Monge & Kantorovich meet Einstein

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Variational Problems in Physics, Economics and Geometry

Fields' Theme Semester: July – December 2014

Organizers:

Wilfrid Gangbo (Atlanta) *optimal transport (OT)*

Young-Heon Kim (Vancouver) *geometry & PDE*

Robert McCann (Toronto) *OT & applications*

Paired with MSRI's Fall 2013 program on Optimal Transport

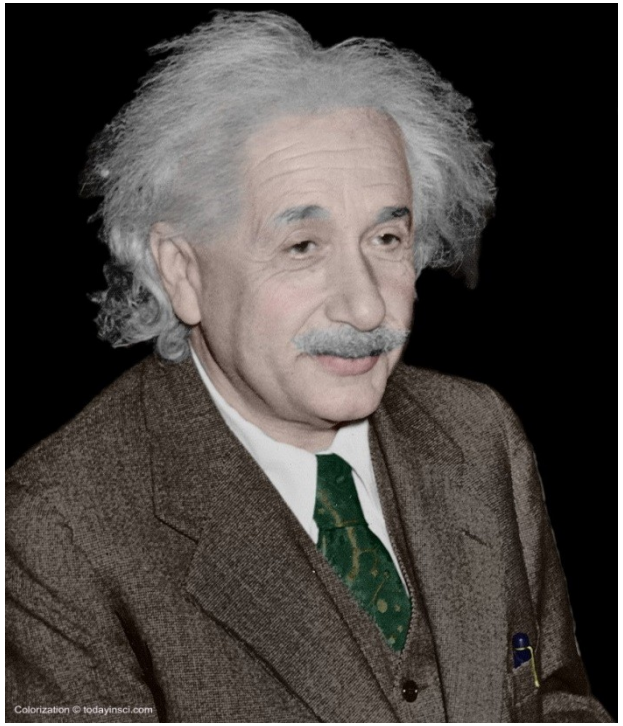
Scientific Committee

- Luigi Ambrosio (Pisa) *Analysis & GMT*
- Yann Brenier (Paris) *Fluids, applied math*
- Pierre-Andre Chiappori (New York) *Economics*
- Toti Daskalopoulos (New York) *geometry*
- L Craig Evans (Berkeley) *Differential equations*
- Felix Otto (Leipzig) *Fluids, pattern formation*
- Neil Trudinger (Canberra) *PDE*

Monge+Kantorovich meet Einstein

Optimal Transport

General Relativity



Gaspard Monge
(1746 – 1818)

-French geometer
-Companion of Napoleon's expedition to Egypt
-Founder École Polytechnique

Leonid Kantorovich
(1912-1986)

-1975 Nobel Memorial Prize in Economics

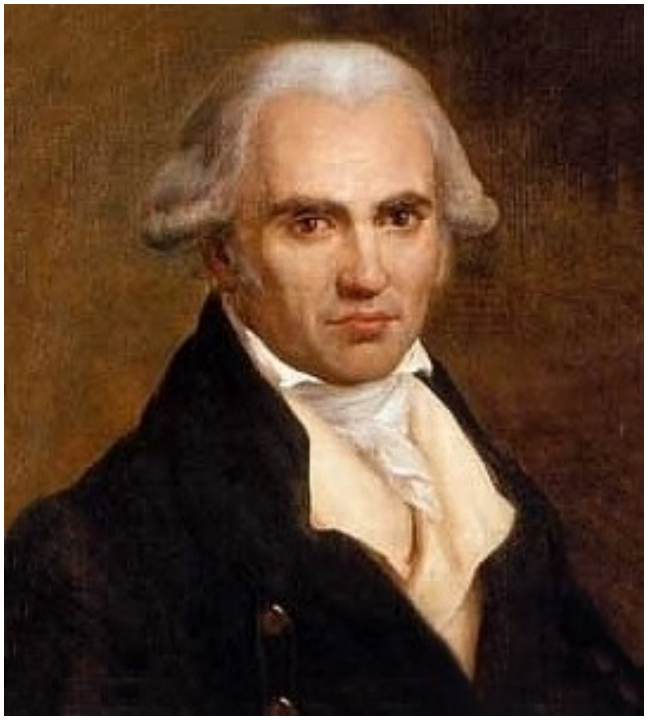
Albert Einstein
(1879-1955)

-1921 Nobel Laureate in Physics

Geometry, economics + physics

Optimal Transport

Gravity



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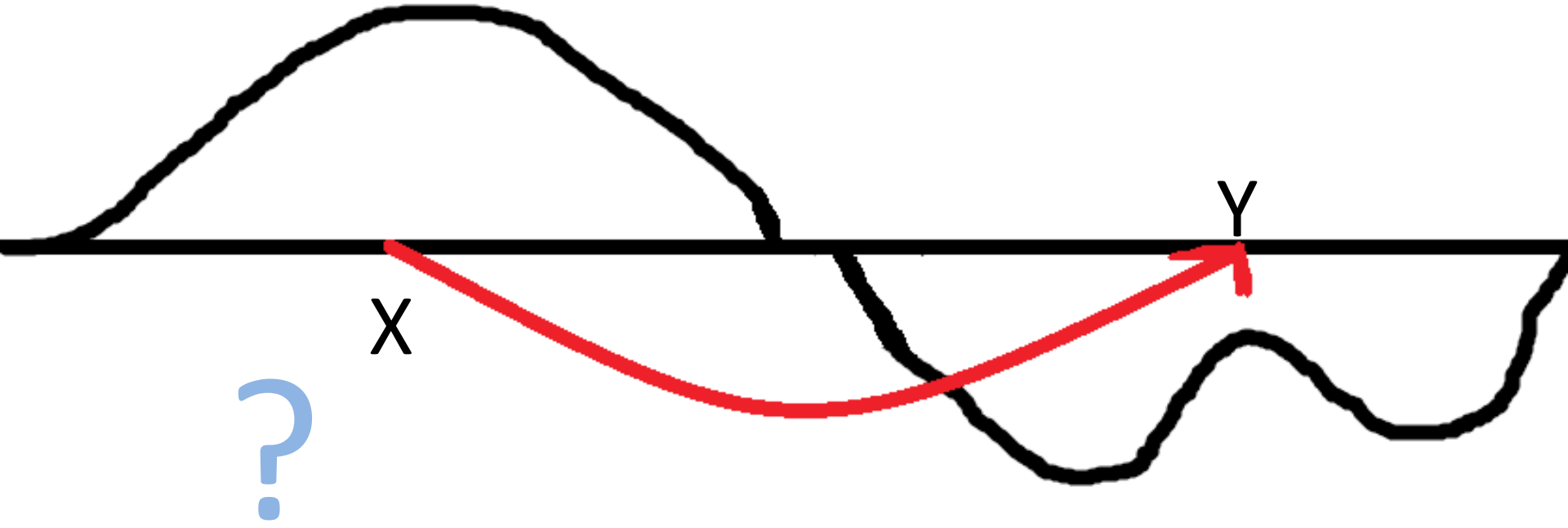
“Deblais et remblais” (Monge, 1781)



