

ON A FAIR COPY OF RIEMANN'S 1859 PUBLICATION CREATED BY ALFRED CLEBSCH

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ABSTRACT. We will verify that the fair copy “*Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*” found in Riemann’s Nachlass is not in Riemann’s hand. Further, we will show that this paper was written by Alfred Clebsch and that it was used after Clebsch’s death with high probability as a setting copy for chapter VII of Riemann’s Collected Works published in 1876.

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1. INTRODUCTION

Most of Bernhard Riemann’s manuscripts, lecture notes, calculation sheets and letters that he left after his early death in 1866 are preserved in 52 compilations at *Niedersächsische Staats- und Universitätsbibliothek Göttingen* (SUB) as “Riemann’s Nachlass”. One of these compilations is one of the most famous compilations of mathematical original papers at all, it is Cod. Ms. B. Riemann 3 (Primzahlen), the Holy Grail of analytic number theory. Some of its papers are commonly considered as the birth of this mathematical discipline. It contains manuscripts connected with Riemann’s 1859 publication “*Über die Anzahl der Primzahlen unter einer gegebenen Grösse*”⁽¹⁾ as well as the derivation of the Riemann-Siegel integral formula, the Riemann-Siegel asymptotic formula, a lot of calculation sheets in order to determine the first zeros of the zeta function, some drafts of letters to other mathematicians⁽²⁾ and much more. Apart from the 1859 publication, none of these findings were published by Riemann, although some of the methods he used were new at that time, like

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- (1) See [9, 10]; with modified title “*Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*” see [7, ch. VII, pp. 136–144] or [11]. The printed version [9] was first published in spring 1860.
- (2) In a letter to Karl Weierstraß, Riemann mentions a new expansion of the zeta function which is thought to be the Riemann-Siegel formula (see [14] and [8, pp. 822–825]).

the saddle point method in [15] to expand a complex integral in an asymptotic series. The method remained unknown until 1909 when Peter Debye [4] rediscovered it in another unpublished paper of Riemann and used it for his asymptotic expansions of the Bessel functions. The integral formula and the asymptotic series mentioned above were unknown too, until C. L. Siegel [19, 20] found them in compilation Cod. Ms. B. Riemann 3 in 1932, more than 70 years after Riemann had derived them.

These opening words show that the Riemann hypothesis alone is not the reason, why mathematicians are fascinated by Riemann’s work about analytic number theory to this day. There are plenty of references about the publication of 1859, the Riemann hypothesis and all other mathematical findings of compilation Cod. Ms. B. Riemann 3 in the literature and on the Internet. Therefore, it is not surprising to find a facsimile of a handwritten version of Riemann’s 1859 paper⁽³⁾ from this compilation on the Internet, where it is thought that the original manuscript of this facsimile not only was created but also written by Riemann himself as a fair copy. Although the compilation came to SUB in 1890 and was inspected by many people since, no one noticed or was interested in the fact that these six pages are not in Riemann’s handwriting. This will be demonstrated in section 2. Further, we will show in section 3 that the unknown writer of the copy is Prof. Alfred Clebsch (1833–1872), successor to Riemann’s Chair at the University of Göttingen from 1868 to 1872. Then we will see in section 4 that the paper was used as a setting copy for chapter VII of Riemann’s Collected Works in 1876 with high probability.

To simplify the references, I have omitted “Cod. Ms. B. Riemann” in the following text; for example I will write “3 (19)” instead of “Cod. Ms. B. Riemann 3 (19)” to reference the first page of Riemann’s draft [10].

Some of the figures in the appendix are not suitable for printing on paper. They should be viewed on the screen using the zoom function of a pdf reader.

2. THE FAIR COPY OF RIEMANN’S PUBLICATION

Manuscript 3 (16–18)⁽³⁾ is a fair copy of Riemann’s publication from 1859 not written in German Kurrent⁽⁴⁾ but in Latin script. Germans usually wrote a German text in Kurrent script at that time but the creator of the fair copy did not. During the examination of 14 compilations of Riemann’s Nachlass I found no other person who wrote a German text in Latin script. This applies for example to Ernst Abbe, Richard Dedekind, Hermann Hankel, Karl Hattendorf, Riemann’s widow Elise, Hermann Amandus Schwarz, Heinrich Weber, Karl Weierstraß and some more. Indeed, the fair copy in Latin script seems to be unusual.

