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L-Convexity and Lattice-Valued Capacities*

Oleh Nykyforchyn
Department of Mathematics and Computer Science
Vasyl' Stefanyk Precarpathian National University,
Shevchenka 57, Ivano-Frankivsk, Ukraine, 76025
oleh.nyk@gmail.com

Dušan Repovš
Faculty of Education, and Faculty of Mathematics and Physics
University of Ljubljana, P.O. Box 2964, Ljubljana, Slovenia, 1001
dusan.repovs@guest.arnes.si

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L-idempotent analogues of convexity are introduced (L is a completely distributive lattice). It is proved that the category of algebras for the monad of L-valued capacities (regular plausibility measures) in the category of compacta is isomorphic to the category of L-idempotent biconvex compacta and their biaffine maps. For the functor of L-valued $\cup$-capacities (L-possibility measures) a family of monads parameterized by monoidal operations $* : L \times L \to L$ is introduced and it is shown that the category of algebras for each of these monads is isomorphic to the category of $(L, \oplus, *)$-convex compacta and their affine maps.

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1. "Conventional" convexity and its closest generalizations

Recall that a set A in a vector space V is called *convex* if it contains all convex combinations of its elements, i.e., $\alpha_1 x_1 + \cdots + \alpha_k x_k \in A$ whenever $k \in \mathbb{N}$, $\alpha_i \in [0, 1]$ and $x_i \in A$ for all $i = 1, \ldots, k$, and $\alpha_1 + \cdots + \alpha_k = 1$. Most important convex sets belong to the following two classes: convex subsets of $\mathbb{R}^n$ and convex compacta, i.e., compact convex subsets of locally convex topological vector spaces. There is no need to emphasize again an exclusive role that convex sets play in optimization, functional analysis, and elsewhere. Therefore it is natural that efforts have been made to find reasonable generalizations of convexity such that analogues of classical results of convex analysis for them are valid.

In particular, Briec and Horvath [3] considered "deformations" of the usual con-

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vex structure on $\mathbb{R}^n_+$ via suitable isotonic bijections, which are applied "forth and back", both to the coordinates of points and to scalars, and they have shown that the limit case of such deformations is the so-called $\mathbb{B}$-convexity, which proved to be useful, e.g., in mathematical economics. Namely, a subset $A \subset \mathbb{R}^n_+$ is said to be $\mathbb{B}$-convex, if for all $k \in \mathbb{N}$, $x_1, \dots, x_k \in A$, and $\alpha_1, \dots, \alpha_k \in [0,1]$ such that $\max\{\alpha_1, \dots, \alpha_k\} = 1$, the point $\alpha_1 x_1 \vee \dots \vee \alpha_k x_k$ is in A (here $\vee$ denotes the coordinatewise maximum). They studied the properties of such sets and obtained analogues of Carathéodory's and Helly's Theorems, proved the connectedness and contractibility, and suggested applications to games and optimization.

On the other hand, Kolokoltsov and Maslov [12] observed that the replacement of the usual operations $+$ and $\cdot$ on the field of reals with "strange" operations $\oplus$ and $\odot$ leads to valuable and interesting analogues. The main such pair is $\oplus = \max$, $\odot = +$, hence the *max-plus* mathematics was obtained. The inventors have shown that it is also a limit case of "distortions" of the conventional analysis (Maslov's dequantization). Another popular version is the *max-min* one: $\oplus = \max$, $\odot = \min$. Observe that such additions are idempotent, i.e., $x \oplus x \equiv x$, hence the entire theory is called *idempotent mathematics*. Of course, the obtained "idempotent analysis" is quite reduced. Nevertheless, it often parallels the usual one. In particular, a subset $A \subset \mathbb{R}^n$ is max-plus convex if for all $k \in \mathbb{N}$, $x_1, \dots, x_k \in A$, and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that $\max\{\alpha_1, \dots, \alpha_k\} = 0$, the point $\alpha_1 \odot x_1 \oplus \ldots \oplus \alpha_k \odot x_k$ is in A (all operations $\oplus = \max$ and $\odot = +$ are performed coordinatewise). Properties are very similar to the ones of convex sets, e.g., Zarichnyi [27] proved that for compact max-plus sets an analogue of Michael's selection theorem holds. Similarly, max-min convex sets were introduced. Geometry of such sets, including max-min convex hulls, was described in [21, 22]. Theorem 23 [9] provides separation of the closed convex sets, etc.

It is straightforward to show that $\mathbb{B}$-convex sets and max-plus sets are the same objects. A correspondence between them is established via the logarithmic function $\ln : \mathbb{R}_+ \to \mathbb{R}$. Such a redundancy, on the one hand, exposes lack of interconnection among researchers in different fields of mathematics; on the other hand, it proves that the suggested notions are necessary and important.

Let us show how such an idempotent convexity is introduced. The field of reals is cut in one place and/or augmented in another, in order to fit "distorted" operations. E.g., for max-plus convexity, $-\infty$ is added to $\mathbb{R}$, and the space $\mathbb{R}^n$ is replaced with $(\mathbb{R} \cup \{-\infty\})^n$. Coefficients are taken from $[-\infty, 0]$, which with the operations $\oplus = \max$ and $\odot = +$ is not a field, but a semiring, with the unit 0. Respectively, the considered space is not a vector space, but an idempotent $([-\infty, 0], \max, +)$-semimodule. A "linear" combination of points is convex if the sum (i.e., the maximum) of its coefficients is equal to the top element of the semiring. Such an approach can be used for any idempotent semiring, e.g., for the semiring $([0,1], \max, \cdot)$, which is isomorphic to $([-\infty, 0], \max, +)$. Hence, having the operations properly defined, we can show that $([0,1], \max, \cdot)$-semimodules and their $([0,1], \max, \cdot)$-convex subsets are essentially the same that $([0,1], \max, \cdot)$-semimodules and $([0,1], \max, \cdot)$-convex (i.e., $\mathbb{B}$-convex) subsets.

Take the least nontrivial idempotent semimodule $\mathbf{2} = \{0,1\}$, with $\oplus = \max$, $\odot = \cdot = \min$. Then a set $(V, \bar{\oplus}, \bar{\odot})$ is a $\mathbf{2}$-semimodule if $\bar{\oplus}$ is commutative and associative, $x \,\bar{\oplus}\, x \equiv x$, $1 \,\bar{\odot}\, x \equiv x$, and $0 \,\bar{\odot}\, x = \bar{0}$ is the unique element such that $\bar{0} \,\bar{\oplus}\, y = y$ for all $y \in V$. By putting $x \leqslant y \iff x \,\bar{\oplus}\, y = y$, we make V an upper semilattice with the bottom element $\bar{0}$. A $\mathbf{2}$-convex subset of V is then simply a subsemilattice.

These examples show that the notion of idempotent convexity should not be reduced only to max-plus, max-min, or $\mathbb{B}$-cases, but investigated in a more general setting. Moreover, it may not be restricted only to subsets of powers of $\mathbb{R}$ or "almost $\mathbb{R}$". The goal of the present paper is also to apply more topology and order theory than in [3], where a more algebraic and analytic approach was used.

2. Idempotent semirings and idempotent semimodules. Linear and affine mappings

Let us return to $\mathbb{B}$-convexity and to the corresponding space $\mathbb{R}^n_+$. It is a $([0,1], \max, \cdot)$-semimodule with $\bar{\oplus}$ and $\bar{\odot}$ being respectively the cordinatewise maximum and the coordinatewise multiplication. Elements $a, b \in \mathbb{B}^n_+$ cannot always be "strictly" compared to see which one is "better", but the rate of preference of a to b can be calculated. For a factor $\lambda \in [0,1]$, we can say that a is preferred to b to a degree at least λ (written $a \succ_\lambda b$) if $a \,\bar{\oplus} (\lambda \,\bar{\odot}\, b) = a$. For example, $a \succ_{\frac{1}{2}} b$ if each coordinate of a is greater than or equal to the half of the corresponding coordinate of b. Such "graded" preference relation is transitive in the sense that, if $a \succ_\lambda b$, $b \succ_\mu c$, then $a \succ_{\lambda\cdot\mu} c$. The element $1 \,\bar{\odot}\, a \,\bar{\oplus}\, \frac{1}{2} \,\bar{\odot}\, b \,\bar{\oplus}\, \frac{1}{5} \,\bar{\odot}\, c$ is the least one that is preferred to a, preferred to b to a degree at least $\frac{1}{2}$, and to c to a degree at least $\frac{1}{5}$. Thus a subset $A \subset \mathbb{R}^n_+$ is $\mathbb{B}$-convex if we can "mix" (partially improve) its elements in the above sense.

Observe that this comparison is sensitive to the "weakest links", i.e., one small coordinate of a makes the result of comparison with b small, irrespective of the other coordinates. Assume that we are interested in different coordinates to a different extent. It would be more interesting to multiply coordinates by *different* factors, i.e., to consider $\mathbb{R}^n_+$ as a semimodule over the semiring $([0,1]^n, \oplus, \odot)$, with $\oplus$ the supremum (i.e., the cordinatewise maximum), and $\odot$ the coordinatewise multiplication. Then $\bar{\oplus}$ is the same, but the multiplication is $(\alpha_1, \ldots, \alpha_n) \,\bar{\odot} (x_1, \ldots, x_n) = (\alpha_1 x_1, \ldots, \alpha_n x_n)$. If $\succ_\alpha$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$ is again defined as $x \succ_\alpha y \iff x \,\bar{\oplus} (\alpha \,\bar{\odot}\, y) = x$, then $(x_1, \ldots, x_n) \succ_{(1, \frac{1}{2}, 0, \ldots, 0)} (y_1, \ldots, y_n)$ if and only if $x_1 \geqslant y_1$, $2x_2 \geqslant y_2$, and other coordinates are ignored. It is easy to understand what $(1, \frac{1}{2}, 1, 1, \ldots, 1) \,\bar{\odot}\, x \,\bar{\oplus} (0, 1, \frac{1}{3}, 1, \ldots, 1) \,\bar{\odot}\, y$ is. Thus it makes sense to consider L-semimodules and L-convex sets not only for $L \subset \mathbb{R}$ or, more generally, linearly ordered L, but also when L is a *lattice* with additionally defined multiplication, as $[0,1]^n$ above.

Now we give the necessary formal definitions. In the sequel $(L, \oplus, \otimes)$ will be a lattice with a bottom and a top element 0 and 1, respectively, and a binary operation $* : L \times L \to L$ such that 1 is a two-sided unit and $*$ is distributive w.r.t. $\oplus$ in both variables. Then $(L, \oplus, *)$ is a semiring.

