PRODUCTS INVOLVING RECIPROCALS OF GIBONACCI POLYNOMIALS

THOMAS KOSHY

Abstract. We explore finite and infinite products involving reciprocals of gibonacci polynomials, and their Pell counterparts.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 4, 6].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [4, 5].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, $\Delta = \sqrt{x^2+4}$, $2\alpha(x) = x + \Delta$, $\gamma(x) = \alpha(2x)$, $\alpha = \alpha(1)$, and $\gamma = \gamma(1)$, and omit a lot of basic algebra. It follows from the *Binet-like formulas* in [4] that $\lim_{m\to\infty} \frac{g_{m+k}}{g_m} = \alpha^k(x)$, and $\lim_{m\to\infty} \frac{b_{m+k}}{b_m} = \gamma^k(x)$.

It follows from the *Cassini-like identities* [4]

$$\begin{aligned} f_{n+k}f_{n-k} - f_n^2 &= (-1)^{n-k+1} f_k^2; \\ l_{n+k}l_{n-k} - l_n^2 &= (-1)^{n-k} \Delta^2 f_k^2 \end{aligned}$$

that $F_{n-2}F_{n-1}F_{n+1}F_{n+2} = F_n^4 - 1$ and $L_{n-2}L_{n-1}L_{n+1}L_{n+2} = L_n^4 - 25$ [4]. They play an important role in our investigations.

2. Products Involving Reciprocals of Fibonacci Polynomials

We begin our explorations with products containing reciprocals of squares of odd-numbered Fibonacci polynomials.

Theorem 2.1.

$$\prod_{n=2}^{m} \left(1 + \frac{x^2}{f_{2n-1}^2}\right) = \frac{1}{x^2+1} \cdot \frac{f_{2m+1}}{f_{2m-1}}. \tag{2.1}$$

Proof. We will establish the formula using recursion [4]. Let A_m denote the left side of (2.1) and B_m denote the right side of (2.1). Then,

$$\begin{aligned} \frac{B_m}{B_{m-1}} &= \frac{f_{2m+1}f_{2m-3}}{f_{2m-1}^2} \\ &= \frac{f_{2m-1}^2 + x^2}{f_{2m-1}^2} \\ &= \frac{A_m}{A_{m-1}}. \end{aligned}$$

This implies, $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \cdots = \frac{A_2}{B_2} = \frac{x^2+1}{x^2+1} = 1$. Consequently, $A_m = B_m$, as desired. □

It then follows that

$$\begin{aligned} \prod_{n=2}^{m} \left(1 + \frac{1}{F_{2n-1}^2}\right) &= \frac{F_{2m+1}}{2F_{2m-1}}; \qquad (2.2) \\ \prod_{n=2}^{\infty} \left(1 + \frac{x^2}{f_{2n-1}^2}\right) &= \frac{\alpha^2(x)}{x^2+1}; \\ \prod_{n=2}^{\infty} \left(1 + \frac{1}{F_{2n-1}^2}\right) &= \frac{\alpha^2}{2}. \end{aligned}$$

Formula (2.1) can be rewritten as

$$\prod_{n=2}^{m} \frac{f_{2n+1}f_{2n-3}}{f_{2n-1}^2} = \frac{1}{x^2+1} \cdot \frac{f_{2m+1}}{f_{2m-1}}.$$

Next, we explore products involving reciprocals of squares of even-numbered Fibonacci polynomials.