We will make a comparison of the handwritings between the fair copy 3 (16–18) and two autographs of Bernhard Riemann. The first autograph is his draft 3 (19–20)⁽⁵⁾ of his communication to the *Berliner Akademie* in October 1859. The date can be seen at the end of line 2 on this page in the area with the water damage. The paper is written in Kurrent script and the handwriting of its creator can be identified to be Riemann’s by comparison with the large number of preserved letters

(3) See [11] and figure 3 for the first page.

(4) The Kurrent script was used in all German countries including some regions in Switzerland, the Russian empire and in the Balkans as standard script for handwritten German text since the 15th century. After a small simplification (Sütterlin) was introduced in 1915, 500 years of German script tradition came to an end in 1941 when the Nazis banned its teaching in a circular directive [6] just like the further use of its printed counterpart, the Fraktur. After the war this was not revised as many other laws and decrees from the Nazi era too.

(5) See [10] and figure 1 for the first page.

that contain Riemann's signature such as [13]. The second autograph is the fragment 3 (22)⁽⁶⁾, a fair copy of his publication in French language and in Latin script, of which only one sheet (two pages) has survived. Because of the different scripts, we cannot compare the German and the French text from 3 (19–20) and 3 (22) directly to prove that fragment 3 (22) is in Riemann's hand. But we can compare mathematical formulas since their notation was already internationally standardised back then, just as it is today. We give two examples in figure 5 and 6 from page 3 (19) and 3 (22). It is not difficult to see that it must be the same person who wrote these formulas as one compares the letters “d”, “n”, “s” and “II”. In both cases the letter “d” looks like the notation of a partial derivative “ ∂ ”, a characteristic of Riemann's handwriting. Also the style of letter “s” has a special feature in both papers: it often looks like digit “5”. Therefore, we can assume that the French fair copy 3 (22) must be in Riemann's hand.

Now, since 3 (22) is an autograph of Riemann in Latin script, we can compare the handwriting of the first page of 3 (16–18) in figure 3 with that of this autograph in figure 2. We can immediately see that the handwritings are completely different. Whereas the handwriting of 3 (16) must be that of a person who used it every day since many years, Riemann's handwriting of 3 (22) shows an unpractised hand. This is not surprising because he used Latin script only under special circumstances, namely when he wrote Latin, French or English texts.

Comparing the style of some letters in figures 5 and 6, the handwriting of mathematical formulas do not match either. The creator of 3 (16) writes letter “s” as it is used today in contrast to Riemann's “5”. Further, his “n” and “II” look different as well as his “dx” which does not contain a “ ∂ ”.

These comparisons show clearly that the handwriting of 3 (16) is not Riemann's so that 3 (16–18) cannot have been written by Riemann himself.

3. ALFRED CLEBSCH, CREATOR OF THE FAIR COPY

If Riemann is not the creator of the fair copy 3 (16–18), then who is? After a search through 13 other compilations of Riemann's Nachlass, I found a letter by Alfred Clebsch (1833–1872), successor to Riemann's Chair at the University of Göttingen from 1868 to 1872, in compilation 1,2⁽⁷⁾. Since this letter, written in Latin script and sent to the publishing company B. G. Teubner at Leipzig, is signed by Alfred Clebsch himself, it is an autograph and can be used for a comparison. If we do so and look at figures 3 and 4, we can immediately see that the handwritings of these documents are identical. To substantiate this observation, it suffices to take a look at figure 7–12, which contains six capital letters from both documents. The result is clear: both documents must have been written by the same person. Thus Alfred Clebsch is the creator of the fair copy 3 (16–18).

4. THE PRESUMED PURPOSE OF THE FAIR COPY

Alfred Clebsch was one of the editors of Riemann's Collected Works by Teubner at Leipzig until his death. The other was Richard Dedekind. This can be gathered for example from Clebsch's letters [2] to Dedekind in 1872. In particular, we see it by his letter dated 17th of September 1872 [2, no. 42] since it contains a proposal for the designation of chapters and their order. Riemann's publication was provided

(6) See [12] and figure 2 for the first page.

