# Lecture 14 — Linear Programming II

("Using Lin. Prog.")

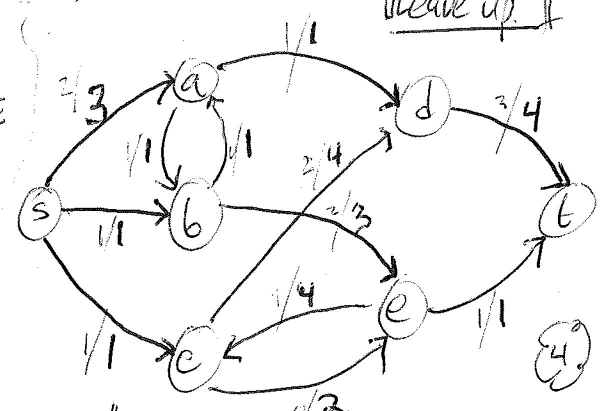
[Scribe]

[Dantzig story from '50s; we'll see that sometimes your prob. is purely lin. & you can just directly apply LP. Even if it's not --- you should still try an "LP relaxation".]

[A problem that's truly linear:] Max-st-Flow:

[Leave up]

Input: Digraph $G=(V,E)$
- "Capacity" $c_{uv} \in \mathbb{Q}^{\geq 0}$ $\forall (u,v) \in E$
- "Source" $s \in V$, "sink" $t \in V$.




[You want to move "stuff" from $s \to t$.
First: Tolstoi, USSR, 1930
Vtcs = towns, edges = railroads
(s) = cement factory, (t) = needs cement.
Caps = amt shippable/day. Want steady state]
How much can s produce/t consume?

[You'll note: all caps in egre in $\mathbb{Z}^+$.
Cris Moore example: s = CMU, t = FOCS, Berkeley, cap = free seats on flights.
How many grad students?]

"Flow conservation": $\forall v \in V \setminus \{s,t\}$, incoming flow = outgoing flow.

[To max flow: is an LP]: Vbls: $f_{uv}$ $\forall (u,v) \in E$

constrs: $f_{uv} \geq 0$ $\forall uv$
$f_{uv} \leq c_{uv}$ $\forall uv$

$$\forall v \neq s,t \quad \sum_{u: u \to v \in E} f_{uv} = \sum_{w: v \to w \in E} f_{vw}$$

"objective": $\max \sum_{v: s \to v} f_{sv} - \sum_{u: u \to s} f_{us}$

FF, TUM
$\Rightarrow$ fact: all $c_{uv} \in \mathbb{N}$
$\Rightarrow$ every basic feas. sol$^n$ (vtx) is integral

[That's it!] $\therefore$ Max-st-Flow $\in$ P. [also solvable by Ford-Fulkerson, etc. '54]

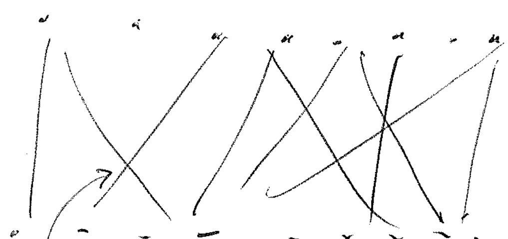
Aside: Solvers: Maple (Rat/Real), Mathematica, Matlab/sedumi, cplex, ... [ask.]

[[FF'54: also used generic railroad example. Credited Ted Harris @ Rand Corp. for the prob. In fact Harris was also working on the (classified) prob. of ~~Russ~~ Soviet railroad network. Why? We'll return...]]

[[What abt. a problem that isn't purely linear?]]

Bipartite Max-Perfect Matching [[We discussed RNC alg, but let's also see LP method. Also soluble using Max-Flow! But never mind...]]