Theorem 2.2.

$$\prod_{n=2}^{m} \left(1 - \frac{x^2}{f_{2n}^2}\right) = \frac{1}{x^2+2} \cdot \frac{f_{2m+2}}{f_{2m}}. \qquad (2.3)$$

Proof. We will confirm the validity of this formula using recursion [4]. Let A_m denote the left side of (2.3) and B_m denote the right side of (2.3). Then,

$$\begin{aligned} \frac{B_m}{B_{m-1}} &= \frac{f_{2m+2}f_{2m-2}}{f_{2m}^2} \\ &= \frac{f_{2m}^2 - x^2}{f_{2m}^2} \\ &= \frac{A_m}{A_{m-1}}. \end{aligned}$$

This yields $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \cdots = \frac{A_2}{B_2} = \frac{f_4}{f_4} = 1$. So, $A_m = B_m$, as desired. □

This theorem implies

$$\prod_{n=2}^{m}\left(1-\frac{1}{F_{2n}^2}\right) = \frac{F_{2m+2}}{3F_{2m}}; \tag{2.4}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{x^2}{f_{2n}^2}\right) = \frac{\alpha^2(x)}{x^2+2};$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n}^2}\right) = \frac{\alpha^2}{3}.$$

We can rewrite formula (2.3) as

$$\prod_{n=2}^{m}\frac{f_{2n-2}f_{2n+2}}{f_{2n}^2} = \frac{1}{x^2+2}\cdot\frac{f_{2m+2}}{f_{2m}}.$$

Next, we investigate products involving reciprocals of odd and even-numbered Fibonacci polynomial squares.

Theorem 2.3.

$$\prod_{n=2}^{m}\left(1-\frac{1}{f_{2n-1}^2}\right)\left(1+\frac{1}{f_{2n}^2}\right) = \frac{x}{x^2+1}\cdot\frac{f_{2m+1}}{f_{2m}}. \tag{2.5}$$

Proof. Again, we will invoke recursion [4] to establish this formula. Let A_m denote the left side of (2.5) and B_m denote the right side of (2.5). Then,

$$\begin{aligned}\frac{A_m}{A_{m-1}} &= \frac{(f_{2m-1}^2-1)(f_{2m}^2+1)}{f_{2m-1}^2 f_{2m}^2}\\ &= \frac{f_{2m}f_{2m-2}\cdot f_{2m+1}f_{2m-1}}{f_{2m-1}^2 f_{2m}^2}\\ &= \frac{f_{2m+1}f_{2m-2}}{f_{2m}f_{2m-1}}\\ &= \frac{B_m}{B_{m-1}}.\end{aligned}$$

This implies, $\dfrac{A_m}{B_m} = \dfrac{A_{m-1}}{B_{m-1}} = \cdots = \dfrac{A_2}{B_2} = 1$. So, $A_m = B_m$, as desired. □

Consequently,

$$\prod_{n=2}^{m}\left(1-\frac{1}{F_{2n-1}^2}\right)\left(1+\frac{1}{F_{2n}^2}\right) = \frac{F_{2m+1}}{2F_{2m}}; \tag{2.6}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{f_{2n-1}^2}\right)\left(1+\frac{1}{f_{2n}^2}\right) = \frac{x}{x^2+1}\alpha(x);$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n-1}^2}\right)\left(1+\frac{1}{F_{2n}^2}\right) = \frac{\alpha}{2},$$

as in [2].

An interesting byproduct: Formulas (2.2), (2.4), and (2.6) can be employed to extract a product containing reciprocals of the fourth powers of Fibonacci numbers. Multiplying these formulas, we get

$$\prod_{n=2}^{m}\left(1-\frac{1}{F_{2n-1}^4}\right)\left(1-\frac{1}{F_{2n}^4}\right) = \frac{F_{2m+1}}{2F_{2m-1}}\cdot\frac{F_{2m+2}}{3F_{2m}}\cdot\frac{F_{2m+1}}{2F_{2m}};$$
$$\prod_{n=3}^{2m}\left(1-\frac{1}{F_n^4}\right) = \frac{F_{2m+2}F_{2m+1}^2}{12F_{2m}^2F_{2m-1}};$$
$$\prod_{n=3}^{\infty}\left(1-\frac{1}{F_n^4}\right) = \frac{\alpha^5}{12}, \tag{2.7}$$

as in [3, 7].