(7) See [1] or figure 4 and subsection 5.1 in the appendix for a German transcription and English translation.

under the title “Primzahlen” as chapter X. But after Clebsch’s unexpected death on 7th November 1872, Heinrich Weber took over the edition of Riemann’s works and changed it to chapter VII⁽⁸⁾. This can be comprehended in the first line of Clebsch’s fair copy 3 (16)⁽⁹⁾ where a Roman X was crossed and replaced by a Roman VII, all written with a pencil. Indeed, Riemann’s publication appeared as chapter VII of the Collected Works so Clebsch’s fair copy must be connected to their edition in some way.

Let us now look at the Collected Works [7] published in 1876 by Teubner at Leipzig, four years after Clebsch’s death. Since it is a scientific book, it was printed in Roman type and not in German Fraktur⁽¹⁰⁾. Although they are defined in Roman type, the capital umlauts “Ä”, “Ö” and “Ü” as well as the “ß” were not used in this book but were replaced by “Ae”, “Oe”, “Ue” and “ss”. This is exactly what Clebsch did in his letters to Dedekind [2], in his letter to Teubner [1]/figure 4 and in his fair copy 3 (16–18) [11]/figure 3, all written in Latin script. So was it Clebsch who had defined the print style of the book? Definitely not, as can be seen in five other mathematical books [3, 5, 16, 17, 18] published between 1868 and 1878 by Teubner that all have the same print style. So the commonly used title of Riemann’s publication

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse

is not correct since it is only due to Teubner’s print style for mathematical books at that time. The correct title

Über die Anzahl der Primzahlen unter einer gegebenen Größe

should be used instead as Riemann defines it in his draft 3 (19–20) on the first line of figure 1 and as it was printed in *Monatsberichte der Akademie* [9]⁽¹¹⁾ in 1860.

The different print style between the original publication [9] of 1860 and the reprint in the Collected Works of 1876 [7, ch. VII, pp. 136–144] gives reason to look at their deviations in detail, including the handwritten versions of Riemann’s draft 3 (19–20) [10] and Clebsch’s fair copy 3 (16–18) [11]. There are only minor deviations which are compiled in the table of figure 13, with the limitation that not all deviations between Riemann’s draft and the print version of 1860 are listed. Up to three non-significant clerical errors, there are only two deviations between Clebsch’s fair copy and the print version of the Collected Works concerning the words “hiedurch”/“hierdurch” and “Hievon”/“Hiervon” in deviation 1 and 3 of the table. Clebsch first wrote “Hiervon” in deviation 3, then he corrected it to “Hievon”. In the Collected Works we find the linguistic modern spelling “hierdurch” and “Hiervon” which may be the result of a lector’s revision. Thus we can consider Clebsch’s fair copy and the print version of 1876 as identical.

Deviation 4 is not significant since it is only a spelling variation, but deviation 2 is worthy of a more detailed investigation. Riemann uses the word “als” – English “as” – in his draft which he also does in the print version of 1860 in *Monatsberichte der Akademie*. Clebsch, however, removes this word in his fair copy. This change is

(8) We cannot exclude that it was Clebsch himself who made this change between 17th September and 25th October 1872 when he sent some of the setting copies to Teubner (see his letter 2,5 (5) in [1] and in figure 4). But this seems to be very unlikely.

(9) See [11] and 3.

(10) The “Fraktur” was the common print style for German text in German countries from the 15th century until 1941. See footnote 4 on page 2.

(11) The word “über” is used on page 671 instead of “Über” because the title is embedded in a larger text passage. But two pages before, on page 669, we find the words “Übersendung” and “Übersetzung” showing that the print style of this book includes the capital umlauts.

also included in the Collected Works of 1876, four years after Clebsch's death. Since it is very unlikely that another person, perhaps the new editor Heinrich Weber or a lector of the Teubner company, made exactly the same modification of Riemann's text at the same location as Clebsch did four years earlier, there is only one possible explanation: Clebsch's fair copy was used as a setting copy for chapter VII of Riemann's Collected Works published in 1876, just as Erwin Neuenschwander suspected in a private communication. Also, it is very likely that it belonged to the "ordered compilation of prints and manuscripts" that Clebsch sent to Teubner at Leipzig on 25th October 1872⁽¹²⁾. Still, what we don't know is, how this document came into compilation Cod. Ms. B. Riemann 3. Possibly, an answer can be found in the archives of the Teubner company, now preserved in the *Staatsarchiv Leipzig*.

