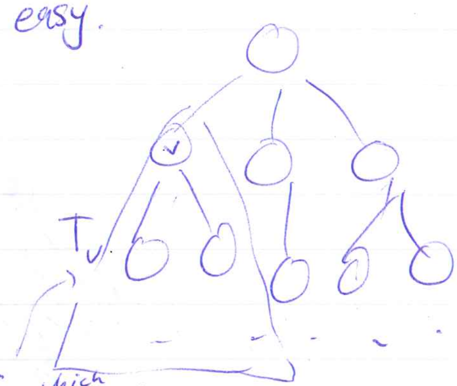
①

Lecture 17 - Treewidth

[Scribe - slides by Marx, Bodlaender]

On trees, NP-hard probs often easy.



eg.①: Max [-weighted]-Indep.-Set

[set of non-adjacent vtcs].

[How?] Recursion/dynamic prog.

Let $MIS^+[v]$ = max indep set in T_v which includes v

$MIS^-[v]$ = ~~~ which doesn't include v.

If v has kids $w_1, \dots, w_d$, $MIS^+[v] = 1 + \sum_{i=1}^{d} M^-[w_i]$

$MIS^-[v] = \sum_{i=1}^{d} \max\{M^+[w_i], M^-[w_i]\}$.

[Leaves are trivial.

End up doing post-order trav. of tree.] $O(n)$ time. [$|V|$]

eg.②: Given CSP inst., (Ω, Ψ) "primal graph" (Gaifman): vtcs = vbls

$(u,v) \in E$ if u,v together in some constr. [scope].

Sps primal graph is a tree ($\Rightarrow \Psi$ arities ≤ 2).

[a bit restrictive, though 3 Col. is a nice eg.]

Then Satisfiability $\in$ P.

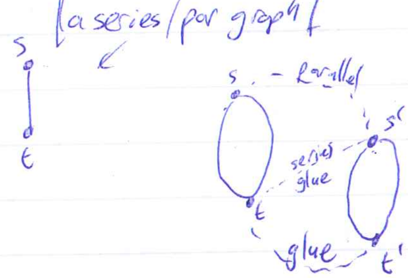
D.P.: Let $E[v, a]$ ($v \in V$, $a \in \Omega$) = $\begin{cases} \text{true if } \exists \text{ partial satisfying asgn. to } T_v \text{ with } v \leftarrow a \\ \text{false else.} \end{cases}$

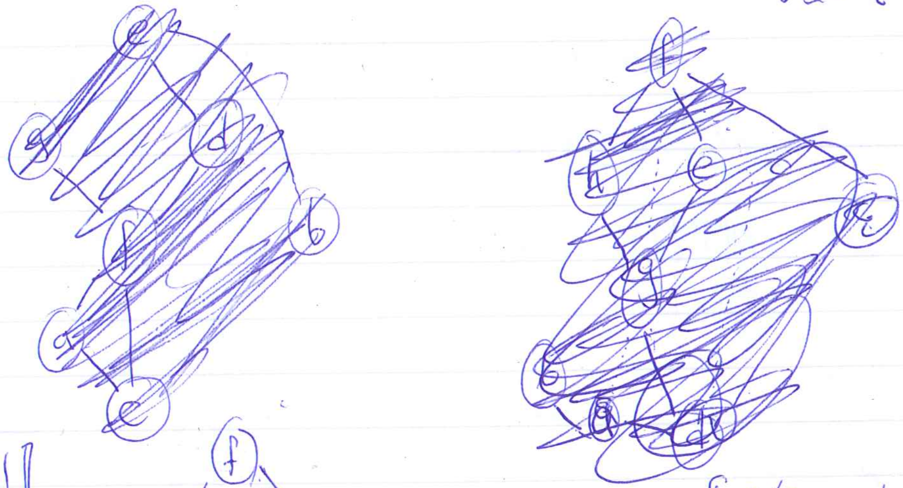
[Leaves triv. (true unless unary constr. forbids). If you have E for v's children it's easy....]

Q: More gen. class of graphs? [In poly time?]

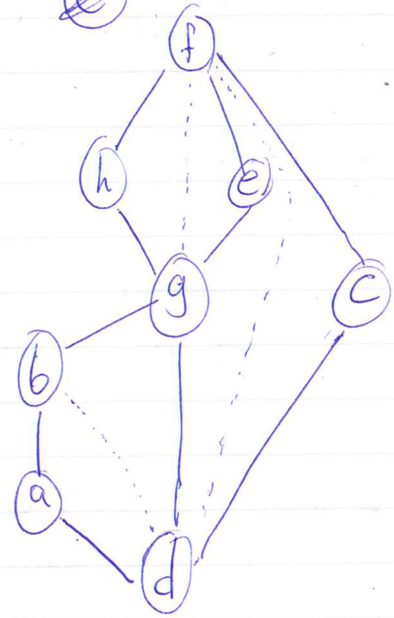
'70s: "Series parallel graphs":
[Recursive def^n . . . .]