Kantorovich 1942

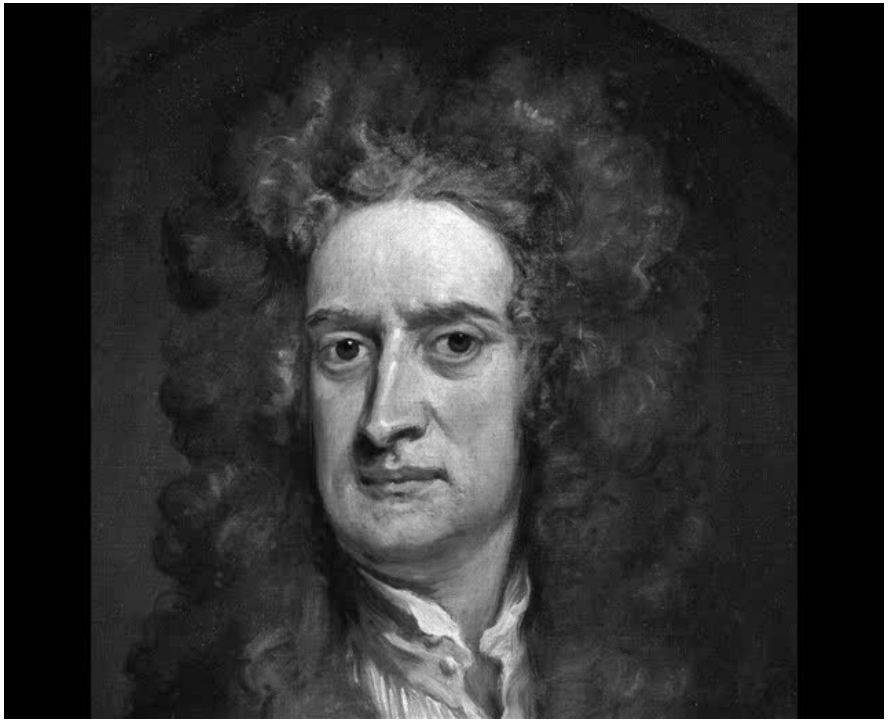
c(x,y) = cost per unit mass transported from x to y

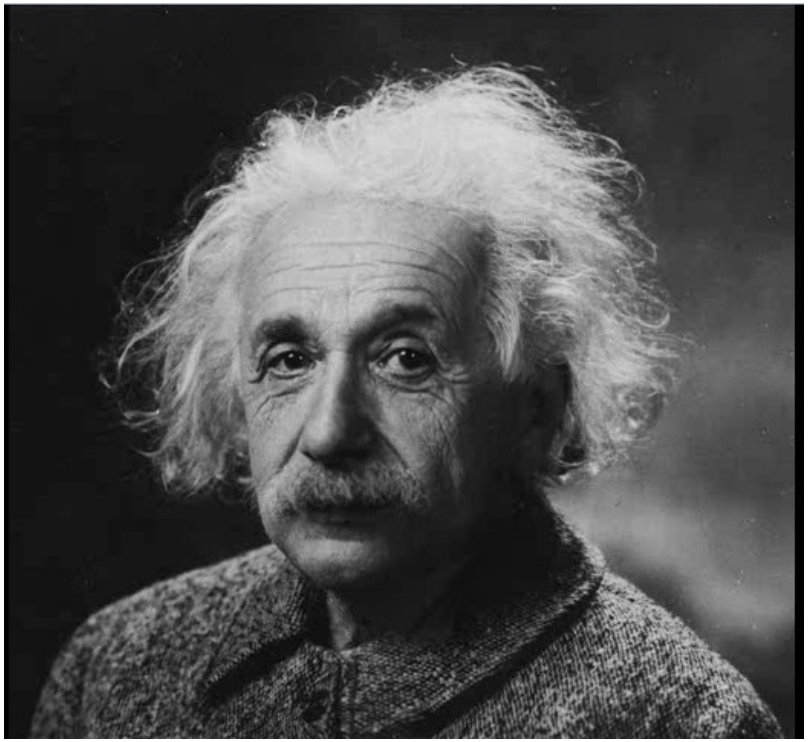
Issues

- Finding a (more than one?) solution
- Recognizing solutions
- Geometric and analytic properties

  e.g. continuous dependence of

  sink on the source supplying it?
- Influence of choice of costs
- Applications





Courtesy of Carla Cedarbaum and Marco Gajardo (MPI Golm)

Applications

- Image processing (Tannenbaum, …)

  (medicine, movies, and data compression)