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(12) See subsection 5.1 on page 7.

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5. APPENDIX

5.1. **Transcription of Clebsch's letter.** The letter is sheet 5 of compilation Cod. Ms. B. Riemann 1,2 (see [1] and figure 4). Alfred Clebsch wrote it on 25th October 1872, 13 days(!) before his tragic death of diphtheria on 7th November 1872, to the publishing company B. G. Teubner at Leipzig in connection with the edition of Bernhard Riemann's Collected Works. This is indubitably an autograph of Alfred Clebsch since it is signed by himself. Presumably, this letter is Clebsch's last preserved document at all.

GERMAN TRANSCRIPTION

Göttingen d. 25^t Oct. 72.

An die Teubnersche Buchhandlung, Leipzig.

Hochgeehrter Herr!

Sie erhalten hiebei die zu der Ausgabe von Riemanns Werken gehörigen Drucke und Manuscripte geordnet; es fehlen nur noch drei nicht sehr umfangreiche Abhandlungen, die ich ihrer Zeit folgen lasse.

Mit vollkommenster Hochachtung

Ihr

ergebenster

A. Clebsch

ENGLISH TRANSLATION

Göttingen, 25th Oct. 72.

To Teubner's bookstore, Leipzig.

Highly esteemed Sir!

Enclosed, you will find prints and manuscripts associated to the edition of Riemann's works in an ordered compilation; there are only three not very extensive treatises missing that I let follow at their time.

