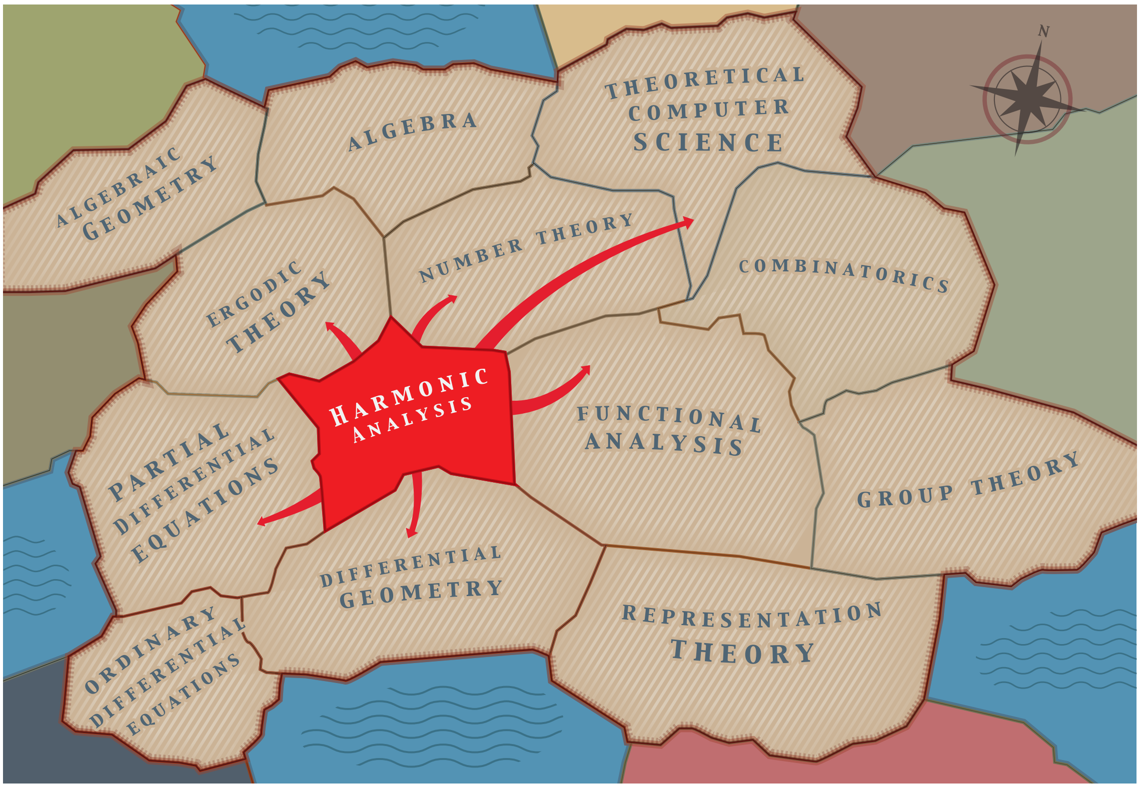
Strides and leaps across challenging mathematical terrain

This year's Crafoord Prize Laureates, ***Jean Bourgain*** *and* ***Terence Tao****, have solved an impressive number of important problems in mathematics. Their deep mathematical erudition and exceptional problem-solving ability have enabled them to discover many new and fruitful connections and to make fundamental contributions to current research in several branches of mathematics.*

To many people, mathematics seems done and dusted. It promises unchanging certainties. Everyone who, at school, toiled through multiplication tables, Pythagoras' theorem or algebraic equations has respect for the capacity of mathematics to deliver an incontrovertible answer to every question. But underlying this apparently sealed edifice is a vast mathematical landscape, open for exploration. For anyone who penetrates it, as researchers do, unknown expanses open up – a vista of mountains, valleys, and paths to follow.

Mathematics has evolved and emerged over millennia. New theories arise; existing ones are streamlined and expanded. New patterns and connections are sought. In fact, the scope of mathematical research in the past century exceeds that of everything done before.



Bourgain and Tao have developed and used the toolbox of harmonic analysis and made fundamental contributions to current research in several branches of mathematics.

Nonetheless, some problems remain unsolved. Some of them deal with *prime numbers*, i.e. numbers that are divisible only by 1 and themselves such as: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47… But how many prime numbers are there? Euclid, who lived and worked in Alexandria some 2,300 years ago, proved that they are infinitely many.

1	43	142	746	595	714	191
2	48	425	980	631	694	091
3	53	709	214	667	673	991
4	58	992	448	703	653	891
5	64	275	682	739	633	791
6	69	558	916	775	613	691
7	74	842	150	811	593	591
8	80	125	384	847	573	491
9	85	408	618	883	553	391
10	90	691	852	919	533	291
11	95	975	086	955	513	191
12	101	258	320	991	493	091
13	106	541	555	027	472	991
14	111	824	789	063	452	891
15	117	108	023	099	432	791
16	122	391	257	135	412	691
17	127	674	491	171	392	591
18	132	957	725	207	372	491
19	138	240	959	243	352	391
20	143	524	193	279	332	291
21	148	807	427	315	312	191
22	154	090	661	351	292	091
23	159	373	895	387	271	991
24	164	657	129	423	251	891
25	169	940	363	459	231	791
26	175	223	597	495	211	691

The longest known arithmetic sequence of prime numbers contains 26 terms with a difference of 5,283,234,035,979,900 between successive numbers.

