```

proof (rule *terminatesI*)

```

  fix s
  from assms
  have s  $\in$  terminates-on g by (simp add: terminates-def)
  then show s  $\in$  terminates-on (map-raw f g)
    by (induction s)(auto intro: terminates-on.intros simp add: map-raw-def)

```

qed

lift-definition *map-trans* :: ('a \Rightarrow 'b) \Rightarrow ('a, 's) generator \Rightarrow ('b, 's) generator **is**
map-raw
by (rule *terminates-map-raw*)

lemma *unstream-map-trans* [*stream-fusion*]: $\text{unstream } (\text{map-trans } f \ g) \ s = \text{map } f \ (\text{unstream } g \ s)$
proof (*induction s taking: g rule: unstream.induct*)
 case (*1 s*)
 show *?case using 1.IH by (cases generator g s)(simp-all add: map-trans.rep-eq map-raw-def)*
qed

2.5.2 drop

fun *drop-raw* :: (*'a, 's*) *raw-generator* \Rightarrow (*'a, (nat \times 's)*) *raw-generator*
where
 drop-raw *g* (*n, s*) = (*case g s of*
 Done \Rightarrow *Done* | *Skip s'* \Rightarrow *Skip* (*n, s'*)
 | *Yield a s'* \Rightarrow (*case n of 0* \Rightarrow *Yield a* (*0, s'*) | *Suc n* \Rightarrow *Skip* (*n, s'*)))

lemma *terminates-drop-raw*:
 assumes *terminates g*
 shows *terminates (drop-raw g)*
proof (*rule terminatesI*)
 fix *st* :: *nat \times 'a*
 obtain *n s* **where** *st = (n, s)* **by**(*cases st*)
 from *assms* **have** *s \in terminates-on g* **by** (*simp add: terminates-def*)
 thus *st \in terminates-on (drop-raw g)* **unfolding** $\langle st = (n, s) \rangle$
 apply(*induction arbitrary: n*)
 apply(*case-tac [!] n*)
 apply(*auto intro: terminates-on.intros*)
 done
qed

lift-definition *drop-trans* :: (*'a, 's*) *generator* \Rightarrow (*'a, nat \times 's*) *generator* **is** *drop-raw*
by (*rule terminates-drop-raw*)

lemma *unstream-drop-trans* [*stream-fusion*]: $\text{unstream } (\text{drop-trans } g) \ (n, s) = \text{drop } n \ (\text{unstream } g \ s)$
proof (*induction s arbitrary: n taking: g rule: unstream.induct*)
 case (*1 s*)
 show *?case using 1.IH(1)[of - n] 1.IH(2)[of - - n] 1.IH(2)[of - - n - 1]*
 by(*cases generator g s n rule: step.exhaust[case-product nat.exhaust]*)
 (*simp-all add: drop-trans.rep-eq*)
qed

2.5.3 *dropWhile*

fun *dropWhile-raw* :: ('a ⇒ bool) ⇒ ('a, 's) raw-generator ⇒ ('a, bool × 's) raw-generator

— Boolean flag indicates whether we are still in dropping phase

where

```

dropWhile-raw P g (True, s) = (case g s of
  Done ⇒ Done | Skip s' ⇒ Skip (True, s')
  | Yield a s' ⇒ (if P a then Skip (True, s') else Yield a (False, s')))
| dropWhile-raw P g (False, s) = (case g s of
  Done ⇒ Done | Skip s' ⇒ Skip (False, s') | Yield a s' ⇒ Yield a (False, s'))

```

lemma *terminates-dropWhile-raw*:

assumes *terminates g*

shows *terminates (dropWhile-raw P g)*

proof (*rule terminatesI*)

fix *st* :: bool × 'a

obtain *b s* **where** *st* = (*b*, *s*) **by** (*cases st*)

from *assms* **have** *s* ∈ *terminates-on g* **by** (*simp add: terminates-def*)

then show *st* ∈ *terminates-on (dropWhile-raw P g)* **unfolding** ⟨*st* = (*b*, *s*)⟩

proof (*induction s arbitrary: b*)

case (*stop s b*)

then show ?*case* **by** (*cases b*)(*simp-all add: terminates-on.stop*)

next

case (*pause s s' b*)

then show ?*case* **by** (*cases b*)(*simp-all add: terminates-on.pause*)

next

case (*unfold s a s' b*)

then show ?*case*

by(*cases b*)(*cases P a*, *auto intro: terminates-on.pause terminates-on.unfold*)

qed

qed

lift-definition *dropWhile-trans* :: ('a ⇒ bool) ⇒ ('a, 's) generator ⇒ ('a, bool × 's)

generator

is *dropWhile-raw* **by** (*rule terminates-dropWhile-raw*)

lemma *unstream-dropWhile-trans-False*:

unstream (dropWhile-trans P g) (False, s) = unstream g s

proof (*induction s taking: g rule: unstream.induct*)

case (*1 s*)

then show ?*case* **by** (*cases generator g s*)(*simp-all add: dropWhile-trans.rep-eq*)

qed

lemma *unstream-dropWhile-trans [stream-fusion]*:

```

    unstream (dropWhile-trans P g) (True, s) = dropWhile P (unstream g s)
proof (induction s taking: g rule: unstream.induct)
  case (1 s)
  then show ?case
  proof(cases generator g s)
    case (Yield a s')
    then show ?thesis using 1.IH(2) unstream-dropWhile-trans-False
      by (cases P a)(simp-all add: dropWhile-trans.rep-eq)
    qed(simp-all add: dropWhile-trans.rep-eq)
qed

```

2.5.4 take

```

fun take-raw :: ('a, 's) raw-generator  $\Rightarrow$  ('a, (nat  $\times$  's)) raw-generator
where
  take-raw g (0, s) = Done
| take-raw g (Suc n, s) = (case g s of
  Done  $\Rightarrow$  Done | Skip s'  $\Rightarrow$  Skip (Suc n, s') | Yield a s'  $\Rightarrow$  Yield a (n, s'))

```

lemma terminates-take-raw:

```

  assumes terminates g
  shows terminates (take-raw g)
proof (rule terminatesI)
  fix st :: nat  $\times$  'a
  obtain n s where st = (n, s) by(cases st)
  from assms have s  $\in$  terminates-on g by (simp add: terminates-def)
  thus st  $\in$  terminates-on (take-raw g) unfolding  $\langle$ st = (n, s) $\rangle$ 
    apply(induction s arbitrary: n)
    apply(case-tac [!] n)
    apply(auto intro: terminates-on.intros)
  done
qed

```

lift-definition take-trans :: ('a, 's) generator \Rightarrow ('a, nat \times 's) generator **is** take-raw
by (rule terminates-take-raw)

lemma unstream-take-trans [stream-fusion]: unstream (take-trans g) (n, s) = take n (unstream g s)

```

proof (induction s arbitrary: n taking: g rule: unstream.induct)
  case (1 s)
  show ?case using 1.IH(1)[of - n] 1.IH(2)
    by(cases generator g s n rule: step.exhaust[case-product nat.exhaust])
      (simp-all add: take-trans.rep-eq)
qed

```

2.5.5 takeWhile

definition *takeWhile-raw* :: ('a ⇒ bool) ⇒ ('a, 's) raw-generator ⇒ ('a, 's) raw-generator
where

takeWhile-raw P g s = (case g s of
 Done ⇒ Done | Skip s' ⇒ Skip s' | Yield a s' ⇒ if P a then Yield a s' else Done)

lemma *terminates-takeWhile-raw*:

assumes *terminates* g

shows *terminates* (*takeWhile-raw* P g)

proof (rule *terminatesI*)

fix s

from *assms* **have** s ∈ *terminates-on* g **by** (*simp* add: *terminates-def*)

thus s ∈ *terminates-on* (*takeWhile-raw* P g)

proof (*induction* s rule: *terminates-on.induct*)

case (*unfold* s a s')

then show ?*case* **by**(*cases* P a)(*auto* *simp* add: *takeWhile-raw-def* *intro*: *terminates-on.intros*)

qed(*auto* *intro*: *terminates-on.intros* *simp* add: *takeWhile-raw-def*)

qed

lift-definition *takeWhile-trans* :: ('a ⇒ bool) ⇒ ('a, 's) generator ⇒ ('a, 's) generator
is *takeWhile-raw* **by** (rule *terminates-takeWhile-raw*)

lemma *unstream-takeWhile-trans* [*stream-fusion*]:

unstream (*takeWhile-trans* P g) s = *takeWhile* P (*unstream* g s)

proof (*induction* s taking: g rule: *unstream.induct*)

case (1 s)

then show ?*case* **by**(*cases* generator g s)(*simp-all* add: *takeWhile-trans.rep-eq* *takeWhile-raw-def*)

qed

2.5.6 (@)

fun *append-raw* :: ('a, 'sg) raw-generator ⇒ ('a, 'sh) raw-generator ⇒ 'sh ⇒ ('a, 'sg + 'sh) raw-generator

where

append-raw g h sh-start (Inl sg) = (case g sg of

Done ⇒ Skip (Inr sh-start) | Skip sg' ⇒ Skip (Inl sg') | Yield a sg' ⇒ Yield a (Inl sg'))

| *append-raw* g h sh-start (Inr sh) = (case h sh of

Done ⇒ Done | Skip sh' ⇒ Skip (Inr sh') | Yield a sh' ⇒ Yield a (Inr sh'))

lemma *terminates-on-append-raw-Inr*:

assumes *terminates h*
shows $Inr\ sh \in terminates\text{-}on\ (append\text{-}raw\ g\ h\ sh\text{-}start)$
proof –
from *assms have sh ∈ terminates-on h by (simp add: terminates-def)*
thus *?thesis by (induction sh)(auto intro: terminates-on.intros)*
qed

lemma *terminates-append-raw:*

assumes *terminates g terminates h*
shows *terminates (append-raw g h sh-start)*
proof (*rule terminatesI*)
fix *s*
show $s \in terminates\text{-}on\ (append\text{-}raw\ g\ h\ sh\text{-}start)$
proof (*cases s*)
case (*Inl sg*)
from $\langle terminates\ g \rangle$ **have** $sg \in terminates\text{-}on\ g$ **by** (*simp add: terminates-def*)
thus $s \in terminates\text{-}on\ (append\text{-}raw\ g\ h\ sh\text{-}start)$ **unfolding** *Inl*
by *induction(auto intro: terminates-on.intros terminates-on-append-raw-Inr[OF*
 $\langle terminates\ h \rangle]$)
qed(*simp add: terminates-on-append-raw-Inr[OF* $\langle terminates\ h \rangle]$)
qed

lift-definition *append-trans :: ('a, 'sg) generator \Rightarrow ('a, 'sh) generator \Rightarrow 'sh \Rightarrow ('a, 'sg + 'sh) generator*
is *append-raw by (rule terminates-append-raw)*

lemma *unstream-append-trans-Inr: unstream (append-trans g h sh) (Inr sh') = unstream h sh'*
proof (*induction sh' taking: h rule: unstream.induct*)
case (*1 sh'*)
then show *?case by (cases generator h sh')(simp-all add: append-trans.rep-eq)*
qed

lemma *unstream-append-trans [stream-fusion]:*

$unstream\ (append\text{-}trans\ g\ h\ sh)\ (Inl\ sg) = append\ (unstream\ g\ sg)\ (unstream\ h\ sh)$
proof(*induction sg taking: g rule: unstream.induct*)
case (*1 sg*)
then show *?case using unstream-append-trans-Inr*
by (*cases generator g sg)(simp-all add: append-trans.rep-eq)*
qed

2.5.7 filter

definition *filter-raw :: ('a \Rightarrow bool) \Rightarrow ('a, 's) raw-generator \Rightarrow ('a, 's) raw-generator*

where

filter-raw P g s = (case g s of
 Done \Rightarrow *Done* | *Skip* s' \Rightarrow *Skip* s' | *Yield* a s' \Rightarrow if P a then *Yield* a s' else *Skip* s')

lemma *terminates-filter-raw*:

assumes *terminates* g

shows *terminates* (*filter-raw* P g)

proof (*rule terminatesI*)

fix s

from *assms* **have** $s \in$ *terminates-on* g **by** (*simp add: terminates-def*)

thus $s \in$ *terminates-on* (*filter-raw* P g)

proof(*induction s*)

case (*unfold s a s'*)

thus ?*case*

by(*cases P a*)(*auto intro: terminates-on.intros simp add: filter-raw-def*)

qed(*auto intro: terminates-on.intros simp add: filter-raw-def*)

qed

lift-definition *filter-trans* :: ($'a \Rightarrow$ *bool*) \Rightarrow ($'a, 's$) *generator* \Rightarrow ($'a, 's$) *generator*

is *filter-raw* **by** (*rule terminates-filter-raw*)

lemma *unstream-filter-trans* [*stream-fusion*]: *unstream* (*filter-trans* P g) s = *filter* P
(*unstream g s*)

proof (*induction s taking: g rule: unstream.induct*)

case (*1 s*)

then show ?*case* **by**(*cases generator g s*)(*simp-all add: filter-trans.rep-eq filter-raw-def*)

qed

2.5.8 *zip*

fun *zip-raw* :: ($'a, 'sg$) *raw-generator* \Rightarrow ($'b, 'sh$) *raw-generator* \Rightarrow ($'a \times 'b, 'sg \times 'sh \times$
 $'a$ *option*) *raw-generator*

— We search first the left list for the next element and cache it in the $'a$ *option* part of the state once we found one

where

zip-raw g h ($sg, sh, None$) = (case g sg of
 Done \Rightarrow *Done* | *Skip* sg' \Rightarrow *Skip* ($sg', sh, None$) | *Yield* a sg' \Rightarrow *Skip* ($sg', sh, Some$
 a))

 | *zip-raw* g h ($sg, sh, Some$ a) = (case h sh of

Done \Rightarrow *Done* | *Skip* sh' \Rightarrow *Skip* ($sg, sh', Some$ a) | *Yield* b sh' \Rightarrow *Yield* (a, b) ($sg,$
 $sh', None$))

lemma *terminates-zip-raw*:

assumes *terminates* g *terminates* h