[a series/par graph]

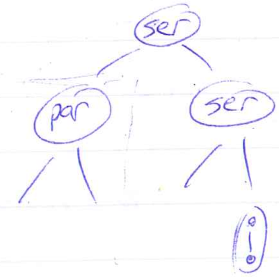


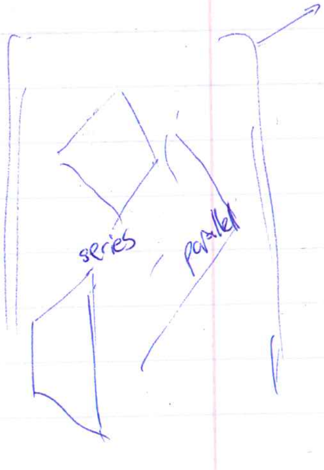


[Leave up!]



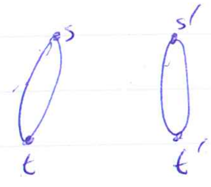
[S/P graphs have a kind of tree decomp.]





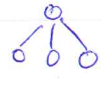
ex: MIS on S/P graphs in O(n) time by D.P.

(Hint



MIS$[s^+,t^+]$
$[s^+,t^-]$
$[s^-,t^+]$, $[s^-,t^-]$).

[Rem: also works for subgraphs of S/P graphs
E.g.



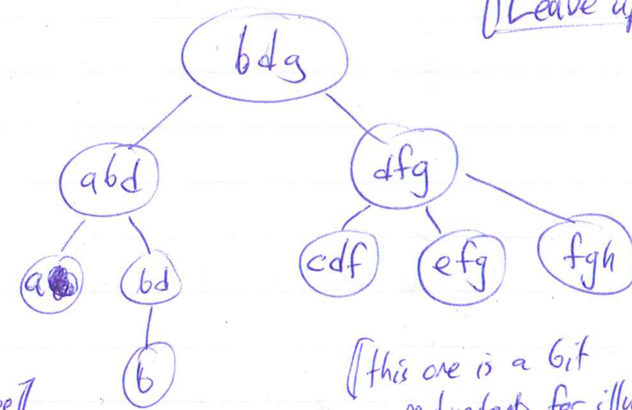
]

[80's: several groups of researchers extended to even more "weakly tree-like" graphs, w/ some "tree decomp"]

def: Tree decomposition of $G=(V,E)$:

[Leave up]

A tree T of "bags" X s.t.:



① if $(u,v) \in E$ then u,v together in some bag

② $\forall v \in V$ the bags containing v are connected in T [form a subtree]

[this one is a bit redundant for illustr. purposes]

[Do example of b, d]

[D.P. will work w/ bags, want them small. Could just shove everything in 1 bag!]

Width: (max bag size) -1.

[-1 is so that trees will have t.w. 1. E.g. has width ②.]

Treewidth of G: least width of a tree decomp.
$tw(G)$

[Introduced in 80's by zillions of indep groups...
Halin '72 Arnborg–Proskurowski, Robertson–Seymour, ...]

[Also, $\exists$ zillions of equivalent definitions....]

facts: • Trees have t.w. 1 $\longrightarrow$ bags are edges
• $tw(G)=1 \iff G$ is a forest [easy]
• $tw(G)=2 \iff G$ is subgraph of a series/par. graph. [not as easy]

"trees: recursive gluing togeth single vertices
s/p: ~~~~~~~~ 2 "
tw k: ~~~~~~~~ k vtcs"

[won't make precise]

• deleting edges cannot incr. tw. [[Same with contracting, actually.]]

facts: • ~~[illegible]~~ • Any clique in G must be ~~[illegible]~~ in a bag.
$\Rightarrow tw(K_n) = n-1$.

def: • G is <u>chordal</u> if every cycle of len ≥ 4 has a "chord"
[[Example graph not chordal.]]

• adding edges to get a chordal graph $\equiv$ <u>triangulation</u>

fact: $tw(G) \leq k \iff$ G has triangulation w/ max clique size $\leq k+1$

[[Example graph: Can you triangulate so no 4^+-cliques?]]
[["Chordal graphs are like maximal treewidth graphs"]]

fact: G chordal $\iff$ has tree decomp w/ each bag a clique
$\iff \exists$ "perfect elim. ordering" [[abc...h in eg.]]
[[for each vtx, higher #'d vtcs form a clique]]
[[start with k+1 clique, keep tacking on cliques]]

~~[[illegible]]~~

<u>ex</u>: any tree decomp easily transformed to a <u>smooth</u> one

def:
all bags have k+1 vtcs
nbring bags have size-k intersection.