But a closely related question has continued to confound mathematicians: how many prime-number 'twins' exist? Twins are pairs of prime numbers that differ from each other by 2, such as: 3 and 5, 5 and 7, 11 and 13, 17 and 19. . . and so forth. Is there an infinite number of twin primes? The answer is not known. The difficulty lies partly in the fact that prime numbers become more sparse as whole numbers increase.

Terence Tao and his British colleague Ben Green jointly solved a difficult problem about sequences of prime numbers. A sequence of numbers is known as an 'arithmetic sequence' if the difference between a number and its immediate successor in the sequence is constant. For example, 5, 11, 17, 23 and 29 make up an arithmetic sequence of prime numbers of length 5, with a difference of 6 between consecutive elements. Green and Tao showed that for any desired length chosen in advance, no matter how high, there exists a finite arithmetic sequence consisting of prime numbers of exactly that length.

Finite arithmetic sequences composed of prime numbers, of arbitrary length, thus exist; on the other hand, no technique for explicitly finding such sequences has been found. So finding an arithmetic sequence of, say, 100 prime numbers is currently beyond our ability, even if Green and Tao have shown that such a sequence exists. Currently, the longest known arithmetic sequence of prime numbers contains 26 terms. It is a sequence of 26 prime numbers, starting with 43,142,746,595,714,191 and with a difference of 5,283,234,035,979,900 between successive terms.

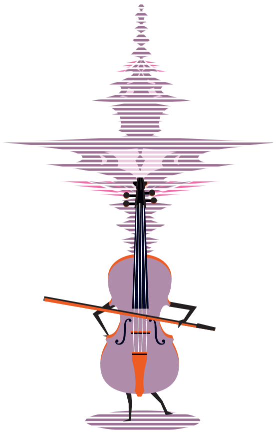
The majority of Jean Bourgain's and Terence Tao's most fundamental results are to be found in the field of *mathematical analysis*. Isaac Newton and Gottfried Wilhelm von Leibniz developed mathematical analysis at the end of the 17th century.

Mathematical analysis studies functions. An example of a function may be a rule assigning a value to each number, as a squaring function, which to each number assigns its square. In this case the value of the function at 2 is 4, at 3 it is 9, at 10 it is 100 and so on.

Some functions can be represented graphically as curves, and the analysis then describes their shapes. It tells us how the function varies: does it change fast or gradually, does it move upward or downward, where is its highest or lowest value?

Newton used analysis to study mechanics and astronomy. Over the past three hundred years, analysis has come to permeate the language of physics and all other natural sciences. It is a key ingredient of quantitative methods used almost everywhere mathematics is applied. Our understanding of the reality around us is to a high degree governed, by its mathematical description.

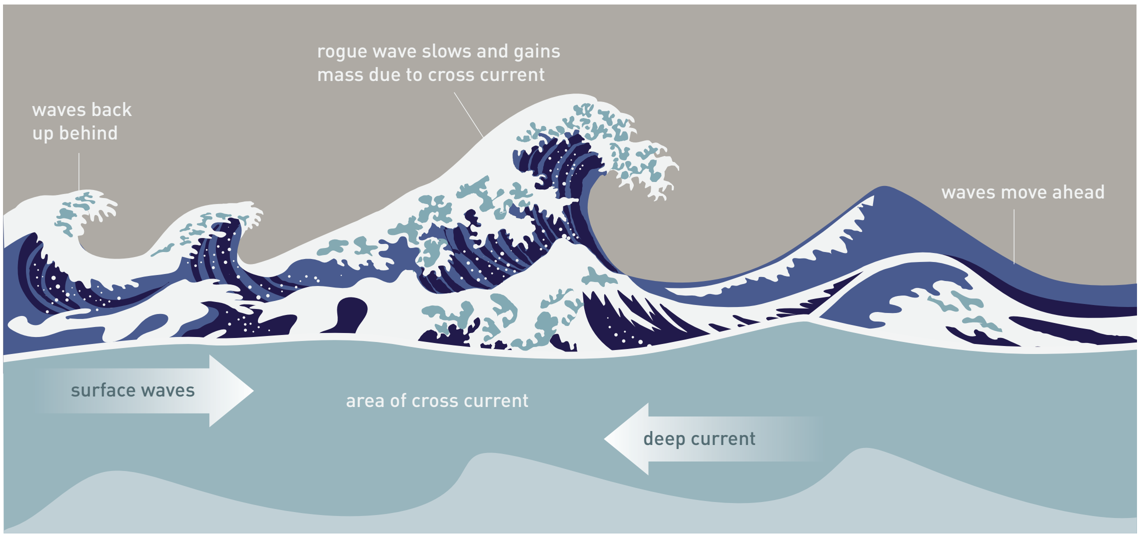
A Frenchman, Jean-Baptiste Joseph Fourier, took an epoch-making step in the development of mathematical analysis nearly 200 years ago. He showed that, in principle, all functions consist of sums of simpler functions. So, for example, the sound (or 'harmonic') of a violin string is composed of a fundamental tone and several overtones. Their frequencies are multiples of the fundamental tone's frequency. Harmonic analysis was born.



