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The Poincaré Conjecture

Clay Research Conference
Resolution of the Poincaré Conjecture
Institut Henri Poincaré
Paris, France, June 8–9, 2010

James Carlson
Editor

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Preface

In 1904, the eminent French mathematician Henri Poincaré formulated the conjecture that bears his name and that motivated much of the research in geometry and topology for the next one hundred years. Among these developments were the theory of knots, homotopy theory, surgery theory, and the formulation by William Thurston of the geometrization conjecture, a sweeping statement that subsumed Poincaré's conjecture and which gave structure and order to the set of all 3-dimensional manifolds.

In 2000, at a meeting in Paris, the Clay Mathematics Institute (CMI), founded by Mr. Landon T. Clay, announced the establishment of the seven Millennium Prize Problems. For the solution of each one, a prize of $1,000,000 was offered. The Poincaré conjecture was one of those seven problems.

In November 2002 came a major development: Grigoriy Perelman posted the first of three papers announcing a proof of the conjecture on arXiv.org. His announcement set off a flurry of excitement in the mathematical world. Perelman gave talks at MIT, SUNY-Stony Brook, Princeton, and the University of Pennsylvania. Seminars were organized to understand what Perelman had done, and several groups of researchers set about the task of carefully verifying and validating his work. CMI supported several of these efforts (Kleiner and Lott, Morgan and Tian), and it organized a working seminar in Princeton in 2004 devoted to Perelman's second paper. On March 10 of 2010, CMI announced award of the Millennium Prize for the Poincaré conjecture to Grigoriy Perelman. The citation read:

> The Clay Mathematics Institute hereby awards the Millennium Prize for resolution of the Poincaré conjecture to Grigoriy Perelman. The Poincaré conjecture is one of the seven Millennium Prize Problems established by CMI in 2000. The Prizes were conceived to record some of the most difficult problems with which mathematicians were grappling at the turn of the second millennium; to elevate in the consciousness of the general public the fact that in mathematics, the frontier is still open and abounds in important unsolved problems; to emphasize the importance of working towards a solution of the deepest, most difficult problems; and to recognize achievement in mathematics of historical magnitude.

The decision to award the prize was made on the basis of deliberations by a Special Advisory Committee appointed to consider the correctness and attribution of the solution, the CMI Scientific Advisory Board, and the CMI Board of Directors. The committee members were Simon Donaldson, David Gabai, Mikhail Gromov, Terence Tao, and Andrew Wiles (Special Advisory Committee), James Carlson, Simon Donaldson, Gregory Margulis, Richard Melrose, Yum-Tong Siu, and Andrew Wiles (Scientific Advisory Board), and Landon T. Clay, Lavinia D. Clay, and Thomas M. Clay (Board of Directors).

On June 8 and 9 of 2010, a conference on the conjecture was held at the Institut Henri Poincaré in Paris. Most of the lectures given there are featured in this volume. They provide an overview of the conjecture—its history, its influence on the development of mathematics, and, finally, its proof. Sadly, there is no article by William Thurston, who passed away in 2012.

Grigoriy Perelman did not accept the Millennium Prize, just as he did not accept the Fields Medal in 2006. What is important, however, is what Perelman gave to mathematics: the solution to a long-standing problem of historical significance, and a set of new ideas and new tools with which better to understand the geometry of manifolds. One chapter of mathematics ends and another begins.

James Carlson

Press Release of March 10, 2010

The Clay Mathematics Institute (CMI) announces today that Dr. Grigoriy Perelman of St. Petersburg, Russia, is the recipient of the Millennium Prize for resolution of the Poincaré conjecture. The citation for the award reads:

> The Clay Mathematics Institute hereby awards the Millennium Prize for resolution of the Poincaré conjecture to Grigoriy Perelman.

The Poincaré conjecture is one of the seven Millennium Prize Problems established by CMI in 2000. The Prizes were conceived to record some of the most difficult problems with which mathematicians were grappling at the turn of the second millennium; to elevate in the consciousness of the general public the fact that in mathematics, the frontier is still open and abounds in important unsolved problems; to emphasize the importance of working towards a solution of the deepest, most difficult problems; and to recognize achievement in mathematics of historical magnitude.

The award of the Millennium Prize to Dr Perelman was made in accord with their governing rules: recommendation first by a Special Advisory Committee (Simon Donaldson, David Gabai, Mikhail Gromov, Terence Tao, and Andrew Wiles), then by the CMI Scientific Advisory Board (James Carlson, Simon Donaldson, Gregory Margulis, Richard Melrose, Yum-Tong Siu, and Andrew Wiles), with final decision by the Board of Directors (Landon T. Clay, Lavinia D. Clay, and Thomas M. Clay).