U n people



V n jobs

edges have weights $w_e \in \mathbb{Q}$ [[not nec. +ve or -ve]],
[[Think: the value (to the army!) of this person doing this job is $w$]]

Goal: Find max-wt. (bipartite) perf. match. (if $\exists$) [[All jobs must get done, want best utility"]]

[[Could you have a 0/1 vbl for... taking edges?]]
[[NOT an LP. But...]] [[Very common paradigm:]]

"Integer linear program" [ILP] $\xrightarrow{\text{"relax"}}$ Linear Program

vbls: $\forall (u,v) \in E$ $X_{uv}$, "intent": 1 if matched, 0 if not.

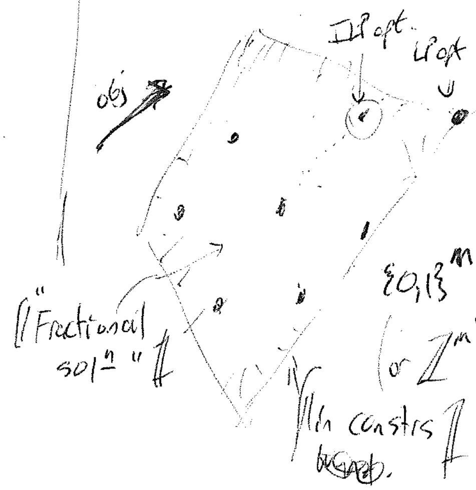
ILP $\xrightarrow{\text{relax}}$ LP

Constrs: $x_{uv} \in \{0,1\}$. [[??]] $0 \le x_{uv} \le 1$.

$\forall u \in U, \quad \sum_{v \sim u} x_{uv} = 1$

$\forall v \in V, \quad \sum_{u \sim v} x_{uv} = 1$

$\max \sum_{uv} w_{uv} x_{uv}$.




Relaxation facts:

- LP infeasible $\Rightarrow$ ILP infeas.
- LP feasible $\Rightarrow$ either ILP infeas or $\underset{\text{of ILP}}{Opt} \le LPOpt$ [[or reverse for a min problem]]

[[Why do this? Well, ILP's are NP complete, but LP's in P. What good is an opt. frac. sol^n though?]] [[easy-to-compute u.b. on Opt]]

[[Can think: [[ $LPOpt = 83.6$ is a proof (poly-time findable) that $Opt \le 83.6$ [[ $\le 83$, by integrality]]

Lucky (?) situation for Max-Wt Bip. P.M.: [[doesn't always happen]]

thm: All extreme pts are integral

$\Rightarrow$ LP feasible means $LPOpt = Opt$, problem $\in P$ [[can find extreme pts]]

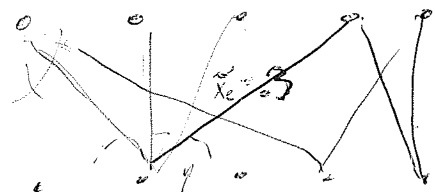
[actually, our proof implicity gives an effic. method converting any LP-opt. sol^n to an optimal (hence integral) extreme pt.]

pf: [Contrapos:] If $\tilde{x}$ feasible & non-integral $\Rightarrow$ $\tilde{x}$ not extreme

$\Updownarrow$

$\tilde{x} = \frac{1}{2}x^+ + \frac{1}{2}x^-$ (*) for some feas. $x^+, x^-$

$\tilde{x}$




sum to 1, $\therefore \exists$ another non-integ. edge

Must exist non integ. cycle.

$\therefore$ $\exists$ $\varepsilon > 0$ s.t.

$\varepsilon < \tilde{x}_e < 1 - \varepsilon$ $\forall e$ on cycle

Let $x^+$ be $\tilde{x}$ with $+\varepsilon$ added to odd edges of cycle,

$x^-$ be $\tilde{x}$ ~ even edges...

