Lecture 16 – CSPs

[[Last time we introduced the Max-Cut problem as a means of motivating S.D.P. Let's return.]]

Max-Cut: Given $G=(V,E)$, find $F: V \to \{-1,1\}$ to maximize $\Pr_{(v,w)\sim E}[F(v) \neq F(w)]$.

$\boxed{\text{SDP}}$ relaxation:

[[poly-time solvable]]

$$\max \operatorname*{avg}_{v\sim w \in E} \left\{ \tfrac{1}{2} - \tfrac{1}{2} y_{vw} \right\}$$

$$\text{s.t.} \quad y_{vv} = 1 \quad \forall v$$

$$Y = \left(y_{vw} \right) \text{ is } \underset{\text{(and symm.)}}{\text{PSD}}$$

[[let's build into def^n of PSD]]

[[attempt to enforce]] relaxation of "$\exists \{x_v\text{'s}\}$ s.t. $y_{vw} = x_v x_w \ \forall v,w$"

rec: $Y \in \mathbb{R}^{n\times n}$ is PSD $\iff \exists\, U \in \mathbb{R}^{m\times n}$ s.t. $Y = U^T U$

$(\iff \exists\, U \in \mathbb{R}^{n\times n})$

$\iff \exists$ joint r.v.'s $X_1, \dots, X_n$ s.t. $y_{vw} = E[X_v X_w] \ \forall v,w$

$\iff \exists$ vectors $\vec{U}_1, \dots, \vec{U}_n \in \mathbb{R}^m$ s.t. $\langle \vec{U}_v, \vec{U}_w \rangle = y_{vw}$

$(\iff \exists$ vectors in $\mathbb{R}^n)$

$\therefore$ $\boxed{\text{SDP}} \iff \max \operatorname*{avg}_{v\sim w \in E} \left\{ \tfrac{1}{2} - \tfrac{1}{2} \langle \vec{U}(v), \vec{U}(w) \rangle \right\}$

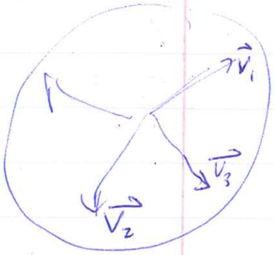
s.t. $\vec{U}: V \to \{\text{unit-len. vecs in } \mathbb{R}^n\}$

[[pretty geometrically natural.]] [[trying to put vtcs of G on sphere to "spread out" edges]]

[max-cut]

Relaxation check: $\mathrm{Opt}(G) \le \mathrm{SDPOpt}(G)$ $\underbrace{\le 1,\ \text{why?}}$

b/c: $\vec{U}(v) = (F^*(v), 0, 0, \dots 0)$ valid for SDP.
Max-cut



Rounding [GW94]

- Solve SDP, get optimal $\vec{U}^*$. Not^n: $\vec{v} := \vec{U}^*(v)$.
- Now what?

Pick rand. hyperplane thru 0, partitions vtcs/vecs into 2 parts.
[rotationally symmetric normal. What rotationally symm. dist do we know??]

→ Pick $\vec{g} = (g_1, \dots, g_n)$, g_i's indep Gaussians, as hyperplane normal. Define cut $F : V \to \{-1, 1\}$

$$v \longmapsto \underbrace{\mathrm{sgn}(\langle \vec{v}, \vec{g} \rangle)}$$

[that's the alg. Analysis...?]

Maybe skip → Handle

rem: Let $Z_v = \langle \vec{v}, \vec{g} \rangle \quad \forall v$ [Joint real Gaussians Lin comb. of $g_1 \dots g_n$ is Gaussian]

$Z_1, \dots, Z_n$ are joint Gaussian r.v.'s

$$E[Z_v Z_w] = E\left[\left(\sum_{i=1}^n \vec{v}_i g_i\right)\left(\sum_{j=1}^n \vec{w}_j g_j\right)\right]$$