```

  shows terminates (zip-raw g h)
proof (rule terminatesI)
  fix s :: 'a × 'c × 'b option
  obtain sg sh m where s = (sg, sh, m) by(cases s)
  show s ∈ terminates-on (zip-raw g h)
proof(cases m)
  case None
  from ⟨terminates g⟩ have sg ∈ terminates-on g by (simp add: terminates-def)
  then show ?thesis unfolding ⟨s = (sg, sh, m)⟩ None
  proof (induction sg arbitrary: sh)
    case (unfold sg a sg')
    from ⟨terminates h⟩ have sh ∈ terminates-on h by (simp add: terminates-def)
    hence (sg', sh, Some a) ∈ terminates-on (zip-raw g h)
    by induction(auto intro: terminates-on.intros unfold.IH)
    thus ?case using unfold.hyps by(auto intro: terminates-on.pause)
  qed(simp-all add: terminates-on.stop terminates-on.pause)
next
  case (Some a')
  from ⟨terminates h⟩ have sh ∈ terminates-on h by (simp add: terminates-def)
  thus ?thesis unfolding ⟨s = (sg, sh, m)⟩ Some
  proof (induction sh arbitrary: sg a')
    case (unfold sh b sh')
    from ⟨terminates g⟩ have sg ∈ terminates-on g by (simp add: terminates-def)
    hence (sg, sh', None) ∈ terminates-on (zip-raw g h)
    by induction(auto intro: terminates-on.intros unfold.IH)
    thus ?case using unfold.hyps by(auto intro: terminates-on.unfold)
  qed(simp-all add: terminates-on.stop terminates-on.pause)
qed

```

lift-definition *zip-trans* :: ('a, 'sg) generator ⇒ ('b, 'sh) generator ⇒ ('a × 'b, 'sg × 'sh × 'a option) generator
 is *zip-raw* by (*rule terminates-zip-raw*)

lemma *unstream-zip-trans* [*stream-fusion*]:
unstream (*zip-trans* *g h*) (*sg, sh, None*) = *zip* (*unstream* *g sg*) (*unstream* *h sh*)
proof (*induction sg arbitrary: sh taking: g rule: unstream.induct*)
 case (*1 sg*)
 then show ?*case*
proof (*cases generator g sg*)
 case (*Yield a sg'*)
 note *IH* = *1.IH*(2)[*OF Yield*]
 have *unstream* (*zip-trans* *g h*) (*sg', sh, Some a*) = *zip* (*a # (unstream* *g sg')*)
 (*unstream* *h sh*)


```

proof(induction sh taking: h rule: unstream.induct)
  case (1 sh)
  then show ?case using IH by(cases generator h sh)(simp-all add: zip-trans.rep-eq)
qed
  then show ?thesis using Yield by (simp add: zip-trans.rep-eq)
qed(simp-all add: zip-trans.rep-eq)
qed

```

2.5.9 *tl*

```

fun tl-raw :: ('a, 'sg) raw-generator ⇒ ('a, bool × 'sg) raw-generator
  — The Boolean flag stores whether we have already skipped the first element
where
  tl-raw g (False, sg) = (case g sg of
    Done ⇒ Done | Skip sg' ⇒ Skip (False, sg') | Yield a sg' ⇒ Skip (True,sg'))
| tl-raw g (True, sg) = (case g sg of
  Done ⇒ Done | Skip sg' ⇒ Skip (True, sg') | Yield a sg' ⇒ Yield a (True, sg'))

```

lemma *terminates-tl-raw*:

```

  assumes terminates g
  shows terminates (tl-raw g)
proof (rule terminatesI)
  fix s :: bool × 'a
  obtain b sg where s = (b, sg) by(cases s)
  { fix sg
    from assms have sg ∈ terminates-on g by(simp add: terminates-def)
    hence (True, sg) ∈ terminates-on (tl-raw g)
    by(induction sg)(auto intro: terminates-on.intros) }
  moreover from assms have sg ∈ terminates-on g by(simp add: terminates-def)
  hence (False, sg) ∈ terminates-on (tl-raw g)
  by(induction sg)(auto intro: terminates-on.intros calculation)
  ultimately show s ∈ terminates-on (tl-raw g) using ⟨s = (b, sg)⟩
  by(cases b) simp-all
qed

```

lift-definition *tl-trans* :: ('*a*, '*sg*) *generator* ⇒ ('*a*, *bool* × '*sg*) *generator*
 is *tl-raw* **by**(*rule terminates-tl-raw*)

lemma *unstream-tl-trans-True*: *unstream (tl-trans g) (True, s) = unstream g s*

```

proof(induction s taking: g rule: unstream.induct)
  case (1 s)
  show ?case using 1.IH by (cases generator g s)(simp-all add: tl-trans.rep-eq)
qed

```

lemma *unstream-tl-trans* [*stream-fusion*]: $\text{unstream } (\text{tl-trans } g) (\text{False}, s) = \text{tl } (\text{unstream } g \ s)$
proof (*induction s taking: g rule: unstream.induct*)
 case (*1 s*)
 then show *?case using unstream-tl-trans-True*
 by (*cases generator g s*)(*simp-all add: tl-trans.rep-eq*)
qed

2.5.10 *butlast*

fun *butlast-raw* :: (*'a, 's*) *raw-generator* \Rightarrow (*'a, 'a option* \times *'s*) *raw-generator*

— The *'a option* caches the previous element we have seen

where

butlast-raw g (None, s) = (case g s of
 Done \Rightarrow *Done* | *Skip s'* \Rightarrow *Skip (None, s')* | *Yield a s'* \Rightarrow *Skip (Some a, s')*)
| *butlast-raw g (Some b, s) = (case g s of*
 Done \Rightarrow *Done* | *Skip s'* \Rightarrow *Skip (Some b, s')* | *Yield a s'* \Rightarrow *Yield b (Some a, s')*)

lemma *terminates-butlast-raw*:

assumes *terminates g*

shows *terminates (butlast-raw g)*

proof (*rule terminatesI*)

fix *st* :: *'b option* \times *'a*

obtain *ma s* **where** *st = (ma, s)* **by** (*cases st*)

from *assms* **have** $s \in \text{terminates-on } g$ **by** (*simp add: terminates-def*)

then show $st \in \text{terminates-on } (\text{butlast-raw } g)$ **unfolding** $\langle st = (ma, s) \rangle$

apply(*induction s arbitrary: ma*)

apply(*case-tac [!] ma*)

apply(*auto intro: terminates-on.intros*)

done

qed

lift-definition *butlast-trans* :: (*'a, 's*) *generator* \Rightarrow (*'a, 'a option* \times *'s*) *generator*

is *butlast-raw* **by** (*rule terminates-butlast-raw*)

lemma *unstream-butlast-trans-Some*:

$\text{unstream } (\text{butlast-trans } g) (\text{Some } b, s) = \text{butlast } (b \# (\text{unstream } g \ s))$

proof (*induction s arbitrary: b taking: g rule: unstream.induct*)

case (*1 s*)

then show *?case by (cases generator g s)*(*simp-all add: butlast-trans.rep-eq*)

qed

lemma *unstream-butlast-trans* [*stream-fusion*]:

$\text{unstream } (\text{butlast-trans } g) (\text{None}, s) = \text{butlast } (\text{unstream } g \ s)$

```

proof (induction s taking: g rule: unstream.induct)
  case (1 s)
  then show ?case using 1 unstream-butlast-trans-Some[of g]
  by (cases generator g s)(simp-all add: butlast-trans.rep-eq)
qed

```

2.5.11 concat

We only do the easy version here where the generator has type $(\text{'a list}, \text{'s})$ generator, not $(\text{'a}, \text{'si})$ generator, $\text{'s})$ generator

```

fun concat-raw :: ( $\text{'a list}, \text{'s})$  raw-generator  $\Rightarrow$  ( $\text{'a}, \text{'a list} \times \text{'s})$  raw-generator
where
  concat-raw g ([], s) = (case g s of
    Done  $\Rightarrow$  Done | Skip s'  $\Rightarrow$  Skip ([], s') | Yield xs s'  $\Rightarrow$  Skip (xs, s'))
| concat-raw g (x # xs, s) = Yield x (xs, s)

```

lemma terminates-concat-raw:

```

assumes terminates g
shows terminates (concat-raw g)
proof (rule terminatesI)
  fix st ::  $\text{'b list} \times \text{'a}$ 
  obtain xs s where st = (xs, s) by (cases st)
  from assms have s  $\in$  terminates-on g by (simp add: terminates-def)
  then show st  $\in$  terminates-on (concat-raw g) unfolding  $\langle$ st = (xs, s) $\rangle$ 
  proof (induction s arbitrary: xs)
    case (stop s xs)
    then show ?case by (induction xs)(auto intro: terminates-on.stop terminates-on.unfold)
  next
    case (pause s s' xs)
    then show ?case by (induction xs)(auto intro: terminates-on.pause terminates-on.unfold)
  next
    case (unfold s a s' xs)
    then show ?case by (induction xs)(auto intro: terminates-on.pause terminates-on.unfold)
  qed
qed

```

lift-definition concat-trans :: $(\text{'a list}, \text{'s})$ generator \Rightarrow $(\text{'a}, \text{'a list} \times \text{'s})$ generator
is concat-raw **by** (rule terminates-concat-raw)

lemma unstream-concat-trans-gen: unstream (concat-trans g) (xs, s) = xs @ (concat (unstream g s))

```

proof (induction s arbitrary: xs taking: g rule: unstream.induct)
  case (1 s)

```

```

then show unstream (concat-trans g) (xs, s) = xs @ (concat (unstream g s))
proof (cases generator g s)
  case Done
    then show ?thesis by (induction xs)(simp-all add: concat-trans.rep-eq)
  next
    case (Skip s')
      then show ?thesis using 1.IH(1)[of s' Nil]
        by (induction xs)(simp-all add: concat-trans.rep-eq)
    next
      case (Yield a s')
        then show ?thesis using 1.IH(2)[of a s' a]
          by (induction xs)(simp-all add: concat-trans.rep-eq)
  qed
qed

```

lemma unstream-concat-trans [stream-fusion]:
 unstream (concat-trans g) ([], s) = concat (unstream g s)
by(simp only: unstream-concat-trans-gen append-Nil)

2.5.12 splice

datatype ('a, 'b) splice-state = Left 'a 'b | Right 'a 'b | Left-only 'a | Right-only 'b

fun splice-raw :: ('a, 'sg) raw-generator \Rightarrow ('a, 'sh) raw-generator \Rightarrow ('a, ('sg, 'sh) splice-state) raw-generator

where