With perfect tribute

your

obedient

A. Clebsch

5.2. Figures.

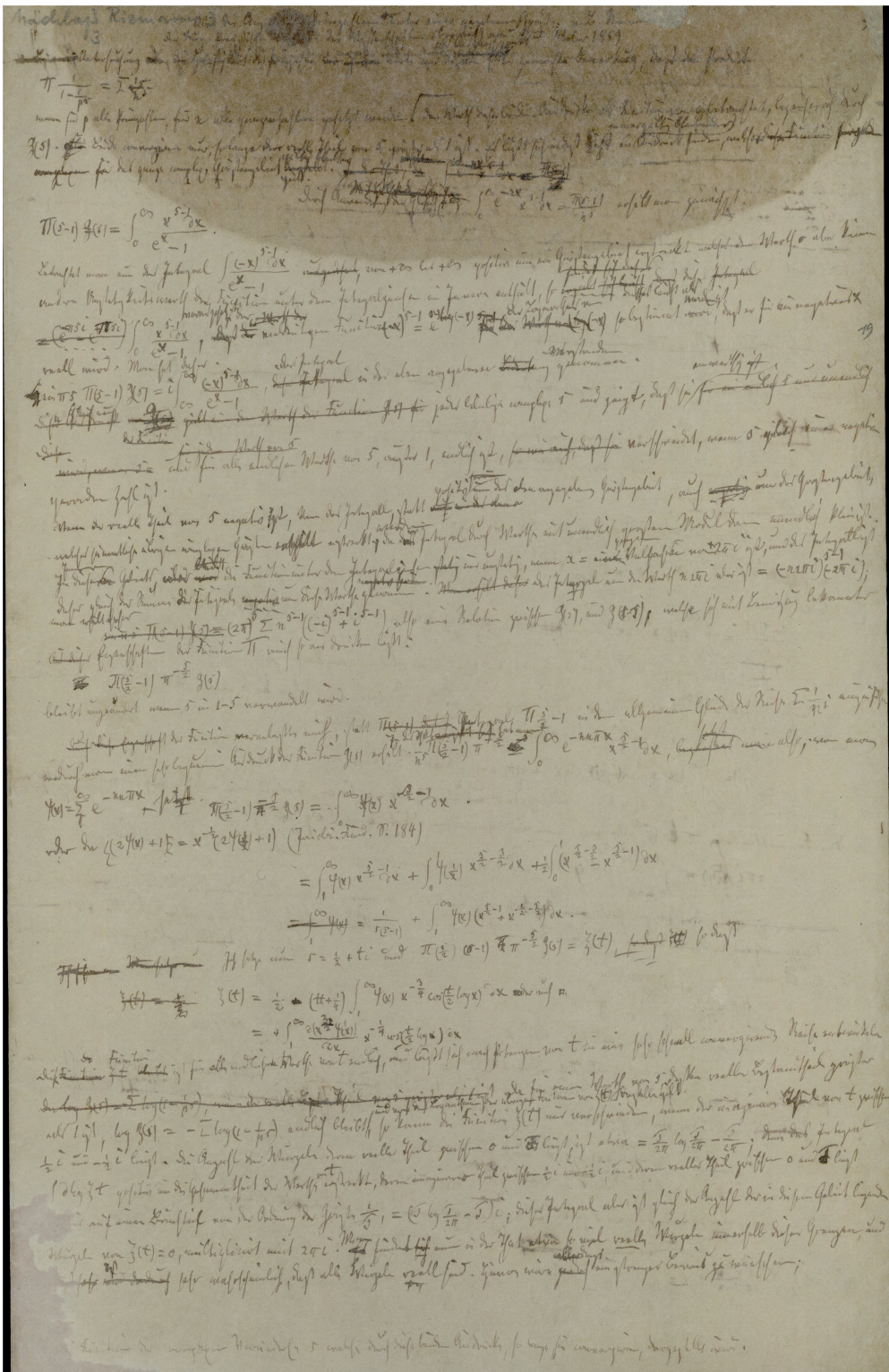


FIGURE 1. Riemann's draft of his publication, first page, October 1859

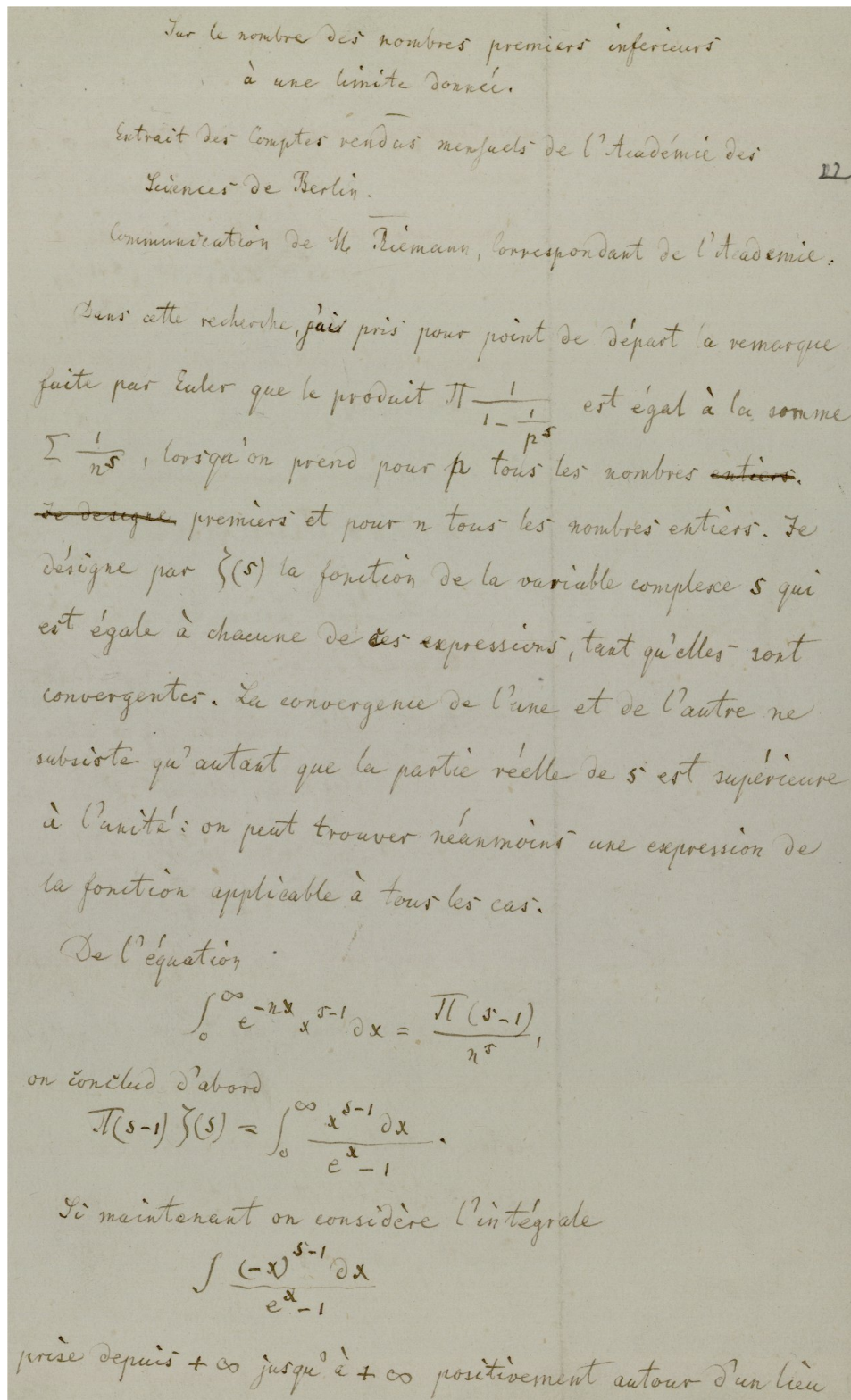


FIGURE 2. Riemann's fair copy in French, first page

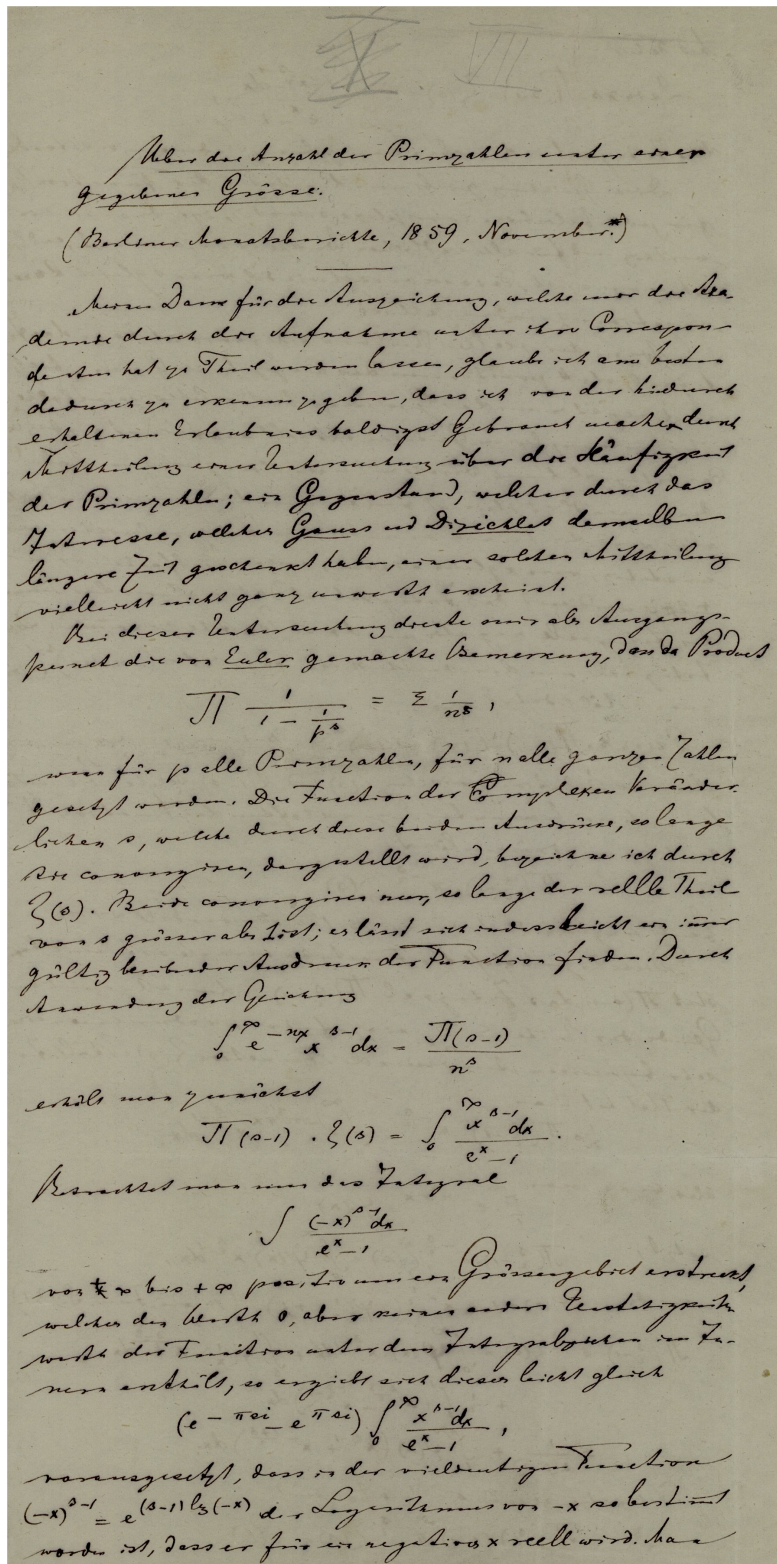


FIGURE 3. Clebsch's fair copy of Riemann's publication, first page

Göppingen d. 25. Oct. 72. 5

An den Teubnerschen Buchhändler, Leipzig.

Hochgeehrter Herr!

Be erhalten hierbei die zu der Ausgabe
von Riemanns Werke gehörigen Drucke
und Manuscripte geordnet; es fehlen nur
noch drei nicht sehr umfangreiche Ab-