Both feasible, and (*) holds. $\square$

[Rem: if you find an optimal non-bfs, this alg lets you get an optimal int. one]

fact: This "integrality" happens for any LP

$$\max/\min \ c \cdot x \ : \quad b' \le Mx \le b, \quad l \le x \le u,$$

where $b', b, l, u \in \mathbb{Z} \cup \{\pm\infty\}$, $M$ "totally unimodular"

[thm ...] $\qquad$ $\equiv$ every sq. submtx has $\det \in \{-1, 0, +1\}$.

[e.g., this prob, max-flow w/ int caps]

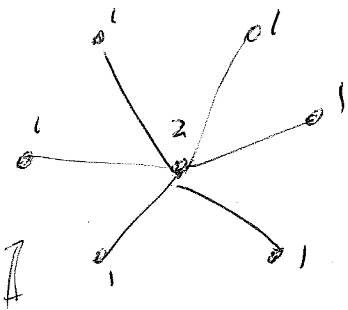
[Next we'll see an eg. where the LP does not exactly solve the prob, but it's still good... ]

## Min-Vertex-Cover

Input: $G=(V,E)$, vtcs have costs $c_v \geq 0$
Output: A "vtx cover": $S \subseteq V$ s.t. each $e \in E$ has an endpt $\in S$.
[[weird name]]

Goal: Min cost.

e.g.




Opt = 2
Greedy = 6
(n-1).

rem: NP-hard, [[So won't get integrally lucky.]]

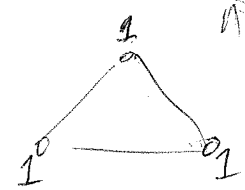
ILP:
$x_v \in \{0,1\} \quad \forall v \in V$
$x_u + x_v \geq 1 \quad \forall (u,v) \in E$
$\min \sum_v c_v x_v$.

$\xrightarrow{\text{relax}}$ LP
$0 \leq x_v \leq 1$.

[[always feasible: $x_v = 1 \ \forall v$]]
or...

$$LPOpt \leq Opt$$

[[can be less]]:




$Opt = 2$, $LPOpt \underset{(=)}{\leq} 3/2$.

$G = K_n$: $Opt = n-1$
$LPOpt \leq \frac{n}{2}$.

[[What good is LP then?]] Proof of $Opt \geq \ldots\ldots$

**LP Rounding:** converting an optimal fractional (LP) sol$^n$ to a (hopefully almost as good) feasible ILP sol$^n$.

[[e.g.]] Say $\tilde{x}$ feasible for LP. [[Think of it as optimal, too, but won't use that]].

define $S = S_{\tilde{x}} = \{v \in V : \tilde{x}_v \geq \frac{1}{2}\}$. "rounding" [[literally, round off $\tilde{x}_v$ to a 0/1 sol$^n$]]

fact: $S$ is a vertex cover (ILP ~~feas~~).

pf: $\underline{\qquad\qquad\qquad}$

fact: $\mathrm{cost}(S) \le 2 \cdot \mathrm{LPcost}(\tilde{x})$

pf: $\quad \sum_v c_v \tilde{x}_v \ge \sum_{v \in S} \frac{1}{2} \cdot c_v = \frac{1}{2}\mathrm{cost}(S)$

cor: Let $x^*$ be opt. LP sol$^n$.

$\mathrm{cost}(S_{x^*}) \le 2\,\mathrm{LPcost}(x^*) = 2\,\mathrm{LPOpt}$.

$\therefore$ Opt ~~...~~ $\le 2 \cdot \mathrm{Opt}$

$\therefore$ can effic. find sol$^n$ w/i factor 2 of Opt

[our first eg. of an] "2-approximation alg.". [there are easier ones, e.g., Max matching]

rem: [thm] For this LP, every vtx $\tilde{x}$ is "half-integral": $\tilde{x}_v \in \{0, \frac{1}{2}, 1\}$

<u>Duality</u>: [Let's go back to Max-st-Flow.]

$$
\begin{array}{lll}
\max & c \cdot x & \\
\text{s.t.} & a^{(1)} \cdot x \le b_1 & \cdot \lambda_1 \\
& a^{(2)} \cdot x \le b_2 & \cdot \lambda_2 \\
& \vdots & \vdots \\
+ & a^{(m)} \cdot x \le b_m. & \cdot \lambda_m \\
\hline
& c \cdot x \le \beta. &
\end{array}
$$

Sps. $\exists\, \lambda_1, \ldots, \lambda_m \ge 0$ s.t.