```

  splice-raw g h (Left-only sg) = (case g sg of
    Done  $\Rightarrow$  Done | Skip sg'  $\Rightarrow$  Skip (Left-only sg') | Yield a sg'  $\Rightarrow$  Yield a (Left-only sg'))
| splice-raw g h (Left sg sh) = (case g sg of
  Done  $\Rightarrow$  Skip (Right-only sh) | Skip sg'  $\Rightarrow$  Skip (Left sg' sh) | Yield a sg'  $\Rightarrow$  Yield a (Right sg' sh))
| splice-raw g h (Right-only sh) = (case h sh of
  Done  $\Rightarrow$  Done | Skip sh'  $\Rightarrow$  Skip (Right-only sh') | Yield a sh'  $\Rightarrow$  Yield a (Right-only sh'))
| splice-raw g h (Right sg sh) = (case h sh of
  Done  $\Rightarrow$  Skip (Left-only sg) | Skip sh'  $\Rightarrow$  Skip (Right sg sh') | Yield a sh'  $\Rightarrow$  Yield a (Left sg sh'))

```

lemma terminates-splice-raw:

assumes g: terminates g **and** h: terminates h
shows terminates (splice-raw g h)

proof (rule terminatesI)

fix s

```

{ fix sg
  from g have sg ∈ terminates-on g by (simp add: terminates-def)
  hence Left-only sg ∈ terminates-on (splice-raw g h)
    by induction(auto intro: terminates-on.intros)
} moreover {
  fix sh
  from h have sh ∈ terminates-on h by (simp add: terminates-def)
  hence Right-only sh ∈ terminates-on (splice-raw g h)
    by induction(auto intro: terminates-on.intros)
} moreover {
  fix sg sh
  from g have sg ∈ terminates-on g by (simp add: terminates-def)
  hence Left sg sh ∈ terminates-on (splice-raw g h)
  proof (induction sg arbitrary: sh)
    case (unfold sg a sg')
    from h have sh ∈ terminates-on h by (simp add: terminates-def)
    hence Right sg' sh ∈ terminates-on (splice-raw g h)
      by induction(auto intro: terminates-on.intros unfold.IH calculation)
    thus ?case using unfold.hyps by (auto intro: terminates-on.unfold)
  qed(auto intro: terminates-on.intros calculation)
} moreover {
  fix sg sh
  from h have sh ∈ terminates-on h by (simp add: terminates-def)
  hence Right sg sh ∈ terminates-on (splice-raw g h)
    by(induction sh arbitrary: sg)(auto intro: terminates-on.intros calculation)
  ultimately show s ∈ terminates-on (splice-raw g h) by(cases s)(simp-all)
}
qed

```

lift-definition $\text{splice-trans} :: ('a, 'sg) \text{ generator} \Rightarrow ('a, 'sh) \text{ generator} \Rightarrow ('a, ('sg, 'sh) \text{ splice-state}) \text{ generator}$

is splice-raw by (rule terminates-splice-raw)

lemma $\text{unstream-splice-trans-Right-only: unstream (splice-trans g h) (Right-only sh) = unstream h sh}$

proof (induction sh taking: h rule: unstream.induct)

case (1 sh)

then show ?case by (cases generator h sh)(simp-all add: splice-trans.rep-eq)

qed

lemma $\text{unstream-splice-trans-Left-only: unstream (splice-trans g h) (Left-only sg) = unstream g sg}$

proof (induction sg taking: g rule: unstream.induct)

case (1 sg)

then show ?case by (cases generator g sg)(simp-all add: splice-trans.rep-eq)

qed

lemma *unstream-splice-trans* [*stream-fusion*]:

unstream (splice-trans g h) (Left sg sh) = splice (unstream g sg) (unstream h sh)

proof (*induction sg arbitrary: sh taking: g rule: unstream.induct*)

case (*1 sg sh*)

then show *?case*

proof (*cases generator g sg*)

case *Done*

with *unstream-splice-trans-Right-only*[*of g h*]

show *?thesis* **by** (*simp add: splice-trans.rep-eq*)

next

case (*Skip sg'*)

then show *?thesis* **using** *1.IH(1)* **by** (*simp add: splice-trans.rep-eq*)

next

case (*Yield a sg'*)

note *IH = 1.IH(2)[OF Yield]*

have *a # (unstream (splice-trans g h) (Right sg' sh)) = splice (unstream g sg) (unstream h sh)*

proof (*induction sh taking: h rule: unstream.induct*)

case (*1 sh*)

show *?case*

proof (*cases generator h sh*)

case *Done*

with *unstream-splice-trans-Left-only*[*of g h sg'*]

show *?thesis* **using** *Yield* **by** (*simp add: splice-trans.rep-eq*)

next

case (*Skip sh'*)

then show *?thesis* **using** *Yield 1.IH(1) 1.prem*s **by**(*simp add: splice-trans.rep-eq*)

next

case (*Yield b sh'*)

then show *?thesis* **using** *IH* *⟨generator g sg = Yield a sg'⟩*

by (*simp add: splice-trans.rep-eq*)

qed

qed

then show *?thesis* **using** *Yield* **by** (*simp add: splice-trans.rep-eq*)

qed

qed

2.5.13 *list-update*

fun *list-update-raw* :: (*'a, 's*) *raw-generator* \Rightarrow *'a* \Rightarrow (*'a, nat \times 's*) *raw-generator*

where

$list\text{-}update\text{-}raw\ g\ b\ (n, s) = (case\ g\ s\ of$
 $\quad Done \Rightarrow Done \mid Skip\ s' \Rightarrow Skip\ (n, s')$
 $\mid Yield\ a\ s' \Rightarrow if\ n = 0\ then\ Yield\ a\ (0, s')$
 $\quad\quad\quad else\ if\ n = 1\ then\ Yield\ b\ (0, s')$
 $\quad\quad\quad else\ Yield\ a\ (n - 1, s^{\wedge}))$

lemma *terminates-list-update-raw*:

assumes *terminates g*
shows *terminates (list-update-raw g b)*
proof (*rule terminatesI*)
fix $st :: nat \times 'a$
obtain $n\ s$ **where** $st = (n, s)$ **by** (*cases st*)
from *assms* **have** $s \in terminates\text{-}on\ g$ **by** (*simp add: terminates-def*)
then show $st \in terminates\text{-}on\ (list\text{-}update\text{-}raw\ g\ b)$ **unfolding** $\langle st = (n, s) \rangle$
proof (*induction s arbitrary: n*)
case (*unfold s a s' n*)
then show $(n, s) \in terminates\text{-}on\ (list\text{-}update\text{-}raw\ g\ b)$
by (*cases n = 0 \vee n = 1*)(*auto intro: terminates-on.unfold*)
qed (*simp-all add: terminates-on.stop terminates-on.pause*)
qed

lift-definition *list-update-trans* :: $('a, 's)\ generator \Rightarrow 'a \Rightarrow ('a, nat \times 's)\ generator$
is *list-update-raw* **by** (*rule terminates-list-update-raw*)

lemma *unstream-lift-update-trans-None*: $unstream\ (list\text{-}update\text{-}trans\ g\ b)\ (0, s) = unstream\ g\ s$

proof (*induction s taking: g rule: unstream.induct*)
case ($1\ s$)
then show *?case* **by** (*cases generator g s*)(*simp-all add: list-update-trans.rep-eq*)
qed

lemma *unstream-list-update-trans [stream-fusion]*:

$unstream\ (list\text{-}update\text{-}trans\ g\ b)\ (Suc\ n, s) = list\text{-}update\ (unstream\ g\ s)\ n\ b$
proof (*induction s arbitrary: n taking: g rule: unstream.induct*)
case ($1\ s$)
then show *?case*
proof (*cases generator g s*)
case *Done*
then show *?thesis* **by** (*simp add: list-update-trans.rep-eq*)
next
case (*Skip s'*)
then show *?thesis* **using** *1.IH(1)* **by** (*simp add: list-update-trans.rep-eq*)
next
case (*Yield a s'*)

then show *?thesis using unstream-lift-update-trans-None*[of $g\ b\ s'$] *1.IH(2)*
by (*cases n*)(*simp-all add: list-update-trans.rep-eq*)
qed
qed

2.5.14 *removeAll*

definition *removeAll-raw* :: $'a \Rightarrow ('a, 's)\ \text{raw-generator} \Rightarrow ('a, 's)\ \text{raw-generator}$
where
removeAll-raw $b\ g\ s = (\text{case } g\ s\ \text{of}$
Done $\Rightarrow \text{Done} \mid \text{Skip } s' \Rightarrow \text{Skip } s' \mid \text{Yield } a\ s' \Rightarrow \text{if } a = b\ \text{then } \text{Skip } s' \ \text{else } \text{Yield } a\ s')$

lemma *terminates-removeAll-raw*:
assumes *terminates* g
shows *terminates* (*removeAll-raw* $b\ g$)
proof (*rule terminatesI*)
fix s
from *assms* **have** $s \in \text{terminates-on } g$ **by** (*simp add: terminates-def*)
then show $s \in \text{terminates-on } (\text{removeAll-raw } b\ g)$
proof(*induction s*)
case (*unfold s a s'*)
then show *?case*
by(*cases a = b*)(*auto intro: terminates-on.intros simp add: removeAll-raw-def*)
qed(*auto intro: terminates-on.intros simp add: removeAll-raw-def*)
qed

lift-definition *removeAll-trans* :: $'a \Rightarrow ('a, 's)\ \text{generator} \Rightarrow ('a, 's)\ \text{generator}$
is *removeAll-raw* **by** (*rule terminates-removeAll-raw*)

lemma *unstream-removeAll-trans* [*stream-fusion*]:
unstream (*removeAll-trans* $b\ g$) $s = \text{removeAll } b\ (\text{unstream } g\ s)$
proof (*induction s taking: g rule: unstream.induct*)
case (*1 s*)
then show *?case*
proof(*cases generator g s*)
case (*Yield a s'*)
then show *?thesis using 1.IH(2)*
by(*cases a = b*)(*simp-all add: removeAll-trans.rep-eq removeAll-raw-def*)
qed(*auto simp add: removeAll-trans.rep-eq removeAll-raw-def*)
qed

2.5.15 *remove1*

fun *remove1-raw* :: $'a \Rightarrow ('a, 's)\ \text{raw-generator} \Rightarrow ('a, \text{bool} \times 's)\ \text{raw-generator}$

where

$remove1\text{-raw } x \ g \ (b, s) = (\text{case } g \ s \ \text{of}$
 $Done \Rightarrow Done \mid Skip \ s' \Rightarrow Skip \ (b, s')$
 $\mid Yield \ y \ s' \Rightarrow \text{if } b \wedge x = y \ \text{then } Skip \ (False, s') \ \text{else } Yield \ y \ (b, s'))$

lemma *terminates-remove1-raw:*

assumes *terminates g*
shows *terminates (remove1-raw b g)*
proof (*rule terminatesI*)
 fix *st :: bool × 'a*
 obtain *c s where st = (c, s) by (cases st)*
 from *assms have s ∈ terminates-on g by (simp add: terminates-def)*
 then show *st ∈ terminates-on (remove1-raw b g) unfolding ⟨st = (c, s)⟩*
 proof (*induction s arbitrary: c*)
 case (*stop s*)
 then show *?case by (cases c)(simp-all add: terminates-on.stop)*
 next
 case (*pause s s'*)
 then show *?case by (cases c)(simp-all add: terminates-on.pause)*
 next
 case (*unfold s a s'*)
 then show *?case*
 by(*cases c*)(*cases a = b, auto intro: terminates-on.intros*)
 qed
qed

lift-definition *remove1-trans :: 'a ⇒ ('a, 's) generator ⇒ ('a, bool × 's) generator*
is *remove1-raw by (rule terminates-remove1-raw)*

lemma *unstream-remove1-trans-False: unstream (remove1-trans b g) (False, s) = unstream g s*

proof (*induction s taking: g rule: unstream.induct*)
 case (*1 s*)
 then show *?case by (cases generator g s)(simp-all add: remove1-trans.rep-eq)*
qed

lemma *unstream-remove1-trans [stream-fusion]:*

$unstream \ (remove1\text{-trans } b \ g) \ (True, s) = remove1 \ b \ (unstream \ g \ s)$
proof(*induction s taking: g rule: unstream.induct*)
 case (*1 s*)
 then show *?case*
 proof (*cases generator g s*)
 case (*Yield a s'*)
 then show *?thesis*

```

    using Yield 1.IH(2) unstream-remove1-trans-False[of b g]
    by (cases a = b)(simp-all add: remove1-trans.rep-eq)
qed(simp-all add: remove1-trans.rep-eq)
qed

```

2.5.16 (#)