James Carlson, President of CMI, said today that "resolution of the Poincaré conjecture by Grigoriy Perelman brings to a close the century-long quest for the solution. It is a major advance in the history of mathematics that will long be remembered". Carlson went on to announce that CMI and the Institut Henri Poincaré (IHP) will hold a conference to celebrate the Poincaré conjecture and its resolution June 8 and 9 in Paris. The program will be posted on www.claymath.org. In addition, on June 7, there will be a press briefing and public lecture by Etienne Ghys at the Institut Oceanographique, near the IHP.

Reached at his office at Imperial College, London for his reaction, Fields Medalist Dr. Simon Donaldson said, "I feel that Poincaré would have been very satisfied to know both about the profound influence his conjecture has had on the development of topology over the last century and the surprising way in which the problem was solved, making essential use of partial differential equations and differential geometry.

Poincaré's conjecture and Perelman's proof

Formulated in 1904 by the French mathematician Henri Poincaré, the conjecture is fundamental to achieving an understanding of three-dimensional shapes (compact manifolds). The simplest of these shapes is the three-dimensional sphere. It is contained in four-dimensional space, and is defined as the set of points at a fixed distance from a given point, just as the two-dimensional sphere (skin of an orange or surface of the earth) is defined as the set of points in three-dimensional space at a fixed distance from a given point (the center).

Since we cannot directly visualize objects in n-dimensional space, Poincaré asked whether there is a test for recognizing when a shape is the three-sphere by performing measurements and other operations inside the shape. The goal was to recognize all three-spheres even though they may be highly distorted. Poincaré found the right test (simple connectivity, see below). However, no one before Perelman was able to show that the test guaranteed that the given shape was in fact a three-sphere.

In the last century, there were many attempts to prove, and also to disprove, the Poincaré conjecture using the methods of topology. Around 1982, however, a new line of attack was opened. This was the Ricci flow method pioneered and developed by Richard Hamilton. It was based on a differential equation related to the one introduced by Joseph Fourier 160 years earlier to study the conduction of heat. With the Ricci flow equation, Hamilton obtained a series of spectacular results in geometry. However, progress in applying it to the conjecture eventually came to a standstill, largely because formation of singularities, akin to formation of black holes in the evolution of the cosmos, defied mathematical understanding.

Perelman's breakthrough proof of the Poincaré conjecture was made possible by a number of new elements. He achieved a complete understanding of singularity formation in Ricci flow, as well as the way parts of the shape collapse onto lower-dimensional spaces. He introduced a new quantity, the entropy, which instead of measuring disorder at the atomic level, as in the classical theory of heat exchange, measures disorder in the global geometry of the space. This new entropy, like the thermodynamic quantity, increases as time passes. Perelman also introduced a related local quantity, the L-functional, and he used the theories originated by Cheeger and Aleksandrov to understand limits of spaces changing under Ricci flow. He showed that the time between formation of singularities could not become smaller and smaller, with singularities becoming spaced so closely—infinitesimally close—that the Ricci flow method would no longer apply. Perelman deployed his new ideas and methods with great technical mastery and described the results he obtained with elegant brevity. Mathematics has been deeply enriched.

Some other reactions

Fields medalist Stephen Smale, who solved the analogue of the Poincaré conjecture for spheres of dimension five or more, commented that: "Fifty years ago I was working on Poincaré's conjecture and thus hold a long-standing appreciation for this beautiful and difficult problem. The final solution by Grigoriy Perelman is a great event in the history of mathematics."

Donal O'Shea, Professor of Mathematics at Mt. Holyoke College and author of *The Poincaré Conjecture*, noted: "Poincaré altered twentieth-century mathematics by teaching us how to think about the idealized shapes that model our cosmos. It is very satisfying and deeply inspiring that Perelman's unexpected solution to the Poincaré conjecture, arguably the most basic question about such shapes, offers to do the same for the coming century.

History and Background

In the latter part of the nineteenth century, the French mathematician Henri Poincaré was studying the problem of whether the solar system is stable. Do the planets and asteroids in the solar system continue in regular orbits for all time, or will some of them be ejected into the far reaches of the galaxy or, alternatively, crash into the sun? In this work he was led to topology, a still new kind of mathematics related to geometry, and to the study of shapes (compact manifolds) of all dimensions.

