Lecture 7: Spectral Graph Theory II

[[SCRIBE]] G: undir graph [[selfloops, mult edges OK]] [[think regular to warm up.]]

Recall: $\{f: V \to \mathbb{R}\}$ dim. $n = |V|$ vec. space, w/ inner prod

$$\langle f, g \rangle := \mathop{E}_{u \sim \pi} [f(u) g(u)].$$

π: . invariant/stationary distrib.: • choose unif. rand edge $u \sim v$ ($\equiv v \sim u$)

• output u.

. $\pi[u]$ prop. to $\deg(u)$: $\pi(u) = \frac{\deg(u)}{2m}$. Unif. dist. iff G regular

For $u_0 \sim \pi$, $u_0 \to u_1 \to u_2 \to \cdots \to u_t$ standard rand. walk

all $\sim \pi$.

$$\mathcal{E}[f] := \frac{1}{2} \mathop{E}_{u \sim v} \left[(f(u) - f(v))^2 \right]$$

$$\boxed{\mathcal{E}[1_S] = \Pr_{u \sim v} [u \in S, v \notin S]}.$$

[[Dirichlet form / local variance]]

$$\mathrm{Var}[f] = \frac{1}{2} \mathop{E}_{\substack{u \sim \pi \\ v \sim \pi \\ \text{indep.}}} \left[(f(u) - f(v))^2 \right]$$

min $\mathcal{E}[f]$? $\Rightarrow$ If G's conn. comps. are $S_1, \ldots, S_\ell$, then $1_{S_1}, \ldots, 1_{S_\ell}$ (1 on S, 0 off S) are lin. indep., their span is all f s.t. $\mathcal{E}[f] = 0$.

max $\mathcal{E}[f]$? [[Not scale inv., $\because \mathcal{E}[cf] = c^2 \mathcal{E}[f]$, so we look at..]]

$$\max \mathcal{E}[f] \quad \text{s.t. } \|f\|_2^2 = E[f^2] \leq 1$$

(equiv.: $E[f^2] = 1$, $\mathrm{Var}[f] \underset{(=)}{\leq} 1$)

fact: $\mathcal{E}[f] \leq 2 \|f\|_2^2$ w/ equality poss. iff G is bipartite.

~~[Let's look more at maximizing $\mathcal{E}[f]$, the minimizing is actually somewhat more interesting]~~ stet: [Will look at max/mining $\mathcal{E}[f]$ some more. Have to start by studying it.]

Pf: $\mathcal{E}[f] = $ [tiny arithmetic] $= \|f\|_2^2 - \underset{u\sim v}{\mathbf{E}}[f(u)\,f(v)]$ [Leave up.]

$|\cdot| \le \|f\|_2^2$ by C.S.

[an interesting qty; let's study]

$$= \underset{u\sim\pi}{\mathbf{E}}\Big[f(u)\,\underset{v\sim u}{\mathbf{E}}[f(v)]\Big]$$

v rand nbr of u

avg val of f on u's nbrs. [a number]

$u \mapsto$ [a fcn on V]

def: $Kf : V \to \mathbb{R}$, $Kf(u) = \underset{v\sim u}{\mathbf{E}}[f(v)]$

$(Kf)(u)$

Markov/transition/normalized adj. operator [think heat diffusion (w/ self-loops)]

rem: K is linear operator [in vec space sense], $K(f+g) = Kf + Kg$ [Why? linearity of expectation]

corresps to matrix

$$u \rightarrow \begin{bmatrix} & & \\ & K & \\ & & \end{bmatrix} \begin{bmatrix} \\ f \\ \\ \end{bmatrix} \begin{array}{l} \leftarrow v_1 \\ \\ \leftarrow v_n \end{array} = \begin{bmatrix} \\ Kf \\ \\ \end{bmatrix} \leftarrow u$$

$$K[u,v] = \begin{cases} \frac{1}{\deg(u)} & \text{if } (u,v) \in E \\ 0 & \text{else} \end{cases} = \Pr[u \to v \mid @u]$$

$\therefore$ K is adj. mtx, normalized so row sums 1 [stoch. mtx]

$= \frac{1}{d} A$ if G d-regular $\longrightarrow$ in which case K is symm [$K = K^T$]

fact: For $f,g: V \to \mathbb{R}$,

$$\langle f, Kg \rangle = \mathop{E}_{u\sim v}[f(u)g(v)] \quad \because \; = \mathop{E}_{u\sim\pi}[f(u)\cdot(Kg)(u)]$$