Recall that a (left idempotent) $(L, \oplus, *)$-semimodule [1] is a set X with operations $\bar{\oplus} : X \times X \to X$ and $\bar{*} : L \times X \to X$ such that for all $x, y, z \in X$, $\alpha, \beta \in L$:

(1) $x \,\bar{\oplus}\, y = y \,\bar{\oplus}\, x$;
(2) $(x \,\bar{\oplus}\, y) \,\bar{\oplus}\, z = x \,\bar{\oplus}(y \,\bar{\oplus}\, z)$;
(3) there is an (obviously unique) element $\bar{0} \in X$ such that $x \,\bar{\oplus}\, \bar{0} = x$ for all x;
(4) $\alpha \,\bar{*}(x \,\bar{\oplus}\, y) = (\alpha \,\bar{*}\, x) \,\bar{\oplus}(\alpha \,\bar{*}\, y)$, $(\alpha \oplus \beta) \,\bar{*}\, x = (\alpha \,\bar{*}\, x) \,\bar{\oplus}(\beta \,\bar{*}\, x)$;
(5) $(\alpha * \beta) \,\bar{*}\, x = \alpha \,\bar{*}(\beta \,\bar{*}\, x)$;
(6) $1 \,\bar{*}\, x = x$; and
(7) $0 \,\bar{*}\, x = \bar{0}$.

Observe that these axioms imply that $(X, \bar{\oplus})$ is an upper semilattice with a bottom element $\bar{0}$, the order is defined by $x \leqslant y \iff x \,\bar{\oplus}\, y = y$, and $\alpha \,\bar{*}\, \bar{0} = \bar{0}$ for all $\alpha \in L$. The operation $\bar{*}$ is isotone in both variables.

Since an $(L, \oplus, *)$-semimodule is an analogue of a vector space, for all $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_n \in L$, it is natural to call the expression $\alpha_1 \,\bar{*}\, x_1 \,\bar{\oplus} \dots \bar{\oplus}\, \alpha_n \,\bar{*}\, x_n$ a *linear combination* of the elements x_i with the coefficients α_i. If $\alpha_1 \oplus \dots \oplus \alpha_n = 1$, then the latter combination is called *convex*.

Obviously, a subset of an L-semimodule that closed under linear combinations is an L-semimodule itself, i.e. a *subsemimodule* of the previous one.

Analogues also exist for linear and affine mappings. A mapping $f : X \to Y$ between $(L, \oplus, *)$-semimodules is called *linear* if it preserves the linear combinations, i.e., for all $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_n \in L$, the equality

$$f(\alpha_1 \,\bar{*}\, x_1 \,\bar{\oplus} \dots \bar{\oplus}\, \alpha_n \,\bar{*}\, x_n) = \alpha_1 \,\bar{*}\, f(x_1) \,\bar{\oplus} \dots \bar{\oplus}\, \alpha_n \,\bar{*}\, f(x_n)$$

is valid. If the latter equality is known only to hold for convex combinations, i.e. whenever $\alpha_1 \oplus \dots \oplus \alpha_n = 1$, then f is called *affine*. Observe that an affine mapping f preserves joins, i.e. $f(x_1 \,\bar{\oplus}\, x_2) = f(x_1) \,\bar{\oplus}\, f(x_2)$ for all $x_1, x_2 \in X$, therefore it is isotone. An affine mapping is linear if and only if it preserves the least element.

Recall that the most famous idempotent semirings are *the max-plus semiring* $(\mathbb{R} \cup \{-\infty\}, \max, +)$ and *the max-min semiring* $(\mathbb{R} \cup \{-\infty, +\infty\}, \max, \min)$. They, especially the first one, form a basis of *tropical* or *idempotent mathematics* [12]. The max-plus semiring does not fit into our scheme because the neutral element for its "multiplication" $+$ is not a top element. However, there is a "good" subsemiring $([-\infty, 0], \max, +)$, which is isomorphic to the semiring $(I, \max, \cdot)$. The latter semiring has an advantage: in it 0 and 1 have their usual order-theoretical meaning. Similarly the max-min semiring is isomorphic to $(I, \max, \min)$. From now on, for the sake of brevity the $(I, \max, \cdot)$-semimodules and the $(I, \max, \min)$-semimodules will be referred to respectively as $(\max, \cdot)$-*semimodules* and $(\max, \min)$-*semimodules*, the same applies also to combinations, affine and linear mappings, etc.

3. L-convex sets

A subset of an L-semimodule that is closed under *convex* combinations is, as usual, called *convex*, or more precisely, *L-convex*.

It is very convenient that we can calculate the usual convex combinations and (due to the linear order on I) the $(\max, \cdot)$-idempotent or the $(\max, \min)$-idempotent convex combinations of a finite number of points "step by step", i.e. by using only pairwise combinations. This is not the case for the convex combinations with lattice-valued coefficients, thus we should simultaneously define the convex combinations of arbitrary finite numbers of points. Now we define sets that contain admissible collections of coefficients.

The n-dimensional[1] L-simplex is the set

$$\Delta^n_\oplus = \{(\alpha_0, \alpha_1, \dots, \alpha_n) \in L^{n+1} \mid \sup\{\alpha_0, \alpha_1, \dots, \alpha_n\} = 1\}.$$

We say that an *L-idempotent convex combination* is given on a set X if for all $n \in \{0, 1, 2, \dots\}$, $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \Delta^n_\oplus$, $x_0, x_1, \dots, x_n \in X$ an element

$$ic(x_0, x_1, \dots, x_n, \alpha_0, \alpha_1, \dots, \alpha_n) \in X,$$

which we denote by $(\alpha_0 \bar{*} x_0) \bar{\oplus} (\alpha_1 \bar{*} x_1) \bar{\oplus} \dots \bar{\oplus} (\alpha_n \bar{*} x_n)$ or simply $\alpha_0 x_0 \bar{\oplus} \alpha_1 x_1 \bar{\oplus} \dots \bar{\oplus} \alpha_n x_n$, is uniquely determined, and the following properties are valid:

(1) $ic(x, 1) = x$ for all $x \in X$;

(2) for all $m, n \in \{0, 1, 2, \dots\}$, $x_0, x_1, \dots, x_n \in X$, $(\alpha_0, \alpha_1, \dots, \alpha_m) \in \Delta^m_\oplus$ and a mapping $\sigma : \{0, 1, \dots m\} \to \{0, 1, \dots, n\}$ we have

$$\alpha_0 x_{\sigma(0)} \bar{\oplus} \alpha_1 x_{\sigma(1)} \bar{\oplus} \dots \bar{\oplus} \alpha_m x_{\sigma(m)} = \beta_0 x_0 \bar{\oplus} \beta_1 x_1 \bar{\oplus} \dots \bar{\oplus} \beta_n x_n,$$

where $\beta_k = \sup\{\alpha_i \mid 0 \leqslant i \leqslant m, \sigma(i) = k\}$ for $k = 0, 1, \dots, n$. This equality means that we can exchange summands, drop summands with zero coefficients and join summands with the same second factor; and

(3) the "big associative law" holds:

$$\begin{aligned}
&\alpha_0(\beta^0_0 x^0_0 \bar{\oplus} \dots \bar{\oplus} \beta^0_{k_0} x^0_{k_0}) \bar{\oplus} \alpha_1(\beta^1_0 x^1_0 \bar{\oplus} \dots \bar{\oplus} \beta^1_{k_1} x^1_{k_1}) \bar{\oplus} \\
&\qquad \dots \bar{\oplus} \alpha_n(\beta^n_0 x^n_0 \bar{\oplus} \dots \bar{\oplus} \beta^n_{k_n} x^n_{k_n}) \\
&= (\alpha_0 \beta^0_0) x^0_0 \bar{\oplus} \dots \bar{\oplus} (\alpha_0 * \beta^0_{k_0}) x^0_{k_0} \bar{\oplus} (\alpha_1 \beta^1_0) x^1_0 \bar{\oplus} \dots \bar{\oplus} (\alpha_1 * \beta^1_{k_1}) x^1_{k_1} \bar{\oplus} \\
&\qquad \dots \bar{\oplus} (\alpha_n * \beta^n_0) x^n_0 \bar{\oplus} \dots \bar{\oplus} (\alpha_n * \beta^n_{k_n}) x^n_{k_n},
\end{aligned}$$

where $x^i_j \in X$, $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \Delta^n_\oplus$, $(\beta^i_0, \beta^i_0, \dots, \beta^i_{k_i}) \in \Delta^{k_i}_\oplus$ for $i = 0, 1, \dots, n$.

In fact, an L-convex combination ic in X is a collection of maps $ic_n : X^{n+1} \times \Delta^n_\oplus \to X$, $n = 0, 1, 2, \dots$, but we will use a common notation ic for all of them.

These properties are obviously valid if $(\alpha_0 \bar{*} x_0) \bar{\oplus} (\alpha_1 \bar{*} x_1) \bar{\oplus} \dots \bar{\oplus} (\alpha_n \bar{*} x_n)$ on X is calculated via the operations $\bar{*}$ and $\bar{\oplus}$ on an L-semimodule N, in which X is contained as a convex subset. In fact, each L-convex combination can be obtained in this way. The following statement is true [16, Proposition 2.2]:

[1] We do not mean any topological dimension here.

Proposition 3.1. *Let an $(L, \oplus, *)$-convex combination be defined on a set X. Then there is an injective mapping e of X into an $(L, \oplus, *)$-semimodule $(N, \bar{\oplus}, \bar{\odot})$ which preserves $(L, \oplus, *)$-convex combinations.*

Thus a set is called *L-convex* if it is equipped with an L-convex combination, and we use the term "*affine mapping*" also for the mappings between L-convex sets which preserve convex combinations in the above sense. Let ic and ic' be L-convex combinations resp. on X and X'. We say that a map $f : (X, ic) \to (X', ic')$ is affine if it preserves L-convex combination, i.e. $f(ic(x_0, \ldots, x_n, \alpha_0, \ldots, \alpha_n)) = ic'(f(x_0), \ldots, f(x_n), \alpha_0, \ldots, \alpha_n)$ for all $x_0, x_1, \ldots, x_n \in X$, $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_\oplus$. This also justifies the use of the usual notation for the L-convex combinations. Observe that, similarly to L-semimodules, such a set X is an upper semilattice with the operation $x \vee y = 1x \oplus 1y$.

The introduced notion is a natural generalization of $\mathbb{B}$-convexity, which is a special case of L-convexity for $L = [0, 1]$, $\oplus = \vee$, $* = \cdot$, and $\otimes = \wedge$.

In this section we also briefly present a result of [16], which shows that L-convex combinations and L-semimodules are closely related to "weakened" versions of L-fuzzy preference relations, an example of which was considered in Section 2.

A family $\succ = (\succ_\alpha)_{\alpha \in L}$ of binary relations on a set X is called an *L-preference* if the following holds for all $x, y, z \in X$, $\alpha, \beta \in L$:

(1) $x \succ_\alpha y$ and $x \succ_\beta y$ if and only if $x \succ_{\alpha \oplus \beta} y$;
(2) $\succ_1$ is a partial order; and
(3) $\succ_0 = X \times X$.

An advantage of such a definition of "graded preference" of x over y is that α can capture both the aspect in which we compare the options and the rate of preference, cf. the example with the set $X = [0, +\infty)^n$ and the lattice $L = [0, 1]^n$.