Using the *Gelin-Cesàro identity* $F_{n-2}F_{n-1}F_{n+1}F_{n+2} = F_n^4 - 1$ [4], we can rewrite formula (2.7) as

$$\prod_{n=3}^{\infty}\frac{F_{n-2}F_{n-1}F_{n+1}F_{n+2}}{F_n^4} = \frac{\alpha^5}{12},$$

as in [3].

2.1. **Alternate Versions.** Using the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [4], we can express the left sides in formulas (2.1), (2.3), and (2.5) in terms of Lucas polynomials.

They yield

$$\prod_{n=2}^{m}\left(1+\frac{\Delta^2x^2}{l_{2n-1}^2+4}\right) = \frac{1}{x^2+1}\cdot\frac{f_{2m+1}}{f_{2m-1}};$$
$$\prod_{n=2}^{\infty}\left(1+\frac{\Delta^2x^2}{l_{2n-1}^2+4}\right) = \frac{\alpha^2(x)}{x^2+1};$$
$$\prod_{n=2}^{\infty}\left(1+\frac{5}{L_{2n-1}^2+4}\right) = \frac{\alpha^2}{2}. \tag{2.8}$$
$$\prod_{n=2}^{m}\left(1-\frac{\Delta^2x^2}{l_{2n}^2-4}\right) = \frac{1}{x^2+2}\cdot\frac{f_{2m+2}}{f_{2m}};$$
$$\prod_{n=2}^{\infty}\left(1-\frac{\Delta^2x^2}{l_{2n}^2-4}\right) = \frac{\alpha^2(x)}{x^2+2};$$
$$\prod_{n=2}^{\infty}\left(1-\frac{5}{L_{2n}^2-4}\right) = \frac{\alpha^2}{3}. \tag{2.9}$$
$$\prod_{n=2}^{m}\left(1-\frac{\Delta^2}{l_{2n-1}^2+4}\right)\left(1+\frac{\Delta^2}{l_{2n}^2-4}\right) = \frac{x}{x^2+1}\cdot\frac{f_{2m+1}}{f_{2m}};$$
$$\prod_{n=2}^{\infty}\left(1-\frac{\Delta^2}{l_{2n-1}^2+4}\right)\left(1+\frac{\Delta^2}{l_{2n}^2-4}\right) = \frac{x\alpha(x)}{x^2+1};$$
$$\prod_{n=2}^{\infty}\left(1-\frac{5}{L_{2n-1}^2+4}\right)\left(1+\frac{5}{L_{2n}^2-4}\right) = \frac{\alpha}{2}. \tag{2.10}$$

It then follows from equations (2.8), (2.9), and (2.10) that

$$\prod_{n=2}^{\infty}\left[1-\frac{25}{(L_{2n-1}^2+4)^2}\right]\left[1-\frac{25}{(L_{2n}^2-4)^2}\right]=\frac{\alpha^5}{12}.$$

3. Pell Versions

Using the relationship $p_n(x)=f_n(2x)$, we can find the Pell counterparts of formulas (2.1), (2.3), and (2.5).

It follows from formula (2.1) that

$$\prod_{n=2}^{m}\left(1+\frac{4x^2}{p_{2n-1}^2}\right)=\frac{p_{2m+1}}{(4x^2+1)p_{2m-1}}.$$

This implies,

$$\begin{aligned}\prod_{n=2}^{m}\left(1+\frac{4}{P_{2n-1}^2}\right) &= \frac{P_{2m+1}}{5P_{2m-1}};\\ \prod_{n=2}^{\infty}\left(1+\frac{4x^2}{p_{2n-1}^2}\right) &= \frac{\gamma^2(x)}{4x^2+1};\\ \prod_{n=2}^{\infty}\left(1+\frac{4}{P_{2n-1}^2}\right) &= \frac{\gamma^2}{5}.\end{aligned}$$

Formula (2.3) yields

$$\prod_{n=2}^{m}\left(1-\frac{4x^2}{p_{2n}^2}\right)=\frac{1}{2(2x^2+1)}\cdot\frac{p_{2m+2}}{p_{2m}}.$$