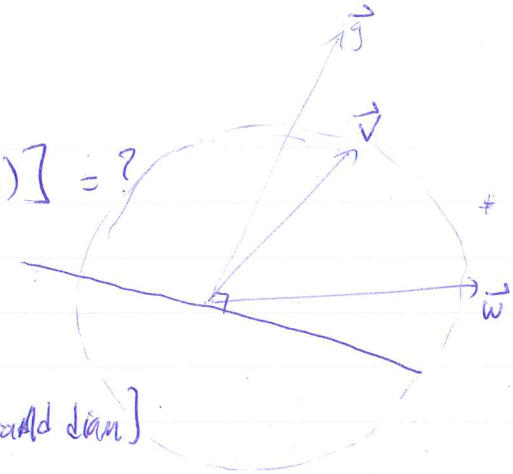
$$= \sum_{i,j} \vec{v}_i \vec{w}_j E[g_i g_j] \to \begin{cases} i \ne j: = E[g_i]E[g_j] = 0 \\ i = j: E[g_i^2] = 1 \end{cases}$$

$$= \sum_i \vec{v}_i, \vec{w}_i = \langle \vec{v}, \vec{w} \rangle = y_{vw}.$$

→ prop: Y is PSD iff $\exists$ jointly Gaussian r.v.'s $Z_1, \dots Z_n$ with $y_{vw} = E[Z_v Z_w]$.

[GW alg: Solve SDP, get corr'd Gaussians $(Z_v)_{v \in V}$. Draw, then take sgn.]

Fix edge v, w. $\Pr_{\vec{g}}[F(v) \neq F(w)] = ?$



By rotational symm. of $\vec{g}$, reduces to 2-d pic.

$$= \Pr[\vec{v}, \vec{w} \text{ split by rand diam}]$$

$$= \frac{\angle(\vec{v}, \vec{w})}{\pi}.$$

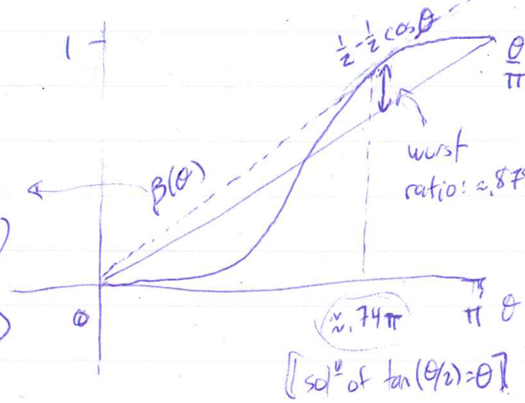
$\therefore$ lin. expec. $\Rightarrow$ $\mathop{E}_{\vec{g}}[\text{frac. edges } F \text{ cuts}] = \operatorname*{avg}_{v \sim w \in E} \left\{ \frac{\angle(\vec{v}, \vec{w})}{\pi} \right\}$

Rec: $\text{SDPOpt}(G) \quad = \operatorname*{avg}_{v \sim w \in E} \left\{ \frac{1}{2} - \frac{1}{2}\langle \vec{v}, \vec{w} \rangle \atop \frac{1}{2} - \frac{1}{2}\cos(\angle(\vec{v}, \vec{w})) \right\}$

skip

$$\beta(\theta) = \begin{cases} \frac{1}{2} - \frac{1}{2}\cos\theta, & \theta \geq \theta^* \approx .74\pi \\ \frac{1}{.878} \cdot \frac{\theta}{\pi}, & \theta \leq \theta^* \end{cases}$$

concave upper envelope



$\leq \operatorname*{avg}_{v \sim w \in E} \{\beta(\angle(\vec{v}, \vec{w}))\}$

concavity $\leq \beta\left(\operatorname*{avg}_{v \sim w \in E} \{\angle(\vec{v}, \vec{w})\}\right)$

numerical fact: $\frac{\theta}{\pi} \geq .878\left(\frac{1}{2} - \frac{1}{2}\cos\theta\right) \quad \forall \theta \in [0, \pi]$

$\therefore$ $E[\text{frac edges } F \text{ cuts}] \geq .878 \, \text{SDPOpt}(G) \geq .878 \cdot \text{Opt}(G)$

[derandable]

"[GW] is a .878-ratio approx. alg for Max-Cut"

Constraint Satisfaction Problems (CSPs)

[[Encompass a wide variety of alg. tasks — Max-Cut, 3-Sat, 3-Coloring, Lin. Eq^{n}'s, ... Basically also Vertex-Cover, etc. Not everything, of course, but quite a lot...]]

def: CSP [[problem]] defined by Domain Ω [[values for vbls]]. Usually $\{0,1,2,\dots,q-1\}$ Often $q=2$.