handlungen, die ich Ihnen fortfolgen lasse.
Mit vollkommener Hochachtung

A. Clebsch

FIGURE 4. Clebsch's letter to Teubner at Leipzig

Figure 5 consists of three vertically stacked panels of handwritten mathematical formulas. The top panel shows the integral $\int_0^{\infty} e^{-nx} x^{s-1} dx = \frac{\Gamma(s)}{n^s}$. The middle panel shows the same formula with a slightly different notation for the gamma function, $\Gamma(s-1)$ in the numerator. The bottom panel shows the formula with $\Gamma(s-1)$ in the numerator and n^s in the denominator.

FIGURE 5. Details of 3 (19), 3 (22) and 3 (16) from top to bottom

Figure 6 consists of three vertically stacked panels of handwritten mathematical formulas. The top panel shows the integral $\int_0^{\infty} \frac{(-x)^{s-1} dx}{e^x - 1}$. The middle panel shows the same formula with a slightly different notation for the gamma function, $\Gamma(s-1)$ in the numerator. The bottom panel shows the formula with $\Gamma(s-1)$ in the numerator and $e^x - 1$ in the denominator.

FIGURE 6. Details of 3 (19), 3 (22) and 3 (16) from top to bottom

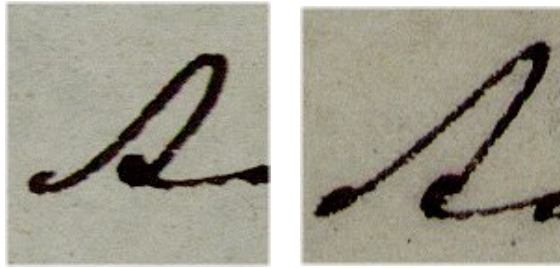


FIGURE 7. "A" in line 4 of 1,2 (5) and in line 5 of 3 (16)

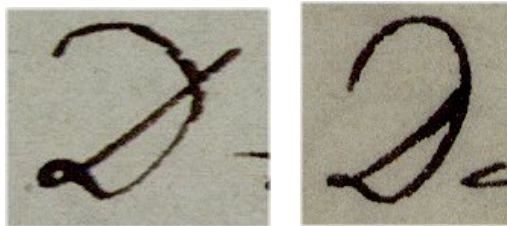


FIGURE 8. "D" in line 5 of 1,2 (5) and in line 24 of 3 (16)

