Designing Efficient and Fair Mechanisms for Multi-Type Resource Allocation

JAAMAS Track

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ABSTRACT

In the *multi-type resource allocation problem* (MTRA), there are $d \geq 2$ *types* of items, and n agents who each demand one unit of items of each type and have *strict linear preferences* over *bundles* consisting of one item of each type. For MTRAs with indivisible items, we first present an impossibility result that no mechanism can satisfy both sd-efficiency and sd-envy-freeness. We show that this impossibility result is circumvented under the natural assumption of lexicographic preferences by providing *lexicographic probabilistic serial* (LexiPS) as an extension of the *probabilistic serial* (PS) mechanism. We also prove that LexiPS satisfies sd-efficiency and sd-envy-freeness. Moreover, LexiPS satisfies sd-weak-strategyproofness when agents are not allowed to misreport their importance orders. The *multi-type probabilistic serial* cannot deal with indivisible items, but provides a stronger efficiency guarantee under the unrestricted domain of strict linear preferences for divisible items, while also retaining desirable fairness guarantees.

KEYWORDS

Multi-type resource allocation; Probabilistic serial; LexiPS; MPS; Fractional assignment; sd-efficiency; sd-envy-freeness

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1 INTRODUCTION

We focus on the *multi-type resource allocation problem* (MTRA) [18] where each item belongs to one of $d \geq 2$ types and each agent demands a *bundle* consisting of one item of each type. Here, items may be either divisible [9–11, 21] or indivisible [17, 22–24].

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Our work follows the line of research initiated by Bogomolnaia and Moulin [5], who proposed the probabilistic serial (PS) mechanism for the classical resource allocation problem [2, 5, 8, 18]. PS is a popular prototype for mechanism designers, which possesses the following strengths: (i) Decomposability: PS can be applied to allocating both divisible and indivisible items, since fractional assignments are always decomposable when $d = 1$, due to the Birkhoff-von Neumann theorem. In other words, a fractional assignment can be represented as a probability distribution over "discrete" assignments, where no item is split among agents. (ii) Efficiency and fairness: PS satisfies sd-efficiency and sd-envy-freeness which are desirable efficiency and fairness properties, respectively. The remarkable properties of PS has encouraged several extensions: to the full preference domain allowing indifferences [14, 15], to multi-unit demands [4, 6, 13, 16], to housing markets [3, 26], and to the bundle assignment [7, 20].

Efficient and fair resource allocation for a single type of items ($d = 1$) has been well studied [1, 5, 19, 27]. However, designing an efficient and fair mechanism for MTRAs with $d \geq 2$ types is more challenging, especially because direct applications of PS to MTRAs fail to simultaneously satisfy the two desirable properties of efficiency and fairness discussed above. Recently, Wang et al. [25] proposed *multi-type probabilistic serial* (MPS) mechanism as an extension of PS for MTRAs, but it does not satisfy decomposability. It is unclear whether similar extensions of the PS mechanism can be applied to the efficient and fair allocation of indivisible items because the outcome may not be decomposable. This leaves the following natural question: *How to design efficient and fair mechanisms for MTRAs with indivisible or divisible items?*

Our results in [12] provide a possible affirmative answer to this question: For indivisible items, the LexiPS mechanism we propose is efficient and fair under the natural restriction of lexicographic preferences. As for divisible items, we prove that the existing MPS mechanism provides a stronger efficiency guarantee under the unrestricted domain of strict linear preferences. This paper summaries the important results of [12].

2 PRELIMINARIES

An MTRA is given by a tuple (N, M) with a preference profile R. Let $N = \{1, \dots, n\}$ be the set of agents and $M = D_1 \cup \cdots \cup D_d$ be the set of all the items where D_i is the set of n items of type i for each $i \leq d$. For all $h \neq i$, we have $D_i \cap D_h = \emptyset$. There is one unit of *supply* of each item in M. We use $\mathcal{D} = D_1 \times \cdots \times D_d$ to denote the set of *bundles*. Each bundle $\mathbf{x} \in \mathcal{D}$ is a d-vector and each component refers to an item of each type. We use $o \in \mathbf{x}$ to indicate that bundle $\mathbf{x}$ contains item o. In an MTRA, each agent demands one unit of item of each type. A *preference profile* is denoted by $R = (\succ_j)_{j \leq n}$, where $\succ_j$ represents agent j's preference as a *strict linear preference*, i.e. the strict linear order over $\mathcal{D}$. Let $\mathcal{R}$ be the set of all the preference profiles.

A *fractional allocation* is a $|\mathcal{D}|$-vector describing the fractional share of each bundle allocated to an agent. Let Π be the set of all the possible fractional allocations. For any $p \in \Pi$, $\mathbf{x} \in \mathcal{D}$, we use $p_{\mathbf{x}}$ to denote the share of $\mathbf{x}$ assigned by p. A *fractional assignment* is a $n \times |\mathcal{D}|$-matrix $P = [p_{j,\mathbf{x}}]_{j \leq n, \mathbf{x} \in \mathcal{D}}$, where (i) $p_{j,\mathbf{x}} \in [0,1]$ is the fractional share of $\mathbf{x}$ allocated to agent j for each $j \leq n, \mathbf{x} \in \mathcal{D}$, (ii) $\sum_{\mathbf{x} \in \mathcal{D}} p_{j,\mathbf{x}} = 1$ for each $j \leq n$, (iii) $\sum_{j \leq n, \mathbf{x} \in \{\hat{\mathbf{x}} \in \mathcal{D} | o \in \hat{\mathbf{x}}\}} p_{j,\mathbf{x}} = 1$ for each $o \in M$. For each $j \leq n$, the j-th row of P, denoted by P_j, represents agent j's fractional allocation in P. We use $\mathcal{P}$ to denote the set of all possible fractional assignments. A *discrete assignment* A is an assignment where each agent is assigned a share of one unit of a bundle, and each item is fully allocated to some agent.