```

fun Cons-raw :: 'a ⇒ ('a, 's) raw-generator ⇒ ('a, bool × 's) raw-generator
where
  Cons-raw x g (b, s) = (if b then Yield x (False, s) else case g s of
    Done ⇒ Done | Skip s' ⇒ Skip (False, s') | Yield y s' ⇒ Yield y (False, s'))

```

lemma *terminates-Cons-raw*:

```

assumes terminates g
shows terminates (Cons-raw x g)
proof (rule terminatesI)
  fix st :: bool × 'a
  obtain b s where st = (b, s) by (cases st)
  from assms have s ∈ terminates-on g by (simp add: terminates-def)
  hence (False, s) ∈ terminates-on (Cons-raw x g)
  by(induction s arbitrary: b)(auto intro: terminates-on.intros)
  then show st ∈ terminates-on (Cons-raw x g) unfolding ⟨st = (b, s)⟩
  by(cases b)(auto intro: terminates-on.intros)
qed

```

lift-definition *Cons-trans* :: 'a ⇒ ('a, 's) generator ⇒ ('a, bool × 's) generator
is Cons-raw **by**(rule terminates-Cons-raw)

lemma *unstream-Cons-trans-False*: $\text{unstream } (\text{Cons-trans } x \ g) \ (False, \ s) = \text{unstream } g \ s$

```

proof(induction s taking: g rule: unstream.induct)
  case (1 s)
  then show ?case by(cases generator g s)(auto simp add: Cons-trans.rep-eq)
qed

```

We do not declare *Cons-trans* as a transformer. Otherwise, literal lists would be transformed into streams which adds a significant overhead to the stream state.

lemma *unstream-Cons-trans*: $\text{unstream } (\text{Cons-trans } x \ g) \ (True, \ s) = x \ \# \ \text{unstream } g \ s$
using *unstream-Cons-trans-False*[of x g s] **by**(simp add: Cons-trans.rep-eq)

2.5.17 List.maps

Stream version based on Coutts [1].

We restrict the function for generating the inner lists to terminating generators because the code generator does not directly supported nesting abstract datatypes in other types.

```
fun maps-raw
  :: ('a ⇒ ('b, 'sg) generator × 'sg) ⇒ ('a, 's) raw-generator
  ⇒ ('b, 's × (('b, 'sg) generator × 'sg) option) raw-generator
where
  maps-raw f g (s, None) = (case g s of
    Done ⇒ Done | Skip s' ⇒ Skip (s', None) | Yield x s' ⇒ Skip (s', Some (f x)))
  | maps-raw f g (s, Some (g'', s'')) = (case generator g'' s'' of
    Done ⇒ Skip (s, None) | Skip s' ⇒ Skip (s, Some (g'', s')) | Yield x s' ⇒ Yield x
    (s, Some (g'', s')))
```

lemma *terminates-on-maps-raw-Some*:

```
  assumes (s, None) ∈ terminates-on (maps-raw f g)
  shows (s, Some (g'', s'')) ∈ terminates-on (maps-raw f g)
```

proof –

```
  from generator[of g''] have s'' ∈ terminates-on (generator g'') by (simp add: termi-
  nates-def)
```

```
  thus ?thesis by(induction)(auto intro: terminates-on.intros assms)
```

qed

lemma *terminates-maps-raw*:

```
  assumes terminates g
  shows terminates (maps-raw f g)
```

proof

```
  fix st :: 'a × (('c, 'd) generator × 'd) option
```

```
  obtain s mgs where st = (s, mgs) by(cases st)
```

```
  from assms have s ∈ terminates-on g by (simp add: terminates-def)
```

```
  then show st ∈ terminates-on (maps-raw f g) unfolding ⟨st = (s, mgs)⟩
```

```
    apply(induction arbitrary: mgs)
```

```
    apply(case-tac [!] mgs)
```

```
    apply(auto intro: terminates-on.intros intro!: terminates-on-maps-raw-Some)
```

```
  done
```

qed

lift-definition *maps-trans* :: ('a ⇒ ('b, 'sg) generator × 'sg) ⇒ ('a, 's) generator

```
  ⇒ ('b, 's × (('b, 'sg) generator × 'sg) option) generator
```

is maps-raw **by**(rule terminates-maps-raw)

lemma *unstream-maps-trans-Some*:

```
  unstream (maps-trans f g) (s, Some (g'', s'')) = unstream g'' s'' @ unstream (maps-trans
  f g) (s, None)
```

proof(induction s'' taking: g'' rule: unstream.induct)

```

    case (1 s'')
  then show ?case by(cases generator g'' s'')(simp-all add: maps-trans.rep-eq)
qed

```

lemma *unstream-maps-trans*:

```

  unstream (maps-trans f g) (s, None) = List.maps (case-prod unstream ∘ f) (unstream
  g s)

```

proof(*induction s taking: g rule: unstream.induct*)

```

  case (1 s)

```

```

  thus ?case

```

proof(*cases generator g s*)

```

  case (Yield x s')

```

```

  with 1.IH(2)[OF this] show ?thesis

```

```

    using unstream-maps-trans-Some[of f g - fst (f x) snd (f x)]

```

```

    by(simp add: maps-trans.rep-eq maps-simps split-def)

```

```

  qed(simp-all add: maps-trans.rep-eq maps-simps)

```

qed

The rule *unstream-map-trans* is too complicated for fusion because of *split*, which does not arise naturally from stream fusion rules. Moreover, according to Farmer et al. [2], this fusion is too general for further optimisations because the generators of the inner list are generated by the outer generator and therefore compilers may think that is was not known statically.

Instead, they propose a weaker version using *flatten* below. (More precisely, Coutts already mentions this approach in his PhD thesis [1], but dismisses it because it requires a stronger rewriting engine than GHC has. But Isabelle's simplifier language is sufficiently powerful.

```

fun fix-step :: 'a ⇒ ('b, 's) step ⇒ ('b, 'a × 's) step

```

where

```

  fix-step a Done = Done

```

```

| fix-step a (Skip s) = Skip (a, s)

```

```

| fix-step a (Yield x s) = Yield x (a, s)

```

```

fun fix-gen-raw :: ('a ⇒ ('b, 's) raw-generator) ⇒ ('b, 'a × 's) raw-generator

```

where *fix-gen-raw* *g* (*a*, *s*) = *fix-step* *a* (*g* *a* *s*)

lemma *terminates-fix-gen-raw*:

```

  assumes  $\bigwedge x. \text{terminates } (g\ x)$ 

```

```

  shows terminates (fix-gen-raw g)

```

proof

```

  fix st :: 'a × 'b

```

```

  obtain a s where st = (a, s) by(cases st)

```

```

  from assms[of a] have s ∈ terminates-on (g a) by (simp add: terminates-def)

```

```

  then show st ∈ terminates-on (fix-gen-raw g) unfolding ⟨st = (a, s)⟩

```

by(*induction*)(*auto intro: terminates-on.intros*)
qed

lift-definition *fix-gen* :: ('a ⇒ ('b, 's) generator) ⇒ ('b, 'a × 's) generator
is *fix-gen-raw* **by**(*rule terminates-fix-gen-raw*)

lemma *unstream-fix-gen*: *unstream* (*fix-gen* *g*) (*a*, *s*) = *unstream* (*g* *a*) *s*
proof(*induction s taking: g a rule: unstream.induct*)
case (*1 s*)
thus ?*case* **by**(*cases generator (g a) s*)(*simp-all add: fix-gen.rep-eq*)
qed

context

fixes *f* :: ('a ⇒ 's')
and *g''* :: ('b, 's') raw-generator
and *g* :: ('a, 's) raw-generator
begin

fun *flatten-raw* :: ('b, 's × 's' option) raw-generator

where

flatten-raw (*s*, *None*) = (*case g s of*
Done ⇒ *Done* | *Skip s'* ⇒ *Skip (s', None)* | *Yield x s'* ⇒ *Skip (s', Some (f x))*)
| *flatten-raw* (*s*, *Some s''*) = (*case g'' s'' of*
Done ⇒ *Skip (s, None)* | *Skip s'* ⇒ *Skip (s, Some s')* | *Yield x s'* ⇒ *Yield x (s,*
Some s'))

lemma *terminates-flatten-raw*:

assumes *terminates g'' terminates g*
shows *terminates flatten-raw*

proof

fix *st* :: 's × 's' option
obtain *s ms* **where** *st* = (*s*, *ms*) **by**(*cases st*)
{ **fix** *s s''*
assume *s*: (*s*, *None*) ∈ *terminates-on flatten-raw*
from ⟨*terminates g''*⟩ **have** *s''* ∈ *terminates-on g''* **by** (*simp add: terminates-def*)
hence (*s*, *Some s''*) ∈ *terminates-on flatten-raw*
by(*induction*)(*auto intro: terminates-on.intros s*) }
note *Some = this*
from ⟨*terminates g*⟩ **have** *s* ∈ *terminates-on g* **by** (*simp add: terminates-def*)
then show *st* ∈ *terminates-on flatten-raw* **unfolding** ⟨*st* = (*s*, *ms*)⟩
apply(*induction arbitrary: ms*)
apply(*case-tac [!] ms*)
apply(*auto intro: terminates-on.intros intro!: Some*)
done

qed

end

lift-definition $flatten :: ('a \Rightarrow 's') \Rightarrow ('b, 's) generator \Rightarrow ('a, 's) generator \Rightarrow ('b, 's \times 's' option) generator$
is $flatten\text{-raw}$ **by**($fact\ terminates\ flatten\text{-raw}$)

lemma $unstream\text{-flatten}\text{-Some}$:

$unstream (flatten f g'' g) (s, Some s') = unstream g'' s' @ unstream (flatten f g'' g) (s, None)$

proof($induction\ s'$ taking: g'' rule: $unstream.induct$)

case ($1\ s'$)

thus $?case$ **by**($cases\ generator\ g''\ s'$)($simp\text{-all}\ add:\ flatten.rep\text{-eq}$)

qed

HO rewrite equations can express the variable capture in the generator unlike GHC rules

lemma $unstream\text{-flatten}\text{-fix}\text{-gen}$ [$stream\text{-fusion}$]:

$unstream (flatten (\lambda s. (s, f s)) (fix\text{-gen}\ g'') g) (s, None) =$

$List.maps (\lambda s'. unstream (g'' s') (f s')) (unstream g s)$

proof($induction\ s$ taking: g rule: $unstream.induct$)

case ($1\ s$)

thus $?case$

proof($cases\ generator\ g\ s$)

case ($Yield\ x\ s'$)

with $1.IH(2)[OF\ this]$ $unstream\text{-flatten}\text{-Some}[of\ \lambda s. (s, f s)\ fix\text{-gen}\ g'']\ g$

show $?thesis$

by($subst\ (1\ 3)\ unstream.simps$)($simp\ add:\ flatten.rep\text{-eq}\ maps\text{-simps}\ unstream\text{-fix}\text{-gen}$)

qed($simp\text{-all}\ add:\ flatten.rep\text{-eq}\ maps\text{-simps}$)

qed

Separate fusion rule when the inner generator does not depend on the elements of the outer stream.

lemma $unstream\text{-flatten}$ [$stream\text{-fusion}$]:

$unstream (flatten f g'' g) (s, None) = List.maps (\lambda s'. unstream g'' (f s')) (unstream g s)$

proof($induction\ s$ taking: g rule: $unstream.induct$)

case ($1\ s$)

thus $?case$

proof($cases\ generator\ g\ s$)

case ($Yield\ x\ s'$)

with $1.IH(2)[OF\ this]$ **show** $?thesis$

using $unstream\text{-flatten}\text{-Some}[of\ f\ g''\ g\ s'\ f\ x]$

by($simp\ add:\ flatten.rep\text{-eq}\ maps\text{-simps}\ o\text{-def}$)

```

qed(simp-all add: maps-simps flatten.rep-eq)
qed

end

```

3 Stream fusion for coinductive lists

```

theory Stream-Fusion-LList imports
  Stream-Fusion-List
  Coinductive.Coinductive-List
begin

```

There are two choices of how many *Skips* may occur consecutively.

- A generator for '*a llist*' may return only finitely many *Skips* before it has to decide on a *Done* or *Yield*. Then, we can define stream versions for all functions that can be defined by corecursion up-to. This in particular excludes *lfilter*. Moreover, we have to prove that every generator satisfies this restriction.
- A generator for '*a llist*' may return infinitely many *Skips* in a row. Then, the *lunstream* function suffers from the same difficulties as *lfilter* with definitions, but we can define it using the least fixpoint approach described in [4]. Consequently, we can only fuse transformers that are monotone and continuous with respect to the ccpo ordering. This in particular excludes *lappend*.

Here, we take the both approaches where we consider the first preferable to the second. Consequently, we define producers such that they produce generators of the first kind, if possible. There will be multiple equations for transformers and consumers that deal with all the different combinations for their parameter generators. Transformers should yield generators of the first kind whenever possible. Consumers can be defined using *lunstream* and refined with custom code equations, i.e., they can operate with infinitely many *Skips* in a row. We just have to lift the fusion equation to the first kind, too.

