Lecture 12 - Expander graphs

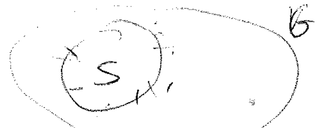
[[Scribe: Hoory Linial Wig~~d~~, Prahladh's course]]

[[I used to be scared of expander graphs, but...]]

Expanders $\approx$ graphs which are sparse, highly connected, explicit.
[[so for rand. dway]]

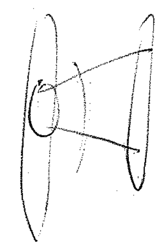
Focus: n-vertex, d-regular graphs, (families)]
d const, $n \to \infty$
$\frac{d}{2} n = O(n)$ edges: sparse ✓.

Highly connected: every $S \subseteq V$ has lots $(\Omega(|S|))$ of edges/vtcs on its boundary



Provided $|S| \le \frac{n}{2}$ or $.01n$ or...

Sometimes: G bip.



only want left-reg.
expansion of sets on left.

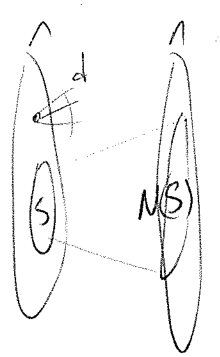
[[explicit?]]
[[Random easy..]]

Random graphs always great. [[ideal, it seems]].

e.g.: ① [[Pinsker '73]] $\forall d \ge 3$ ~~...~~ $\forall n \ge n_0$,

a random left-d-reg $n+n$ bip. graph has, whp,

$|N(S)| \ge (d-2)|S|$ $\forall$ left sets S, $|S| \ge c_d n$
[[nbrhood]] [[$d \ge 3$]] [[$\frac{1}{20d^4}$]]



[[makes life tougher]]

[[Bassalygo]] 2 n left vtcs, $\frac{3}{4}n$ right vtcs,
$d \geq 64$, $|N(S)| \geq .8d|S|$
$\forall\ |S| \leq \frac{.02}{d} n$. [[I don't 100% guarantee this]]

Pf: [[straightforward]] Prob. Method. ∎

[[Illustrates that often you need slightly diff things; hard to force one param setting]]

[[But not good for algs/want explicit...]]

<u>Explicit</u>: Constructible in poly(n) time.

[[what you often need]] → <u>Strongly Explicit</u>: Given $u \in [n]$, $i \in [d]$, i^{th} nbr of u can be output in polylog(n) time [[in input/output size]]

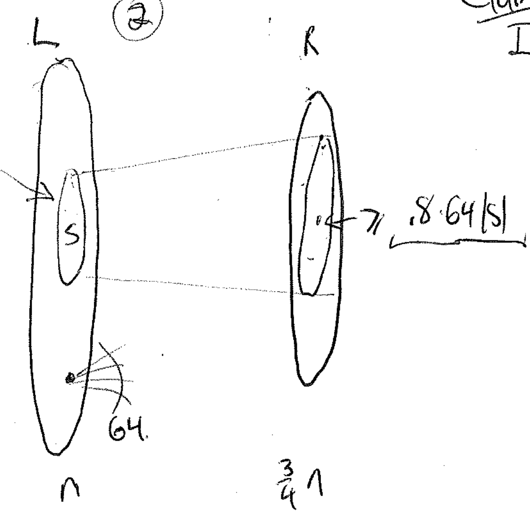
<u>Roughly</u>: For "almost" expander variant, we know explicit constrs. "almost" matching what random graphs get.

[[2 is pretty hard; don't know the analogous non-bip. version]].

E.g.: 2 with $d = 64$, "$\frac{3}{4}$" $= .000\ldots01$, "$.02$", $\boxed{\gamma = .8}$

[[Roughly: you want an explicit rand graph? You got one.]]
[[Plan: applics, constructions, more applics...]]

App 1: Good Codes: [[Sipser-Spielman '96]]



Claim: If $\emptyset \neq |S| \leq \frac{.02}{64} n$, $\exists\, v \in N(S) \subseteq R$ with exactly 1 nbr in S.

Pf: [all have ≥ 1] If they all had ≥ 2, there'd be $\geq 2 \cdot .8 \cdot 64|S| \geq 64|S|$ edges betw. S & $N(S)$, $\Rightarrow\Leftarrow$ $\square$

Consider code w/ parity check mtx $H \in \mathbb{F}_2^{R \times L}$ the adj. mtx. [[explicit]]

$L = |n|$

$$\frac{3}{4}n \; R \begin{bmatrix} & & H & & \end{bmatrix} \begin{bmatrix} | \\ z \\ | \end{bmatrix} S, L$$

dim $= n - \frac{3}{4}n = \frac{1}{4}n$, $\Rightarrow$ rate $\frac{1}{4}$:)

Claim: D = min dist $\geq \frac{.02}{64} n$:) [const. rel dist.]