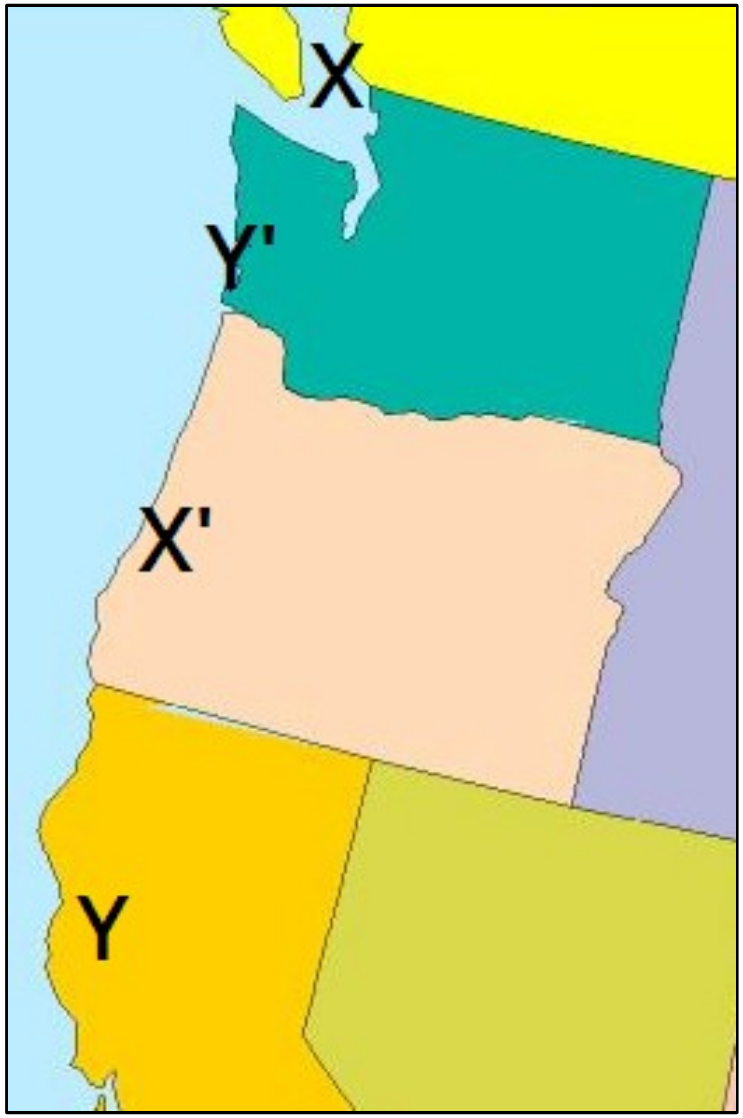
- Weather prediction (Cullen …)

- Early universe reconstruction (BRENIER, Frisch, …)

- Dissipative dynamics (OTTO, AMBROSIO,…)

Influence of choice of cost

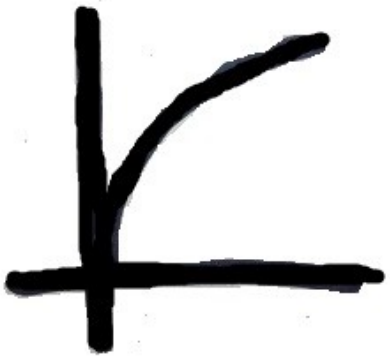
?



$c(x,y)=|x-y|2$

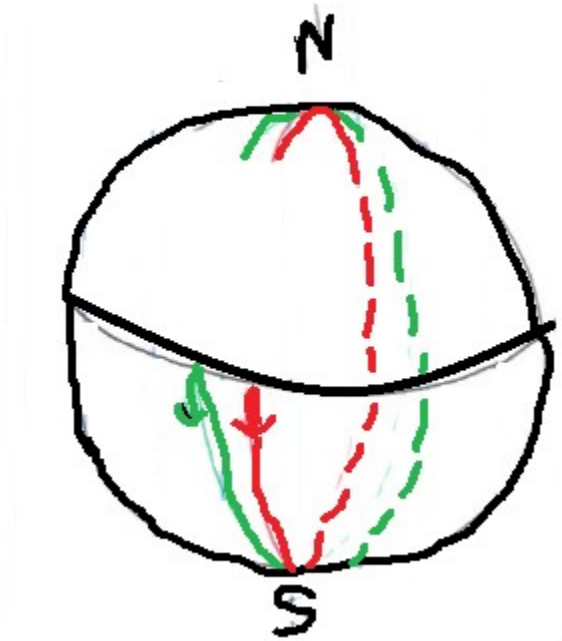
vs.

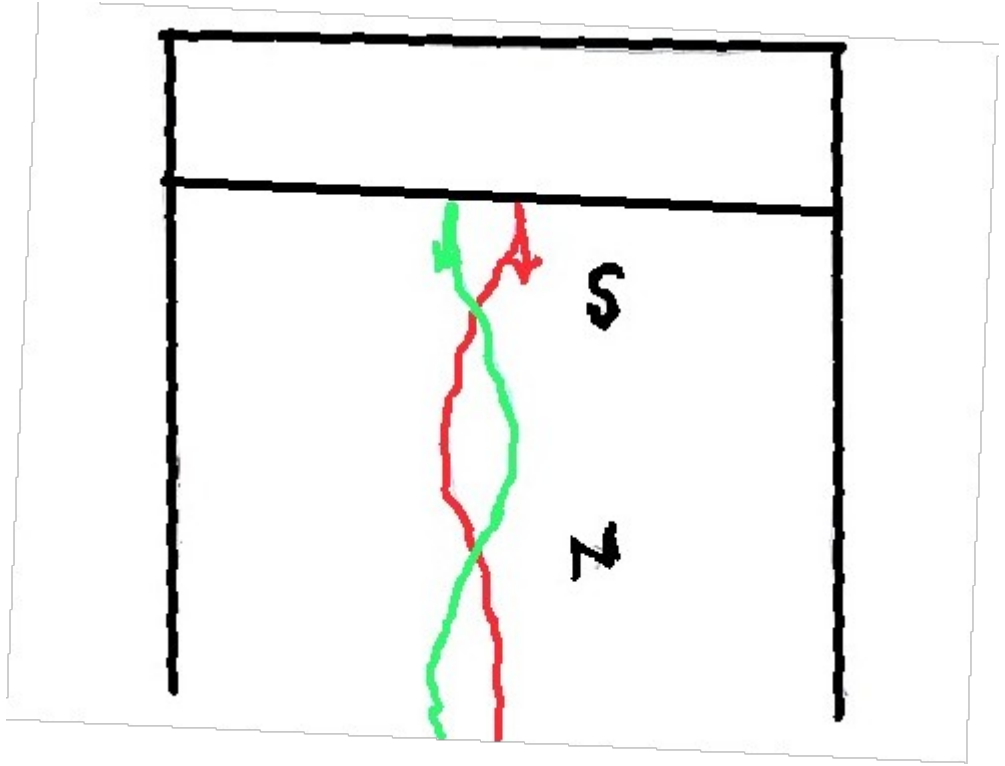
$c(x,y)=|x-y|1/2$



Mathematical General Relativity (Einstein's theory of gravity)