[[almost think of this as def^n]]

$$= \mathop{E}_{u\sim\pi}[f(u) \mathop{E}_{v\sim u}[g(v)]]$$

$$= \mathop{E}_{u\sim v}[f(u)g(v)]$$

$*$

$\quad\parallel$

$\langle Kf, g\rangle$

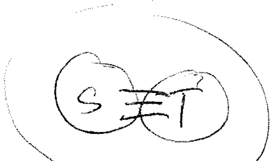
[[but also...]]

$$= \mathop{E}_{u\sim v}[f(v)g(u)]$$

$$= \cdots \langle g, Kf\rangle.$$

"K is self-adjoint"

[[this is the equiv. of "K is "symmetric" in the G-is-regular case.]]

e.g.: 

[[don't have to be disj. actually]]

Let $S, T \subseteq V$, $f = 1_S$, $g = 1_T$.

$$\langle f, Kg\rangle = \mathop{E}_{u\sim v}[1_S(u)1_T(v)]$$

$$(=\langle g, Kf\rangle) = \mathop{\Pr}_{\substack{u\sim v\\ \text{rand edge}}}[u\in S, v\in T]$$

[[prob. you go from S to T in one step]]

Aside: Let ρ be a prob. distr. on V: $[\text{———}\rho\text{——}]$

[[think a row vector]]

rho, haha

- Draw $u \sim \rho$
- Do 1 step rand walk $\to v$.

Let ρ' be prob. dist. of v.

Ex: [[trivial]] $[\ \rho\ ]\,[\ K\ ] = [\ \rho'\ ]$

$\therefore$ [[in partic]] $\pi K = \pi$.

[[K is a mtx: can multiply mtcs / compose operators]]

Q: What is $K^2 = K\circ K$

A: $(K^2 f)(u) = \mathop{E}_{u \xrightarrow{2\text{ steps}} w}[f(w)]$

Pf: $(K(\underbrace{Kf}_{g}))(u) \triangleq \mathop{E}_{v\sim u}[\underbrace{g}_{(Kf)}(v)] = \mathop{E}_{v\sim u}\left[\mathop{E}_{w\sim v}[f(w)]\right] \left(= \mathop{E}_{\substack{v\sim u\\ w\sim v}}[f(w)]\right)$. □

Cor: $\forall t\in\mathbb{N}$, $K^t f(u) = \mathop{E}_{u \xrightarrow{t\text{ steps}} w}[f(w)]$

⟦mtx power or t-fold composition⟧

$t=0$? ✓ $\underset{\text{id. mtx}}{I} f(u) = \mathop{E}_{w:=u}[f(w)] = f(u)$. ✓

Back to $\mathcal{E}[f] = \frac{1}{2}\mathop{E}_{u\sim v}[(f(u)-f(v))^2]$ ← ⟦as we saw by simple arith.⟧

$= \langle f,f\rangle - \mathop{E}_{u\sim v}[f(u)f(v)]$ → $\langle f, Kf\rangle$

$\|f\|_2^2$

$= \langle f, \underset{I}{f} - Kf\rangle = \langle f, \underbrace{(I-K)}_{L} f\rangle$ ⟦ah, the Laplacian⟧

def: L: (normalized) Laplacian operator/mtx is $L = I - K$.

i.e., $Lf: V\to\mathbb{R}$, $Lf(u) = f(u) - \mathop{E}_{v\sim u}[f(v)]$.

⟦Aside: elsewhere you'll see all this tortured stuff about $D^{-1/2}$, D is diag(deg) etc. All b/c they don't use $\langle,\rangle_\pi$:(⟧

Q: What is the "meaning" of L?

A: $\langle f, Lf\rangle = \mathcal{E}[f]$ ← awesomely meaningful. $= \frac{1}{2}\mathop{E}_{u\sim v}[(f(u)-f(v))^2]$

⟦($\langle f, Lg\rangle \overset{\text{Not?}}{=} \mathcal{E}[f,g]$)⟧

Let $S \subseteq V$, let $f = 1_S$. $\langle f, Lf\rangle = \mathcal{E}[f] = \Pr_{u\sim v}[u\in S, v\notin S]$ $\left(= \frac{1}{2}\text{"vol"}(\partial S)\right)$

$\langle f, f\rangle = E[f^2] = \Pr_{u\sim\pi}[u\in S]$ $\quad u\sim\pi,\ v\sim u$

$\therefore$ ratio is $\Pr[v\notin S \mid u\in S]$

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$= \Pr[$ pick rand $u \sim S$, do 1 step, that you go outside $S]$

[[choose w. prob. prop'l to deg(u)]]

def: $\Phi[S]$ = <u>Conductance of S</u> $\in [0,1]$ "escape prob. of S"

<u>Sparsest Cut</u>: Given G, find S with $vol(S) \le \frac{1}{2}$ <u>minimizing</u> $\Phi[S]$

[[NP-hard, majorly important for divide & conquer algs on graphs approximability wide open.]].