Pf: AFSOC $z \neq 0$ a codeword with Ham wt $\leq \frac{.02}{64} n$. $z \equiv$ subset of S. Let v be unique nbr.

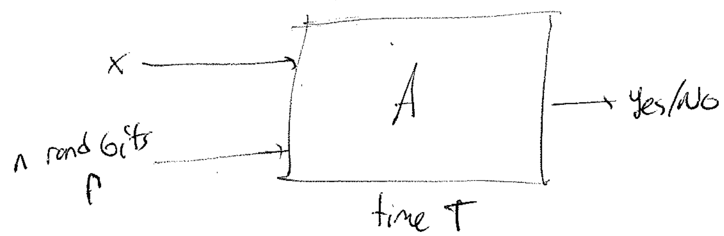
$$[\text{— } v \text{ —}] \begin{bmatrix} | \\ z \\ | \end{bmatrix} = 1 \Rightarrow Hz \not\equiv 0 \Rightarrow \text{not a codeword.}$$

$\square$

Prop: [easy, but we'll defer] [(crucially uses $\boxed{.8 > .75}$)] poly(n)-time decoding from $< \frac{D}{2}$ errors. $O(n)$ if very careful.

Alg: Given z: [While $|Hz| \neq 0$, flip any z_i which decreases $|Hz|$.] [["Belief propagation"]]

App 2: Error Reduction: Say A is one-sided err. rand. alg deciding membership in P [[e.g. Primality]]



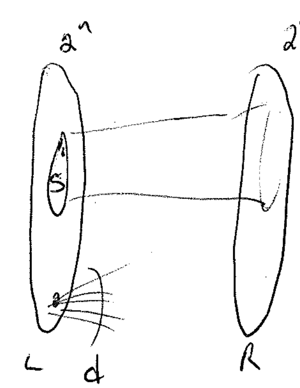
s.t.: $x \in P \Rightarrow \Pr_r[\text{YES}] = 1$

$x \notin P \Rightarrow \Pr_r[\text{YES}] \le 1\%$.

Reduce error while conserving rand bits?

(i) Repeat d times: dT time, dn rand bits, err prob $.01^d$,

(ii) "Take" any expander like ② with $2^n + 2^n$ vertices [[!!]]
strongly explicit



$|N(S)| \ge .8d|S|$ if $|S| \le \frac{.02}{d} 2^n$.

- Pick $l \in L$: cost: n rand bits
- Let $r_1, \dots, r_d \in \{0,1\}^n$ be its nbrs. Cost: $\text{poly}(\log 2^n) =$ poly(n) time.
- Run A with $r_1, \dots, r_d$, say YES iff all cause YES.

∴ derand alg. has error $\le \frac{.02}{d}$, using $\text{poly}(n) + dT$ time and n rand bits.

Err reduc. w/ NO extra randomness!!!

Rem: Using len-t rand walk on [[a non bip ?]] 8-reg, use $n + O(t)$ rand bits, error $\frac{1}{2^{\Omega(t)}}$ [[spectral analysis, easy]].

Analysis: Let $B \subseteq R$ be the $\le 1\%$ "bad" rand strings for A.
Let S be the "bad" choices for l: those with $N(l) \subseteq B$.

[[works for 2-sided err, too]]

Claim: $|S| < \frac{.02}{d} 2^n$.

Pf: Otherwise, $|S| \ge \frac{.02}{d} 2^n$. Choose any $S' \subseteq S$ with $|S'| = \frac{.02}{d} 2^n$

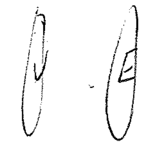
$$|N(S')| \ge .8d|S'| = .8 \cdot .02 \cdot 2^n > 1\% \cdot 2^n$$

But $N(S') \subseteq B$ & $|B| \le 1\% \cdot 2^n$, $\Rightarrow\Leftarrow$.

Expansion Params / Constructions

- Typically construct non-bip. graphs. $G=(V,E)$

  To get bip.:
  - double cover $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$
  - line graph



- Edge expansion vs. vtx expansion

$\forall |S| \le \frac{n}{2}$, vs. $\forall |S| \le \frac{n}{2}$,

$\Pr_{u \sim v}[u \in S, v \notin S] \geq \varepsilon$. vs. $|N(S)| \geq \varepsilon \cdot d \cdot |S|$.

More Typical to study b/c by Cheeger's ineq., related to 2nd-smallest eig, ← . poly time computable.

[Alon-Milman '85]: Can also connect eigs & vtx expansion [Not as tight...]

def: (n,d,ε)-spectral expander: $\lambda_1 \geq \varepsilon$. ($\Rightarrow \Omega(\frac{\varepsilon}{2})$-edge exp; $\Leftarrow O(\varepsilon^2)$-edge exp.) [Cheeger.]