The simplest such shape was the circle, or distorted versions of it such as the ellipse or something much wilder: lay a piece of string on the table, tie one end to the other to make a loop, and then move it around at random, making sure that the string does not touch itself. The next simplest shape is the two-sphere, which we find in nature as the idealized skin of an orange, the surface of a baseball, or the surface of the earth, and which we find in Greek geometry and philosophy as the "perfect shape". Again, there are distorted versions of the shape, such as the surface of an egg, as well as still wilder objects. Both the circle and the two-sphere can be described in words or in equations as the set of points at a fixed distance from a given point (the center). Thus it makes sense to talk about the three-sphere, the four-sphere, etc. These shapes are hard to visualize, since they naturally are contained in four-dimensional space, five-dimensional space, and so on, whereas we live in three-dimensional space. Nonetheless, with mathematical training, shapes in higher-dimensional spaces can be studied just as well as shapes in dimensions two and three.

In topology, two shapes are considered the same if the points of one correspond to the points of another in a continuous way. Thus the circle, the ellipse, and the wild piece of string are considered the same. This is much like what happens in the geometry of Euclid. Suppose that one shape can be moved, without changing lengths or angles, onto another shape. Then the two shapes are considered the same (think of congruent triangles). A round, perfect two-sphere, like the surface of a ping-pong ball, is topologically the same as the surface of an egg.

In 1904 Poincaré asked whether a three-dimensional shape that satisfies the "simple connectivity test" is the same, topologically, as the ordinary round three-sphere. The round three-sphere is the set of points equidistant from a given point in four-dimensional space. His test is something that can be performed by an imaginary being who lives inside the three-dimensional shape and cannot see it from"outside." The test is that every loop in the shape can be drawn back to the point of departure without leaving the shape. This can be done for the two-sphere and the three-sphere. But it cannot be done for the surface of a doughnut, where a loop may get stuck around the hole in the doughnut.

The question raised became known as the Poincaré conjecture. Over the years, many outstanding mathematicians tried to solve it—Poincaré himself, Whitehead, Bing, Papakirioukopolos, Stallings, and others. While their efforts frequently led to the creation of significant new mathematics, each time a flaw was found in the proof. In 1961 came astonishing news. Stephen Smale, then of the University of

California at Berkeley (now at the City University of Hong Kong) proved that the analogue of the Poincaré conjecture was true for spheres of five or more dimensions. The higher-dimensional version of the conjecture required a more stringent version of Poincaré's test; it asks whether a so-called homotopy sphere is a true sphere. Smale's theorem was an achievement of extraordinary proportions. It did not, however, answer Poincaré's original question. The search for an answer became all the more alluring.

Smale's theorem suggested that the theory of spheres of dimensions three and four was unlike the theory of spheres in higher dimension. This notion was confirmed a decade later, when Michael Freedman, then at the University of California, San Diego, now of Microsoft Research Station Q, announced a proof of the Poincaré conjecture in dimension four. His work used techniques quite different from those of Smale. Freedman also gave a classification, or kind of species list, of all simply connected four-dimensional manifolds.

Both Smale (in 1966) and Freedman (in 1986) received Fields medals for their work. There remained the original conjecture of Poincaré in dimension three. It seemed to be the most difficult of all, as the continuing series of failed efforts, both to prove and to disprove it, showed. In the meantime, however, there came three developments that would play crucial roles in Perelman's solution of the conjecture.

Geometrization

The first of these developments was William Thurston's geometrization conjecture. It laid out a program for understanding all three-dimensional shapes in a coherent way, much as had been done for two-dimensional shapes in the latter half of the nineteenth century. According to Thurston, three-dimensional shapes could be broken down into pieces governed by one of eight geometries, somewhat as a molecule can be broken into its constituent, much simpler atoms. This is the origin of the name, "geometrization conjecture."

A remarkable feature of the geometrization conjecture was that it implied the Poincaré conjecture as a special case. Such a bold assertion was accordingly thought to be far, far out of reach—perhaps a subject of research for the twenty-second century. Nonetheless, in an imaginative *tour de force* that drew on many fields of mathematics, Thurston was able to prove the geometrization conjecture for a wide class of shapes (Haken manifolds) that have a sufficient degree of complexity. While these methods did not apply to the three-sphere, Thurston's work shed new light on the central role of Poincaré's conjecture and placed it in a far broader mathematical context.