$\rightarrow$ Proof that $\mathrm{LPOpt} \le \beta$. Farkas Lemma $\Rightarrow$ $\exists$ optimal proof of this form.

I.e., $\left\{\begin{array}{ll} \min & b \cdot \lambda \\ \text{s.t.} & A^T\lambda = c \end{array}\right\}$ ← an "dual" LP
= primal LP's max.

life rule: Given any LP, always take its dual, try to "interpret" it.

Max st-Flow LP

$\max \sum_{v: s \to v} f_{sv} - \sum_{u: u \to s} f_{us}$ —take dual→ min ___ s.t. ___ "clean up" (mess around) a little.

s.t.

dual $\equiv$ equiv to

$\lambda_{uv}$ vbl $\forall\, uv \in E$ [cap. constrs]

$\mu_v$ vbl $\forall\, v \in V$ [flow conserv. constrs]

"interp"

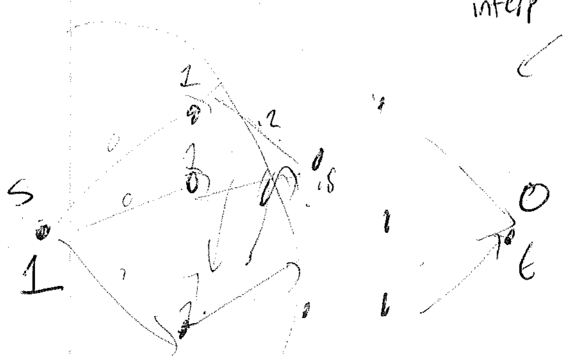
$$\min \sum_{uv \in E} c_{uv} \lambda_{uv}$$

s.t. $\mu_s = 0$

$\mu_t = 1$

$\lambda_{uv} \geq 0 \quad \forall uv \in E$

$\lambda_{uv} \geq \mu_u - \mu_v$




Natural Relaxation of Min-st-Cut ILP!

Find $S \subseteq V$, $S \ni s$, $S \not\ni t$,

$\min \ \text{cost}(S) = \sum_{\substack{u \in S \\ v \notin S}} c_{uv}$

obvious [Given a cut S, only have so much cap. coming out]

Duality: max flow $=$ min "fractional" s-t-cut $\leq$ min-st-cut.

Actually, min-st-cut LP is exact.

Max-Flow = Min-Cut.

Fact: [[like for ~~max-cut~~ bip-matching]]. [[Proof not hard;]]

~~Extreme pts LP~~

In fact, ~~[illegible]~~ LPs ~~are~~ totally unimodular.

[[Proves the fact that if caps all integral, so ~~are~~ all bfs's.]]

[[ F.F. proved this (so did P. Elias-Feinstein-Shannon)

$\downarrow$

Ted Harris @ Rand

$\downarrow$ soviet rail network

Gen. Ross @ airforce $\to$ cared abt min-cut for bombing purposes]]

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Hmm?

Let $(\lambda_{uv}, \mu_v)$ be dual opt., $f_{uv}$ be primal opt.

Let $S = \{v : \mu_v > 0\}$.

is min cut

If $u \in S, v \notin S$, $\lambda_{uv} \geq \mu_u - \mu_v > 0$.

$\Downarrow$ "complem. slackness"

$f_{uv} = c_{uv}$

Sim, $u \notin S, v \in S$

$\Rightarrow f_{uv} = 0$.

$\therefore$ . . . . . .

Q: Max-Cut . . . . . . ?