Kaprekar Number

Santanu Bandyopadhyay

Department of Energy Science and Engineering

Indian Institute of Technology Bombay

Powai, Mumbai, 400076, India

8 June 2020

In 1980, Indian mathematician D. R. Kaprekar introduced some interesting numbers; today, these numbers are known as *Kaprekar Numbers* (Kaprekar, 1980). What makes the Kaprekar numbers curious and interesting? Let's consider an example. 297 is a Kaprekar number: square of 297 is 88209; divided it into two portions, 88 and 209; now total them, 88 + 209 to get 297 back. Consider an n–digit number k (take another example of 45). Square it ($45^2 = 2025$) and add the right n digits (that is 25) to the remaining n or n-1 digits (in this case, 20). If the resultant quantity is k, then k is a Kaprekar number (45 is a Kaprekar number as 20 + 25 = 45). See these additional examples of these numbers:

$9^2 = 81 \Rightarrow 8 + 1 = 9$

$55^2 = 3025 \Rightarrow 30 + 25 = 55$

$7272^2 = 52881984 \Rightarrow 5288 + 1984 = 7272$

There are infinitely many Kaprekar numbers and first few of them are 1, 9, 45, 55, 99, 297, 703, 999, 2223, 2728, 4879, 4950, 5050, 5292, 7272, 7777, 9999, ... (A006886 in OEIS).

In his original paper, Kaprekar included 9 as a Kaprekar number but did not list 99, 999, 9999, ... etc. Interestingly, the string of 9s or $10^n - 1$ (for all $n \geq 1$) are always Kaprekar numbers. As $(10^n - 1)^2 = (10^n - 2) \times 10^n + 1$ and this implies $(10^n - 2) + 1 = 10^n - 1$ (Iannucci, 2000). We can visualize this in the following way:

$$(\underbrace{9...9}_{n\text{ nines}})^2 = \underbrace{9...9}_{n-1\text{ nines}}8\underbrace{0..0}_{n-1\text{ nines}}1 \Rightarrow \underbrace{9...9}_{n-1\text{ nines}}8 + \underbrace{0..0}_{n-1\text{ nines}}1 = \underbrace{9...9}_{n\text{ nines}}$$

Similarly, Kaprekar did not include 181819 and 818181; however, these are also Kaprekar numbers (Charosh, 1981).

It is not difficult to generate all *n*-digit Kaprekar numbers, for any given *n*. Let's demonstrate this for $n = 3$. First, determine two factors of $10^n - 1$ in such a way that their product is $10^n - 1$, but they are prime to each other. Such factors are called *unitary divisors* (Iannucci, 2000). For $n = 3$, unitary divisors of 999 are: $999 = 1 \times 999 = 27 \times 37$. Let's consider the second factors, i.e., unitary divisors 27 and 37. Now determine a multiplier of 27 that leaves a remainder of unity when divided by 37. Note that $27 \times 11 = 297$ and when we divide 297 by 37, we get: $297 = 8 \times 37 + 1$. 297 is a Kaprekar number. Similarly, $37 \times 19 = 703 = 27 \times 26 + 1$ and 703 is a Kaprekar number.

There are also higher-power Kaprekar numbers. For example, 45 is a cubic Kaprekar number as $45^3 = 91125$, and $9 + 11 + 25 = 45$. The first few cubic Kaprekar numbers are 1, 8, 10, 45, 297, 2322, 2728, 4445, 4544, 4949, 5049, 5455, 5554, 7172, 27100, 44443, 55556, 60434, 77778, 143857, ... (A006887 in OEIS). Iannucci and Foster (2005) noted that numbers of the form $5 \times 10^{m-1} \times (10^m \pm 1)$ for $n \geq 3$, are always cubic Kaprekar numbers. Therefore, 499500, 500500, 49995000, 50005000, … are all cubic Kaprekar numbers. It may be interesting to note that numbers such as 1, 45, 297, 2728, 77778, 329967, 461539, … etc. are both quadratic and cubic Kaprekar numbers.

Interestingly, 45 is also a biquadratic Kaprekar number as $45^4 = 4100625$, and $4 + 10 + 06 + 25 = 45$. The sequence of biquadratic Kaprekar numbers run as follows 1, 7, 45, 55, 67, 100, 433, 4950, 5050, 38212, 65068, ... (A171493 in OEIS). Other than 1, 45 is the only number that is simultaneously quadratic, cubic, and biquadratic Kaprekar number (Spencer, 2014).

Kaprekar numbers prompted us to ask another interesting question. Is there an integer *m* (>1) such that for each integer exponent *n*, the sum of the decimal digits of m^n is equal to *m*? Please note the difference. Instead of splitting the resultant number in terms of the number of digits of the original number, we are simply adding all digits of m^n. There are plenty of such numbers. For example, 2^1 (= 2), 9^2 ($= 81 \rightarrow 8+1 = 9$), 8^3 ($= 512 \rightarrow 5+1+2 = 8$), … 46^8 ($= 20047612231936 \rightarrow 2+0+0+4+7+6+1+2+2+3+1+9+3+6 = 46$), ... etc. For $n \leq 104$, such a solution exists. However, there is no solution for $n = 105$, as shown by Norman Megill (Guy, 1990).

D. R. Kaprekar

Dattatreya Ramchandra Kaprekar (1905–1986) was born on 17th January 1905 in Dahanu, Maharashtra. His mother, Janakibai, expired when he was only eight years old.

His father, Ramchandra, was a clerk. Kaprekar went to school in Thane, Maharshtra. He studied at Fergusson College in Pune. In 1927, he received the Wrangler R P Paranjpe Mathematical Prize for his essay on the theory of envelopes (Gupta, 2006). He received his bachelor's degree in 1929 from Bombay University and never did any postgraduate education. He was a school teacher in the small town of Devlali, near Nashik, Maharashtra, till his retirement in 1962. After his retirement, he was given a monthly grant of Rs. 500 for five years by the University Grants Commission, India.

He became well known recreational mathematician for his investigations and discoveries related to different exciting properties pertaining to number theory. He was a good school teacher. He used to motivate his pupils using his fascination for numbers. He used to deliver lectures at near-by colleges about unique properties of some impressive numbers. He used to say about himself, "*A drunkard wants to go on drinking wine to remain in that pleasurable state. The same is the case with me in so far as numbers are concerned*." He used to call himself *ankamitra* (meaning "friend of numbers"). Although Kaprekar was mostly unknown outside the field of recreational mathematics in his lifetime, his work and its impact on number theory have become widely recognized in recent years.

References

Charosh, M., 1981. Some Applications of Casting Out 999...'s, *Journal of Recreational Mathematics*, 14(2), 111-118.

Gupta, R.C., 2006. D. R. Kaprekar-A Birth Centenary Tribute (1905-1986), *Indian Journal of History of Science*, 41(2), 223-226, 2006.

Guy, R.K., 1990. The second strong law of small numbers, *Mathematics Magazine*, 63(1), 3-20.

Iannucci, D.E., 2000. The Kaprekar Numbers. *Journal of Integer Sequences*, 3, Article 1.2.

Iannucci, D.E. and Foster, B., 2005. Kaprekar Triples. *Journal of Integer Sequences*, 8, Article 5.4.8.

Kaprekar, D., 1980. On Kaprekar Numbers. *Journal of Recreational Mathematics*, 13(2), 1980-81.

OEIS, The On-Line Encyclopedia of Integer Sequences (https://oeis.org/).

Spencer, A., 2014. *Adam Spencer's Big Book of Numbers: Everything you wanted to know about the numbers 1 to 100*. Brio Books.