[[Though we're somewhat more interested in ~~minimizing~~ $\mathcal{E}[f]$ let's think about maximizing (cf. Max-Cut), as eigenvalues will finally enter the pic.]]

$$\max \ \mathcal{E}[f] = \langle f, Lf \rangle = \tfrac{1}{2} \mathop{E}_{u \sim v}[(f(u)-f(v))^2]$$

$$\text{s.t. } \|f\|_2^2 = \langle f, f \rangle = 1.$$

$(\Leftrightarrow Var[f]=1)$

← compact set [[ellipsoid in $\mathbb{R}^n$]]

continuous fcn [[$f: \mathbb{R}^n \to \mathbb{R}$]]

$\therefore$ a maximizer $\varphi: V \to \mathbb{R}$ exists. [[also, roughly speaking, efficiently findable]]

Claim: $L\phi = \lambda\phi$ for some $\lambda \in \mathbb{R}$ $\Leftrightarrow$ $L\phi$ parallel to ϕ

[[assuming that:]] cor: $\mathcal{E}[\phi] = \langle \varphi, \lambda\varphi \rangle = \lambda \|\varphi\|_2^2 = \lambda$ $\leftarrow \left(\begin{smallmatrix} \ge 0 \\ \le 2 \end{smallmatrix}\right)$

fact: $E[\phi] = 0$. Pf: maximizer will have $Var[\phi] = 1$ [[replace ϕ with $\phi - \mu$, rescale]].

<u>Proof of claim</u>: [[Lagrange mults]] Suppose

$L\phi$ (not parallel)

ϕ

Let ψ be unit vector in this dir. [[exists, $\because$ not $\parallel$]]

Let $\varepsilon \ne 0$ be small. Let $f = \phi + \varepsilon\psi$. $\|f\|_2^2 = 1 + \varepsilon^2$ (Pythagoras)

$$\langle f, Lf \rangle = \langle \phi + \varepsilon\psi, L\phi + \varepsilon L\psi \rangle$$
$$= \langle \phi, L\phi \rangle + 2\varepsilon \langle \psi, L\phi \rangle + O(\varepsilon^2)$$

[[used self-adj.]]

$$\therefore \frac{\langle f, Lf\rangle}{\langle f,f\rangle} = \langle \phi, L\phi\rangle + 2\varepsilon \langle \psi, L\phi\rangle \pm O(\varepsilon^2)$$

not 0 [see pic]

$> \langle \phi, L\phi\rangle$ for small enough $\pm\varepsilon$, $\Rightarrow\Leftarrow$. $\square$

[Great, so ϕ is a maximizer. Let's disallow it now — restrict to subspace orthog]

Now consider $\max \langle f, Lf\rangle$

s.t. $\langle f,f\rangle = 1$, $f \perp \phi$ $\leftarrow$ compact set [$\equiv \langle f, \mathbb{1}\rangle = 0$]

Repeat: Gives ϕ' with $L\phi' = \lambda'\phi'$, $E[\phi'] = 0 \Leftrightarrow \langle \phi', \mathbb{1}\rangle = 0$

$\phi' \perp \phi$. $\lambda' \le \lambda$.

Now do s.t. $f \perp \phi, \phi'$. Repeat, etc. [All will be $\perp$ to $\mathbb{1}$; last one will be $\mathbb{1}$ then.]

$\boxed{\text{thm:}}$ Given G, $\exists$ orthonormal fcns ("eigenfunctions" / vectors) $\phi_0, \phi_1, \dots, \phi_{n-1} : V \to \mathbb{R}$ [Change not^{n} to $\phi_1 \dots \phi_n$]

$\langle \phi_i, \phi_j\rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else}\end{cases}$ $\quad \phi_0 = \mathbb{1}$ $\quad$ form basis

& reals ("eigenvalues") $0 = \lambda_0 \le \lambda_1 \le \dots \lambda_{n-1} \le 2$

our first λ. our first ϕ

s.t. $L\phi_i = \lambda_i \phi_i$.