"gravity not a force, merely a manifestation of curvature in the underlying geometry of spacetime"



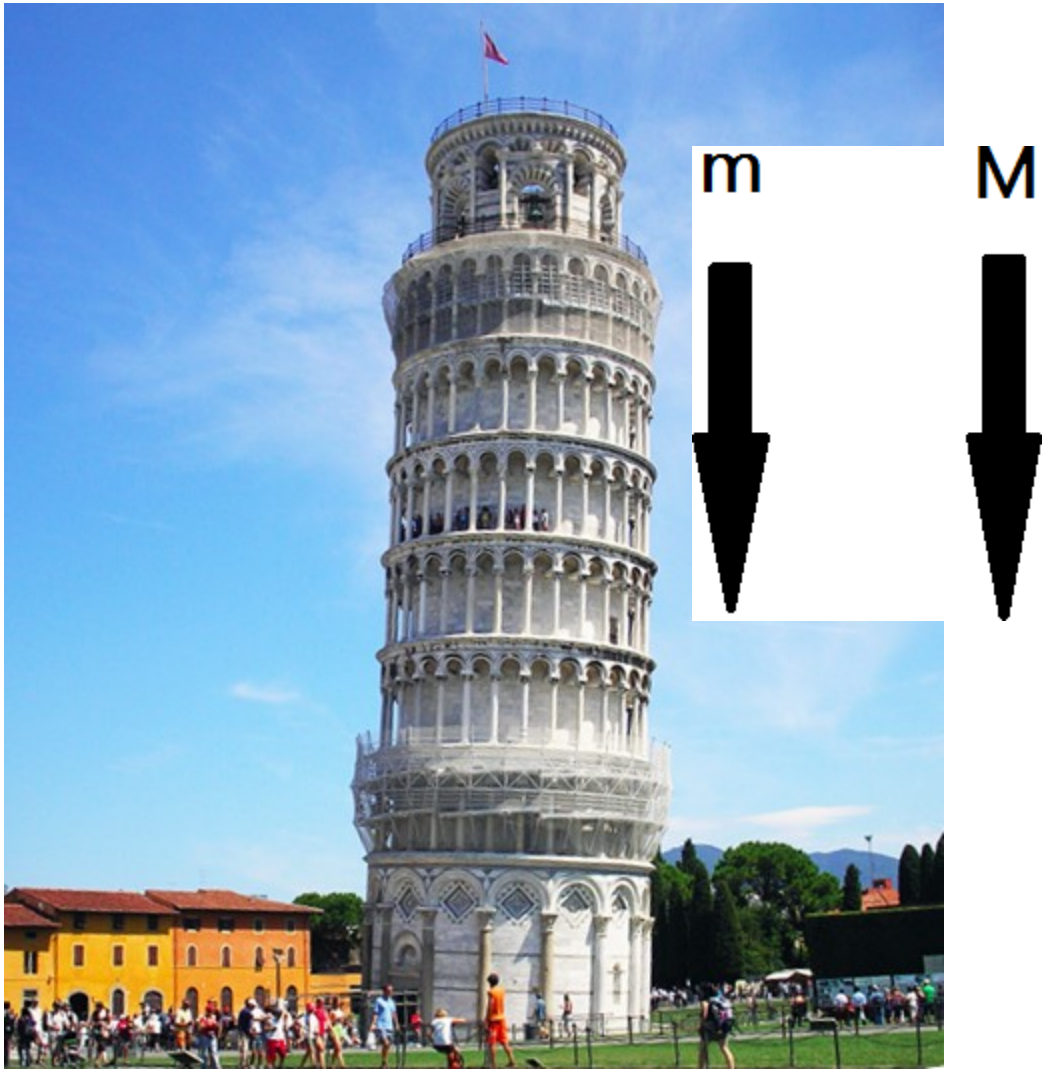


Flat Earth Society

Resolved several philosophical and practical problems:

- perihelion precession of Mercury
- bending of light around the sun
- GPS (global positioning system)
- inertial mass = gravitational mass

Caricature of Galileo's Falling Mass Experiment



The Einstein Equation

"geometry = physics"

$$G_{ab} = 8\pi T_{ab}$$

average sectional curvature in a given direction MINUS the same quantity averaged over all directions