Consequently,

$$\begin{aligned}\prod_{n=2}^{m}\left(1-\frac{4}{P_{2n}^2}\right) &= \frac{P_{2m+2}}{6P_{2m}};\\ \prod_{n=2}^{\infty}\left(1-\frac{4x^2}{p_{2n}^2}\right) &= \frac{\gamma^2(x)}{2(2x^2+1)};\\ \prod_{n=2}^{\infty}\left(1-\frac{4}{P_{2n}^2}\right) &= \frac{\gamma^2}{6}.\end{aligned}$$

From formula (2.5), we get

$$\prod_{n=2}^{m}\left(1-\frac{1}{p_{2n-1}^2}\right)\left(1+\frac{1}{p_{2n}^2}\right)=\frac{2x}{4x^2+1}\cdot\frac{p_{2m+1}}{p_{2m}}.$$

This implies,

$$\begin{aligned}
\prod_{n=2}^{m}\left(1-\frac{1}{P_{2n-1}^2}\right)\left(1+\frac{1}{P_{2n}^2}\right) &= \frac{2}{5}\cdot\frac{P_{2m+1}}{P_{2m}};\\
\prod_{n=2}^{\infty}\left(1-\frac{1}{p_{2n-1}^2}\right)\left(1+\frac{1}{p_{2n}^2}\right) &= \frac{2x}{4x^2+1}\gamma(x);\\
\prod_{n=2}^{\infty}\left(1-\frac{1}{P_{2n-1}^2}\right)\left(1+\frac{1}{P_{2n}^2}\right) &= \frac{2\gamma}{5}.
\end{aligned}$$

Using the formula $q_n^2 - 4(x^2+1)p_n^2 = 4(-1)^n$ [5], we can extract the Pell-Lucas versions of formulas (2.1), (2.3), and (2.5). In the interest of brevity, we omit them.

4. Products Involving Reciprocals of Lucas Polynomials

We now explore the Lucas counterparts of formulas (2.1), (2.3), and (2.5). Again, we will employ recursion [4] to establish them.

Theorem 4.1.

$$\prod_{n=2}^{m}\left(1-\frac{\Delta^2x^2}{l_{2n-1}^2}\right) = \frac{x}{x^3+3x}\cdot\frac{l_{2m+1}}{l_{2m-1}}. \tag{4.1}$$

Proof. Let A_m denote the left side of (4.1) and B_m denote the right side of (4.1). Then,

$$\begin{aligned}
\frac{B_m}{B_{m-1}} &= \frac{l_{2m+1}l_{2m-3}}{l_{2m-1}^2}\\
&= \frac{l_{2m-1}^2-\Delta^2x^2}{l_{2m-1}^2}\\
&= \frac{A_m}{A_{m-1}}.
\end{aligned}$$

This implies,

$$\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \cdots = \frac{A_2}{B_2} = \frac{l_5l_1}{l_3^2}\cdot\frac{l_3^2}{xl_5} = 1.$$

Thus $A_m = B_m$, as expected. □

Formula (4.1) yields

$$\begin{aligned}
\prod_{n=2}^{m}\left(1-\frac{5}{L_{2n-1}^2}\right) &= \frac{L_{2m+1}}{4L_{2m-1}}; \qquad (4.2)\\
\prod_{n=2}^{\infty}\left(1-\frac{\Delta^2x^2}{l_{2n-1}^2}\right) &= \frac{x}{x^3+3x}\alpha^2(x);\\
\prod_{n=2}^{\infty}\left(1-\frac{5}{L_{2n-1}^2}\right) &= \frac{\alpha^2}{4}.
\end{aligned}$$

Because $l_{2n+1}l_{2n-3} = l_{2n-1}^2 - \Delta^2x^2$, we can rewrite formula (4.1) as

$$\prod_{n=2}^{m}\frac{l_{2n+1}l_{2n-3}}{l_{2n-1}^2} = \frac{x}{x^3+3x}\cdot\frac{l_{2m+1}}{l_{2m-1}}.$$

Next, we investigate the Lucas version of Theorem 2.2.