The following property of preferences is often considered:

(4') if $x \succ_\alpha y$, $y \succ_\beta z$, then $x \succ_{\alpha * \beta} z$ (*$*$-transitivity*).

Then Proposition 2.3 [16] implies:

Proposition 3.2. (a) *For an injective mapping e from a set X into an $(L, \oplus, *)$-convex set C the family of relations $\succ = (\succ_\alpha)_{\alpha \in L}$ on X, defined as $x \succ_\alpha y \iff 1 \bar{*} e(x) \bar{\oplus} \alpha \bar{*} e(y) = e(x)$ in C for all $x, y \in X$, $\alpha \in L$, is a $*$-transitive L-preference on X.*

(b) *If $\succ = (\succ_\alpha)_{\alpha \in L}$ is a $*$-transitive L-preference on a set X, then there is an injective mapping e from X into an $(L, \oplus, *)$-semimodule $(N, \bar{\oplus}, \bar{*})$ such that $x \succ_\alpha y \iff e(x) \bar{\oplus} (\alpha \bar{*} e(y)) = e(x)$ for all $x, y \in X$, $\alpha \in L$.*

See the latter citation for more information on realization of L-preferences via L-convex combinations.

4. L-biconvex sets

Consider an L-semimodule $(X, \bar\oplus, \bar*)$ such that:

(a) X is a distributive lattice with a bottom element $\bar 0$, a top element $\bar 1$, and a meet denoted $\bar\otimes$.

(b) the multiplication $\bar*$ on X satisfies the equality $(\alpha \bar* \bar 1) \bar\otimes x = \alpha \bar* x$ for all $\alpha \in L$, $x \in X$.

Remark 4.1. The property (b) implies:

$$(\alpha \bar* \bar 1) \bar\otimes (\beta \bar* \bar 1) = \alpha \bar*(\beta \bar* \bar 1) = (\alpha * \beta) \bar* \bar 1 \leqslant (\alpha \otimes \beta) \bar* \bar 1.$$

On the other hand, $(\alpha \otimes \beta) \bar* \bar 1 \leqslant \alpha \bar* \bar 1$, $(\alpha \otimes \beta) \bar* \bar 1 \leqslant \gamma \bar* \bar 1$, hence

$$(\alpha \bar* \bar 1) \bar\otimes (\beta \bar* \bar 1) \geqslant (\alpha \otimes \beta) \bar* \bar 1,$$

consequently

$$(\alpha \bar* \bar 1) \bar\otimes (\beta \bar* \bar 1) = (\alpha * \beta) \bar* \bar 1 = (\alpha \otimes \beta) \bar* \bar 1,$$

therefore the correspondence $\alpha \mapsto \alpha \bar* \bar 1$ preserves not only the pairwise joins, which follows from the definition of semimodule, but also pairwise meets. Observe that such an $(L, \oplus, *)$-semimodule is simultaneously an $(L, \oplus, \otimes)$-semimodule with the same operations, and:

$$\begin{aligned}(\alpha \bar* x) \bar\otimes (\beta \bar* y) &= (\alpha \bar* \bar 1) \bar\otimes x \bar\otimes (\beta \bar* \bar 1) \bar\otimes y \\ &= (\alpha \bar* \bar 1) \bar\otimes (\beta \bar* \bar 1) \bar\otimes (x \bar\otimes y) = (\alpha \otimes \beta) \bar*(x \bar\otimes y).\end{aligned}$$

On the other hand, the equality

$$(\alpha \bar* x) \bar\otimes (\beta \bar* y) = (\alpha \otimes \beta) \bar*(x \bar\otimes y)$$

implies $(\alpha \bar* \bar 1) \bar\otimes (\beta \bar* \bar 1) = (\alpha \otimes \beta) \bar* \bar 1$ and $(\alpha \bar* \bar 1) \bar\otimes x = \alpha \bar* x$ for all values of the variables.

Probably the most important example of an idempotent semiring is an arbitrary distributive lattice $L = (L, \oplus, \otimes)$, where 0 and 1 are resp. the bottom and the top elements, and multiplication coincides with meet. Note that reversing the order on L results in the distributive lattice $\tilde L = (\tilde L, \otimes, \oplus)$, with the bottom and the top elements $\tilde 0 = 1$ and $\tilde 1 = 0$, respectively, which is an idempotent semiring as well.

Proposition 4.2. *If an $(L, \oplus, \otimes)$-semimodule $(X, \bar\oplus, \bar*)$ satisfies conditions* (a) *and* (b)*, then, for the operation $\underline{*} : L \times X \to X$ defined by the formula $\alpha \underline{*} x = (\alpha \bar* \bar 1) \bar\oplus x$ for all $\alpha \in L$ and $x \in X$, the triple $(\tilde X, \bar\otimes, \underline{*})$ is an $(\tilde L, \otimes, \oplus)$-semimodule which also satisfies* (a), (b).

Proof is a simple calculation.

Observe that $\alpha \bar* \bar 1 = \alpha \underline{*} \bar 0$, hence the formula $(\alpha \underline{*} \bar 0) \bar\oplus x = \alpha \underline{*} x$ is valid, which is dual in the obvious sense to the formula $(\alpha \bar* \bar 1) \bar\otimes x = \alpha \bar* x$ (cf. condition (b)).

Thus we obtain two dual structures on X, that of an L-semimodule and that of a $\tilde{L}$-semimodule. In addition to the L-convex combinations, i.e., to the expressions of the form $(\alpha_1 \bar{*} x_1) \bar{\oplus} \ldots \bar{\oplus} (\alpha_n \bar{*} x_n)$, with $\alpha_1 \oplus \ldots \oplus \alpha_n = 1$, we define the $\tilde{L}$-*convex* (or *dually L-convex*) *combinations* of the form $(\alpha_1 \underline{*} x_1) \bar{\otimes} \ldots \bar{\otimes} (\alpha_n \underline{*} x_n)$, with $\alpha_1 \otimes \ldots \otimes \alpha_n = 0$.

Therefore we call such $(X, \bar{\oplus}, \bar{\otimes}, \bar{*}, \underline{*})$ an *L-biconvex set*. It is easy to see that the mapping $p : L \to X$ that takes each α to the product $\alpha \bar{*} \bar{1}$ (or, equivalently, to $\alpha \underline{*} \bar{0}$), is a lattice morphism which preserves the bottom and the top elements. Conversely, each lattice morphism $p : (L, \oplus, \otimes) \to (X, \bar{\oplus}, \bar{\otimes})$ that preserves the bottom and the top elements determines the L-biconvex set $(X, \bar{\oplus}, \bar{\otimes}, \bar{*}, \underline{*})$ by the formulae $\alpha \bar{*} x = p(\alpha) \bar{\otimes} x$, $\alpha \underline{*} x = p(\alpha) \bar{\oplus} x$ for all $\alpha \in L$, $x \in X$. Thus there is a one-to-one correspondence between the L-biconvex sets and top- and bottom-preserving lattice morphisms from L. From now on we shall assume that the corresponding lattice morphism p is fixed for each L-biconvex set $(X, \bar{\otimes}, \bar{\oplus}, \bar{*}, \underline{*})$.

Proposition 4.3. *Let a subset Y of an L-biconvex set $(X, \bar{\otimes}, \bar{\oplus}, \bar{*}, \underline{*})$ be closed w.r.t. the L-convex and the dual L-convex combinations, contain a bottom element $\bar{0}'$ and a top element $\bar{1}'$. Then $(Y, \bar{\oplus}', \bar{\otimes}', \bar{*}', \underline{*}')$ is an L-biconvex set as well, if $\bar{\oplus}'$ and $\bar{\otimes}'$ are the restrictions of $\bar{\oplus}$ and $\bar{\otimes}$ to Y and $\alpha \bar{*}' x = \bar{0}' \bar{\oplus} (\alpha \bar{*} x)$, $\alpha \underline{*}' x = \bar{1}' \bar{\otimes} (\alpha \underline{*} x)$ for all $\alpha \in L$, $x \in L$.*

In particular, such Y is a sublattice X, and the structure of the L-biconvex set $(Y, \bar{\oplus}', \bar{\otimes}', \bar{*}', \underline{*}')$ is determined by the lattice morphism $p' : L \to Y$, $p'(\alpha) = (p(\alpha) \bar{\oplus} \bar{0}') \bar{\otimes} \bar{1}'$ for all $\alpha \in L$. A simple example of such Y is a *closed interval* $[a, b] = \{x \in X \mid a \leqslant x \leqslant y\}$, for $a, b \in X$, $a \leqslant b$.

Although L-biconvex sets seem to be almost trivial, they have a decision making interpretation and will appear in our exposition in the sequel. We say that a map $f : (X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus) \to (X', \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ between L-biconvex sets is *biaffine* if it satisfies the equality

$$
\begin{aligned}
& f((\alpha_0 \otimes x_0) \bar{\oplus} (\alpha_1 \otimes x_1) \bar{\oplus} \ldots \bar{\oplus} (\alpha_n \otimes x_n)) \\
&= (\alpha_0 \otimes f(x_0)) \bar{\oplus} (\alpha_1 \otimes f(x_1)) \bar{\oplus} \ldots \bar{\oplus} (\alpha_n \otimes f(x_n))
\end{aligned}
$$

whenever $x_0, x_1, \ldots, x_n \in X$, $\alpha_0, \alpha_1, \ldots, \alpha_n \in L$, $\alpha_0 \oplus \alpha_1 \oplus \ldots \oplus \alpha_n = 1$, and the equality

$$
\begin{aligned}
& f((\beta_0 \oplus x_0) \bar{\otimes} (\beta_1 \oplus x_1) \bar{\otimes} \ldots \bar{\otimes} (\beta_n \oplus x_n)) \\
&= (\beta_0 \oplus f(x_0)) \bar{\otimes} (\beta_1 \oplus f(x_1)) \bar{\otimes} \ldots \bar{\otimes} (\beta_n \oplus f(x_n))
\end{aligned}
$$

whenever $x_0, x_1, \ldots, x_n \in X$, $\beta_0, \beta_1, \ldots, \beta_n \in L$, $\beta_0 \otimes \beta_1 \otimes \ldots \otimes \beta_n = 0$.

If L-biconvex structures on X and X' are determined with bottom- and top-preserving lattice morphisms $p : L \to X$ and $p' : L \to X'$, respectively, then a mapping $f : X \to X'$ is biaffine if and only if f is a lattice morphism and $f \circ p = p'$.

5. L-convex compacta and L-biconvex compacta

Having defined idempotent analogues of convex sets in vector spaces, we are going to topologize them to obtain objects similar to the compact closed sets in locally convex topological vector spaces. We do not need to invoke additional tools like local bases or (semi-)norms because the operations on L-semimodules and L-convex sets naturally determine partial orders. Under certain assumptions, these orders lead to the required topologies.

