Lecture 13 – Linear Programming I

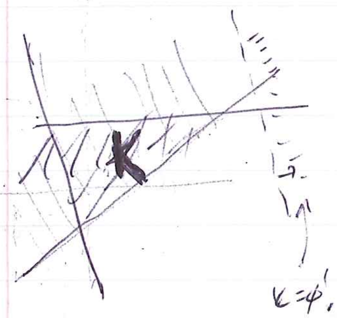
[[Scribe → Matousek & Gartner]]

– The greatest polynomial time alg.

[[There I said it. Actually, inaccurate to say; L.P. is a problem. The fact it's in P is the greatest thing ever.]]

L.P.: Input:

$$K := \left\{ \begin{array}{l} \text{list of linear ineqs. [[over } \mathbb{Q}\text{]]} \\ a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \geq b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \geq b_m \end{array} \right\}$$



[[You want $\leq$? Just negate coeffs.
You want $=$? Put in two inequalities $\leq, \geq$.
You want $>$? Sorry, no.
You want: "min $c^T x$ s.t. $x \in K$"?
You can do it, just hold on.]]

Output: • if $K \neq \emptyset$ → a pt $x \in \mathbb{Q}^n$ sat'ing all ineqs.
• if $K = \emptyset$ → a "proof" [[!?]]

[[!? Why must one exist? of poly size?!]]

thm: [[NZ Shor '70-72, Nemirovski-Yudin '70s]] Khachiyan '79 [[Russian PhD]] [[Gacs-Lovasz '80]] Solvable in polytime. [[Front page of NYT, Guardian, Der Spiegel]]

[[GLS...]] + goodies [[Get a vtx, not just any sol'n. Don't need K explicitly; just sep oracle...]]

[[Return to poly-sizeness of sol'ns later, what abt. "proof"?]]

[What would be a proof that K is unsatisfiable? Suppose I pointed and say: hey, if you add up all the ineqs the LHS becomes 0 (everything miraculously cancels) but RHS becomes 1 (or any other pos #). Clear proof of unsat. More generally...?]

Proof of unsat: nonneg multipliers $\lambda_1, \dots, \lambda_m \in \mathbb{Q}^{\geq 0}$ s.t.

$$\begin{array}{rll} & \lambda_1 \cdot (a_{11}x_1 + \dots a_{1n}x_n) & \lambda_1 \cdot b_1 \\ + & \lambda_2 \cdot (\quad) & + \vdots \\ & \dots & \vdots \\ + & \lambda_m \cdot (\quad) & + \lambda_m \cdot b_m \\ \hline = & 0x_1 + \dots + 0x_n & = 1 \end{array}$$

[or any pos #]

thm: Farkas Lemma / LP duality $K = \emptyset \Rightarrow$ such a proof exists

[$\Rightarrow$ LP $\in$ NP $\cap$ coNP, why people believed it in P for long time.]

pf: (sketch) [Sps you wanted to manually solve...]

$$\left\{\begin{array}{l} x - 5y + 2z \geq 7 \\ 3x - 2y - 6z \geq -12 \\ -2x + 5y - 4z \geq -10 \\ -3x + 6y - 3z \geq -9 \\ -10y + z \geq -15 \end{array}\right\}$$

pos mult [changes nothing]

Fourier-Motzkin elim.

① elim first vbl (x)

$$\left\{\begin{array}{l} x - 5y + 2z \geq 7 \\ x - \frac{2}{3}y - 2z \geq -4 \\ -x + \frac{5}{2}y - 2z \geq -5 \\ -x + 2y - z \geq -3 \\ -10y + z \geq -15 \end{array}\right\} \overset{\text{mentally}}{\Longleftrightarrow} \left\{\begin{array}{l} x \geq 5y - 2z + 7 \\ x \geq \frac{2}{3}y + 2z - 4 \\ x \leq \frac{5}{2}y - 2z + 5 \\ x \leq 2y - z + 3 \end{array}\right\}$$

ignore for now

[Think of y, z as numeric consts. Then this system is satisfiable iff all the lower bounds $\leq$ all the upper bounds]

sat'bility $\equiv$

[not the same sol^n set, but sat'bility is equiv]

$$\left\{\begin{array}{l} \frac{5}{2}y - 2z + 5 \geq 5y - 2z + 7 \\ 2y - z + 3 \geq 5y - 2z + 7 \\ \frac{5}{2}y - 2z + 5 \geq \frac{2}{3}y + 2z - 4 \\ 2y - z + 3 \geq \frac{2}{3}y + 2z - 4 \end{array}\right\}$$

& $-10y + z \geq -15$

② Elim $y, z, \ldots$ (not effic! [# ineqs can square at each step])

actually, take all pairwise sums of $+x$ ~~'s & $-x$ ~~'s.