~~[illegible]~~ ~~Seymour Thomas~~

[[in example, just contract up junk on bottom left; sometimes you need to split edges]]

<u>~~rem~~</u>: smooth tree decomps have $n - tw(G)$ edges
[[clear from picture. Less than n, in partic.]]

[[One more fun characterization...]]

[Seymour-Thomas]: $tw(G) \le k \iff k+1$ cops can win the Cops & Robber game.

[[Explain game in words. Robber has great speed, can walk along edges arbit. fast. Cops can be on nodes or hovering in helicopters. Cops decide to land at v, robber can see them coming, run.]]

[2 cops nec. & suff. on a tree is easy.]]

[Illustrate 3 cops winning on eg. graph. Uses smoothness.]]

($\Rightarrow$) easy. ($\Leftarrow$) hard.

cor: $m \times m$ grid has $tw \ge m-1$ ($m-1$ cops can't win)

hint: $\exists$ a cop-free row, col.

ex: $tw(m \times m$ grid$) = m$ [need slightly better robber strat.]]

[Rem: shows that even a deg. 4 planar graph can have $\sqrt{n}$ t.w.]]

[[In a second we'll see that $O(1)$ tw $\Rightarrow$ fast algs for NP-hard probs. These typically req. the tree decomp itself... How to find?]

[Tree Decomp algs when $tw(G) = k$.]

[ACP'87] : exact in time $n^{k+O(1)}$ ← (2). Hint: only n^{k+1} cop locales [can enum. all strats...]].

[Bod'96] —— $2^{\tilde{O}(k^3)} n$. [Horrible consts. Also, not great for $k = O(\log n)$, a case we'll care a lot]]

⋮

[BDDFLP'13]: width $5k+4$ in time $2^{O(k)} n$. [Cool. $O(k)$ approx is fine.]]

[FHL'08] : $O(k\sqrt{\log k})$ in time poly(n).

[Finally, algs on bdd treewidth graphs.]

thm: [easy] If the primal graph of a CSP inst. has treewidth $\le k$, sat'bility decidable in $n^{k+O(1)}$ time.
$(\not\Omega(\ldots)^{k})$

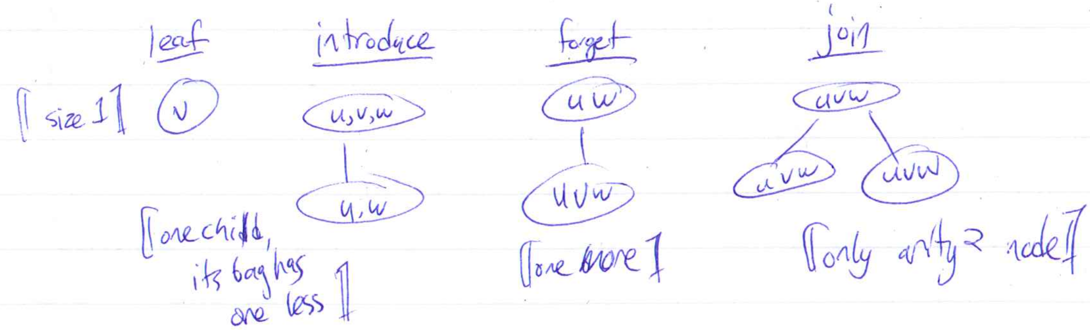
[E.g. 3Sat on $O(1)$-tw graphs $\to$ n.p.]

[Rem: every constr $\to$ clique $\to$ in single bag]