Firstly, we recall some definitions and notation from the domain theory. All "triple-numbered" statements below refer to [7], which is an excellent reference book. For a partial order $\leqslant$ on a set X, the relation $\tilde{\leqslant}$, defined as $x \tilde{\leqslant} y \iff y \leqslant x$, for $x, y \in X$, is a partial order called *opposite* to $\leqslant$, and $(X, \leqslant)^{op}$ denotes the poset $(X, \tilde{\leqslant})$. If the original order $\leqslant$ is obvious, we write simply X^{op} for the *reversed* poset. We also apply $\tilde{(\ )}$ to all notation to denote passing to the opposite order, i.e. write $\tilde{X} = X^{op}$, $\tilde{\sup} = \inf$, $\tilde{0} = 1$ etc. This was already done for the lattice L in the description of the L-biconvex sets.

For a subset A of a poset $(X, \leqslant)$, we denote

$$A\uparrow = \{x \in X \mid a \leqslant x \text{ for some } a \in A\}, \quad A\downarrow = \{x \in X \mid x \leqslant a \text{ for some } a \in A\}.$$

If $A = A\uparrow$ ($A = A\downarrow$), then a set A is called *upper* (resp. *lower*).

A *topological meet* (or *join*) *semilattice* is a semilattice L carrying a topology such that the mapping $\wedge : L \times L \to L$ (resp. $\vee : L \times L \to L$) is continuous. A lattice L with a topology such that both $\wedge : L \times L \to L$ and $\vee : L \times L \to L$ are continuous is called a *topological lattice.*

A set A in a poset $(X, \leqslant)$ is *directed* (*filtered*) if, for all $x, y \in A$, there is $z \in A$ such that $x \leqslant z$, $y \leqslant z$ (resp. $z \leqslant x$, $z \leqslant y$). A poset is called *directed complete* (*dcpo* for short) if it has lowest upper bounds for all its directed subsets.

Fix a partial order $\leqslant$ on a set X. The *Scott topology* $\sigma(X)$ consists of all those $U \subseteq X$ that satisfy $x \in U \ \Leftrightarrow\ U \cap D \neq \varnothing$ for every $\leqslant$ −directed $D \subseteq X$ with a least upper bound x. Note that "$\Leftarrow$" above implies $U = U\uparrow$.

A mapping f between dcpos X and Y is *Scott continuous*, i.e. continuous w.r.t. $\sigma(X)$ and $\sigma(Y)$, if and only if it preserves suprema of directed sets (cf. Proposition II.2-1).

The *lower topology* $\omega(X)$ on a poset $(X, \leqslant)$ is the least topology such that all sets of the form $\{x\}\downarrow$ are closed. The join of (i.e. the least topology that contains) $\sigma(X)$ and $\omega(X)$ is called the *Lawson topology* on X and denoted by $\lambda(X)$. The space $(X, \lambda(X))$ is denoted by ΛX.

In a dcpo X, a lower set is Lawson closed if and only if it is Scott closed, which is equivalent to the closedness under suprema of directed subsets.

Let L be a poset. We say that x is *way below* y and write $x \ll y$ if and only if, for all directed subsets $D \subseteq L$ such that $\sup D$ exists, the relation $y \leq \sup D$ implies the existence of $d \in D$ such that $x \leq d$. The "way-below" relation is transitive and antisymmetric. An element satisfying $x \ll x$ is said to be *compact* or *isolated from below* and in this case the set $\{x\}\uparrow$ is Scott open (hence Lawson open).

A poset L is called *continuous* if each element $y \in L$ is a least upper bound of a directed set of all $x \in L$ such that $x \ll y$. A *domain* is a continuous dcpo. If domain is a semilattice (a complete lattice) it is called a *continuous semilattice* (resp. a continuous lattice). Obviously a continuous lattice with a bottom element is a complete lattice.

By Theorem III.1-9 the Lawson topology on a complete semilattice L is a compact T_0-topology. Theorem III.1-10 asserts that for a domain the Lawson topology is Hausdorff. Hence the Lawson topology on a complete continuous semilattice S is compact Hausdorff, and by Theorems II.1-14, III-2.28 the mapping $\wedge : \Lambda S \times \Lambda S \to \Lambda S$ is continuous, i.e. $(S, \lambda(S))$ is a topological semilattice.

Theorem II.1-14 and Proposition III.2-6 imply that, for a dcpo S and a domain L, the topologies $\lambda(S \times L)$ and $\lambda(S) \times \lambda(L)$ on $S \times L$ are equal.

A topological semilattice is called a *Lawson semilattice* or said to have *small subsemilattices* if, at each point, it possesses a local base consisting of subsemilattices. A topological lattice L is called a *Lawson lattice* if, in each point, it has a local base consisting of sublattices, or, equivalently, if L and L^{op} are Lawson semilattices.

By the Fundamental Theorem on Compact Semilattices (Theorem VI.3-4), each complete continuous semilattice with the Lawson topology is a compact Hausdorff Lawson semilattice, and each compact Hausdorff Lawson semilattice is a complete continuous semilattice such that the given topology agrees with the Lawson topology.

Similarly, by Proposition VII.2-10, a complete lattice L admits a compact Hausdorff topology making it a Lawson lattice if and only if both L and L^{op} are continuous semilattices and the Lawson topologies on L and L^{op} agree (and provide a unique such topology). Such a lattice L is called *linked bicontinuous*, and the Lawson topology on L coincides with the lower topology on L^{op}, and vice versa, hence the topology in question on L is the *interval topology*, i.e. the join of the lower topologies on L and L^{op}.

A complete lattice X is called *completely distributive* if, for each collection $\{A_i \mid i \in \mathcal{I}\}$ of its subsets, either of the equivalent equalities

$$\inf\Big\{\sup_{i\in\mathcal{I}} a_i \;\Big|\; (a_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} A_i\Big\} = \sup_{i\in\mathcal{I}} \inf A_i$$

and

$$\sup\Big\{\inf_{i\in\mathcal{I}} a_i \;\Big|\; (a_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} A_i\Big\} = \inf_{i\in\mathcal{I}} \sup A_i$$

is valid. This property implies linked bicontinuity and the infinite distributivity of join w.r.t. infimum and meet w.r.t. supremum in each variable.

The Scott topology, the lower topology, and the Lawson topology on a poset X^{op} are called the *dual Scott topology*, the *upper topology*, and the *dual Lawson topology* on X, respectively. If the set X^{op} is a continuous (semi-)lattice, then X is called a *dually continuous (semi-)lattice*.

Now we are ready to define an appropriate category for L-convex sets with the infinite L-convex combinations (cf. [2, 14] for the definition of category and related notions).

From now on $(L, \oplus, \otimes)$ will be a completely distributive lattice, and $* : L \times L \to L$ will have 1 as its two-side unit and will be distributive w.r.t. all suprema and filtered infima. Therefore $*$ is continuous w.r.t. the Scott (= the upper), the dual Scott (= the lower), and the Lawson topologies on L. We call such $(L, \oplus, *)$ a *completely distributive quantale*. From now on, L will be used as a shorthand for $(L, \oplus, *)$ wherever this does not lead to a confusion.

Recall that an $(L, \oplus, *)$-semimodule $(K, \bar{\oplus}, \bar{*})$ is called *a dually continuous semimodule* [18] if K^{op} is a continuous lattice, and $\bar{*} : L \times K \to K$ distributes in each variable w.r.t. the finite suprema and the filtered infima. If K is complete and $\bar{*}$ is distributive w.r.t. all suprema, then K with the dual Lawson topology is a compact Hausdorff Lawson topological semilattice, which means that each point of K possesses a local basis which consists of subsemilattices. This also implies that $\bar{*}$ is continuous w.r.t. the Lawson topologies on L and K^{op}. Hence such a semimodule is called *a compact Hausdorff Lawson* $(L, \oplus, *)$*-semimodule*.

Let $* : L \times L \to L$ be Lawson continuous. We call an L-convex set X an *L-convex compactum* if X^{op} is a complete continuous lower semilattice, which implies compactness in the Lawson topology on X^{op}, and all finite L-convex combinations of the form $(\alpha_0 \bar{*} x_0) \bar{\oplus} (\alpha_1 \bar{*} x_1) \bar{\oplus} \ldots \bar{\oplus} (\alpha_n \bar{*} x_n)$ are jointly continuous.

Of course, this is the case if X is a closed under dual Lawson topology L-convex subset of a dually continuous L-semimodule. This characterization is exhaustive:

Theorem 5.1. *Let $(L, \oplus, *)$ be a completely distributive quantale. A pair of a compactum X and a collection ic of continuous mappings $ic_n : X^{n+1} \times \Delta^n_{\oplus} \to X$, $n = 0, 1, 2, \ldots$, is an L-convex compactum if and only if X is a closed convex subset of a compact Hausdorff Lawson L-semimodule $(N, \bar{\oplus}, \bar{*})$ such that $ic_n(x_0, \ldots, x_n, \alpha_0, \ldots, \alpha_n) \equiv \underbrace{\alpha_0 \bar{*} x_0 \bar{\oplus} \ldots \bar{\oplus} \alpha_n \bar{*} x_n}_{in\ N}$ whenever $n \in \{0, 1, 2 \ldots\}$, $x_0, x_1, \ldots, x_n \in X$, $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_{\oplus}$.*

Thus, similarly to the usual convex sets and convex compacta, L-convex sets and L-convex compacta have equivalent external (via embeddings into vector or "vector-like" spaces) and intrinsic (via operations of convex combination) characterization, and we can use whatever is more convenient for a particular purpose.

Therefore we can correctly define an L-convex combination of an infinite number of points using only finite L-convex combinations. If $\{(x_i, \alpha_i) \mid i \in \mathcal{I}\} \subset X \times L$ for an $(L, \oplus, *)$-convex compactum X that is embedded into a compact Hausdorff

Lawson $(L,\oplus,*)$-semimodule $(N,\bar\oplus,\bar *)$, and $\sup\limits_{i\in\mathcal{I}}\alpha_i=1$, then

$$\underbrace{\sup\{\alpha_i\,\bar*\,x_i\mid i\in\mathcal{I}\}}_{\text{in } N}$$
$$=\inf\big\{\underbrace{\sup_{i\in\mathcal{I}_1}\alpha_i\,\bar*\sup_{i\in\mathcal{I}_1}x_i\,\bar\oplus\ldots\bar\oplus\sup_{i\in\mathcal{I}_k}\alpha_i\,\bar*\sup_{i\in\mathcal{I}_k}x_i}_{\text{in } X}\;\big|\;k\in\mathbb{N},\mathcal{I}=\mathcal{I}_1\cup\dots\cup\mathcal{I}_k\big\}.$$

The right side is the infimum of a filtered set, therefore it is preserved by a dually Lawson continuous embedding of X into a compact Hausdorff Lawson $(L,\oplus,*)$-semimodule N. This implies that the left side does not depend on the embedding of X into N. We call this expression *the L-convex combination* of the points x_i with the coefficients α_i and denote it $\bar\bigoplus_{i\in\mathcal{I}}\alpha_i\,\bar*\,x_i$.

We denote by $(L,\oplus,*)$-$\mathcal{C}onv$ the category that consists of all L-convex compacta and of all dually Lawson continuous affine mappings. We call it the *category of L-convex compacta*. Clearly, both the finite and the infinite L-convex combinations are preserved by the arrows in $(L,\oplus,*)$-$\mathcal{C}onv$.