[[kinds of constrs allowed]] Set of preds/relations Ψ. Each $\psi \in \Psi$ is $\psi : \Omega^{r} \to \{0,1\}$ (r = arity)

CSP(Ψ) [[prob. name]]

def: Instance [[input]]: on n-vbl set V is a list of constrs: Each constr $C = (S, \psi)$ where $\psi \in \Psi$ and S ("scope") is a tuple of vbls $(v_1, \dots, v_r)$ from V, $r = \text{arity}(\psi)$

[[Goal: assign vbls values in the domain so as to max # of sat. constrs.]]

eg: Max-Cut: $\Omega = \{-1,1\}$, $\Psi = \{\neq\}$ $\quad$ ($\neq : \{-1,1\}^2 \to \{0,1\}$)

Max-3Sat: $\Omega = \{0,1\}$, $\Psi = \{ x \vee y \vee z,\ x \vee y \vee \bar{z}, \dots, \bar{x} \vee \bar{y} \vee \bar{z},\ x \vee y, \dots, \bar{x} \vee \bar{y},\ x, \bar{x} \}$ [[14 preds allowed]]

Max-E3Sat: just these 8 $\quad$ "$x \vee y \vee z$ + literals allowed"

Max-NAE3Sat: $\Omega = \{0,1\}$, $\Psi = \{NAE_3 + \text{literals}\}$

$$NAE_3(x,y,z) = \begin{cases} 1 & \text{if } x,y,z \text{ not all equal} \\ 0 & \text{if } x=y=z. \end{cases}$$

Max-3Cut (-3Coloring) : $\Omega = \{red, green, blue\}$, $\Psi = \{\neq\}$.

Max-Bij(q) : $\Omega = \{0, 1, \dots, q-1\}$, $\Psi = \{$ any "bijective" $\psi : \Omega^2 \to \{0,1\}\}$

$\forall a \in \Omega, \exists! b \in \Omega$ s.t. $\psi(a,b) = 1$.

AKA "Unique-Games (q)"

def: Given instance $\mathcal{U}$, assignment is map $F : V \to \Omega$.
Constr. $C = (S, \psi)$ is satisfied if $\psi(F(S)) = 1$

$F(v_1, .., v_r)$

Value of F is $Val_{\mathcal{U}}(F)$ = frac. constrs sat $= \mathop{E}_{(S,\psi)\sim\mathcal{U}} [\psi(F(S))]$ [!]

$\in [0,1]$

Optimum is $Opt(\mathcal{U}) = \max_{F : V \to \Omega} \{Val_{\mathcal{U}}(F)\}$

"satisfiable": $Opt(\mathcal{U}) = 1$.

task 1: "Satisfiability": Given $\mathcal{U}$, decide if $Opt(\mathcal{U}) = 1$. → And find satisfying F. [[Easy reduct.]]

alg task 2: Given $\mathcal{U}$, find F with $Val_{\mathcal{U}}(F)$ as large as poss. Ideally $Opt(\mathcal{U})$
"Optimization".

task 3: [Just a #/val. ver of 2?] Given $\mathcal{U}$, output a "proof" of "$Opt(\mathcal{U}) \le \beta$".
[interesting prob...] Ideally β as small as poss.

Satisfiability: Which CSPs in P, which NP-complete?

Max-1Cut ← testing bip.

Bij ← propagation alg.

3-Sat

NAE3-Sat

3Cut/Color

[How can you tell? If I give you an unfamiliar one, ~~often~~ in P or NPC?]

P ← 2Sat

P ← 3-Lin mod 2 ($x_i + x_j + x_k = 0/1$)

1-out-of-3 Sat → NPC.

P ← Order(10)

[topo sort] [$\Omega = [10]$, constrs "$v<w$"]

[FV93]: Dichotomy Conjecture: Every CSP is in P or NPC.
[Reminder of Ladner's Thm: $P \neq NP \Rightarrow \exists$ probs in NP which are neither.]

$\exists$ rich algebraic theory of this ["universal"]

[BJK'05]: Alg. Dich. Conj.: CSP(Ψ) satisfiability in P if Ψ has "Taylor polymorphism" [WOLOG core]
else NPC.

proved [easy] open.