energy and momentum fluxes of matter in system

a,b = t, x, y, z

GR: Seeks solutions to this equation and their properties

Spaceship near a black hole

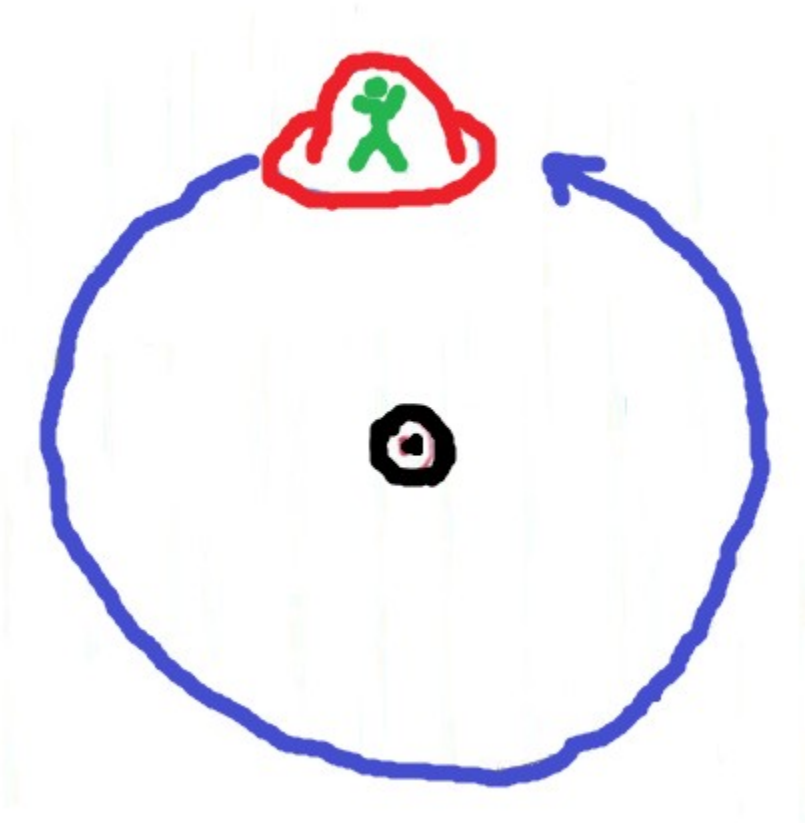
$$0 = 8\pi Ttt = Gtt$$

$$= Rtxtx + Rtyty + Rtztz$$

side to side

front to back

top to bottom



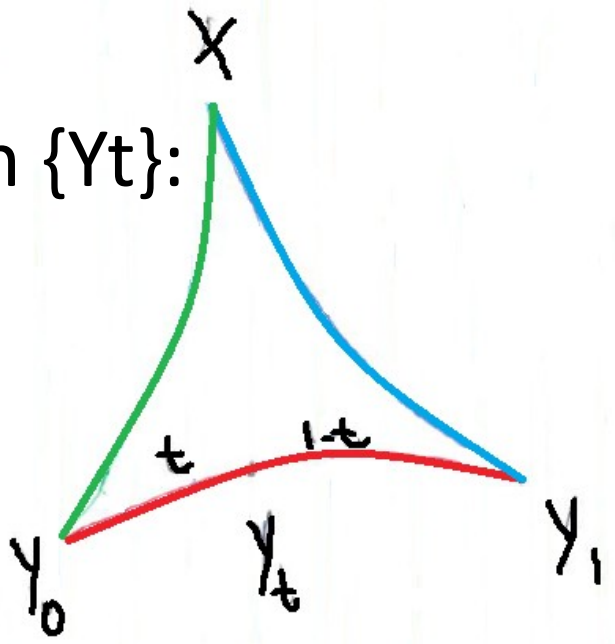
Shortest paths and triangles in a curved world

- A distance $d(Y0,Y1) \leq d(Y0,X) + d(X,Y1)$

- Induces a notion of shortest path $\{Yt\}$:

$$d(Y0,Y1) = d(Y0,Yt) = d(Yt,Y1)$$

$$t \qquad\qquad 1-t$$



Sectional curvature via comparison triangles

$d(X,Y_{1/2}) > |\tilde{X}\text{-}\tilde{Y}_{1/2}|$ ``positive sectional curvature''

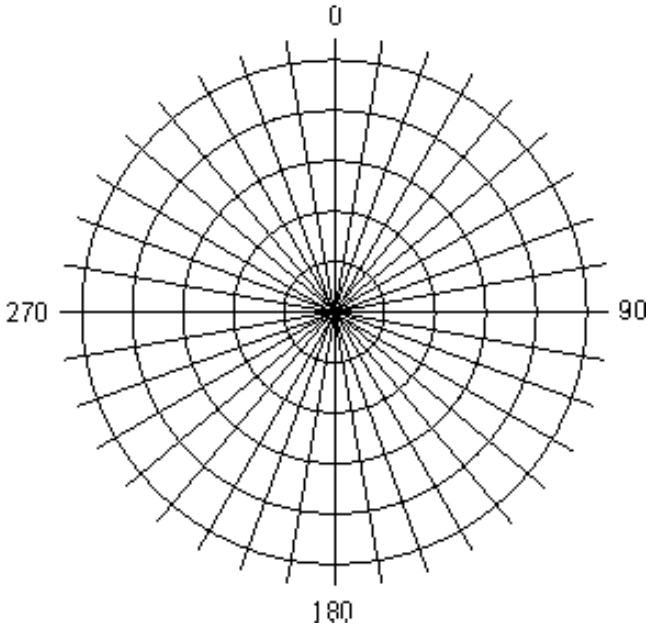
$<$ ``negative sectional curvature''

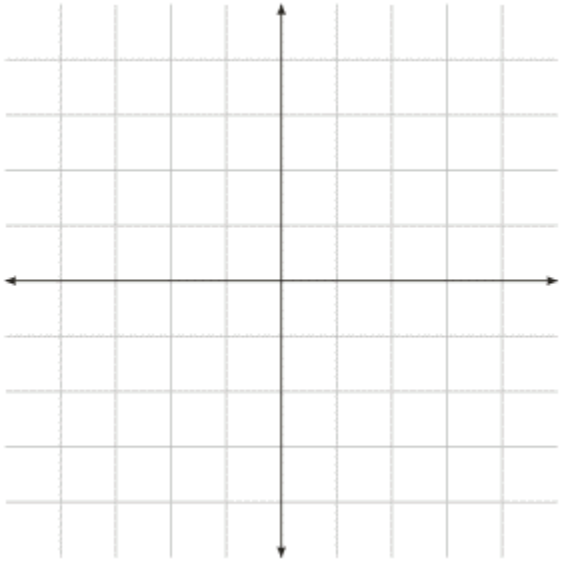
Caveat:

- in general relativity, d(x,y)2 must be replaced by a quantity that can be positive or negative, depending on whether the separation of the events x=(X,t) and y=(Y,T) in spacetime is sufficient for them to be able to influence each other, or not

Observer independence

- The precise form of Einstein's equation was derived based upon the idea of observer independence, (i.e. physics the same in all coordinate systems)





What in the world might this have to do with Optimal Transportation?!?