To get some feeling what the L-convex compacta are, let us consider the simplest case $L=\mathbf{2}=\{0,1\}$, with the unique appropriate multiplication $\bar*:\mathbf{2}\times X\to X$, namely $1\,\bar*\,x\equiv x$, $0\,\bar*\,x\equiv\bar 0$. Hence each linear combination is either trivial (with only zero coefficients) or affine, which in turn is a finite supremum. Thus, the $\mathbf{2}$-convex compacta are precisely the complete dually continuous semilattices, i.e., the compact Hausdorff Lawson upper semilattices (with or without bottom elements), and the affine mappings are simply the join-preserving ones.

Note that even $(\max,\cdot)$-convex or $(\max,\min)$-convex sets, which are the closest to the usual notion of convexity, may visually appear strange (see [21, 22]). Nevertheless, as it was mentioned earlier, the compact max-plus convex sets in $\mathbb{R}^n$, which are equivalent to the $(\max,\cdot)$-convex compacta, have nice topological and geometric properties, similar to the properties of convex compacta.

An $(L,\oplus,\otimes)$-semimodule $(X,\bar\oplus,\bar*)$ is called an L-biconvex compactum if it is an L-biconvex set and a completely distributive lattice such that $\bar*$ is Lawson continuous. It is obviously also an $(L,\otimes,\oplus)$-biconvex compactum with the operations "$\bar\otimes$" (lattice meet) and "$\underline{*}$", with $\alpha\,\underline{*}\,x=x\,\bar\oplus(\alpha\,\bar*\,\bar 1)$ for $\alpha\in L$, $x\in X$. Equivalently, the multiplication "$\bar*$" is determined by a complete top- and bottom-preserving lattice morphism $p:L\to X$ as follows: $\alpha\,\bar*\,x=p(\alpha)\,\bar\otimes\,x$. The *category L-$\mathcal{B}i\mathcal{C}onv$* of L-biconvex compacta consists of all L-biconvex compacta and their Lawson continuous biaffine mappings.

6. Semimodules and convex sets of L-valued capacities

Following [10], for a domain D we call the elements of the set $[D\to L^{op}]^{op}$ *L-fuzzy monotonic predicates* on D. The elements of D are considered as pieces of information about the state of a certain system or process, and $a\leqslant b$ in D means that b contains more information than a (is more specific/restrictive). For

$m \in [D \to L^{op}]^{op}$ and $a \in D$, we regard $m(a)$ as the truth value of a, hence it is required that $m(b) \leqslant m(a)$ for all $a \leqslant b$. The second ^{op} means that we order fuzzy predicates pointwisely, i.e. $m_1 \leqslant m_2$ iff $m_1(a) \leqslant m_2(a)$ in L (not in L^{op} !) for all $a \in D$. We denote $\underline{M}_{[L]}D = [D \to L^{op}]^{op}$, and, for D with a least element 0, consider also the subset $M_{[L]}D \subset \underline{M}_{[L]}D$ of all *normalized* predicates that take $0 \in D$ (no information) to $1 \in L$ (complete truth). See [10] also for applications of monotonic predicates to denotational semantics of programming languages. We mention them because they provide nontrivial and important examples of complete dually continuous L-semimodules and L-convex compacta, which, as will be shown below, are also free in categorical sense.

Both $\underline{M}_{[L]}D$ and $M_{[L]}D$ are completely distributive lattices, hence they are continuous and dually continuous lattices. It is also clear that all *infima* and *finite suprema* of functions in $\underline{M}_{[L]}D$ and $M_{[L]}D$, including the pairwise joins $m_1 \bar{\oplus} m_2$, are calculated argumentwise, whereas the supremum of a collection $\{m_i \mid i \in \mathcal{I}\}$ of elements of these lattices is equal to

$$(\sup_{i\in\mathcal{I}} m_i)(d) = \inf\{\sup_{i\in\mathcal{I}} m_i(d') \mid d' \in D, d' \ll d\}, \quad d \in D.$$

If the multiplication $* : L \times L \to L$ is Lawson continuous, then by [18, Corollaries 5.8, 5.9] $(\underline{M}_{[L]}, \bar{\oplus}, \bar{\odot})$ with the operations

$$(m_1 \bar{\oplus} m_1)(d) = m_1(d) \oplus m_2(op), \quad m_1, m_2 \in \underline{M}_{[L]}D, d \in D,$$

$$(\alpha \bar{\circledast} m)(d) = \alpha * m(d), \quad m \in \underline{M}_{[L]}D, d \in D,$$

is a compact Hausdorff Lawson $(L, \oplus, *)$-semimodule. The same applies to $(M_{[L]}, \bar{\oplus}, \bar{\circledast})$, where $\bar{\oplus}$ is identical, but

$$(\alpha \bar{\odot} m)(d) = \begin{cases} \alpha * m(d), & d \neq 0, \\ 1, & d = 0, \end{cases} \quad m \in M_{[L]}D, d \in D.$$

For a special, but important, case $* = \otimes$, the semimodules $(\underline{M}_{[L]}D, \bar{\oplus}, \bar{\odot})$ and $(M_{[L]}D, \bar{\oplus}, \bar{\circledast})$ are even L-biconvex compacta [18, Proposition 8.4].

Let X be a compactum, $\operatorname{Exp} X$ the set of all closed subsets of X, and $\exp X = \operatorname{Exp} X \setminus \{\varnothing\}$. If $\operatorname{Exp} X$ and $\exp X$ are ordered by the reverse inclusion, i.e., $A \leqslant B \iff B \subset A$, we obtain the poset $\operatorname{Exp}_{\supset} X$, which is a continuous lattice, hence a domain with a bottom element, and a complete continuous semilattice $\exp_{\supset} X$. The Lawson topologies on $\operatorname{Exp} X$ and $\exp X$ are the famous *Vietoris topologies* [24]. Now we substitute $\exp_{\supset} X$ for D in $\underline{M}_{[L]}D$ and $\operatorname{Exp}_{\supset} X$ for D in $M_{[L]}D$. What follows is a description of the obtained posets.

Let X be a set, $\mathcal{F}$ be an algebra of subsets of X, L be a partially ordered set with a bottom element 0 and a top element 1. A function $c : \mathcal{F} \to L$ is a *plausibility measure* [8] if:

(1) $c(\varnothing) = 0$, $c(X) = 1$; and

(2) $A \subset B$, $A, B \in \mathcal{F}$ imply $c(A) \leqslant c(B)$ in L.

We impose additional restriction on X, L and c, and we obtain the definition of a lattice-valued capacity. In the sequel let L be a compact Lawson lattice [13] with $\alpha \oplus \beta$ and $\alpha \otimes \beta$ being the pairwise supremum and infimum of $\alpha, \beta \in L$, a bottom element 0 and a top element 1 (although some of the following definitions and statements are valid under weaker restrictions). If f, g are L-valued function with the same domain M and $\alpha \in L$, then functions $f \oplus g$, $f \otimes g$, $\alpha \oplus f$ and $\alpha \otimes f$ with the domain M are defined by the formulae $(f \oplus g)(x) = f(x) \oplus g(x)$, $(f \otimes g)(x) = f(x) \otimes g(x)$, $(\alpha \oplus f)(x) = \alpha \oplus f(x)$, $(\alpha \otimes f)(x) = \alpha \otimes f(x)$ for all $x \in M$.

Let X be a compact Hausdorff space. A function $c : \operatorname{Exp} X \to L$ is called an *L-valued capacity* [17] (or an *L-capacity* for short) on a compactum X if the following hold:

1. $c(\varnothing) = 0$; and
2. for each closed subsets F, G in X, the inclusion $F \subset G$ implies $c(F) \leqslant c(G)$ (monotonicity);
3. if the value $c(F)$ for a closed set $F \subset X$ belongs to an open set $V \subset L$, then there is an open set $W \supset F$ such that $c(G) \in V{\downarrow}$ for all closed sets $G \subset X$, $G \subset W$ (upper semicontinuity).

A capacity c on X is called *normalized* if $c(X) = 1$.

Denote the set of all L-valued capacities on a compactum X by $\underline{M}_L X$, and let $M_L X$ be the subset of all normalized L-capacities. Observe that $\underline{M}_L X = \underline{M}_{[L]}(\exp_{\supset} X)$, $M_L X = M_{[L]}(\operatorname{Exp}_{\supset} X)$, i.e., a (normalized) L-valued capacity is the same as a (normalized) L-fuzzy monotonic predicate on respectively $\exp_{\supset} X$ and $\operatorname{Exp}_{\supset} X$. This immediately implies that $\underline{M}_L X$ and $M_L X$ are compact Hausdorff Lawson semimodules with $\bar{\oplus}$ being the argumentwise supremum and the multiplications

$$(\alpha \,\bar{\circledast}\, c)(F) = \alpha * c(F), \quad c \in \underline{M}_L X, F \in \operatorname{Exp} X,$$

and

$$(\alpha \,\bar{\odot}\, c)(F) = \begin{cases} \alpha * c(F), & F \neq X, \\ 1, & F = X, \end{cases} \quad c \in M_L X, F \in \operatorname{Exp} X,$$

respectively, and, for $* = \oplus$, they are L-biconvex compacta.

It was also proved in [17] that a subbase that consists of all sets of the form

$$\begin{aligned} O_+(U, V) &= \{c \in M_L X \mid \text{ there is } F \underset{\text{cl}}{\subset} U \text{ such that } c(F) \geqslant \alpha \text{ for some } \alpha \in V\} \\ &= \{c \in M_L X \mid \text{ there is } F \underset{\text{cl}}{\subset} U, c(F) \in V{\uparrow}\}, \end{aligned}$$

where $U \underset{\text{op}}{\subset} X$, $V \underset{\text{op}}{\subset} L$, and

$$\begin{aligned} O_-(F, V) &= \{c \in M_L X \mid c(F) \leqslant \alpha \text{ for some } \alpha \in V\} \\ &= \{c \in M_L X \mid c(F) \in V{\downarrow}\}, \end{aligned}$$

where $F \underset{\text{cl}}{\subset} X$, $V \underset{\text{op}}{\subset} L$, determines a compact Hausdorff topology on $M_L X$ such that the latter set is a completely distributive lattice. This implies that this topology coincides with the Lawson topology and the dual Lawson topology on $M_L X$.

We call an L-valued capacity c on X a $\cup$-capacity (a $\cap$-capacity) if $c(A \cup B) = c(A) \oplus c(B)$ (resp. $c(A \cap B) = c(A) \otimes c(B)$) for all $A, B \underset{\text{cl}}{\subset} X$. For an L-valued $\cup$-capacity c and $F \underset{\text{cl}}{\subset} X$ we have $c(F) = \sup\{c(\{x\}) \mid x \in F\}$. In the sequel we shall simplify the notation by writing $c(x)$ for $c(\{x\})$, if c is a $\cup$-capacity. The obtained function $c : X \to L$ is upper semicontinuous, and $\sup c = 1$ if and only if c is normalized. Conversely, given an upper normalized semicontinuous function $c : X \to L$ with $\sup c = 1$, we determine a normalized $\cup$-capacity by the formula $c(F) = \sup\{c(x) \mid x \in F\}$. Therefore we will identify normalized L-valued $\cup$-capacities with u.s.c. functions $c : X \to L$ such that $\sup c = 1$.