Invar.: all ineqs are nonneg lincombs of orig. ineqs. [how we get the λ_i's]

At end:

$x_n \geq -3$

$x_n \geq -6$

$\vdots$

$x_n \leq 10 \quad (-x_n \geq -10)$

$x_n \leq -1$

if $\exists$ sol^n, can pick rational one else infeas.

$\vdots$

$0 \geq -1$

[maybe $0 \geq 1$ already $\Rightarrow$ infeas.]

$\vdots$

$0 \geq -2$

can backtrack to get feasible sol^n in $\mathbb{Q}^n$ [if exists]

[Okay, so either a rat. sol^n exists or a rational "proof" of unsolvability exists. Still not clear why in poly time. Why not? Maybe all sol^n's need huge # of bits!]

thm: Given $K = \left\{ \begin{array}{l} \text{---} \geq b \\ \text{---} \end{array} \right\}$ with <u>input size</u> L $= \langle K \rangle$ [[trad. for L.P.]]

[[# bits in input rep, $\geq m, n$]]

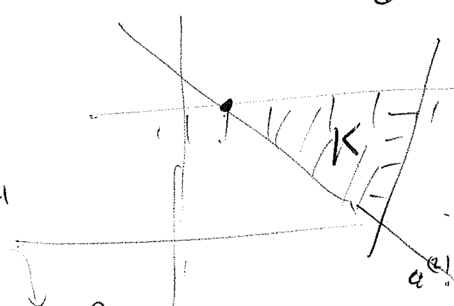
① $K \neq \emptyset \Rightarrow \exists$ "feasible sol^n" in $\mathbb{Q}^n$ expressible w/ poly(L) bits

② $K = \emptyset \Rightarrow \exists$ proof (λ_i's) " " " "

pf: Suffices to show ①. <u>Why?</u> The question "$\exists$? proof of unsat" is an LP in vars λ_i !! [[size $\approx L$]]

"dual LP".

Proof essence:



$K \neq \emptyset$ "$\Rightarrow$"

ignoring annoyances

$\exists$ feasible vertex / extreme point / "basic feasible sol^n" x^*

↳ the unique sol^n of <u>some</u> (linindep) n out of the m ineqs [[when converted to eqs]].

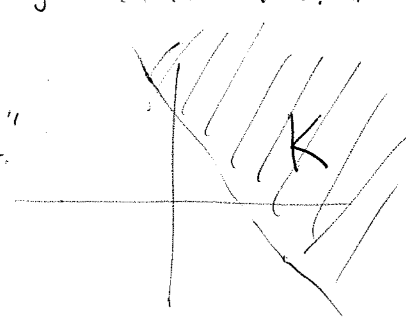
$\Rightarrow x^*$ sol^n to $n \times n$ subsys. of eqs

$\Rightarrow$ "Gauss elim. in P" [[Hmwk]]

$\Rightarrow \langle x^* \rangle = \text{poly}(L)$.

[[e.g. what if $m<n$??]] "$\square$"

e.g. $n=2$
$m=1$: "$x_1 + x_2 \geq 1$"



no vertex!

Rem: Above "proof" would be fine if K included...

"def": <u>Big box constraints</u> : $\forall i \in [n]$ $\beta_i^- \le x_i \le \beta_i^+$

for $\langle \beta_i^\pm \rangle = \mathrm{poly}(L)$

[Give you n lin. indep. [I] constraints !]

[But that's what we're trying to prove you can do!!] Why possible...?

Replace each $a^{(i)} \cdot x \geq b_i$ with $a^{(i)} \cdot x - s_i = b_i$, $s_i \geq 0$ ← new "slack vbl".

each orig. x_i with $x_i = x_i^+ - x_i^-$, $x_i^\pm \geq 0$.