- (Ma, TRUDINGER, Wang ‘05)
- Local and global necessary and sufficient analytic conditions on c(x,y) for smoothness of optimal maps

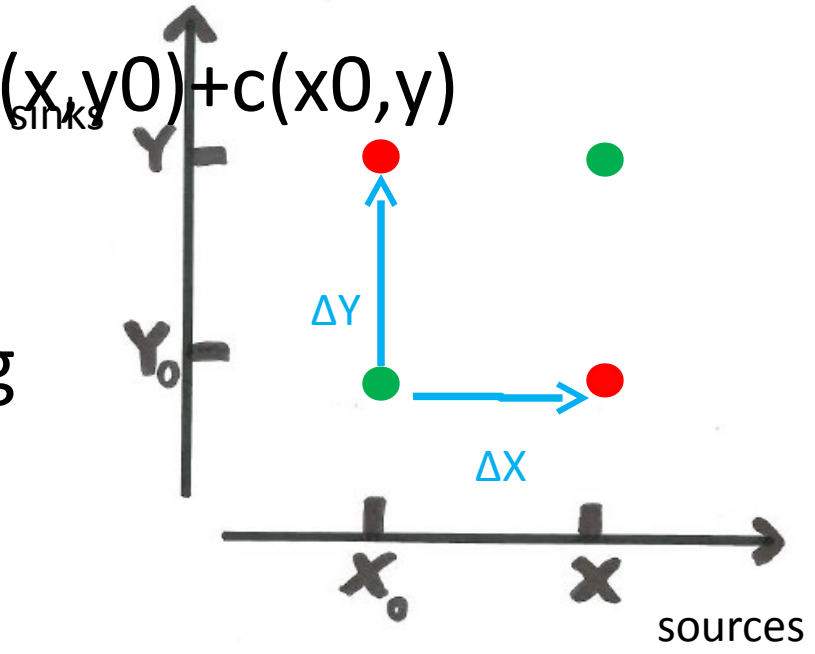
- (KIM, McCANN ‘10)

Optimal Transport

Given a pair of pairs (x0,y0) and (x,y),

$$\Delta 0(x,y) = -c(x,y) - c(x0,y0) + c(x,y0) + c(x0,y)$$

replaces d(x,y)2 in defining an analogous notion of sectional curvature at each point.



Transportation on manifolds

(A1) Assume c is C4 and for each (x0,y0) in X x Y

$\Delta 0(x,y) = -c(x,y) - c(x0,y0) + c(x,y0) + c(x0,y)$

has no critical points save (x,y) = (x0,y0).

THEOREM (Gangbo '96, Levin '99): For sources given by densities on the manifolds X and Y, Kantorovich's problem has a unique solution. Moreover, it's a Monge map: almost every mine supplies a single

The geometry of optimal transport

(A2) Assume for each (x0, y0) the Hessian

$$h = \mathrm{Hess}\ \Delta 0\ (x0, y0)$$

has no zero eigenvalues. It can then be used

as a metric tensor to define a geometry and the associated geodesics, curvature, etc.

Note this metric has signature (n,n) on Xn x Yn

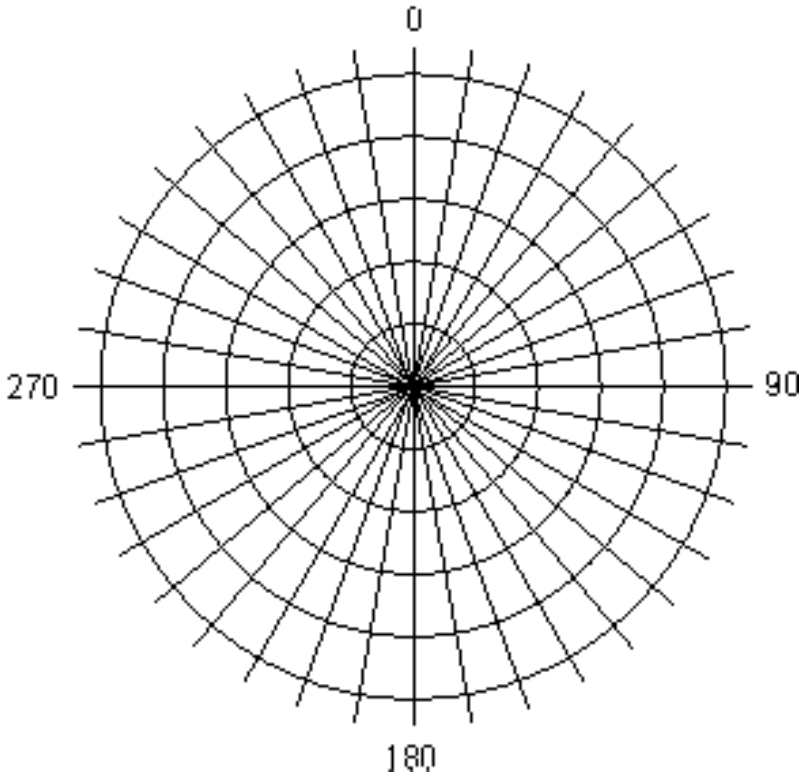
since $\Delta 0(x0 + \delta x, y0 - \delta y) = -\Delta 0(x0+\delta x, y0+\delta y) +$