Theorem 4.2.

$$\prod_{n=2}^{m}\left(1+\frac{\Delta^2x^2}{l_{2n}^2}\right)=\frac{x^2+2}{x^4+4x^2+2}\cdot\frac{l_{2m+2}}{l_{2m}}. \tag{4.3}$$

Proof. Letting A_m denote the left side of (4.3) and B_m denote the right side of (4.3), we get

$$\begin{aligned}\frac{B_m}{B_{m-1}} &= \frac{l_{2m+2}l_{2m-2}}{l_{2m}^2}\\ &= \frac{l_{2m}^2+\Delta^2x^2}{l_{2m}^2}\\ &= \frac{A_m}{A_{m-1}}.\end{aligned}$$

This implies, $\frac{A_m}{B_m}=\frac{A_{m-1}}{B_{m-1}}=\cdots=\frac{A_2}{B_2}=\frac{l_6l_2}{l_4^2}\cdot\frac{l_4l_4}{l_2l_6}=1$. Consequently, $A_m=B_m$, as desired. □

It follows from formula (4.3) that

$$\begin{aligned}\prod_{n=2}^{m}\left(1+\frac{5}{L_{2n}^2}\right) &= \frac{3}{7}\cdot\frac{L_{2m+1}}{L_{2m-1}}; \qquad (4.4)\\ \prod_{n=2}^{\infty}\left(1+\frac{\Delta^2x^2}{l_{2n}^2}\right) &= \frac{x^2+2}{x^4+4x^2+2}\alpha^2(x);\\ \prod_{n=2}^{\infty}\left(1+\frac{5}{L_{2n}^2}\right) &= \frac{3\alpha^2}{7}.\end{aligned}$$

Using the identity $l_{2n+2}l_{2n-2}=l_{2n}^2+\Delta^2x^2$, we can rewrite formula (4.3) as

$$\prod_{n=2}^{m}\frac{l_{2n+2}l_{2n-2}}{l_{2n}^2}=\frac{x^2+2}{x^4+4x^2+2}\cdot\frac{l_{2m+2}}{l_{2m}}.$$

Next, we present the Lucas version of Theorem 2.3.

Theorem 4.3.

$$\prod_{n=2}^{m}\left(1+\frac{\Delta^2}{l_{2n-1}^2}\right)\left(1-\frac{\Delta^2}{l_{2n}^2}\right)=\frac{x^2+2}{x^3+3x}\cdot\frac{l_{2m+1}}{l_{2m}}. \tag{4.5}$$

Proof. Again, letting A_m denote the left side of (4.5) and B_m denote the right side of (4.5), we have

$$\begin{aligned}\frac{A_m}{A_{m-1}} &= \frac{(l_{2m-1}^2+\Delta^2)(l_{2m}^2-\Delta^2)}{l_{2m-1}^2l_{2m}^2}\\ &= \frac{l_{2m}l_{2m-2}\cdot l_{2m+1}l_{2m-1}}{l_{2m-1}^2l_{2m}^2}\\ &= \frac{l_{2m+1}l_{2m-2}}{l_{2m}l_{2m-1}}\\ &= \frac{B_m}{B_{m-1}}.\end{aligned}$$

This yields $\frac{A_m}{B_m} = \frac{A_{m-1}}{B_{m-1}} = \cdots = \frac{A_2}{B_2} = \frac{l_5 l_3}{l_4^2} \cdot \frac{l_4 l_2}{l_3^2} \cdot \frac{l_3 l_4}{l_2 l_5} = 1$. Consequently, $A_m = B_m$, as desired. □