```

type-synonym ('a, 's) lgenerator = 's  $\Rightarrow$  ('a, 's) step

```

```

inductive-set productive-on :: ('a, 's) lgenerator  $\Rightarrow$  's set
for g :: ('a, 's) lgenerator

```

```

where

```

```

  Done: g s = Done  $\Longrightarrow$  s  $\in$  productive-on g
| Skip:  $\llbracket$  g s = Skip s'; s'  $\in$  productive-on g  $\rrbracket$   $\Longrightarrow$  s  $\in$  productive-on g
| Yield: g s = Yield x s'  $\Longrightarrow$  s  $\in$  productive-on g

```

```

definition productive :: ('a, 's) lgenerator  $\Rightarrow$  bool
where productive g  $\longleftrightarrow$  productive-on g = UNIV

```

```

lemma productiveI [intro?]:

```

```

  ( $\bigwedge$  s. s  $\in$  productive-on g)  $\Longrightarrow$  productive g

```

by(*auto simp add: productive-def*)

lemma *productive-onI* [*dest?*]: *productive g* \implies *s* \in *productive-on g*
by(*simp add: productive-def*)

A type of generators that eventually will yield something else than a skip.

typedef (*'a, 's*) *lgenerator'* = {*g* :: (*'a, 's*) *lgenerator*. *productive g*}
morphisms *lgenerator Abs-lgenerator'*

proof

show (λ -. *Done*) \in ?*lgenerator'* **by**(*auto intro: productive-on.intros productiveI*)
qed

setup-lifting *type-definition-lgenerator'*

3.1 Conversions to '*a* *l*list

3.1.1 Infinitely many consecutive *Skips*

context *fixes g* :: (*'a, 's*) *lgenerator*
notes [[*function-internals*]]
begin

partial-function (*l*list) *lunstream* :: '*s* \Rightarrow '*a* *l*list

where

lunstream s = (case *g s* of
 Done \Rightarrow *LNil* | *Skip s'* \Rightarrow *lunstream s'* | *Yield x s'* \Rightarrow *LCons x (lunstream s')*)

declare *lunstream.simps*[*code*]

lemma *lunstream-simps*:

g s = *Done* \implies *lunstream s* = *LNil*
g s = *Skip s'* \implies *lunstream s* = *lunstream s'*
g s = *Yield x s'* \implies *lunstream s* = *LCons x (lunstream s')*

by(*simp-all add: lunstream.simps*)

lemma *lunstream-sels*:

shows *lnull-lunstream*: *lnull (lunstream s)* \longleftrightarrow
(case *g s* of *Done* \Rightarrow *True* | *Skip s'* \Rightarrow *lnull (lunstream s')* | *Yield - -* \Rightarrow *False*)
and *lhd-lunstream*: *lhd (lunstream s)* =
(case *g s* of *Skip s'* \Rightarrow *lhd (lunstream s')* | *Yield x -* \Rightarrow *x*)
and *ltl-lunstream*: *ltl (lunstream s)* =
(case *g s* of *Done* \Rightarrow *LNil* | *Skip s'* \Rightarrow *ltl (lunstream s')* | *Yield - s'* \Rightarrow *lunstream s'*)

by(*simp-all add: lhd-def lunstream-simps split: step.split*)

end

3.1.2 Finitely many consecutive *Skips*

lift-definition $\text{lunstream}' :: ('a, 's) \text{lgenerator}' \Rightarrow 's \Rightarrow 'a \text{llist}$
is lunstream .

lemma $\text{lunstream}'\text{-simps}$:

$\text{lgenerator } g \ s = \text{Done} \Longrightarrow \text{lunstream}' \ g \ s = \text{LNil}$
 $\text{lgenerator } g \ s = \text{Skip } s' \Longrightarrow \text{lunstream}' \ g \ s = \text{lunstream}' \ g \ s'$
 $\text{lgenerator } g \ s = \text{Yield } x \ s' \Longrightarrow \text{lunstream}' \ g \ s = \text{LCons } x \ (\text{lunstream}' \ g \ s')$

by(transfer , $\text{simp add: lunstream-simps}$)+

lemma $\text{lunstream}'\text{-sels}$:

shows $\text{lnull-lunstream}'$: $\text{lnull } (\text{lunstream}' \ g \ s) \longleftrightarrow$
 $(\text{case } \text{lgenerator } g \ s \text{ of } \text{Done} \Rightarrow \text{True} \mid \text{Skip } s' \Rightarrow \text{lnull } (\text{lunstream}' \ g \ s') \mid \text{Yield } - \Rightarrow$
 $\text{False})$

and $\text{lhd-lunstream}'$: $\text{lhd } (\text{lunstream}' \ g \ s) =$
 $(\text{case } \text{lgenerator } g \ s \text{ of } \text{Skip } s' \Rightarrow \text{lhd } (\text{lunstream}' \ g \ s') \mid \text{Yield } x \ - \Rightarrow x)$

and $\text{ltl-lunstream}'$: $\text{ltl } (\text{lunstream}' \ g \ s) =$
 $(\text{case } \text{lgenerator } g \ s \text{ of } \text{Done} \Rightarrow \text{LNil} \mid \text{Skip } s' \Rightarrow \text{ltl } (\text{lunstream}' \ g \ s') \mid \text{Yield } - \ s' \Rightarrow$
 $\text{lunstream}' \ g \ s')$

by(transfer , $\text{simp add: lunstream-sels}$)+

setup $\langle \text{Context.theory-map } (\text{fold}$

$\text{Stream-Fusion.add-unstream } [\text{@}\{\text{const-name } \text{lunstream}\}, \text{@}\{\text{const-name } \text{lunstream}'\}]\rangle$

3.2 Producers

3.2.1 Conversion to streams

fun $\text{lstream} :: ('a, 'a \text{llist}) \text{lgenerator}$

where

$\text{lstream } \text{LNil} = \text{Done}$
 $\mid \text{lstream } (\text{LCons } x \ xs) = \text{Yield } x \ xs$

lemma $\text{case-lstream-conv-case-llist}$:

$(\text{case } \text{lstream } xs \text{ of } \text{Done} \Rightarrow \text{done} \mid \text{Skip } xs' \Rightarrow \text{skip } xs' \mid \text{Yield } x \ xs' \Rightarrow \text{yield } x \ xs') =$
 $(\text{case } xs \text{ of } \text{LNil} \Rightarrow \text{done} \mid \text{LCons } x \ xs' \Rightarrow \text{yield } x \ xs')$

by($\text{simp split: llist.split}$)

lemma $\text{mcont2mcont-lunstream}$ [$\text{THEN } \text{lstream.mcont2mcont}$, simp , cont-intro]:

shows mcont-lunstream : $\text{mcont } \text{lSup } \text{lprefix } \text{lSup } \text{lprefix } (\text{lunstream } \text{lstream})$

by($\text{rule } \text{lstream.fixp-preserves-mcont1}$ [$\text{OF } \text{lunstream.mono } \text{lunstream-def}$])($\text{simp add: case-lstream-conv-case-l}$)

lemma lunstream-lstream : $\text{lunstream } \text{lstream } xs = xs$

by($\text{induction } xs$)($\text{simp-all add: lunstream-simps}$)

lift-definition $lstream' :: ('a, 'a\ llist)\ lgenerator'$

is $lstream$

proof

fix $s :: 'a\ llist$

show $s \in productive-on\ lstream$ **by**(cases s)(auto intro: productive-on.intros)

qed

lemma $lunstream'-lstream: lunstream'\ lstream'\ xs = xs$

by(transfer)(rule $lunstream-lstream$)

3.2.2 iterates

definition $iterates-raw :: ('a \Rightarrow 'a) \Rightarrow ('a, 'a)\ lgenerator$

where $iterates-raw\ f\ s = Yield\ s\ (f\ s)$

lemma $lunstream-iterates-raw: lunstream\ (iterates-raw\ f)\ x = iterates\ f\ x$

by(coinduction arbitrary: x)(auto simp add: iterates-raw-def $lunstream-sels$)

lift-definition $iterates-prod :: ('a \Rightarrow 'a) \Rightarrow ('a, 'a)\ lgenerator'$ **is** $iterates-raw$

by(auto 4 3 intro: productiveI productive-on.intros simp add: iterates-raw-def)

lemma $lunstream'-iterates-prod\ [stream-fusion]: lunstream'\ (iterates-prod\ f)\ x = iterates\ f\ x$

by transfer(rule $lunstream-iterates-raw$)

3.2.3 unfold-llist

definition $unfold-llist-raw :: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'a) \Rightarrow ('b, 'a)\ lgenerator$

where

$unfold-llist-raw\ stop\ head\ tail\ s = (if\ stop\ s\ then\ Done\ else\ Yield\ (head\ s)\ (tail\ s))$

lemma $lunstream-unfold-llist-raw:$

$lunstream\ (unfold-llist-raw\ stop\ head\ tail)\ s = unfold-llist\ stop\ head\ tail\ s$

by(coinduction arbitrary: s)(auto simp add: $lunstream-sels\ unfold-llist-raw-def$)

lift-definition $unfold-llist-prod :: ('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'a) \Rightarrow ('b, 'a)\ lgenerator'$

is $unfold-llist-raw$

proof(rule $productiveI$)

fix $stop$ **and** $head :: 'a \Rightarrow 'b$ **and** $tail\ s$

show $s \in productive-on\ (unfold-llist-raw\ stop\ head\ tail)$

by(cases $stop\ s$)(auto intro: productive-on.intros simp add: $unfold-llist-raw-def$)

qed

lemma *lunstream'-unfold-llist-prod* [*stream-fusion*]:

lunstream' (*unfold-llist-prod stop head tail*) *s* = *unfold-llist stop head tail s*
by *transfer*(*rule lunstream-unfold-llist-raw*)

3.2.4 *inf-llist*

definition *inf-llist-raw* :: (*nat* \Rightarrow '*a*) \Rightarrow ('*a*, *nat*) *lgenerator*

where *inf-llist-raw f n* = *Yield (f n) (Suc n)*

lemma *lunstream-inf-llist-raw*: *lunstream (inf-llist-raw f) n* = *ldropn n (inf-llist f)*

by (*coinduction arbitrary: n*)(*auto simp add: lunstream-sels inf-llist-raw-def*)

lift-definition *inf-llist-prod* :: (*nat* \Rightarrow '*a*) \Rightarrow ('*a*, *nat*) *lgenerator'* **is** *inf-llist-raw*

by (*auto 4 3 intro: productiveI productive-on.intros simp add: inf-llist-raw-def*)

lemma *inf-llist-prod-fusion* [*stream-fusion*]:

lunstream' (*inf-llist-prod f*) 0 = *inf-llist f*
by *transfer*(*simp add: lunstream-inf-llist-raw*)

3.3 Consumers

3.3.1 *lhd*

context fixes *g* :: ('*a*, '*s*) *lgenerator* **begin**

definition *lhd-cons* :: '*s* \Rightarrow '*a*

where [*stream-fusion*]: *lhd-cons s* = *lhd (lunstream g s)*

lemma *lhd-cons-code*[*code*]:

lhd-cons s = (*case g s of Done* \Rightarrow *undefined* | *Skip s'* \Rightarrow *lhd-cons s'* | *Yield x -* \Rightarrow *x*)
by (*simp add: lhd-cons-def lunstream-simps lhd-def split: step.split*)

end

lemma *lhd-cons-fusion2* [*stream-fusion*]:

lhd-cons (lgenerator g) s = *lhd (lunstream' g s)*
by *transfer*(*rule lhd-cons-def*)

3.3.2 *llength*

context fixes *g* :: ('*a*, '*s*) *lgenerator* **begin**

definition *gen-llength-cons* :: *enat* \Rightarrow '*s* \Rightarrow *enat*

where *gen-llength-cons n s* = *n + llength (lunstream g s)*

lemma *gen-llength-cons-code* [code]:
 $gen-llength-cons\ n\ s = (case\ g\ s\ of$
 $\quad Done \Rightarrow n \mid Skip\ s' \Rightarrow gen-llength-cons\ n\ s' \mid Yield\ -\ s' \Rightarrow gen-llength-cons\ (eSuc\ n)$
 $\quad s')$
by(*simp add: gen-llength-cons-def lunstream-simps iadd-Suc-right iadd-Suc split: step.split*)