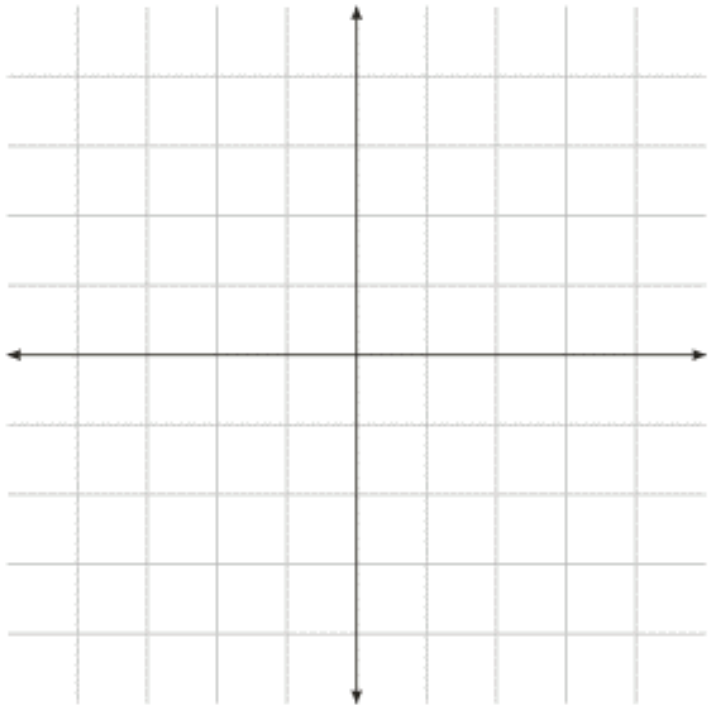
(A3) Assume each horizontal and vertical fibre is h-geodesically convex

THEOREM (Ma-Trudinger-Wang, Loeper, Kim-Mc) Non-negative sectional curvatures of each right-angled triangle with vertices (x,y), $(x+\delta x,y)$ and $(x,y+\delta y)$ is then necessary and sufficient for the smoothness of optimal maps between all

Continuity of Transport

- holds in all coordinate systems if it holds in one coordinate system!





Link to minimal surface theory

Modify the metric with a conformal factor

$$h' = k(x,y)\, h$$

chosen so the volume element of h' coincides with the product density f(x)g(y).

THEOREM (Kim-Mc-Warren) The support (=smallest closed set of full mass) of any Kantorovich optimizer is locally volume

Where's the economics?

- Monopolist nonlinear pricing (Figalli-Kim-Mc)

Where's the economics?

- Monopolist nonlinear pricing (Figalli-Kim-Mc)
- One principal vs. many agents $x \in X$ in R^m
- $0 \leq f(x)$ = relative frequency of agent type x
- $b(x,y) = -c(x,y)$ = value of car $y \in Y$ to buyer x
- $a(y)$ = principal's cost to manufacture $y \in R^n$
- PRINCIPAL: seeks to design a price menu

$v : Y \rightarrow R$ so as to maximize her profits

Participation constraint

- There is a distinguished point $y_0 \in Y$ called the `null product', which the agent is free to select and the principal is not permitted to profit from
- i.e. $v(y_0) = a(y_0)$ (= 0 WLOG)

Agent x's problem

- facing price menu v, to select that product $y_{b,v}(x)$ which attains the maximum

$$u(x) = \max \{ b(x,y) - v(y) \mid y \in Y \}$$

(whose value $u = vb$ is called x's *indirect utility*)

Principal's objective

- Find a price menu v which maximizes profits,
- or equivalently, minimizes expected losses

$$L = - Ef[\, v(yb,v(x)) - a(yb,v(x)) \,]$$

- This depends on v in a very nonlinear way.
- Life would be much better if L were convex…

Some history

- SPENCE-MIRRLEES: If X,Y are real intervals and the mixed partial derivative bxy doesn't change sign, then yb,v (x) depends monotonically on x

  hence the problem can be reduced to an ODE

- Mirrlees '71 was working on optimal taxation

- Spence '73 on educational attainment as a

- ROCHET-CHONE: if X,Y are convex domains in some Euclidean space and $b(x,y) = x \bullet y$ is bilinear, then $y=Du(x)$ and u is convex(!)

- also L is a convex function of u if a(y) is convex (but the problem cannot be reduced to a PDE)

Reformulating L as a function of u

$$u(x) = \max \{ b(x,y) - v(y) \mid y \in Y \} =: vb(x)$$

The set U of functions u which can arise in this way (meaning u=vb for some v) has enough compactness properties to know that L(u) attains its minimum (CARLIER), but not much more than that generally.

Some outstanding questions

- Are the interesting phenomena (uniqueness, exclusion, bunching,…) discovered by (ARMSTRONG and) ROCHET-CHONE, specific to their setting or more general?

- How would competition (duopoly) affect the conclusions of the model? (in this case, there is no obvious variational principle for the

CONVEXITY RESULTS (with Figalli and Kim)

THEOREM 1: Assuming b=-c satisfies (A1)-(A3),

the set U := { u = vb | v : Y ➔ R} of b-convex utilities is a *convex* set if and only if the following variant of (A4) holds:

(A5) Non-negative h-sectional curvature of each *(not necessarily right-angled)* triangle with vertices (x,y), (x+δx,y) and (x,y+δy)

- THEOREM 2: Assuming (A1-A5), the net losses L(u) form a convex function on U

  provided the principal's cost a(y) is itself

  b*-convex. (Strict convexity of L(u) follows from

  stronger variants of (A5) or the b*-convexity of a)

- Many conclusions (uniqueness, stability) follow.

Examples and non-examples

- $b(x,y) = x \bullet y + A(x)B(y)$
- $b(x,y) = d2(x,y)$ on a Riemannian manifold X=Y
- ARMSTRONG's desirability of exclusion
- maximizing societal welfare subject to a

Returning to our upcoming program…

Special long-term visitors include

ORGANIZERS: W Gangbo, Y-H Kim, R McCann

SEVERAL MEMBERS OF SCIENTIFIC COMMITTEE:

L Ambrosio, Y Brenier, F Otto, N Trudinger,

…

CLAY SCHOLARS:

DISTINGUISHED LECTURE SERIES

Sept 8-10 Simon Brendle (Stanford) Geometry

Oct (~ 14-16) Sylvia Serfaty (NYU)

Singular perturbation problems in physics

Nov 3 Keyfitz Lecturer Paul Milgrom (Stanford) Economics

CONFERENCES:

Variational problems in…

15-19 September: ECONOMICS

Optimization, transportation and equilibrium

6-10 October: PHYSICS

Nonlinearity, transport, physics and patterns

10-14 November: GEOMETRY

MINISCHOOLS FOR STUDENTS

11-12 September ECONOMICS: PA Chiappori, RMc, A Galichon (Paris), X Shi (Toronto), R Vohra (Penn)

2-3 October PHYSICS: L Ambrosio, F Otto,

R Jerrard, M Pugh, (Toronto) R Seiringer (Vienna)

GRADUATE COURSES

- MAT 1501 *Self-dual variational calculus in PDE and mass transport* NASSIF GHOUSSOUB

- MAT 1901 *Variational problems in physics, economics and geometry*

Consisting of the three minischools + projects (coordinated by RMc).