Consequently,

$$\begin{aligned} \prod_{n=2}^{m} \left(1 + \frac{5}{L_{2n-1}^2}\right)\left(1 - \frac{5}{L_{2n}^2}\right) &= \frac{3}{4} \cdot \frac{L_{2m+1}}{L_{2m}}; \qquad (4.6) \\ \prod_{n=2}^{\infty} \left(1 + \frac{\Delta^2}{l_{2n-1}^2}\right)\left(1 - \frac{\Delta^2}{l_{2n}^2}\right) &= \frac{x^2+2}{x^3+3x}\alpha(x); \\ \prod_{n=2}^{\infty} \left(1 + \frac{5}{L_{2n-1}^2}\right)\left(1 - \frac{5}{L_{2n}^2}\right) &= \frac{3\alpha}{4}. \end{aligned}$$

Using the Cassini-like identity for Lucas polynomials, we can rewrite formula (4.5) as

$$\prod_{n=2}^{m} \frac{l_{2n-2} l_{2n-1} l_{2n} l_{2n+1}}{l_{2n-1}^2 l_{2n}^2} = \frac{x^2+2}{x^3+3x} \cdot \frac{l_{2m+1}}{l_{2m-1}}.$$

Another interesting byproduct: Using equations (4.2), (4.4), and (4.6), we can extract a formula for the product involving reciprocals of the fourth powers of Lucas numbers. Multiplying these formulas, we get

$$\begin{aligned} \prod_{n=2}^{m} \left(1 - \frac{25}{L_{2n-1}^4}\right)\left(1 - \frac{25}{L_{2n}^4}\right) &= \frac{L_{2m+1}}{4L_{2m-1}} \cdot \frac{3L_{2m+2}}{7L_{2m}} \cdot \frac{3L_{2m+1}}{4L_{2m}}; \\ \prod_{n=3}^{2m} \left(1 - \frac{25}{L_n^4}\right) &= \frac{9}{112} \cdot \frac{L_{2m+2} L_{2m+1}^2}{L_{2m}^2 L_{2m-1}}; \\ \prod_{n=3}^{\infty} \left(1 - \frac{25}{L_n^4}\right) &= \frac{9}{112}\alpha^5. \qquad (4.7) \end{aligned}$$

Using the *Gelin-Cesàro-like identity* $L_{n-2}L_{n-1}L_{n+1}L_{n+2} = L_n^4 - 25$ [4], we can rewrite formula (4.7) as

$$\prod_{n=3}^{\infty} \frac{L_{n-2}L_{n-1}L_{n+1}L_{n+2}}{L_n^4} = \frac{9}{112}\alpha^5.$$

4.1. **Alternate Versions.** Using the identity $l_n^2 - \Delta^2 f_n^2 = 4(-1)^n$ [4], we can express formulas (4.1), (4.3), and (4.5) and their implications in terms of Fibonacci polynomials:

$$\begin{aligned} \prod_{n=2}^{m} \left(1 - \frac{\Delta^2 x^2}{\Delta^2 f_{2n-1}^2 - 4}\right) &= \frac{x}{x^3+3x} \cdot \frac{l_{2m+1}}{l_{2m-1}}; \\ \prod_{n=2}^{\infty} \left(1 - \frac{\Delta^2 x^2}{\Delta^2 f_{2n-1}^2 - 4}\right) &= \frac{x\alpha^2(x)}{x^3+3x}; \end{aligned}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{5}{5F_{2n-1}^2-4}\right) = \frac{\alpha^2}{4}. \tag{4.8}$$

$$\prod_{n=2}^{m}\left(1+\frac{\Delta^2x^2}{\Delta^2f_{2n}^2+4}\right) = \frac{x^2+2}{x^4+4x^2+2}\cdot\frac{l_{2m+2}}{l_{2m}};$$

$$\prod_{n=2}^{\infty}\left(1+\frac{\Delta^2x^2}{\Delta^2f_{2n}^2+4}\right) = \frac{x^2+2}{x^4+4x^2+2}\alpha^2(x);$$

$$\prod_{n=2}^{\infty}\left(1+\frac{5}{5F_{2n}^2+4}\right) = \frac{3\alpha^2}{7}. \tag{4.9}$$