We denote by $M_{\cup L} X$ and $M_{\cap L} X$ the spaces of all normalized L-valued $\cup$-capacities and all normalized L-valued $\cap$-capacities. It is straightforward to prove that $M_{\cup L} X$ and $M_{\cap L} X$ are closed in $M_L X$. The set $M_{\cup L} X$ in the $(L, \oplus, *)$-semimodule $(M_L X, \bar{\oplus}, \bar{\odot})$ is not a subsemimodule but an $(L, \oplus, *)$-convex subset, therefore is an $(L, \oplus, *)$-convex compactum.

For a compactum X, there is an embedding $\eta_L X : X \hookrightarrow M_L X$:

$$\eta_L X(x) = \delta_x,\ \delta_x(F) = \begin{cases} 1, & \text{if } x \in F \\ 0, & \text{if } x \notin F \end{cases}, \quad x \in X, F \underset{\text{cl}}{\subset} X,$$

here δ_x is the Dirac measure concentrated at x. Since $\eta_L X(X) \subset M_{\cup L} X$, we restrict $\eta_L X$ to the embedding $\eta_{\cup L} X : X \hookrightarrow M_{\cup L} X$. Hence we consider X as a subspace of an L-biconvex compactum $(M_L X, \bar{\oplus}, \bar{\odot})$ and of an $(L, \oplus, *)$-convex compactum $M_{\cup L} X$.

Propositions 8.1 and 9.4 [18] imply (if we substitute $\exp_{\supset} X$ for D) respectively:

Proposition 6.1. *Let $(L, \oplus, \otimes)$ be a completely distributive lattice. For a compactum X the L-biconvex compactum $(M_L X, \bar{\oplus}, \bar{\odot})$ is free over X, i.e., for a continuous mapping $\varphi : X \to K$ into an L-biconvex compactum its unique continuous biaffine extension $\Phi : M_L X \to K$ is determined by the formula*

$$\Phi(c) = \sup_{F \in \exp X} c(F) \,\bar{*}\, \inf \varphi(F)$$

for all $c \in M_L X$.

Proposition 6.2. *Let $(L, \oplus, *)$ be a completely distributive quantale with the Lawson continuous multiplication. For a compactum X the $(L, \oplus, *)$-convex compactum $M_{\cup L} X$ is free over X, i.e., for a continuous mapping $\varphi : X \to K$ into an $(L, \oplus, *)$-convex compactum its unique continuous affine extension Φ :*

$M_{\cup L}X \to K$ is determined by the formula

$$\Phi(c) = \bar{\bigoplus}_{x\in X} c(\{x\}) \bar{*} \varphi(x)$$

for all $c \in M_{\cup L}X$.

The two latter statements are analogues of the well known fact that the space PX of all probability measures on a compactum X is a free convex compactum over X. Recall that

$$PX = \{m \in M_{[0,1]}X \mid m(A \cup B) + m(A \cap B) = m(A) + m(B) \text{ for all } A, B \underset{\mathrm{cl}}{\subset} X\}$$

consists of all normalized σ-additive regular measures on X and is a closed subset of $M_{[0,1]}X$. The subspace PX becomes a convex compactum in the obvious way: the probability measure $\lambda_1 m_1 + \ldots \lambda_n m_n$, for $\lambda_1, \ldots, \lambda_n \in [0, 1]$, $\lambda_1 + \cdots + \lambda_n = 1$, and $m_1, \ldots, m_n \in PX$, is the set function that sends each $F \underset{\mathrm{cl}}{\subset} X$ to $\lambda_1 m_1(F) + \ldots \lambda_n m_n(F)$. Given a continuous mapping φ from X to a convex compact set K in a locally convex vector topological space N, we can calculate its unique affine continuous extension $\Phi : PX \to K$ as $\Phi(m) = \int_X \varphi(x)\, dm(x)$ for all $m \in PX$, where the latter expression is the integral in N of the vector function φ.

7. Monad of lattice-valued capacities, monads of lattice-valued ∪-capacities, and their algebras

The above mentioned constructions of function spaces are functorial, cf. [2, 14] for the definitions of functor and natural transformation. We extend M_L to a *functor of (normalized) L-valued capacities* in the category $\mathcal{C}omp$ of compact Hausdorff spaces and their continuous mappings in an obvious way: if $f : X \to Y$ is a continuous map of compacta, $c \in M_L X$ and $F \underset{\mathrm{cl}}{\subset} Y$, then the mapping $M_L f : M_L X \to M_L Y$ is defined as $M_L f(c)(F) = c(f^{-1}(F))$. Since $M_L f(M_{\cup L}X) \subset M_{\cup L}Y$, $M_L f(M_{\cap L}X) \subset M_{\cap L}Y$, we denote by $M_{\cup L}f : M_{\cup L}X \to M_{\cup L}Y$ and $M_{\cap L}f : M_{\cap L}X \to M_{\cap L}Y$ the restrictions of $M_L f$ for each $f : X \to Y$ in $\mathcal{C}omp$ and obtain also *subfunctors* $M_{\cup L}, M_{\cap L}$ of the functor M_L. If $L = [0, 1]$, then similarly the subfunctor P of $M_{[0,1]}$ appears, which is the famous *probability measure functor* [24].

As we know from the category theory, each construction of free objects leads to a monad [2, 14]. A *monad* $\mathbb{T}$ in a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ (the *functorial part* of the monad) and natural transformations $\eta : \mathbf{1}_{\mathcal{C}} \to T$ (the *unit*), $\mu : T^2 \to T$ (the *multiplication*) such that the equalities $\mu X \circ \eta TX = \mu X \circ T\eta X = \mathbf{1}_{TX}$, $\mu X \circ \mu TX = \mu X \circ T\mu X$ are valid for all objects X in the category $\mathcal{C}$. It is often convenient to visualize these equalities as commutative diagrams:

$$\begin{array}{ccc} TX & \xrightarrow{\eta TX} & T^2X \\ {\scriptstyle T\eta X}\downarrow & \searrow^{\mathbf{1}_{TX}} & \downarrow{\scriptstyle \mu X} \\ T^2X & \xrightarrow[\mu X]{} & TX \end{array} \qquad \begin{array}{ccc} T^3X & \xrightarrow{T\mu X} & T^2X \\ {\scriptstyle \mu TX}\downarrow & & \downarrow{\scriptstyle \mu X} \\ T^2X & \xrightarrow[\mu X]{} & TX \end{array}$$

Now we briefly describe monads for M_L, $M_{\cup L}$ and P. The units $\eta_L : \mathbf{1}_{\mathcal{C}omp} \to M_L$ and $\eta_{\cup L} : \mathbf{1}_{\mathcal{C}omp} \to M_{\cup L}$ are respectively the collections of embeddings $\eta_L X : X \to M_L X$ and $\eta_{\cup L} X : X \to M_{\cup L} X$ for all compacta X. Since $\eta_{[0,1]} X(X) \subset PX$, the restriction $\eta_P X : X \to PX$ of $\eta_{[0,1]} X$ exists for each compactum X, thus the required unit $\eta_P : \mathbf{1}_{\mathcal{C}omp} \to P$ is obtained.

To construct a mapping ${M_L}^2 X \to M_L X$, observe that $\mathbf{1}_{M_L X} : M_L X \to M_L X$ is a continuous mapping from the compactum $M_L X$ into the L-biconvex compactum $M_L X$, hence by Proposition 6.1 there is its unique continuous L-biaffine extension $\mu_L X : {M_L}^2 X \to M_L X$. Similarly $\mu^*_{\cup L} X : {M_{\cup L}}^2 X \to M_{\cup L} X$ and $\mu_P X : P^2 X \to PX$ are respectively the unique continuous $(L, \oplus, *)$-affine extension of $\mathbf{1}_{M_{\cup L} X} : M_{\cup L} X \to M_{\cup L} X$ and the unique continuous affine (in the usual sense) extension of $\mathbf{1}_{PX} : PX \to PX$.

It is routine to verify that

$$\begin{aligned} &\mu_L X(C)(F) \\ = \ &\sup\{\alpha \in L \mid C(\{c \in M_L X \mid c(F) \geqslant \alpha\}) \geqslant \alpha\}, \quad C \in {M_L}^2 X, F \underset{\text{cl}}{\subset} X. \end{aligned}$$

Sometimes it is more convenient to use an equivalent formula:

$$\begin{aligned} &\mu_L X(C)(F) \\ = \ &\sup\{C(\mathcal{F}) \otimes \inf\{c(F) \mid c \in \mathcal{F}\} \mid \mathcal{F} \underset{\text{cl}}{\subset} M_L X\}, \quad C \in {M_L}^2 X, F \underset{\text{cl}}{\subset} X. \end{aligned}$$

The component of the natural transformation $\mu^*_{\cup L} : M^2_{\cup L} \to M_{\cup L}$ is equal to

$$\begin{aligned} &\mu^*_{\cup L} X(C)(F) \\ = \ &\sup\{C(\mathcal{F}) * \inf\{c(F) \mid c \in \mathcal{F}\} \mid \mathcal{F} \underset{\text{cl}}{\subset} M_{\cup L} X\}, \quad C \in {M_{\cup L}}^2 X, F \underset{\text{cl}}{\subset} X. \end{aligned}$$

The components of μ_P were described and studied in detail in [24].

Thus the monad of L-valued capacities $\mathbb{M}_L$, the monad $\mathbb{M}^*_{\cup L}$ (which depends on the multiplication $*$), and the probability measure monad $\mathbb{P}$ are constructed.

Observe that for $C \in M_{\cup L}(M_{\cup L} X)$ and $F \underset{\text{cl}}{\subset} X$ we have

$$\mu_L X(C)(F) = \sup\{C(c) \otimes \sup_F c \mid c \in M_{\cup L} X\}.$$

This implies

$$\mu_L X(C)(F \cup G) = \mu_L X(C)(F) \oplus \mu_L X(C)(G)$$

for closed $F, G \subset X$. Thus $\mu_L X(M_{\cup L}(M_{\cup L}X)) \subset M_{\cup L}X$. We define the mapping $\mu_{\cup L}X : X \to M_{\cup L}X$ for each compactum X as the restriction of $M_{\cup L}X$ and obtain the *submonad* $\mathbb{M}_{\cup L} = (M_{\cup L}, \eta_{\cup L}, \mu_{\cup L})$ $\mathbb{M}$ [11]. It is straightforward to verify that the monad $\mathbb{M}_{\cup L}$ is a special case of $\mathbb{M}^*_{\cup L} = (M_{\cup L}, \eta_{\cup L}, \mu^*_{\cup L})$ for $* = \otimes$.