→ equiv. <u>Equational Form</u>:

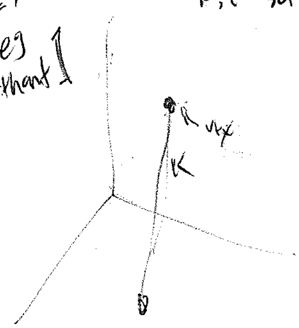
$$K \text{ "}\equiv\text{" } K' = \left\{ \begin{array}{l} Ax' = b, \\ x' \geq 0 \end{array} \right\}$$

, A is $m \times n$, $m \le n$ [rank m WOLOG]

$x' \geq 0$ ← ptwise

$\langle L' \rangle = \mathrm{poly}(\langle L \rangle)$
[effic. transf. betw. K, K' sol^{n}s]

affine $n-m$ dim subspace.



- e.g. $m = n$: K = a pt., fall nonneg
  → sol^n to $n \times n$ sys, GE ... poly($\langle L \rangle$) bits ✓
- $m = n-1$: K = line ∩ pos orth
  can't be ∥ to <u>all</u> coord hyperplanes $x_i = 0$
  ∴ $\exists x^* \in K$ sol^n to $\left\{ \begin{array}{l} Ax = b \\ x_i = 0 \end{array} \right\}$ for some i
  GE... ✓
- $m = n-2$: K = 2-d flat ∩ pos orth.
  - must hit some $\{x_i = 0\}$. Throw in, reduce to $m = n-1$ case.

etc. ▦

summary: Given LP prob, "$K=\emptyset$?", can WOLOG add "big box constrs" $-2^{\mathrm{poly}\langle K\rangle} \le x_i \le 2^{\mathrm{poly}\langle K\rangle}$ $\forall i\in[n]$.

[[quadratic okay]]

$K\neq\emptyset \Rightarrow \exists x^*\in K$ a "vertex".

Q: Sps deciding $K=\emptyset$? in P. [[Which it is...]]
Can you [[black-box]] also find $x\in K$?

A: Yes: ① Find $q_1\in\mathbb{Q}$ s.t. $K\cap\{x_1=q_1\}\neq\emptyset$.
How? Binary search with $K\cap\{x_1\ge q_1\}$!
$\log(2^{\mathrm{poly}\langle K\rangle}) = \mathrm{poly}\langle K\rangle$ "emptiness checks"

* have to argue $\exists$ vtx of form $\frac{\text{int}}{\text{int}}$ where this int you know as same determinant

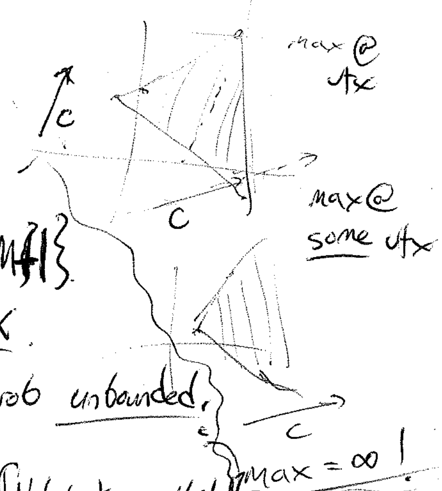
② Find $q_2\in\mathbb{Q}$ s.t. $K\cap\{x_1=q_1, x_2=q_2\}\neq\emptyset$.
etc.

Q: What abt. solving "LP" $\in$ [[trad. formulated]] $\max\{c\cdot x : x\in K\}$?

A: [[Careful about adding big-box constrs!]]
Obs: if max $\neq\infty$ ("unbounded"), max occurs @ a vertex.
$\Rightarrow$ max $\le 2^{\mathrm{poly}\langle K\rangle} =: M$.

Step ①: Compute $\max\{c\cdot x : x\in K \cap c\cdot x\le M+1\}$ by binary searching on $c\cdot x$.

②: If max val is $M+1$, orig. prob unbounded.



Q: Find $x\in K$ $(\neq\emptyset)$ which is a vertex. [[Not just any old pt]]
Simplicity: assume K has box constrs. [[not nec; a little exercise]]

A: Artificially solve $\max\{c\cdot x : x\in K\}$ for any c not parallel to a constr!