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rem/ex: [λ_i's uniquely def'd but] ϕ_i's not uniquely def'd. When $\exists$ "multiplicity"/eigval;

e.g.; $\lambda_5 = \lambda_6 = \lambda$ $\quad \phi_5, \phi_6$ any rotation

Even w/o mult.: $\pm\phi_i$. [$\mathbb{1}$ will always be one, we'll stick w/ $\mathbb{1}$.]

[What is the importance of this?

The basis $\{\phi_0, \dots, \phi_{n-1}\}$ makes understanding $L, K, \mathcal{E}$ easy!!]

def: Given $\phi_0, \ldots, \phi_{n-1}$ [o.n. basis],

can write any $f: V \to \mathbb{R}$ uniquely as lin comb.:

$$f = \underbrace{\hat{f}(0)}\phi_0 + \underbrace{\quad}\phi_1 + \cdots + \underbrace{\hat{f}(n-1)}\phi_{n-1} = \sum_{i<n} \hat{f}(i)\phi_i$$

real coeffs

for real consts $\hat{f}(0) \ldots \hat{f}(n-1) \in \mathbb{R}$. [$\phi_i$s are like world's most interesting fcns]

prop: $Lf = \underset{\text{lin op}}{L}\left(\sum_{i<n} \hat{f}(i)\phi_i\right) = \sum_{i<n} \hat{f}(i) L\phi_i = \sum_{i<n} \underbrace{\lambda_i \hat{f}(i)}\phi_i$

$\widehat{Lf}(i)$: L mults $\hat{f}(i)$ by λ_i.

prop: $\langle f, g \rangle = \sum_{i<n} \hat{f}(i)\hat{g}(i)$ [dot-prod formula]

pf: $\langle \sum_i \hat{f}(i)\phi_i, \sum_j \hat{g}(j)\phi_j \rangle = \sum_{ij} \hat{f}(i)\hat{g}(j)\langle \phi_i, \phi_j \rangle$: o.n.ality.

cor: $\boxed{\|f\|_2^2 = \langle f, f \rangle = \sum_{i<n} \hat{f}(i)^2.}$ $E[f] = \langle f, \mathbb{1} \rangle = \langle f, \phi_0 \rangle = \hat{f}(0)$. $\boxed{\therefore \mathrm{Var}[f] = \sum_{0<i<n} \hat{f}(i)^2}$ (†)

$\mathcal{E}[f] = \langle f, Lf \rangle = \sum_{i<n} \lambda_i \hat{f}(i)^2$ (‡) e.g. $0 \cdot \hat{f}(0)^2 + .1\hat{f}(1)^2 + .3\hat{f}(2)^2 + .9\hat{f}(3)^2 + \cdots + 1.8\hat{f}(n-2)^2 + 2\hat{f}(n-1)^2$

2 ⇒ bipartite.

Q: What is $\min \dfrac{\mathcal{E}[f]}{\mathrm{Var}[f]}$ s.t. $\mathrm{Var}[f] \neq 0 \iff f$ not const

~~(WLOG E[f] = 0 ⇒ ... E[f²] = 1)~~

A: λ_1 : follows immed. from (†), (‡)

achieved by ϕ_1. "Poincaré Ineq.": $\lambda_1 \mathrm{Var}[f] \leq \mathcal{E}[f]$.

If $\phi_1 : V \to \mathbb{R}$ had range $\{0,1\}$, then $\mathcal{E}[\phi] =$ edge bdry size

i.e., 1_S for some $S \subseteq V$, $\mathrm{Var}[\phi] = E[\phi^2] - E[\phi]^2 = \mathrm{vol}(S)(1 - \mathrm{vol}(S))$

$= \mathrm{vol}(S)\,\mathrm{vol}(S^c)$,

$$\frac{\mathcal{E}[f]}{\mathrm{Var}[f]} = \frac{\mathcal{E}[1_S]}{\|1_S\|_2^2 \|1_{\bar{S}}\|_2^2} \overset{\text{factor 2}}{\approx} \Phi(S) \text{ or } \Phi(\bar{S})$$

for whichever has vol $\leq \frac{1}{2}$.

Would get S with $\Phi[S] = \Theta(\lambda_1)$. Const factor approx to sparsest cut. What if not $\{0,1\}$? Cheeger, ...

Random walks