PROGRAM POSTDOCS

Manuel Gnann (’14 Leipzig – Otto)

Thin film dynamics / 4th order nonlinear diffusions

- Arjun Krishnan (’14 New York – Varadhan)

Variational principles governing percolation shapes

- Jean Louet (’14 Paris Orsay – Santambrogio)

Optimal transport with gradient penalization

Affiliated postdoctoral fellows

Fields-Perimeter Africa Postdoctoral Fellow

Cyril Batkam (14 Sherbrooke – Colin/Kaczynski) *variational PDE*

- *On leave from Paris (Serfaty)*

Mircea Petrache ('13 ETHZ – Riviere) *geometric measure theory*

FIELDS MEDAL SYMPOSIUM

17-20 November

The many faces of entropy: kinetic theory, optimal transport, geometry

In honor of CEDRIC VILLANI

Organized by: A Figalli (Austin) RMc (Toronto)

OUTREACH EVENTS

Include a screening of 2014's d'Alembert prize winning film-maker Olivier Peyon's film

Comment j'ai deteste les maths…

(How I came to hate math)

at the Alliance Francais, 24 Spadina Rd

To be followed by a discussion panel including both Peyon and Villani.

HOPE TO SEE YOU ALL THERE!

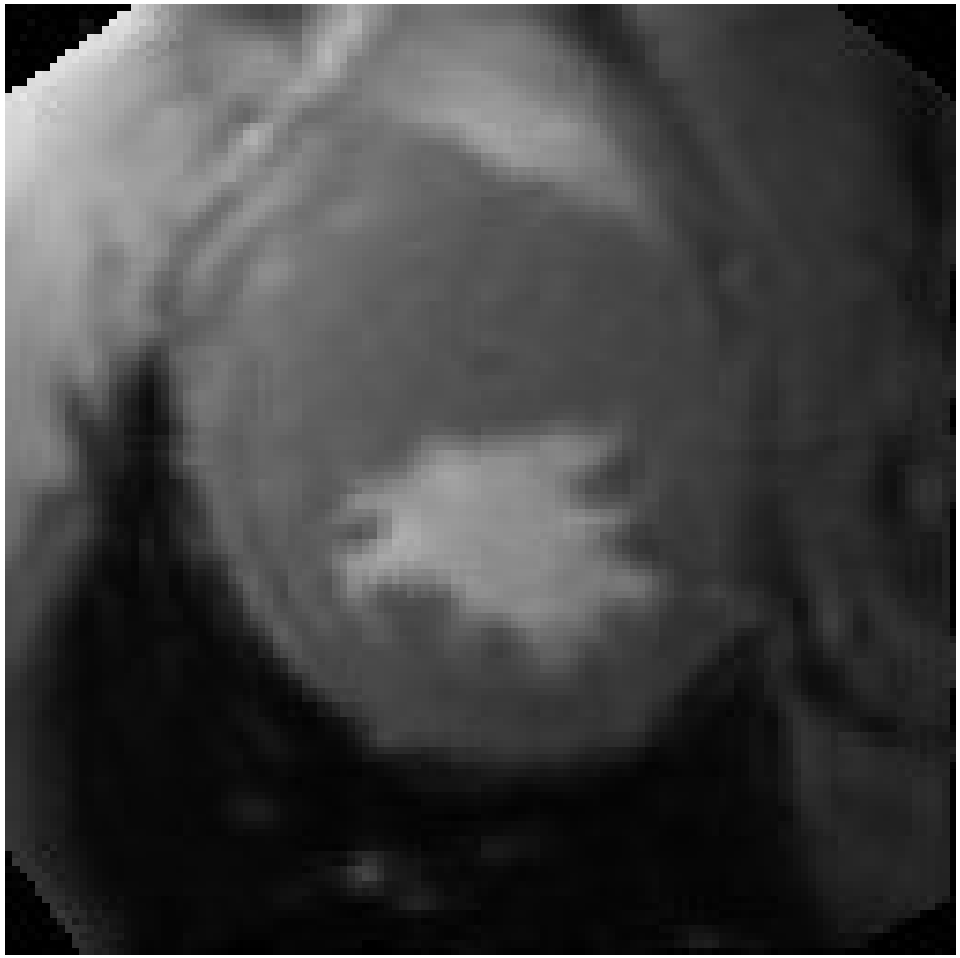
Images courtesy of:

- Boomvisits.com
- Carla Cedarbaum and Marco Gajardo (MPI Golm)
- Yann Brenier (MSRI)
- Media.techtarget.org
- Mmfd.nsu.ru
- Steve Haker, Allen Tannenbaum et al
- Theorganicprepper.ca
- Users.cm.jmu.edu
- Wallpapers.vah.in
- Wikimedia.org
- Yellowmaps.com

For further information:

- Robert McCANN: surveys at www.math.toronto.edu/mccann
- Kip Thorne: *Black Holes and Time Warps: Einstein's Outrageous Legacy* (1995)
- Cedric VILLANI: *Topics in Optimal Transportation* (2003); *Optimal Transport, Old and New* (2008)
- Robert Wald: *General Relativity* (1984)

Thanks also to:



Courtesy of Steve Haker, Allen Tannenbaum, et al