lemma *gen-llength-cons-fusion* [stream-fusion]:
 $gen-llength-cons\ 0\ s = llength\ (lunstream\ g\ s)$
by(*simp add: gen-llength-cons-def*)

end

context **fixes** $g :: ('a, 's)\ lgenerator'$ **begin**

definition *gen-llength-cons'* :: $enat \Rightarrow 's \Rightarrow enat$
where $gen-llength-cons' = gen-llength-cons\ (lgenerator\ g)$

lemma *gen-llength-cons'-code* [code]:
 $gen-llength-cons'\ n\ s = (case\ lgenerator\ g\ s\ of$
 $\quad Done \Rightarrow n \mid Skip\ s' \Rightarrow gen-llength-cons'\ n\ s' \mid Yield\ -\ s' \Rightarrow gen-llength-cons'\ (eSuc\ n)$
 $\quad s')$
by(*simp add: gen-llength-cons'-def cong: step.case-cong*)(*rule gen-llength-cons-code*)

lemma *gen-llength-cons'-fusion* [stream-fusion]:
 $gen-llength-cons'\ 0\ s = llength\ (lunstream'\ g\ s)$
by(*simp add: gen-llength-cons'-def gen-llength-cons-fusion lunstream'.rep-eq*)

end

3.3.3 *lnull*

context **fixes** $g :: ('a, 's)\ lgenerator$ **begin**

definition *lnull-cons* :: $'s \Rightarrow bool$
where [stream-fusion]: $lnull-cons\ s \longleftrightarrow lnull\ (lunstream\ g\ s)$

lemma *lnull-cons-code* [code]:
 $lnull-cons\ s \longleftrightarrow (case\ g\ s\ of$
 $\quad Done \Rightarrow True \mid Skip\ s' \Rightarrow lnull-cons\ s' \mid Yield\ -\ - \Rightarrow False)$
by(*simp add: lnull-cons-def lunstream-simps split: step.split*)

end

context fixes $g :: ('a, 's) \text{ lgenerator}'$ **begin**

definition $\text{lnull-cons}' :: 's \Rightarrow \text{bool}$

where $\text{lnull-cons}' = \text{lnull-cons} (\text{lgenerator } g)$

lemma $\text{lnull-cons}'\text{-code}$ [code]:

$\text{lnull-cons}' s \longleftrightarrow (\text{case } \text{lgenerator } g \text{ s of}$
 $\text{Done} \Rightarrow \text{True} \mid \text{Skip } s' \Rightarrow \text{lnull-cons}' s' \mid \text{Yield } - \Rightarrow \text{False})$

by(simp add: $\text{lnull-cons}'\text{-def}$ cong: step.case-cong)(rule lnull-cons-code)

lemma $\text{lnull-cons}'\text{-fusion}$ [stream-fusion]:

$\text{lnull-cons}' s \longleftrightarrow \text{lnull} (\text{lunstream}' g s)$

by(simp add: $\text{lnull-cons}'\text{-def}$ lnull-cons-def $\text{lunstream}'\text{-rep-eq}$)

end

3.3.4 $\text{l\text{list-all2}}$

context

fixes $g :: ('a, 'sg) \text{ lgenerator}$

and $h :: ('b, 'sh) \text{ lgenerator}$

and $P :: 'a \Rightarrow 'b \Rightarrow \text{bool}$

begin

definition $\text{l\text{list-all2-cons}} :: 'sg \Rightarrow 'sh \Rightarrow \text{bool}$

where [stream-fusion]: $\text{l\text{list-all2-cons}} sg sh \longleftrightarrow \text{l\text{list-all2}} P (\text{lunstream } g sg) (\text{lunstream } h sh)$

definition $\text{l\text{list-all2-cons1}} :: 'a \Rightarrow 'sg \Rightarrow 'sh \Rightarrow \text{bool}$

where $\text{l\text{list-all2-cons1}} x sg' sh = \text{l\text{list-all2}} P (\text{LCons } x (\text{lunstream } g sg')) (\text{lunstream } h sh)$

lemma $\text{l\text{list-all2-cons-code}}$ [code]:

$\text{l\text{list-all2-cons}} sg sh =$
(case $g \text{ sg}$ of
 $\text{Done} \Rightarrow \text{lnull-cons } h sh$
 $\mid \text{Skip } sg' \Rightarrow \text{l\text{list-all2-cons}} sg' sh$
 $\mid \text{Yield } a \text{ sg}' \Rightarrow \text{l\text{list-all2-cons1}} a \text{ sg}' sh)$

by(simp split: step.split add: $\text{l\text{list-all2-cons-def}}$ lnull-cons-def $\text{l\text{list-all2-cons1-def}}$ lunstream-simps lnull-def)

lemma $\text{l\text{list-all2-cons1-code}}$ [code]:

$\text{l\text{list-all2-cons1}} x sg' sh =$
(case $h \text{ sh}$ of

$Done \Rightarrow False$
 $| Skip\ sh' \Rightarrow llist\text{-}all2\text{-}cons1\ x\ sg'\ sh'$
 $| Yield\ y\ sh' \Rightarrow P\ x\ y \wedge llist\text{-}all2\text{-}cons\ sg'\ sh'$
by(*simp split: step.split add: llist-all2-cons-def lnull-cons-def lnull-def llist-all2-cons1-def lunstream-simps*)

end

lemma *llist-all2-cons-fusion2* [*stream-fusion*]:
 $llist\text{-}all2\text{-}cons\ (lgenerator\ g)\ (lgenerator\ h)\ P\ sg\ sh \longleftrightarrow llist\text{-}all2\ P\ (lunstream'\ g\ sg)\ (lunstream'\ h\ sh)$
by *transfer(rule llist-all2-cons-def)*

lemma *llist-all2-cons-fusion3* [*stream-fusion*]:
 $llist\text{-}all2\text{-}cons\ g\ (lgenerator\ h)\ P\ sg\ sh \longleftrightarrow llist\text{-}all2\ P\ (lunstream\ g\ sg)\ (lunstream'\ h\ sh)$
by *transfer(rule llist-all2-cons-def)*

lemma *llist-all2-cons-fusion4* [*stream-fusion*]:
 $llist\text{-}all2\text{-}cons\ (lgenerator\ g)\ h\ P\ sg\ sh \longleftrightarrow llist\text{-}all2\ P\ (lunstream'\ g\ sg)\ (lunstream\ h\ sh)$
by *transfer(rule llist-all2-cons-def)*

3.3.5 *lnth*

context *fixes* $g :: ('a, 's)\ lgenerator$ **begin**

definition *lnth-cons* :: $nat \Rightarrow 's \Rightarrow 'a$
where [*stream-fusion*]: $lnth\text{-}cons\ n\ s = lnth\ (lunstream\ g\ s)\ n$

lemma *lnth-cons-code* [*code*]:
 $lnth\text{-}cons\ n\ s = (case\ g\ s\ of$
 $\quad Done \Rightarrow undefined\ n$
 $\quad | Skip\ s' \Rightarrow lnth\text{-}cons\ n\ s'$
 $\quad | Yield\ x\ s' \Rightarrow (if\ n = 0\ then\ x\ else\ lnth\text{-}cons\ (n - 1)\ s'))$
by(*cases n*)(*simp-all add: lnth-cons-def lunstream-simps lnth-LNil split: step.split*)

end

lemma *lnth-cons-fusion2* [*stream-fusion*]:
 $lnth\text{-}cons\ (lgenerator\ g)\ n\ s = lnth\ (lunstream'\ g\ s)\ n$
by *transfer(rule lnth-cons-def)*

3.3.6 *lprefix*

context

fixes $g :: ('a, 'sg) \text{ lgenerator}$

and $h :: ('a, 'sh) \text{ lgenerator}$

begin

definition $lprefix-cons :: 'sg \Rightarrow 'sh \Rightarrow \text{bool}$

where [*stream-fusion*]: $lprefix-cons\ sg\ sh \longleftrightarrow lprefix\ (\text{lunstream}\ g\ sg)\ (\text{lunstream}\ h\ sh)$

definition $lprefix-cons1 :: 'a \Rightarrow 'sg \Rightarrow 'sh \Rightarrow \text{bool}$

where $lprefix-cons1\ x\ sg'\ sh \longleftrightarrow lprefix\ (LCons\ x\ (\text{lunstream}\ g\ sg'))\ (\text{lunstream}\ h\ sh)$

lemma *lprefix-cons-code* [*code*]:

$lprefix-cons\ sg\ sh \longleftrightarrow (\text{case}\ g\ sg\ \text{of}$

$Done \Rightarrow True \mid Skip\ sg' \Rightarrow lprefix-cons\ sg'\ sh \mid Yield\ x\ sg' \Rightarrow lprefix-cons1\ x\ sg'\ sh)$

by (*simp add: lprefix-cons-def lprefix-cons1-def lunstream-simps split: step.split*)

lemma *lprefix-cons1-code* [*code*]:

$lprefix-cons1\ x\ sg'\ sh \longleftrightarrow (\text{case}\ h\ sh\ \text{of}$

$Done \Rightarrow False \mid Skip\ sh' \Rightarrow lprefix-cons1\ x\ sg'\ sh'$

$\mid Yield\ y\ sh' \Rightarrow x = y \wedge lprefix-cons\ sg'\ sh')$

by (*simp add: lprefix-cons-def lprefix-cons1-def lunstream-simps split: step.split*)

end

lemma *lprefix-cons-fusion2* [*stream-fusion*]:

$lprefix-cons\ (\text{lgenerator}\ g)\ (\text{lgenerator}\ h)\ sg\ sh \longleftrightarrow lprefix\ (\text{lunstream}'\ g\ sg)\ (\text{lunstream}'\ h\ sh)$

by *transfer(rule lprefix-cons-def)*

lemma *lprefix-cons-fusion3* [*stream-fusion*]:

$lprefix-cons\ g\ (\text{lgenerator}\ h)\ sg\ sh \longleftrightarrow lprefix\ (\text{lunstream}\ g\ sg)\ (\text{lunstream}'\ h\ sh)$

by *transfer(rule lprefix-cons-def)*

lemma *lprefix-cons-fusion4* [*stream-fusion*]:

$lprefix-cons\ (\text{lgenerator}\ g)\ h\ sg\ sh \longleftrightarrow lprefix\ (\text{lunstream}'\ g\ sg)\ (\text{lunstream}\ h\ sh)$

by *transfer(rule lprefix-cons-def)*

3.4 Transformers

3.4.1 *lmap*

definition $lmap-trans :: ('a \Rightarrow 'b) \Rightarrow ('a, 's) \text{ lgenerator} \Rightarrow ('b, 's) \text{ lgenerator}$

where $lmap-trans = \text{map-raw}$

lemma *lunstream-lmap-trans* [*stream-fusion*]: **fixes** $f\ g\ s$
defines [*simp*]: $g' \equiv \text{lmap-trans } f\ g$
shows $\text{lunstream } g'\ s = \text{lmap } f\ (\text{lunstream } g\ s)$ (**is** $?lhs = ?rhs$)
proof(*rule lprefix-antisym*)
show *lprefix ?lhs ?rhs*
proof(*induction g' arbitrary: s rule: lunstream.fixp-induct*)
case ($\exists \text{lunstream-g}'$)
then show $?case$
by(*cases g s*)(*simp-all add: lmap-trans-def map-raw-def lunstream-simps*)
qed *simp-all*
next
note [*cont-intro*] = *ccpo.admissible-leI[OF llist-ccpo]*
show *lprefix ?rhs ?lhs*
proof(*induction g arbitrary: s rule: lunstream.fixp-induct*)
case ($\exists \text{lunstream-g}$)
thus $?case$ **by**(*cases g s*)(*simp-all add: lmap-trans-def map-raw-def lunstream-simps*)
qed *simp-all*
qed

lift-definition $\text{lmap-trans}' :: ('a \Rightarrow 'b) \Rightarrow ('a, 's)\ \text{lgenerator}' \Rightarrow ('b, 's)\ \text{lgenerator}'$
is *lmap-trans*

proof
fix $f :: 'a \Rightarrow 'b$ **and** $g :: ('a, 's)\ \text{lgenerator}$ **and** s
assume *productive g*
hence $s \in \text{productive-on } g$..
thus $s \in \text{productive-on } (\text{lmap-trans } f\ g)$
by *induction(auto simp add: lmap-trans-def map-raw-def intro: productive-on.intros)*
qed

lemma *lunstream'-lmap-trans'* [*stream-fusion*]:
 $\text{lunstream}' (\text{lmap-trans}' f\ g)\ s = \text{lmap } f\ (\text{lunstream}' g\ s)$
by *transfer(rule lunstream-lmap-trans)*

3.4.2 *ltake*

fun *ltake-trans* :: $('a, 's)\ \text{lgenerator} \Rightarrow ('a, (\text{enat} \times 's))\ \text{lgenerator}$

where

$\text{ltake-trans } g\ (n, s) =$
(if $n = 0$ *then* *Done* *else case* $g\ s$ *of*
 $\text{Done} \Rightarrow \text{Done} \mid \text{Skip } s' \Rightarrow \text{Skip } (n, s') \mid \text{Yield } a\ s' \Rightarrow \text{Yield } a\ (\text{epred } n, s')$ *)*

lemma *ltake-trans-fusion* [*stream-fusion*]:
fixes $g'\ g$