[The missing piece is algorithm]

[Schaefer '78]: ✓ if $|\Omega| = 2$.
[Bulatov '06]: ✓ 3. [Also, whenever Ψ contains all unary preds.]

Markovic++13? ✓? 4

[They'll probably solve it soon.]

[[Nothing's perfect. What if you get $\mathcal{I}$ and not sat'ble. Want to solve the opt. prob.]]

☹ Opt version is NP-hard for "all" (nontriv) CSPs. [[E.g., even <u>Max-Cut</u>.]]

[[Often turn to approx. algs.]]

def: [[re task 2]]: Efficient alg. A is an <u>(α,β)-approx. alg.</u> if: $\forall \mathcal{I}$ with $\mathrm{Opt}(\mathcal{I}) \geq \beta$, A finds. asgn F with $\mathrm{Val}_{\mathcal{I}}(F) \geq \alpha$.

[[in expec., if A randomized]] [[alg. gets]] [[best asgn]]

[[Sat. prob. is $\alpha = \beta = 1$]]

rem: One alg. can be (α,β)-approx simult. for many β.
Eg.: [GW] is $(.878\beta, \beta)$-approx. alg. for Max-Cut $\forall \beta$. [[doesn't need to know β]]

def: [[task 3]]: <u>(α,β)-refutation alg.</u>: $\forall \mathcal{I}$ with $\mathrm{Opt}(\mathcal{I}) < \alpha$, alg. outputs "proof" of "$\mathrm{Opt}(\mathcal{I}) < \beta$".

rem: Weaker than (α,β)-approx alg. [[Why?]]
[[Say you had an (α,β)-approx alg. Given instance $\mathcal{I}$ with $\mathrm{Opt}(\mathcal{I}) < \alpha$, run your alg., and say it "fails" if it fails to return an asgn. of Val $\geq \alpha$. Of course, it will fail but this constitutes a proof of $\mathrm{Opt} < \beta$, b/c if $\mathrm{Opt} \geq \beta$, your alg. <u>will</u> find a val-α sol^{n}!]]

eg: Given CSP inst. $\mathcal{I}$, write ILP, relax to LP, compute LP Opt $=: \beta^*$ output "$\mathrm{Opt}(\mathcal{I}) \leq \beta^*$".
Even better pf: dual LP sol^{n} of val. β^*.

[Thy of approximability in P or NP-C much less well understood. Very active area.]

Max-E3Sat: $(1,1)$-approx [= Sat. check]: NP-C.

$(1-10^{-10}, 1)$-approx: NPC "PCP Thm" [ALMSS 94]

$(\frac{7}{8}, 1)$-approx: $\in P$ [Joh '72]

— pick a rand. asgn. [derand by condit expec.]

[E3-Sat killed]  $(\frac{7}{8}+\varepsilon, 1)$-approx: NPC $\forall \varepsilon > 0$ [Hås '99].

Max-3Sat $(\frac{7}{8}, 1)$-approx: $\in P$ [KZ'97] [very difficult; uses SDP]

[all approx-hard is also refutation-hard]

Max-Cut: $(\frac{3}{4}, \frac{3}{4})$-approx: NPC.

$(\frac{3}{4}, \frac{4}{5}-\varepsilon)$-approx: NPC $\forall \varepsilon > 0$ [Hås '99, TSSW'00]

$(\frac{3}{4}, \frac{1}{2}+\frac{1}{2\sqrt{2}})$-approx: $\in P$ [GW]

$\approx .8536$

$(\frac{3}{4}, \frac{5}{6})$-approx: NOT KNOWN

like Factoring/GIso!

[Not only that, in light of recent research, sort of reasonable to guess "neither"]

Max-Bij(q): $(1,1)$-approx: P

$(1-\varepsilon, 1-\varepsilon)$- " : NPC $\forall \varepsilon > 0$.

$(\frac{1}{2}, 1-\varepsilon)$- " : P if $q=2$

?? if $q \geq 3$.

Unique-Games Conj. [Khot]: $\forall \varepsilon > 0\ \exists q$ s.t. $(\varepsilon, 1-\varepsilon)$-approx is NP-hard

$\overset{\text{known}}{\Longleftrightarrow}$ $\forall \varepsilon\ \exists q$ s.t. $(\frac{1}{2}, 1-\varepsilon)$.