A *mechanism* f is a mapping from preference profiles to fractional assignments. For any profile $R \in \mathcal{R}$, we use $f(R)$ to refer to the fractional assignment output by f and $f(R)_j$ refer to agent j's fractional allocation in $f(R)$ for any agent $j \leq n$ accordingly.

stochastic dominance. Given a preference $\succ$ over $\mathcal{D}$, the *stochastic dominance* relation associated with $\succ$, denoted by $\succeq^{sd}$, is a partial ordering over Π such that for any pair of fractional allocations $p, q \in \Pi$, p (weakly) *stochastically dominates* q, denoted by $p \succeq^{sd} q$, if for any $\mathbf{y} \in \mathcal{D}$, $\sum_{\mathbf{x} \in U(\succ, \mathbf{y})} p_{\mathbf{x}} \geq \sum_{\mathbf{x} \in U(\succ, \mathbf{y})} q_{\mathbf{x}}$, where $U(\succ, \mathbf{y}) = \{\mathbf{x} \in \mathcal{D} | \mathbf{x} \succ \mathbf{y}\} \cup \{\mathbf{y}\}$.

We discuss the following desirable properties for assignments in this paper, and we say the mechanism f satisfies a property X if $f(R)$ satisfies X for any $R \in \mathcal{R}$.

sd-efficiency. Given a preference profile R, a fractional assignment P is sd-efficient if there is no other fractional assignment $Q \neq P$ such that $Q \succeq_j^{sd} P$ for any $j \leq n$.

sd-envy-freeness. Given a preference profile R, a fractional assignment P is sd-envy-free if $P_j \succeq_j^{sd} P_k$ for any two agents $j, k \leq n$.

sd-weak-strategyproofness. Given a preference profile R, a mechanism f satisfies sd-weak-strategyproofness if for any profile $R \in \mathcal{R}$ and agent $j \leq n$, it holds that $f(R') \succeq_j^{sd} f(R) \implies f(R')_j = f(R)_j$. for any $R' \in \mathcal{R}$ where $R' = (\succ'_j, \succ_{-j})$ and $\succ_{-j}$ denotes the preferences of agents in the set $N \setminus \{j\}$.

3 MAIN RESULTS

In order for a mechanism to deal with indivisible items, it must always output a decomposable assignment. However, Theorem 3.1 shows that no such mechanism can guarantee both efficiency (sd-efficiency) and fairness (sd-envy-freeness) simultaneously.

THEOREM 3.1. *No mechanism that satisfies sd-efficiency and sd-envy-freeness always outputs decomposable assignments for MTRAs.*

Faced with this impossibility result, a natural question to ask is whether this impossibility can be circumvented under a reasonable restriction. The domain of lexicographic preferences provides one such avenue, as we show: The output of our LexiPS mechanism designed for MTRAs with lexicographic preferences is always decomposable, and LexiPS retains the desirable properties of PS. Besides, LexiPS is sd-weak-strategyproof when agents may misreport the ranking of items in each type, but cannot misreport their importance orders.

Lexicographic preference. A strict preference $\succ$ over $\mathcal{D} = D_1 \times \cdots \times D_d$ is *lexicographic* if there exist (i) an *importance order*, i.e. a strict linear order $\rhd$ over types $\{1, \dots, d\}$ and (ii) for each type $i \leq d$, a strict linear order $\succ^i$ over D_i such that for any two bundles $\mathbf{x}, \mathbf{y} \in \mathcal{D}$, $\mathbf{x} \succ \mathbf{y}$ if there exists a type i satisfying $D_i(\mathbf{x}) \succ^i D_i(\mathbf{y})$ and $D_h(\mathbf{x}) = D_h(\mathbf{y})$ for any $h \rhd i$.

LexiPS. The algorithm of LexiPS consists of two parts, consuming single items and computing assignments over bundles. The consumption runs in d phases. In each phase, each agent identifies current most important type and only consumes items of that type. The time for each phase is one unit. At the beginning of each phase, every agent decides her most preferred unexhausted item and then consumes the item at a uniform rate of one unit per unit of time. The consumption pauses whenever one of the items being consumed becomes exhausted, and it continues after all the agent have decided their current most preferred unexhausted items. After consumption, we obtain the assignment over items for each type. We view an agent's share of an item as the probability that she is assigned that item in the final output, which does not depend on what she is assigned in other types. In this way, we can compute the assignment over bundles as the final output of LexiPS.

THEOREM 3.2. *For MTRAs with lexicographic preferences, LexiPS satisfies sd-efficiency and sd-envy-freeness. Especially, LexiPS outputs decomposable assignments.*

THEOREM 3.3. *For MTRAs with lexicographic preferences, LexiPS satisfies sd-weak-strategyproofness when agents report importance orders truthfully.*

The MPS mechanism [25] is not guaranteed to output a decomposable assignment for MTRAs even under lexicographic preferences. However, MPS can deal with the unrestricted domain of strict linear preferences unlike LexiPS, and is still a useful mechanism for divisible items. In fact, we show that MPS satisfies lexi-efficiency, a stronger efficiency guarantee than sd-efficiency than LexiPS, while retaining sd-envy-freeness. Under lexicographic preferences, MPS is sd-weak-strategyproof even when agents are allowed to misreport their importance orders. We also present the family of eating algorithms, of which MPS is a member, and show that it characterizes the set of all lexi-efficient assignments.

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