```

defines [simp]:  $g' \equiv \text{ltake-trans } g$ 
shows  $\text{lunstream } g' (n, s) = \text{ltake } n (\text{lunstream } g s)$  (is ?lhs = ?rhs)
proof(rule lprefix-antisym)
show lprefix ?lhs ?rhs
proof(induction  $g'$  arbitrary:  $n$   $s$  rule: lunstream.fixp-induct)
  case ( $\exists$  lunstream- $g'$ )
  thus ?case
  by(cases  $g$   $s$ )(auto simp add: lunstream-simps neq-zero-conv-eSuc)
qed simp-all
show lprefix ?rhs ?lhs
proof(induction  $g$  arbitrary:  $s$   $n$  rule: lunstream.fixp-induct)
  case ( $\exists$  lunstream- $g$ )
  thus ?case by(cases  $g$   $s$   $n$  rule: step.exhaust[case-product enat-coexhaust])(auto simp
add: lunstream-simps)
  qed simp-all
qed

```

lift-definition $\text{ltake-trans}' :: ('a, 's) \text{lgenerator}' \Rightarrow ('a, (\text{enat} \times 's)) \text{lgenerator}'$
is ltake-trans

```

proof
fix  $g :: ('a, 's) \text{lgenerator}$  and  $s :: \text{enat} \times 's$ 
obtain  $n$   $sg$  where  $s = (n, sg)$  by(cases  $s$ )
assume productive  $g$ 
hence  $sg \in \text{productive-on } g ..$ 
then show  $s \in \text{productive-on } (\text{ltake-trans } g)$  unfolding  $\langle s = (n, sg) \rangle$ 
  apply(induction arbitrary:  $n$ )
  apply(case-tac [!]  $n$  rule: enat-coexhaust)
  apply(auto intro: productive-on.intros)
  done
qed

```

lemma $\text{ltake-trans}'\text{-fusion}$ [stream-fusion]:
 $\text{lunstream}' (\text{ltake-trans}' g) (n, s) = \text{ltake } n (\text{lunstream}' g s)$
by transfer(rule $\text{ltake-trans-fusion}$)

3.4.3 ldropn

abbreviation (input) $\text{ldropn-trans} :: ('b, 'a) \text{lgenerator} \Rightarrow ('b, \text{nat} \times 'a) \text{lgenerator}$
where $\text{ldropn-trans} \equiv \text{drop-raw}$

lemma $\text{ldropn-trans-fusion}$ [stream-fusion]:
fixes g **defines** [simp]: $g' \equiv \text{ldropn-trans } g$
shows $\text{lunstream } g' (n, s) = \text{ldropn } n (\text{lunstream } g s)$ (**is** ?lhs = ?rhs)
proof(rule lprefix-antisym)

```

show lprefix ?lhs ?rhs
proof(induction g' arbitrary: n s rule: lunstream.fixp-induct)
  case (∃ lunstream-g')
    thus ?case
      by(cases g s n rule: step.exhaust[case-product nat.exhaust])
        (auto simp add: lunstream-simps elim: meta-allE[where x=0])
    qed simp-all
note [cont-intro] = ccpo.admissible-leI[OF llist-ccpo]
show lprefix ?rhs ?lhs
proof(induction g arbitrary: n s rule: lunstream.fixp-induct)
  case (∃ lunstream-g)
    thus ?case by(cases n)(auto split: step.split simp add: lunstream-simps elim: meta-allE[where
x=0])
    qed simp-all
qed

```

lift-definition *ldropn-trans'* :: (*'a, 's*) *lgenerator'* \Rightarrow (*'a, nat \times 's*) *lgenerator'*
is *ldropn-trans*

```

proof
  fix g :: ('a, 's) lgenerator and ns :: nat  $\times$  's
  obtain n s where ns: ns = (n, s) by(cases ns)
  assume g: productive g
  show ns  $\in$  productive-on (ldropn-trans g) unfolding ns
  proof(induction n arbitrary: s)
    case 0
      from g have s  $\in$  productive-on g ..
      thus ?case by induction(auto intro: productive-on.intros)
    next
      case (Suc n)
        from g have s  $\in$  productive-on g ..
        thus ?case by induction(auto intro: productive-on.intros Suc.IH)
    qed
  qed

```

lemma *ldropn-trans'-fusion* [*stream-fusion*]:
lunstream' (ldropn-trans' g) (n, s) = ldropn n (lunstream' g s)
by *transfer(rule ldropn-trans-fusion)*

3.4.4 *ldrop*

```

fun ldrop-trans :: ('a, 's) lgenerator  $\Rightarrow$  ('a, enat  $\times$  's) lgenerator
where
  ldrop-trans g (n, s) = (case g s of
    Done  $\Rightarrow$  Done | Skip s'  $\Rightarrow$  Skip (n, s')

```

| *Yield x s' ⇒ (if n = 0 then Yield x (n, s') else Skip (epred n, s'))*)

lemma *ldrop-trans-fusion* [*stream-fusion*]:
fixes *g g'* **defines** [*simp*]: *g' ≡ ldrop-trans g*
shows *lunstream g' (n, s) = ldrop n (lunstream g s) (is ?lhs = ?rhs)*
proof(*rule lprefix-antisym*)
show *lprefix ?lhs ?rhs*
by(*induction g' arbitrary: n s rule: lunstream.fixp-induct*)
(auto simp add: lunstream-simps neq-zero-conv-eSuc elim: meta-allE[where x=0]
split: step.split)
show *lprefix ?rhs ?lhs*
proof(*induction g arbitrary: n s rule: lunstream.fixp-induct*)
case (*∃ lunstream-g*)
thus *?case*
by(*cases n rule: enat-coexhaust*)(*auto simp add: lunstream-simps split: step.split*
elim: meta-allE[where x=0])
qed *simp-all*
qed

lemma *ldrop-trans-fusion2* [*stream-fusion*]:
lunstream (ldrop-trans (lgenerator g)) (n, s) = ldrop n (lunstream' g s)
by *transfer (rule ldrop-trans-fusion)*

3.4.5 *ltakeWhile*

abbreviation (*input*) *ltakeWhile-trans* :: (*a ⇒ bool*) ⇒ (*a, 's*) *lgenerator* ⇒ (*a, 's*)
lgenerator
where *ltakeWhile-trans ≡ takeWhile-raw*

lemma *ltakeWhile-trans-fusion* [*stream-fusion*]:
fixes *P g g'* **defines** [*simp*]: *g' ≡ ltakeWhile-trans P g*
shows *lunstream g' s = ltakeWhile P (lunstream g s) (is ?lhs = ?rhs)*
proof(*rule lprefix-antisym*)
show *lprefix ?lhs ?rhs*
by(*induction g' arbitrary: s rule: lunstream.fixp-induct*)(*auto simp add: lunstream-simps*
takeWhile-raw-def split: step.split)
show *lprefix ?rhs ?lhs*
by(*induction g arbitrary: s rule: lunstream.fixp-induct*)(*auto split: step.split simp*
add: lunstream-simps takeWhile-raw-def)
qed

lift-definition *ltakeWhile-trans'* :: (*a ⇒ bool*) ⇒ (*a, 's*) *lgenerator'* ⇒ (*a, 's*) *lgener-*
ator'
is *ltakeWhile-trans*

proof

fix P **and** $g :: ('a, 's)$ *lgenerator* **and** s
assume *productive* g
hence $s \in$ *productive-on* g ..
thus $s \in$ *productive-on* (*ltakeWhile-trans* P g)
apply(*induction*)
apply(*case-tac* [3] P x)
apply(*auto intro: productive-on.intros simp add: takeWhile-raw-def*)
done

qed

lemma *ltakeWhile-trans'-fusion* [*stream-fusion*]:

lunstream' (*ltakeWhile-trans'* P g) $s =$ *ltakeWhile* P (*lunstream'* g s)

by *transfer(rule ltakeWhile-trans-fusion)*

3.4.6 *ldropWhile*

abbreviation (*input*) *ldropWhile-trans* :: $('a \Rightarrow \text{bool}) \Rightarrow ('a, 'b)$ *lgenerator* $\Rightarrow ('a, \text{bool} \times 'b)$ *lgenerator*

where *ldropWhile-trans* \equiv *dropWhile-raw*

lemma *ldropWhile-trans-fusion* [*stream-fusion*]:

fixes P g g' **defines** [*simp*]: $g' \equiv$ *ldropWhile-trans* P g

shows *lunstream* g' (*True*, s) = *ldropWhile* P (*lunstream* g s) (**is** *?lhs = ?rhs*)

proof –

have *lprefix* *?lhs* *?rhs* *lprefix* (*lunstream* g' (*False*, s)) (*lunstream* g s)

by(*induction* g' *arbitrary: s rule: lunstream.fixp-induct*)(*simp-all add: lunstream-simps split: step.split*)

moreover **have** *lprefix* *?rhs* *?lhs* *lprefix* (*lunstream* g s) (*lunstream* g' (*False*, s))

by(*induction* g *arbitrary: s rule: lunstream.fixp-induct*)(*simp-all add: lunstream-simps split: step.split*)

ultimately show *?thesis* **by**(*blast intro: lprefix-antisym*)

qed

lemma *ldropWhile-trans-fusion2* [*stream-fusion*]:

lunstream (*ldropWhile-trans* P (*lgenerator* g)) (*True*, s) = *ldropWhile* P (*lunstream'* g s)

by *transfer(rule ldropWhile-trans-fusion)*

3.4.7 *lzip*

abbreviation (*input*) *lzip-trans* :: $('a, 's1)$ *lgenerator* $\Rightarrow ('b, 's2)$ *lgenerator* $\Rightarrow ('a \times 'b, 's1 \times 's2 \times 'a)$ *option* *lgenerator*

where *lzip-trans* \equiv *zip-raw*