We also need the notion of algebra for a monad $\mathbb{T} = (T, \eta, \mu)$ in a category $\mathcal{C}$. It is a pair of an object X of $\mathcal{C}$ and a mapping $b : TX \to X$ such that $b \circ \eta X = \mathbf{1}_X$, $b \circ Tb = b \circ \mu X$. A morphism from $\mathbb{T}$-algebra (X, b) to a $\mathbb{T}$-algebra (X', b') is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $b' \circ Tf = f \circ b$. Thus $\mathbb{T}$-algebras and their morphisms form a category, which is called the *Eilenberg-Moore category* [6] and denoted $\mathcal{C}^{\mathbb{T}}$.

Consider the monad $\mathbb{P}$. If K is a convex compactum, then the identity mapping from a compactum K to a convex compactum K has a unique continuous affine extension $b : PK \to K$, which is called *barycenter mapping.* Then (K, b) is a $\mathbb{P}$-algebra, and a theorem by Świrszcz [23] asserts that all $\mathbb{P}$-algebras are obtained in this way. Hence the category of convex compacta $\mathcal{C}onv$ is "the same" as the category of $\mathbb{P}$-algebras, or, categorically speaking, $\mathcal{C}onv$ is *monadic* over $\mathcal{C}omp$.

The categories of algebras for the monad of real-valued (i.e., for $L = [0, 1]$) regular normed capacities [28] and two its submonads were analogously described in [19] in terms of $(\max, \min)$-idempotent versions of convexity. Our work also follows the line of similar results of Day [5], Radul [20], Wyler [25], Zarichyi [26], and others.

This paper presents generalizations of results of [19] to capacities with values in compact Hausdorff Lawson lattices. We are going to show that the Eilenberg-Moore categories of algebras for the monads $\mathbb{M}^*_{\cup L}$ and $\mathbb{M}_L$ are respectively the categories of $(L, \oplus, *)$-convex compacta and L-biconvex compacta.

Theorem 7.1. *Let X be a compactum and L a compact Lawson lattice. Then there exists a one-to-one correspondence between continuous maps $\xi : M_{\cup L}X \to X$ such that the pair (X, ξ) is an $\mathbb{M}^*_{\cup L}$-algebra, and continuous L-convex combinations ic that make X an L-convex compactum.*

If (X, ic) is an L-convex compactum, then the property of local convexity *holds: for a neighborhood U of any element $x \in X$ there is a neighborhood V of x, $V \subset U$, such that $\alpha_0 y_0 \oplus \dots \alpha_n \alpha n \in V$ for all $y_0, \dots, y_n \in V$, $(\alpha_0, \dots, \alpha_n) \in \Delta^n_{\oplus}$.*

Theorem 7.2. *Let (X, ξ), (X', ξ') be $\mathbb{M}_{\cup L}*$-algebras, and ic and ic′ the respective idempotent convex combinations. Then a continuous map $f : X \to Y$ is a morphism of $\mathbb{M}_{\cup}$-algebras $(X, \xi) \to (X', \xi')$ if and only if $f : (X, ic) \to (X', ic')$ is $(L, \oplus, *)$-affine.*

In particular, such results are valid for $\mathbb{M}_{\cup L}$-algebras.

Theorem 7.3. *Let X be a compactum. Then there is a one-to-one correspondence between:*

(*1*) *continuous maps $\xi : M_L X \to X$ such that the pair (X, ξ) is an $\mathbb{M}_L$-algebra;*

(*2*) *quadruples* $(\bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ *of continuous operations* $\bar{\oplus} : X \times X \to X$, $\otimes : L \times X \to X$, $\bar{\otimes} : X \times X \to X$, $\oplus : L \times X \to X$ *such that* $(X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ *is an L-idempotent biconvex compactum; and*

(*3*) *triples* $(\bar{\oplus}, \otimes, p)$ *of continuous maps* $\bar{\oplus} : X \times X \to X$, $\bar{\otimes} : X \times X \to X$, $p : L \to X$ *such that*
   (*a*) $(X, \bar{\oplus}, \bar{\otimes})$ *is a Lawson lattice; and*
   (*b*) $p : (L, \oplus) \to (X, \bar{\oplus})$ *is a complete bottom- and top-preserving lattice morphism.*

In the case (*2*) *the following property of* local biconvexity *holds: for a neighborhood* U *of any element* $x \in X$ *there is a neighborhood* V *of* x, $V \subset U$, *such that* $(\alpha_0 \otimes y_0) \bar{\oplus} (\alpha_1 \otimes y_1) \bar{\oplus} \ldots \bar{\oplus} (\alpha_n \otimes y_n) \in V$, $(\beta_0 \oplus y_0) \bar{\otimes} (\beta_1 \oplus y_1) \bar{\otimes} \ldots \bar{\otimes} (\beta_n \oplus y_n) \in V$ *whenever* $y_0, y_1, \ldots, y_n \in V$, $\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n \in L$, $\alpha_0 \oplus \alpha_1 \oplus \ldots \oplus \alpha_n = 1$, $\beta_0 \otimes \beta_1 \otimes \ldots \otimes \beta_n = 0$.

Theorem 7.4. *Let* (X, ξ), (X', ξ') *be* $\mathbb{M}_L$*-algebras and quadruples* $(\bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ *of continuous operations be determined on* X *and* X' *by* ξ *and* ξ' *resp. (in the sense of Theorem* 7.3*). Then a continuous map* $f : X \to Y$ *is a morphism of* $\mathbb{M}_L$*-algebras* $(X, \xi) \to (X', \xi')$ *if and only if* $f : (X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus) \to (X', \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ *is biaffine.*

We conclude that the category $(L, \oplus, *)$-$\mathcal{C}onv$ of L-convex compacta and the category L-$\mathcal{B}i\mathcal{C}onv$ of L-idempotent biconvex compacta are monadic over the category $\mathcal{C}omp$ of compacta. Thus L-idempotent versions of convexity are proper analogues of the usual convexity in algebraic, topological and categorical aspects.

8. Proofs of theorems

Proof of Theorem 5.1. *Sufficiency* is obvious. We briefly outline a construction of N for a given (X, ic) which is required to show *necessity.* Let for $K \in \exp(X \times L)$ a mapping $gr_K : X \to X$ be defined by the formula

$$gr_K(y) = \sup\{1y \bar{\oplus} \alpha x \mid (x, \alpha) \in K\}, \quad y \in X.$$

We write $K \sim K'$ for $K, K' \in \exp(X \times L)$ if $gr_K = gr_{K'}$. The equivalence relation "$\sim$" is closed in $\exp(X \times L) \times \exp(X \times L)$, therefore the quotient set $N = \exp(X \times L)/_\sim$ is a compactum. We define operations $\bar{\oplus} : N \times N \to N$ and $\bar{*} : L \times N \to N$ by the formulae (by $[K]$ the equivalence class of $K \in \exp(X \times L)$ is denoted): $[K] \bar{\oplus} [K'] = [K \cup K']$, $\alpha \bar{*} [K] = [\{(x, \alpha * \beta) \mid (x, \beta) \in K\}]$ for $K, K' \in \exp(X \times L)$, $\alpha \in L$. The embedding $e : X \hookrightarrow N$ is defined by the formula $e(x) = [\{(x, 1)\}]$, $x \in X$. It is straightforward to show that N satisfies the requirements. $\square$

Proof of Theorem 7.1. Let $\xi : M_{\cup L} X \to X$ be such that the pair (X, ξ) is an $\mathbb{M}^*_{\cup L}$-algebra. For $x_0, \ldots, x_n \in X$, $(\alpha_0, \ldots, \alpha_n) \in \Delta^n_\oplus$ we put

$$\begin{aligned} &ic(x_0, \ldots, x_n, \alpha_0, \ldots, \alpha_n) = \alpha_0 x_0 \oplus \alpha_1 x_1 \oplus \ldots \oplus \alpha_n x_n \\ &= \xi((\alpha_0 * \delta_{x_0}) \oplus (\alpha_1 * \delta_{x_1}) \oplus \ldots \oplus (\alpha_0 * \delta_{x_n})). \end{aligned}$$

Then ic is continuous for all $n \in \{0,1,2,\dots\}$, and $ic(x,1) = \xi(\delta_x) = \xi \circ \eta_L X(x) = x$. Let $m, n \in \{0,1,2,\dots\}$, $x_0, x_1, \dots, x_n \in X$, $(\alpha_0, \alpha_1, \dots, \alpha_m) \in \Delta^m_\oplus$ and σ be a mapping $\{0,1,\dots m\} \to \{0,1,\dots,n\}$. If $\beta_k = \sup\{\alpha_i \mid 0 \leqslant i \leqslant m, \sigma(i) = k\}$ for $k = 0,1,\dots,n$, then

$$\begin{aligned}&(\alpha_0 * \delta_{x_{\sigma(0)}}) \oplus (\alpha_0 * \delta_{x_{\sigma(0)}}) \oplus \dots \oplus (\alpha_0 * \delta_{x_{\sigma(0)}})\\ &= (\beta_0 * \delta_{x_0}) \oplus (\beta_0 * \delta_{x_0}) \oplus \dots \oplus (\beta_0 * \delta_{x_0}),\end{aligned}$$

therefore

$$\alpha_0 x_{\sigma(0)} \oplus \alpha_1 x_{\sigma(1)} \oplus \dots \oplus \alpha_m x_{\sigma(m)} = \beta_0 x_0 \oplus \beta_1 x_1 \oplus \dots \oplus \beta_n x_n.$$

Now let $x^i_j \in X$, $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \Delta^n_\oplus$, $(\beta^i_0, \beta^i_1, \dots, \beta^i_{k_i}) \in \Delta^{k_i}_\oplus$ for $i = 0,1,\dots,n$. We define a capacity $C \in M_{\cup L}(M_{\cup L}X)$ to be equal to

$$\begin{aligned}&(\alpha_0 * \delta_{(\beta^0_0 * \delta_{x^0_0}) \oplus \dots \oplus (\beta^0_{k_0} * \delta_{x^0_{k_0}})}) \oplus (\alpha_1 * \delta_{(\beta^1_0 * \delta_{x^1_0}) \oplus \dots \oplus (\beta^1_{k_1} * \delta_{x^1_{k_1}})}) \oplus \dots\\ &\oplus (\alpha_n * \delta_{(\beta^n_0 * \delta_{x^n_0}) \oplus \dots \oplus (\beta^n_{k_n} * \delta_{x^n_{k_n}})}).\end{aligned}$$

By the definition of $\mathbb{M}^*_{\cup L}$-algebra we have $\xi \circ M_{\cup L}\xi(C) = \xi \circ \mu_{\cup L}(C)$, which implies the required "big associative law". We have proved that ic is a continuous L-idempotent convex combination. For any $A \underset{\mathrm{cl}}{\subset} X$ we have $\sup A = \xi(\delta_A)$, where

$$\delta_A(F) = \begin{cases} 1, & \text{if } F \cap A \neq \varnothing,\\ 0, & \text{if } F \cap A = \varnothing, \end{cases} \quad F \underset{\mathrm{cl}}{\subset} X,$$

and δ_A depends on A continuously w.r.t. the Vietoris topology. Therefore $\sup A$ continuously depends on A, which implies that X is an upper Lawson semilattice and thus is an $(L, \oplus, *)$-convex compactum.