The harmonic sound of a violin string is composed of a fundamental tone and several overtones.

Harmonic analysis became a key tool for solving differential equations, which are at the core of mathematical analysis. In turn, differential equations are a key tool for physics, engineering and other fields of science. Today, there is no limit to the applications of this branch of mathematics, which is constantly developing.

With the fundamental contributions of Jean Bourgain and Terence Tao, some of the most difficult, non-linear differential equations can now be studied successfully. These describe more "messy" processes, such as turbulent currents, tsunami waves and chaos. Who knows how these equations can be used in the future?



Both Crafoord Laureates contributed to the study of some of the most difficult, non-linear differential equations which describe such processes as turbulent currents, tsunami waves and chaos.

What sets mathematics apart is that its major advances can long lie hidden from the world, intelligible only to a handful of experts. One such example is Bernhard Riemann's geometry, which only after several decades found its application in physics, and became the basis of Albert Einstein's general theory of relativity.

Within mathematics itself, specialized areas may be concealed from other mathematicians' eyes. In modern mathematical research, dialogue and communication with other mathematicians have been increasingly crucial for progress. On their own and jointly with others, Jean Bourgain and Terence Tao have made astounding contributions to many fields of mathematics. They have developed and used the toolbox of harmonic analysis in groundbreaking and surprising ways, attracting a great deal of attention among researchers worldwide.

Ideas from harmonic analysis, an area whose tools have the capacity to find hidden patterns in seemingly random data, have proved extremely useful for research on prime numbers as well. Studying them is like finding the music in a noisy recording. Prime numbers appear to crop up randomly among all the whole numbers and thus, in a way, can be interpreted as 'noise'.

Fascination with hidden patterns among prime numbers was long regarded as a mathematician's 'art for art's sake'. Nowadays, our best encryption methods used for secure data transmission rely on the difficulty of dealing with very large prime numbers.

Another indispensable tool for modern cryptography, and for other parts of computer science as well, is the ability to obtain high-quality random numbers. A decent level of randomness can, for example, be obtained from noise in a computer's microphone or pictures of falling leaves in a webcam. By using methods from harmonic analysis, Jean Bourgain has shown how two good, independent sources of randomness can be used to create an almost perfect random number sequence.

One problem studied by both Jean Bourgain and Terence Tao, together and separately in cooperation with others, is what is known as *the Kakeya problem*. In 1917 Soichi Kakeya, a Japanese mathematician, posed a question that may be considered fairly bizarre: what is the minimum area on which a needle can be completely turned around? This might be described as making a U-turn with a car in the smallest possible area, assuming that the car is as thin as a needle. Ten years later, in 1927, came a surprising answer: a needle can be rotated on an arbitrarily small area.

The original question, relating to two dimensions (an area), was thereby answered. But in more dimensions a modified version lives on. This Kakeya problem has proved to have a fundamental bearing on a number of areas in mathematics, and has been a challenge taken on by both Crafoord Prize Laureates. However odd the original Kakeya needle problem may appear, attempts to solve the higher-dimensional Kakeya problem have sustained increasingly active mathematical attention over the past three decades. There exists no solution yet, but the concepts created in order to solve the problem may turn out to have more significance in mathematics than the answer to the original question may bring.

The study of the Kakeya problem has uncovered profound connections with harmonic analysis and issues relating to whole numbers. By changing the perspective and viewing the problem from new angles Bourgain and Tao have shown many surprising insights.

Yet again, mathematics has shown interconnections among its diverse branches. The fact that methods developed in one area become tools for solving problems in entirely different, and apparently unrelated areas, shows the underlying unity of mathematics.

LINKS AND FURTHER READING

More information about this year's prize is available on the Royal Swedish Academy of Sciences' website, *http://kva.se/crafoordprize* and at *www.crafoordprize.se*

Popular-scientific article

Journeys to the Distant Fields of Prime, 2007, by Kenneth Chang, New York Times, March 2007:
http://www.nytimes.com/2007/03/13/science/13prof.html?pagewanted=all

Scientific articles

Jean Bourgain (a list of publications for the years 1976–2011):
http://www.math.ias.edu/files/bourgain/Bourgain.pdf

Terence Tao (a list of publications for the years 1996–2011):
http://www.math.ucla.edu/~tao/preprints/

Websites

Terence Tao's blog: http://terrytao.wordpress.com/
Wikipedia- article about Terence Tao: *http://en.wikipedia.org/wiki/Terence_Tao*
Wikipedia-article about Jean Bourgain: *http://en.wikipedia.org/wiki/Jean_Bourgain*

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