```

lemma lzip-trans-fusion [stream-fusion]:
  fixes g h gh defines [simp]: gh  $\equiv$  lzip-trans g h
  shows lunstream gh (sg, sh, None) = lzip (lunstream g sg) (lunstream h sh)
  (is ?lhs = ?rhs)
proof –
  have lprefix ?lhs ?rhs
    and  $\wedge x.$  lprefix (lunstream gh (sg, sh, Some x)) (lzip (LCons x (lunstream g sg))
(lunstream h sh))
  proof(induction gh arbitrary: sg sh rule: lunstream.fixp-induct)
  case ( $\exists$  lunstream)
  { case 1 show ?case using  $\exists$ 
    by(cases g sg)(simp-all add: lunstream-simps) }
  { case 2 show ?case using  $\exists$ 
    by(cases h sh)(simp-all add: lunstream-simps) }
qed simp-all
moreover
note [cont-intro] = ccpo.admissible-leI[OF llist-ccpo]
have lprefix ?rhs ?lhs
  and  $\wedge x.$  lprefix (lzip (LCons x (lunstream g sg)) (lunstream h sh)) (lunstream gh
(sg, sh, Some x))
  proof(induction g arbitrary: sg sh rule: lunstream.fixp-induct)
  case ( $\exists$  lunstream-g)
  note IH =  $\exists.IH$ 
  { case 1 show ?case using  $\exists$ 
    by(cases g sg)(simp-all add: lunstream-simps fun-ord-def) }
  { case 2 show ?case
    proof(induction h arbitrary: sh sg x rule: lunstream.fixp-induct)
    case ( $\exists$  unstream-h)
    thus ?case
    proof(cases h sh)
    case (Yield y sh')
    thus ?thesis using  $\exists$ .prems IH  $\exists$ .hyps
      by(cases g sg)(auto 4  $\exists$  simp add: lunstream-simps fun-ord-def intro: mono-
tone-lzip2[THEN monotoneD] lprefix-trans)
      qed(simp-all add: lunstream-simps)
      qed simp-all }
    next
    case 2 case 2
    show ?case
    by(induction h arbitrary: sh rule: lunstream.fixp-induct)(simp-all add: lunstream-simps
split: step.split)
    qed simp-all
    ultimately show ?thesis by(blast intro: lprefix-antisym)
  }

```

qed

lemma *lzip-trans-fusion2* [*stream-fusion*]:

$\text{lunstream} (\text{lzip-trans} (\text{lgenerator } g) h) (sg, sh, \text{None}) = \text{lzip} (\text{lunstream}' g sg) (\text{lunstream}' h sh)$

by *transfer*(rule *lzip-trans-fusion*)

lemma *lzip-trans-fusion3* [*stream-fusion*]:

$\text{lunstream} (\text{lzip-trans } g (\text{lgenerator } h)) (sg, sh, \text{None}) = \text{lzip} (\text{lunstream } g sg) (\text{lunstream}' h sh)$

by *transfer*(rule *lzip-trans-fusion*)

lift-definition *lzip-trans'* :: ('a, 's1) *lgenerator'* \Rightarrow ('b, 's2) *lgenerator'* \Rightarrow ('a \times 'b, 's1 \times 's2 \times 'a *option*) *lgenerator'*

is *lzip-trans*

proof

fix $g :: ('a, 's1) \text{ lgenerator}$ **and** $h :: ('b, 's2) \text{ lgenerator}$ **and** $s :: 's1 \times 's2 \times 'a \text{ option}$
assume *productive g and productive h*

obtain $sg \ sh \ mx$ **where** $s = (sg, sh, mx)$ **by** (*cases s*)

{ **fix** $x \ sg$

from $\langle \text{productive } h \rangle$ **have** $sh \in \text{productive-on } h \ ..$

hence $(sg, sh, \text{Some } x) \in \text{productive-on} (\text{lzip-trans } g \ h)$

by (*induction*)(*auto simp add: intro: productive-on.intros*) }

moreover

from $\langle \text{productive } g \rangle$ **have** $sg \in \text{productive-on } g \ ..$

then have $(sg, sh, \text{None}) \in \text{productive-on} (\text{lzip-trans } g \ h)$

by *induction*(*auto intro: productive-on.intros calculation*)

ultimately show $s \in \text{productive-on} (\text{lzip-trans } g \ h)$ **unfolding** s

by(*cases mx*) *auto*

qed

lemma *lzip-trans'-fusion* [*stream-fusion*]:

$\text{lunstream}' (\text{lzip-trans}' g h) (sg, sh, \text{None}) = \text{lzip} (\text{lunstream}' g sg) (\text{lunstream}' h sh)$

by *transfer*(rule *lzip-trans-fusion*)

3.4.8 *lappend*

lift-definition *lappend-trans* :: ('a, 'sg) *lgenerator'* \Rightarrow ('a, 'sh) *lgenerator* \Rightarrow 'sh \Rightarrow ('a, 'sg + 'sh) *lgenerator*

is *append-raw* .

lemma *lunstream-append-raw*:

fixes $g \ h \ sh \ gh$ **defines** [*simp*]: $gh \equiv \text{append-raw } g \ h \ sh$

assumes *productive g*

shows $\text{lunstream } gh \text{ (Inl } sg) = \text{lappend (lunstream } g \text{ } sg) \text{ (lunstream } h \text{ } sh)$
proof(*coinduction arbitrary: sg rule: llist.coinduct-strong*)
case (*Eq-llist sg*)
{ **fix** sh'
have $\text{lprefix (lunstream } gh \text{ (Inr } sh')) \text{ (lunstream } h \text{ } sh')$
by(*induction gh arbitrary: sh' rule: lunstream.fixp-induct*)(*simp-all add: lunstream-simps split: step.split*)
moreover have $\text{lprefix (lunstream } h \text{ } sh') \text{ (lunstream } gh \text{ (Inr } sh'))$
by(*induction h arbitrary: sh' rule: lunstream.fixp-induct*)(*simp-all add: lunstream-simps split: step.split*)
ultimately have $\text{lunstream } gh \text{ (Inr } sh') = \text{lunstream } h \text{ } sh'$
by(*blast intro: lprefix-antisym*) }
note $\text{Inr} = \text{this[unfolded gh-def]}$
from $\langle \text{productive } g \rangle$ **have** $sg : sg \in \text{productive-on } g \dots$
then show $?case$ **by** *induction(auto simp add: lunstream-sels Inr)*
qed

lemma *lappend-trans-fusion [stream-fusion]:*
 $\text{lunstream (lappend-trans } g \text{ } h \text{ } sh) \text{ (Inl } sg) = \text{lappend (lunstream' } g \text{ } sg) \text{ (lunstream } h \text{ } sh)$
by *transfer(rule lunstream-append-raw)*

lift-definition $\text{lappend-trans}' :: ('a, 'sg) \text{lgenerator}' \Rightarrow ('a, 'sh) \text{lgenerator}' \Rightarrow 'sh \Rightarrow ('a, 'sg + 'sh) \text{lgenerator}'$

is *append-raw*

proof

fix $g :: ('a, 'sg) \text{lgenerator}$ **and** $h :: ('a, 'sh) \text{lgenerator}$ **and** $sh \text{ } s$
assume *productive g productive h*
{ **fix** sh'
from $\langle \text{productive } h \rangle$ **have** $sh' \in \text{productive-on } h \dots$
then have $\text{Inr } sh' \in \text{productive-on (append-raw } g \text{ } h \text{ } sh)$
by *induction (auto intro: productive-on.intros)*
} **moreover** {
fix sg
from $\langle \text{productive } g \rangle$ **have** $sg \in \text{productive-on } g \dots$
then have $\text{Inl } sg \in \text{productive-on (append-raw } g \text{ } h \text{ } sh)$
by *induction(auto intro: productive-on.intros calculation)* }
ultimately show $s \in \text{productive-on (append-raw } g \text{ } h \text{ } sh)$ **by**(*cases s*) *auto*
qed

lemma *lappend-trans'-fusion [stream-fusion]:*
 $\text{lunstream' (lappend-trans' } g \text{ } h \text{ } sh) \text{ (Inl } sg) = \text{lappend (lunstream' } g \text{ } sg) \text{ (lunstream' } h \text{ } sh)$
by *transfer(rule lunstream-append-raw)*

3.4.9 *lfilter*

definition *lfilter-trans* :: ('a ⇒ bool) ⇒ ('a, 's) lgenerator ⇒ ('a, 's) lgenerator
where *lfilter-trans* = *filter-raw*

lemma *lunstream-lfilter-trans* [*stream-fusion*]:
fixes *P g g'* **defines** [*simp*]: *g' ≡ lfilter-trans P g*
shows *lunstream g' s = lfilter P (lunstream g s)* (**is** ?lhs = ?rhs)
proof(rule *lprefix-antisym*)
show *lprefix ?lhs ?rhs*
by(*induction g' arbitrary: s rule: lunstream.fixp-induct*)
(*simp-all add: lfilter-trans-def filter-raw-def lunstream-simps split: step.split*)
show *lprefix ?rhs ?lhs*
by(*induction g arbitrary: s rule: lunstream.fixp-induct*)
(*simp-all add: lfilter-trans-def filter-raw-def lunstream-simps split: step.split*)
qed

lemma *lunstream-lfilter-trans2* [*stream-fusion*]:
lunstream (lfilter-trans P (lgenerator g)) s = lfilter P (lunstream' g s)
by *transfer*(rule *lunstream-lfilter-trans*)

3.4.10 *llist-of*

lift-definition *llist-of-trans* :: ('a, 's) generator ⇒ ('a, 's) lgenerator'
is $\lambda x. x$

proof
fix *g :: ('a, 's) raw-generator* **and** *s*
assume *terminates g*
hence *s ∈ terminates-on g* **by**(*simp add: terminates-def*)
then show *s ∈ productive-on g*
by(*induction*)(*auto intro: productive-on.intros*)
qed

lemma *lunstream-llist-of-trans* [*stream-fusion*]:
lunstream' (llist-of-trans g) s = llist-of (unstream g s)
apply(*induction s taking: g rule: unstream.induct*)
apply(rule *llist.expand*)
apply(*auto intro: llist.expand simp add: llist-of-trans.rep-eq lunstream-sels lunstream'.rep-eq split: step.split*)
done

We cannot define a stream version of *list-of* because we would have to test for finiteness first and therefore traverse the list twice.

end

4 Examples and test cases for stream fusion

theory *Stream-Fusion-Examples* **imports** *Stream-Fusion-LList* **begin**

lemma fixes *rhs z*

defines *rhs* \equiv *nth-cons* (*flatten* ($\lambda s'. s'$) (*upto-prod* 17) (*upto-prod* z)) (2, *None*) 8

shows *nth* (*List.maps* ($\lambda x. \text{upto } x \text{ } 17$) (*upto* 2 z)) 8 = *rhs*

using [[*simplproc add: stream-fusion, stream-fusion-trace*]]

apply(*simp del: id-apply*) — fuses

by(*unfold rhs-def*) *rule*

lemma fixes *rhs z*

defines *rhs* \equiv *nth-cons* (*flatten* ($\lambda s. (s, 1)$) (*fix-gen* ($\lambda x. \text{upto-prod } (id \ x)$)) (*upto-prod* z)) (2, *None*) 8

shows *nth* (*List.maps* ($\lambda x. \text{upto } 1 \ (id \ x)$) (*upto* 2 z)) 8 = *rhs*

using [[*simplproc add: stream-fusion, stream-fusion-trace*]]

apply(*simp del: id-apply*) — fuses

by(*unfold rhs-def*) *rule*

lemma fixes *rhs n*

defines *rhs* \equiv *List.maps* ($\lambda x. [Suc \ 0..<sum-list-cons \ (replicate-prod \ x) \ x]$) [2..*n*]

shows (*concat* (*map* ($\lambda x. [1..<sum-list \ (replicate \ x \ x)]$) [2..*n*])) = *rhs*

using [[*simplproc add: stream-fusion, stream-fusion-trace*]]

apply(*simp add: concat-map-maps*) — fuses partially

by(*unfold rhs-def*) *rule*

4.1 Micro-benchmarks from Farmer et al. [2]

definition *test-enum* :: *nat* \Rightarrow *nat* — *id* required to avoid eta contraction

where *test-enum* *n* = *foldl* (+) 0 (*List.maps* ($\lambda x. \text{upt } 1 \ (id \ x)$) (*upt* 1 *n*))

definition *test-nested* :: *nat* \Rightarrow *nat*

where *test-nested* *n* = *foldl* (+) 0 (*List.maps* ($\lambda x. \text{List.maps } (\lambda y. \text{upt } y \ x)$) (*upt* 1 *x*)) (*upt* 1 *n*))

definition *test-merge* :: *integer* \Rightarrow *nat*

where *test-merge* *n* = *foldl* (+) 0 (*List.maps* ($\lambda x. \text{if } 2 \ \text{dvd } x \ \text{then } \text{upt } 1 \ x \ \text{else } \text{upt } 2 \ x$) (*upt* 1 (*nat-of-integer* *n*)))

This rule performs the merge operation from [2, §5.2] for *if*. In general, we would also need it for all case operators.

lemma *unstream-if* [*stream-fusion*]:

unstream (*if* *b* *then* *g* *else* *g'*) (*if* *b* *then* *s* *else* *s'*) =
(if *b* *then* *unstream* *g* *s* *else* *unstream* *g'* *s'*)

by *simp*

lemma *if-same* [code-unfold]: (if b then x else x) = x
by *simp*

code-thms *test-enum*

code-thms *test-nested*

code-thms *test-merge*

4.2 Test stream fusion in the code generator

definition *fuse-test* :: integer

where *fuse-test* =

integer-of-int (lhd (lfilter ($\lambda x. x < 1$) (lappend (lmap ($\lambda x. x + 1$) (llist-of (map ($\lambda x. \text{if } x = 0 \text{ then undefined else } x$) [-3..5]))) (repeat 3))))))

ML-val $\langle \text{val } \sim 2 = @\{\text{code } \textit{fuse-test}\} \rangle$ — If this test fails with exception Fail, then the stream fusion simproc failed. This test exploits that stream fusion introduces laziness.

end

References

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