$$\prod_{n=2}^{m}\left(1+\frac{\Delta^2}{\Delta^2f_{2n-1}^2-4}\right)\left(1-\frac{\Delta^2}{\Delta^2f_{2n}^2+4}\right) = \frac{x^2+2}{x^3+3x}\cdot\frac{l_{2m+1}}{l_{2m}};$$

$$\prod_{n=2}^{\infty}\left(1+\frac{\Delta^2}{\Delta^2f_{2n-1}^2-4}\right)\left(1-\frac{\Delta^2}{\Delta^2f_{2n}^2+4}\right) = \frac{x^2+2}{x^3+3x}\alpha(x);$$

$$\prod_{n=2}^{\infty}\left(1+\frac{5}{5F_{2n-1}^2-4}\right)\left(1-\frac{5}{5F_{2n}^2+4}\right) = \frac{3\alpha}{4}. \tag{4.10}$$

It then follows from equations (4.8), (4.9), and (4.10) that

$$\prod_{n=2}^{\infty}\left[1-\frac{25}{(5F_{2n-1}^2-4)^2}\right]\left[1-\frac{25}{(5F_{2n}^2+4)^2}\right]=\frac{9}{112}\alpha^5.$$

Next, we find the Pell-Lucas consequences of formulas (4.1), (4.3), and (4.5).

4.2. **Pell-Lucas Implications.** Because $q_n(x) = l_n(2x)$ and $\gamma(x) = \alpha(2x)$, it follows from equations (4.1), (4.3), and (4.5) that

$$\prod_{n=2}^{m}\left[1-\frac{16x^2(x^2+1)}{q_{2n-1}^2}\right] = \frac{x}{4x^3+3x}\cdot\frac{q_{2m+1}}{q_{2m-1}};$$

$$\prod_{n=2}^{m}\left[1+\frac{16x^2(x^2+1)}{q_{2n}^2}\right] = \frac{2x^2+1}{8x^4+8x^2+1}\cdot\frac{q_{2m+2}}{q_{2m}};$$

$$\prod_{n=2}^{m}\left[1+\frac{4(x^2+1)}{q_{2n-1}^2}\right]\left[1-\frac{4(x^2+1)}{q_{2n}^2}\right] = \frac{2x^2+1}{4x^3+3x}\cdot\frac{q_{2m+1}}{q_{2m}},$$

respectively.

It then follows that

$$\prod_{n=2}^{m}\left(1-\frac{8}{Q_{2n-1}^2}\right) = \frac{1}{7}\cdot\frac{Q_{2m+1}}{Q_{2m-1}};$$

$$\prod_{n=2}^{m}\left(1+\frac{8}{Q_{2n}^2}\right) = \frac{3}{17}\cdot\frac{Q_{2m+2}}{Q_{2m}};$$

$$\prod_{n=2}^{m}\left(1+\frac{2}{Q_{2n-1}^2}\right)\left(1-\frac{2)}{Q_{2n}^2}\right] = \frac{3}{7}\cdot\frac{Q_{2m+1}}{Q_{2m}},$$

respectively. In addition, we have

$$\begin{aligned}\prod_{n=2}^{\infty}\left[1-\frac{16x^2(x^2+1)}{q_{2n-1}^2}\right] &= \frac{x}{4x^3+3x}\gamma^2(x);\\ \prod_{n=2}^{\infty}\left[1+\frac{16x^2(x^2+1)}{q_{2n}^2}\right] &= \frac{2x^2+1}{8x^4+8x^2+1}\gamma^2(x);\\ \prod_{n=2}^{\infty}\left[1+\frac{4(x^2+1)}{q_{2n-1}^2}\right]\left[1-\frac{4(x^2+1)}{q_{2n}^2}\right] &= \frac{2x^2+1}{4x^3+3x}\gamma(x).\end{aligned}$$

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Department of Mathematics, Framingham State University, Framingham, MA 01701
Email address: tkoshy@emeriti.framingham.edu