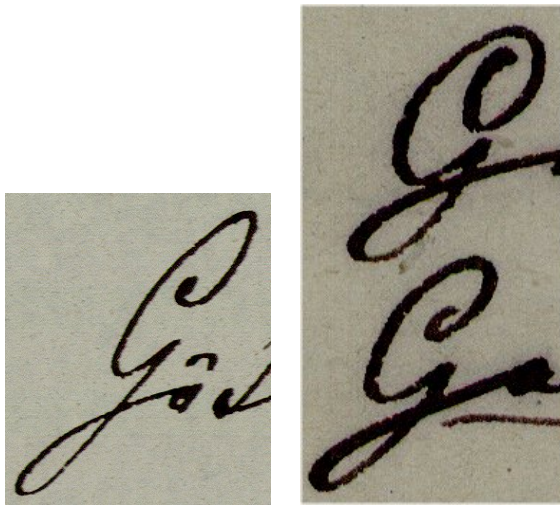


FIGURE 9. "G" in line 1 of 1,2 (5) and in line 11/12 of 3 (16)

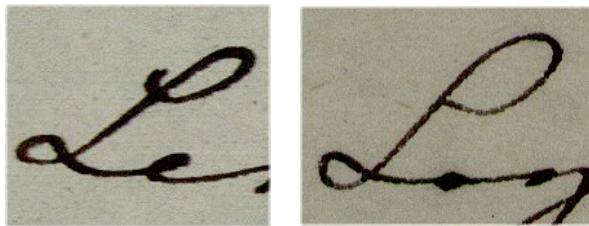


FIGURE 10. "L" in line 2 of 1,2 (5) and in line 37 of 3 (16)

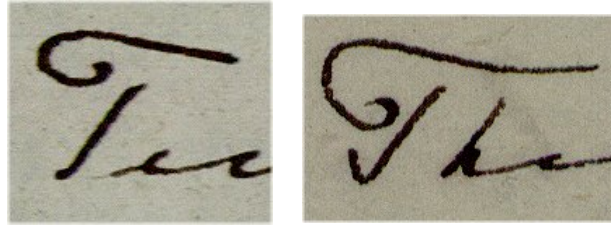


FIGURE 11. “T” in line 2 of 1,2 (5) and in line 7 of 3 (16)

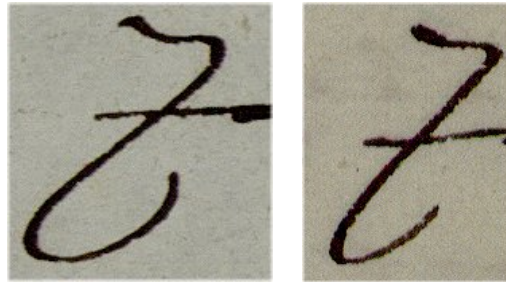


FIGURE 12. “Z” in line 8 of 1,2 (5) and in line 13 of 3 (16)

No.	Location	Riemann's draft (October 1859)	Monatsberichte der Akademie (spring 1860)	Clebsch's fair copy (before November 1872)	Collected Works (1876)
1	—; p. 676, l. 10 f. b.; 3 (16), l. 8-9; p. 136, l. 8	—	von der <u>hiedurch</u> erhaltenen	von der <u>hiedurch</u> erhaltenen	von der <u>hierdurch</u> erhaltenen
2	3 (19), l. 11; p. 672, l. 16; 3 (16), l. 5 f. b.; p. 137, l. 5-6	so ergibt sich dieses <u>leicht als</u>	so ergibt sich dieses <u>leicht als gleich</u>	so ergibt sich dieses <u>leicht gleich</u>	so ergibt sich dieses <u>leicht gleich</u>
3	3 (19), l. 2 f. b.; p. 674, l. 5 f. b.; 3 (17), l. 23; p. 139, l. 9	<u>Hievon</u> wäre allerdings	<u>Hievon</u> wäre allerdings	<u>Hievon</u> wäre allerdings (after correction from “Hiervon” to “Hievon”)	<u>Hievon</u> wäre allerdings
4	3 (19) rear, l. 7 f. b.; p. 676, l. 3 f. b.; 3 (17) rear, l. 12 f. b.; p. 141, l. 5	aber dann <u>in's</u> Unendliche	aber dann <u>in's</u> Unendliche	aber dann <u>ins</u> Unendliche	aber dann <u>ins</u> Unendliche

l.: line
f. b.: from bottom

FIGURE 13. Deviations between the four versions of Riemann's publication