If a continuous mapping $\xi' : M_{\cup L}X \to X$ determines the same L-convex combination by the formula

$$\begin{aligned}&ic(x_0, \dots, x_n, \alpha_0, \dots, \alpha_n) = \alpha_0 x_0 \bar{\oplus} \alpha_1 x_1 \bar{\oplus} \dots \bar{\oplus} \alpha_n x_n\\ &= \xi'((\alpha_0 * \delta_{x_0}) \oplus (\alpha_1 * \delta_{x_1}) \oplus \dots \oplus (\alpha_0 * \delta_{x_n})),\end{aligned}$$

then it coincides with ξ at the set of all L-capacities of the form $(\alpha_0 * \delta_{x_0}) \oplus (\alpha_1 * \delta_{x_1}) \oplus \dots \oplus (\alpha_0 * \delta_{x_n})$, which are dense in $M_{\cup L}X$, hence $\xi' = \xi$.

Now let X be an $(L, \oplus, *)$-convex compactum with the continuous L-convex combination ic. Since $M_{\cup L}X$ is a free $(L, \oplus, *)$-convex compactum over X, there is a unique continuous affine extension $\xi : M_{\cup L}X \to X$ of the identity mapping on X. Because of the way the monad $\mathbb{M}^*_{\cup L}$ is constructed, the pair (X, ξ) is a $\mathbb{M}^*_{\cup L}$-algebra.

Let U be a neighborhood of x in X. The continuity of ξ and the equality $\xi(\delta_x) = x$ imply the existence of a neighborhood $\tilde{U} \subset M_{\cup L}X$, of the capacity

δ_x, such that for all $c \in \tilde{U}$ we have $\xi(c) \in U$. There is also a neighborhood $\tilde{V} \ni x$ such that for all $y_0, y_1, \dots, y_n \in \tilde{V}$, $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \Delta^n_L$ the inclusion $\alpha_0\delta_{y_0} \bar{\oplus} \alpha_1\delta_{y_1} \bar{\oplus} \dots \bar{\oplus} \alpha_n\delta y_n \in \tilde{U}$ is valid. It is easy to verify that the set

$$V = \{\alpha_0 y_0 \bar{\oplus} \alpha_1 y_1 \bar{\oplus} \dots \bar{\oplus} \alpha_n y_n \mid n \in \{0, 1, \dots\},$$
$$(\alpha_0, \alpha_1, \dots, \alpha_n) \in \Delta^n_L, y_0, y_1, \dots, y_n \in \tilde{V}\}$$

is a neighborhood of x that is required for the local convexity of X. □

Proof of Theorem 7.2. Recall that a mapping $f : (X, ic) \to (X', ic')$ is affine if and only if

$$f(ic(x_0, \dots, x_n, \alpha_0, \dots, \alpha_n)) = ic'(f(x_0), \dots, f(x_n), \alpha_0, \dots, \alpha_n)$$

for all $x_0, x_1, \dots, x_n \in X$, $(\alpha_0, \alpha_1, \dots, \alpha_n) \in \Delta^n_L$. Taking into account the formulae for ic, ic' via ξ, ξ', we obtain an equivalent equality

$$f \circ \xi\big((\alpha_0 * \delta_{x_0}) \oplus (\alpha_1 * \delta_{x_1}) \oplus \dots \oplus (\alpha_0 * \delta_{x_n})\big)$$
$$= \xi'\big((\alpha_0 * \delta_{f(x_0)}) \oplus (\alpha_1 * \delta_{f(x_1)}) \oplus \dots \oplus (\alpha_0 * \delta_{f(x_n)})\big),$$

i.e., $f \circ \xi(c) = \xi' \circ M_{\cup L} f(c)$ for all capacities of the form $c = (\alpha_0 * \delta_{x_0}) \oplus (\alpha_1 * \delta_{x_1}) \oplus \dots \oplus (\alpha_0 * \delta_{x_n})$. Since such capacities are dense in $M_{\cup L} X$, f being affine is equivalent to the equality $f \circ \xi(c) = \xi' \circ M_{\cup L} f(c)$ for all $c \in M_{\cup L} X$, i.e., to the statement that f is a morphism of $\mathbb{M}^*_{\cup L}$-algebras. □

Proof of Theorem 7.3. The equivalence between (2) and (3) has already been explained at the end of Section 5.

Assume (2), then X is an L-biconvex compactum, and $M_L X$ is a free L-biconvex compactum over X. Hence there is a unique continuous biaffine extension $\xi : M_L X \to X$ of the identity mapping on X. Recall how the monad $\mathbb{M}_L$ was constructed and conclude that the pair (X, ξ) is a $\mathbb{M}_L$-algebra, i.e., (1) holds.

Now let (X, ξ) be a $\mathbb{M}_L$-algebra. We use the fact that the *inclusion hyperspace monad* $\mathbb{G}$ [20] is a submonad of the monad $\mathbb{M}_L$. The components of the embedding $i_L : \mathbb{G} \hookrightarrow \mathbb{M}_L$ are of the form

$$i_L X(\mathcal{A})(F) = \begin{cases} 1, & \text{if } F \in \mathcal{A}, \\ 0 & \text{otherwise,} \end{cases} \quad F \underset{\text{cl}}{\subset} X.$$

Hence $(X, \xi \circ i_L X)$ is a $\mathbb{G}$-algebra. By Theorem 2 of [20] the operations $\bar{\oplus} : X \times X \to X$, $\bar{\otimes} : I \times X \to X$ that are defined by the formulae $x \bar{\oplus} y = \xi(\delta_x \vee \delta_y) = i_L X(\eta_G X(x) \cap \eta_G X(y))$ and $x \bar{\otimes} y = \xi(\delta_x \wedge \delta_y) = i_L X(\eta_G X(x) \cup \eta_G X(y))$ are such that $(X, \bar{\oplus}, \bar{\otimes})$ is a completely distributive lattice. Its bottom and top elements are equal respectively to $\bar{0} = \xi(c_0)$ and $\bar{1} = \xi(c_1)$, where $c_0 = i_L(\{X\})$ and $c_1 = i_L(\exp X)$ are respectively the least and the greatest of the normalized L-capacities.

The monad $\mathbb{M}_{\cup L}$ is also a submonad of $\mathbb{M}_L$, therefore X is a $\mathbb{M}_{\cup L}$-algebra, i.e., is an $(L, \oplus, \otimes)$-convex compactum with the following $(L, \oplus, \otimes)$-convex combination:

$$\begin{aligned} ic(x_0, \dots, x_n, \alpha_0, \dots, \alpha_n) &= \alpha_0 x_0 \bar{\oplus} \alpha_1 x_1 \bar{\oplus} \dots \bar{\oplus} \alpha_n x_n \\ &= \xi((\alpha_0 \otimes \delta_{x_0}) \oplus (\alpha_1 \otimes \delta_{x_1}) \oplus \dots \oplus (\alpha_0 \otimes \delta_{x_n})) \end{aligned}$$

for all $x_0, \dots, x_n \in X$, $(\alpha_0, \dots, \alpha_n) \in \Delta_L^n$. Note that $1x \bar{\oplus} 1y$ coincides with the expression $x \bar{\oplus} y$ defined above, and X contains the bottom element $\bar{0}$, hence X is complete dually continuous $(L, \oplus, \otimes)$-semimodule with the multiplication $\bar{\odot} : L \times X \to X$ defined as follows:

$$\alpha \bar{\odot} x = 1\bar{0} \bar{\oplus} \alpha x = \xi(c_0 \oplus \alpha\delta_x) = \xi(\alpha \bar{\odot} \delta_x), \quad \alpha \in L, x \in X,$$

the latter product is in $M_L X$.

Observe that

$$c_0 \oplus \alpha\delta_x = \mu_L X(\delta_{c_0} \oplus \delta_{\alpha c_1 \otimes \delta_x}),$$

hence

$$\begin{aligned} \alpha \bar{\odot} x &= \xi(c_0 \bar{\oplus} \alpha\delta_x) = \xi \circ \mu_L X(\delta_{c_0} \oplus \delta_{\alpha c_1 \otimes \delta_x}) \\ &= \xi \circ M_L \xi(\delta_{c_0} \oplus \delta_{\alpha c_1 \otimes \delta_x}) = \xi(\delta_{\bar{0}} \oplus \delta_{\alpha \bar{1} \bar{\otimes} x}) = \bar{0} \bar{\oplus} (\alpha \bar{1} \bar{\otimes} x) = \alpha \bar{1} \bar{\otimes} x, \end{aligned}$$

therefore $(X, \bar{\oplus}, \bar{\odot})$ is an L-biconvex compactum. It is straightforward to verify that the obtained operations $\bar{\oplus}, \bar{\odot}$ are the unique ones that determine a structure of an L-biconvex compactum on X such that ξ is a continuous biaffine extension of the identity mapping $X \to X$. Thus (1) implies (2).

To show the local biconvexity for operations $\bar{\oplus}, \bar{\odot}, \bar{\otimes}, \underline{\odot}$ that satisfy (2), observe that for each neighborhood U of a point x the local convexity of $(X, \bar{\oplus}, \bar{\odot})$ implies the existence of a neighborhood U' of this point such that $U' \subset U$ and $(\alpha_0 \bar{\odot} x_0) \bar{\oplus} \dots \bar{\oplus} (\alpha_n \bar{\odot} x_n) \in U'$ for all $n \in \{0, 1, \dots\}$, $x_0, \dots, x_n \in U'$, $\alpha_0, \dots, \alpha_n \in L$, $\alpha_0 \oplus \dots \oplus \alpha_n = 1$. Next, this neighborhood by the local convexity of $(X, \bar{\otimes}, \underline{\odot})$ contains a neighborhood U'' of the point x such that $(\beta_0 \underline{\odot} x_0) \bar{\otimes} \dots \bar{\otimes} (\beta_n \underline{\odot} x_n) \in U'$ for all $n \in \{0, 1, \dots\}$, $x_0, \dots, x_n \in U'$, $\beta_0, \dots, \beta_n \in L$, $\beta_0 \otimes \dots \otimes \beta_n = 0$. Then the set

$$\begin{aligned} V = \{&(\alpha_0 \bar{\odot} x_0) \bar{\oplus} \dots \bar{\oplus} (\alpha_n \bar{\odot} x_n) \mid \\ &n \in \{0, 1, \dots\}, \alpha_0, \dots, \alpha_n \in L, \alpha_0 \oplus \dots \oplus \alpha_n = 1, x_0, x_1, \dots, x_n \in \tilde{U}''\} \end{aligned}$$

is a required neighborhood of the point x. □

It is also straightforward to modify the proof of Theorem 7.2 to obtain *Theorem* 7.4.

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