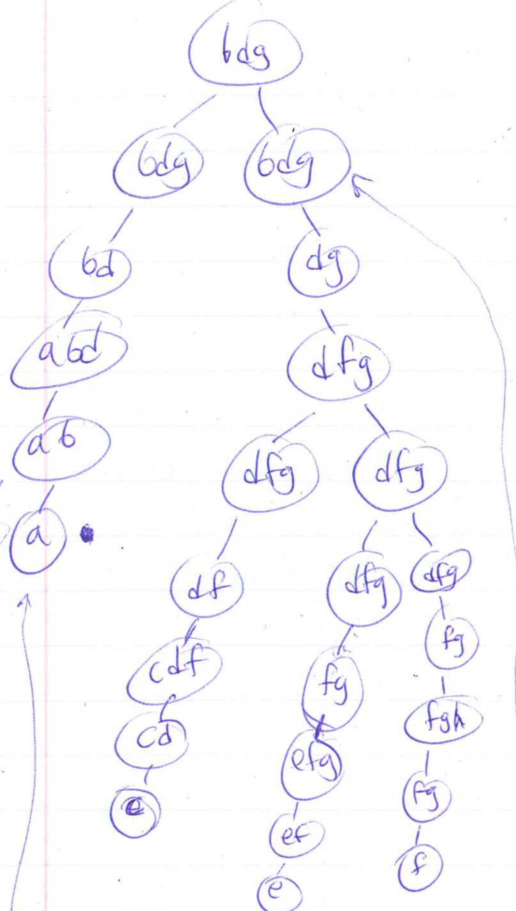
thm: ~~Other probs~~ Also solvable in poly-time when $tw(G) \le O(1)$:
- ~~...~~ colorability, Ham-Cycle/TSP, Clique, ...
- vtx-disj. paths.. [not edge-]

Courcelle Thm: Let F be formula in extended monadic 2^{nd}-order logic, like "$\exists C \subseteq V \ \forall v \in C \ \exists u_1, u_2 \in C . (u_1 \ne u_2 \wedge adj(u_1, v) \wedge adj(u_2, v))$"
[quantif. over sets of edges, and "incident(e,v)" OK too] "$G=(V,E)$ has a cycle."
Then deciding F on G with $tw \le k$ in $f(k,F) \cdot O(n)$ time [consts. is truly horrible!]

~~Class~~ Basic illustration ① [Not strictly nec., but helpful]
Convert to "nice" tree decomp:
rooted $\le$binary tree,
4 types of node:



ex: Convertible to $O(kn)$-node decomp in $O(k^2 n)$ time.

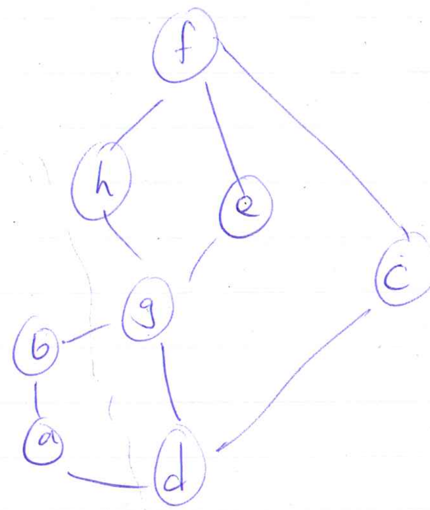


② $\boxed{\text{D.P.}}$

. 3-Colorability Example.

For bag X, coloring $c: X \to \{R, G, B\}$

Let $E[X, c] = \begin{cases} \text{true if } c \text{ can be extended to a proper col. of } V_X := \text{vtcs in } X \text{ and descendent bags} \end{cases}$



leaf (easy): $a \to R$, TRUE

$a \to G$ ✓

$a \to B$ ✓.

intro nodes: Say

$\Leftarrow$ introducing v.

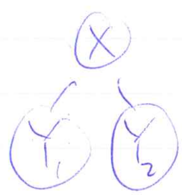
$$E[X, c] = \begin{cases} E[Y, c|_Y] & \text{if } c(v) \neq c(u) \ \forall \text{ nbrs } u \text{ of } v. \\ \text{else FALSE.} \end{cases}$$

$b \to R$, $d \to G$, $g \to R$. ✗

$b \to R$, $d \to B$, $g \to G$? deps if $d \to B$, $g \to G$ okay

forget nodes: $E[X, c]$ = true iff $E[Y, c']$ true for ≥ 1 of the three exts c' of c.

join nodes:



$E[X, c] = E[Y_1, c] \wedge E[Y_2, c]$.

TIME: $\tilde{O}(3^k) \cdot n$

[Finally, although planar graphs can have huge treewidth can often upgrade treewidth algs to planar PTASes.]

thm: [Eppstein] A planar graph w/ diam D has tw. $\le 3D-2$.

thm: Let G be planar. [higher genus vers too.] Let $k \ge 2$.
In poly(n) time can partition G's edges into k sets such that when any 1 set is
→ deleted [Brenda Baker '94]
→ contracted [DHM '07]
the result has t.w. $O(k)$.

→ Set $k = O(\frac{1}{\varepsilon})$ → "neglig."

eg. cor: Can find $(1-\varepsilon)$-approx. max-indep.-set in planar graph in time $2^{O(1/\varepsilon)} \cdot \text{poly}(n)$
in fact, n.