(1) [Margulis '73] Gabber-Galil '81 [Jimbo-Maruoka analysis, see also Lee]

Let $V = \mathbb{Z}_m^2$. E: connect (x,y) to $(x \pm y, y)$, $(x \pm y+1, y)$, $(x, y \pm x)$, $(x, y \pm x+1)$. [elementary analysis, motiv. by basic 2d geom]

[$d=8$.] [Partic nice it's power of 2, use 3 bits to choose nbr.]

thm: $(m^2, 8, \varepsilon)$-spectral exp., $\varepsilon \geq \varepsilon_0 > 0$ ($\uparrow \approx .1$)

Strongly explicit. [Outstanding for error reduction applics.]

[actually, having all eigvals of L a bit big not so awesome: in a rand graph, they're all close to 1

thm [Friedman]: In random d-regular graph,
$$\lambda_1, \dots, \lambda_{n-1} \in \left[1 \pm \frac{2}{\sqrt{d}} \cdot \left(\sqrt{1-\tfrac{1}{d}+\varepsilon}\right)\right] \quad \text{w.p.} \geq 1 - o_n(1).$$

Better look at K's eigvals $1 = \kappa_0 \geq \underbrace{\kappa_1 \cdots \geq \kappa_{n-1}}_{\text{all } |\cdot| \lesssim \frac{2}{\sqrt{d}} \xrightarrow[d \to \infty]{} 0\,!}$

easy ex: "Expander Mixing Lemma": if $|\kappa_1|, \dots, |\kappa_{n-1}| \leq \kappa$,
$$\forall S, T \subseteq V, \quad \left| \Pr_{u \sim v}[u \in S, v \in T] - \mathrm{vol}(S)\mathrm{vol}(T) \right| \leq \kappa \sqrt{\mathrm{vol}(S)\mathrm{vol}(T)}.$$

[key for exp. code decoding analysis] $\qquad \Pr_{\substack{u,v\\ \text{indep}}}[u \in S, v \in T]$

(2) def: "Ramanujan Graph": d-reg., $\kappa \leq \frac{2}{\sqrt{d}}\sqrt{1 - 1/d}$.

thm: [LPS '88, Morgenstern '94]: [NUMBER THY, Hard! Deep!] [PGL(2, q) involved ...]

$\exists$ explicit [not strong.] Ramanujan graphs if d = prime power + 1,
$n = 1 + p$, p prime $\equiv 1 \bmod 4$.

need a prime...

"cor": Let $V = \mathbb{Z}_p$. E: connect a to $a+1$, $a-1$, $\frac{1}{a} \bmod p$. $\quad \frac{1}{0} := 0.$

deg. 3, and ε_0-spectral expander, for $\varepsilon_0 \approx \frac{1}{100}$. [all $\lambda_i \geq \varepsilon$]

(3) [RVW] Zig-zag / replacement product iter. constr.]

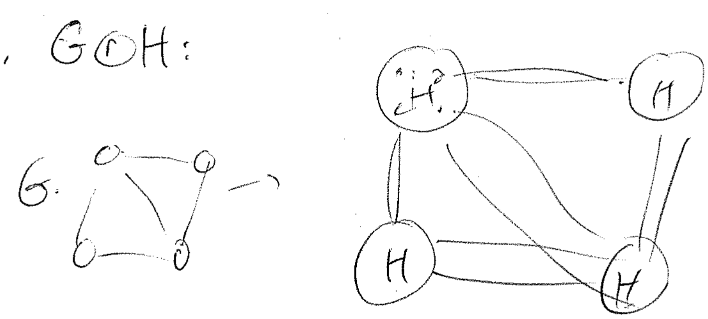
HW3.7 [Jerrum-Son-Tetali-Vigoda]

Let G be (n, D, ε_G)-spectral exp.

Let H be (D, d, ε_H) - " "

Repl. Prod. $G ⓡ H$:

[Look it up in Hmwk].



← $G ⓡ H$

↓

$\left(Dn,\ 2d,\ \frac{\varepsilon_H \varepsilon_G}{16}\right)$-spectral exp.

[??]

#vts ↑D :) deg ↓ small :) eigval ↓ $\frac{\varepsilon_H}{16}$:(

[How to get eigval back up?] G^t: same vtcs, connect $u \to v$ if ∃ t-step walk.

#vtcs same, deg ↑ $(2d)^t$ eigval ↑ by factor ≈ t [easy].

Iterate this! If H has $d = 4$, #vtcs 8^t, 2nd smallest eig ≳ $\frac{16}{t}$

then interleaving ⓡ, ^{t} means vtcs ↑ arbit, degree $= 8$, eig stays const.!

Such H exist w/ $t = O(1)$ [30?].

Finite-size obj, can find by brute force!

rem: Zigzag techniques only known way to get explicit bip. expanders like the ones we used @ start of lecture!