Limits of Spaces

The second current of ideas did not appear to have a connection with the Poincaré conjecture until much later. While technical in nature, the work, in which the names of Cheeger and Perelman figure prominently, has to do with how one can take limits of geometric shapes, just as we learned to take limits in beginning calculus class. Think of Zeno and his paradox: you walk half the distance from where you are standing to the wall of your living room. Then you walk half the remaining distance. And so on. With each step you get closer to the wall. The wall is your "limiting position," but you never reach it in a finite number of steps. Now imagine a shape changing with time. With each "step" it changes shape, but can nonetheless be a "nice" shape at each step—smooth, as the mathematicians say. For the limiting shape the situation is different. It may be nice and smooth, or it may have special points that are different from all the others, that is, singular points, or "singularities." Imagine a Y-shaped piece of tubing that is collapsing: as time increases, the diameter of the tube gets smaller and smaller. Imagine further that one second after the tube begins its collapse, the diameter has gone to zero. Now the shape is different: it is a Y shape of infinitely thin wire. The point where the arms of the Y meet is different from all the others. It is the singular point of this shape. The kinds of shapes that can occur as limits are called Aleksandrov spaces, named after the Russian mathematician A. D. Aleksandrov who initiated and developed their theory.

Differential Equations

The third development concerns differential equations. These equations involve rates of change in the unknown quantities of the equation, e.g., the rate of change of the position of an apple as it falls from a tree towards the earth's center. Differential equations are expressed in the language of calculus, which Isaac Newton invented in the 1680s in order to explain how material bodies (apples, the moon, and so on) move under the influence of an external force. Nowadays physicists use differential equations to study a great range of phenomena: the motion of galaxies and the stars within them, the flow of air and water, the propagation of sound and light, the conduction of heat, and even the creation, interaction, and annihilation of elementary particles such as electrons, protons, and quarks.

In our story, conduction of heat and change of temperature play a special role. This kind of physics was first treated mathematically by Joseph Fourier in his 1822 book, *Théorie Analytique de la Chaleur*. The differential equation that governs change of temperature is called the heat equation. It has the remarkable property that as time increases, irregularities in the distribution of temperature decrease.

Differential equations apply to geometric and topological problems as well as to physical ones. But one studies not the rate at which temperature changes, but rather the rate of change in some geometric quantity as it relates to other quantities such as curvature. A piece of paper lying on the table has curvature zero. A sphere has positive curvature. The curvature is a large number for a small sphere, but is a small number for a large sphere such as the surface of the earth. Indeed, the curvature of the earth is so small that its surface has sometimes mistakenly been thought to be flat. For an example of negative curvature, think of a point on the bell of a trumpet. In some directions the metal bends away from your eye; in others it bends towards it.

An early landmark in the application of differential equations to geometric problems was the 1963 paper of J. Eells and J. Sampson. The authors introduced the "harmonic map equation," a kind of nonlinear version of Fourier's heat equation. It proved to be a powerful tool for the solution of geometric and topological problems. There are now many important nonlinear heat equations—the equations for mean curvature flow, scalar curvature flow, and Ricci flow.

Also notable is the Yang-Mills equation, which came into mathematics from the physics of quantum fields. In 1983 this equation was used to establish very strong restrictions on the topology of four-dimensional shapes on which it was possible to do calculus [**2**]. These results helped renew hopes of obtaining other strong geometric results from analytic arguments—that is, from calculus and differential equations. Optimism for such applications had been tempered to some extent by the examples of René Thom (on cycles not representable by smooth submanifolds) and Milnor (on diffeomorphisms of the six-sphere).

Ricci Flow

The differential equation that was to play a key role in solving the Poincaré conjecture is the Ricci flow equation. It was discovered two times, independently. In physics, the equation originated with the thesis of Friedan [**3**], although it was perhaps implicit in the work of Honerkamp [**7**]. In mathematics it originated with the 1982 paper of Richard Hamilton [**4**]. The physicists were working on the renormalization group of quantum field theory, while Hamilton was interested in geometric applications of the Ricci flow equation itself. Hamilton, now at Columbia University, was then at Cornell University.

On the left-hand side of the Ricci flow equation is a quantity that expresses how the geometry changes with time—the derivative of the metric tensor, as the mathematicians like to say. On the right-hand side is the Ricci tensor, a measure of the extent to which the shape is curved. The Ricci tensor, based on Riemann's theory of geometry (1854), also appears in Einstein's equations for general relativity (1915). Those equations govern the interaction of matter, energy, curvature of space, and the motion of material bodies.

The Ricci flow equation is the analogue, in the geometric context, of Fourier's heat equation. The idea, *grosso modo*, for its application to geometry is that, just as Fourier's heat equation disperses temperature, the Ricci flow equation disperses curvature. Thus, even if a shape was irregular and distorted, Ricci flow would gradually remove these anomalies, resulting in a very regular shape whose topological nature was evident. Indeed, in 1982 Hamilton showed that for positively curved, simply connected shapes of dimension three (compact three-manifolds) the Ricci flow transforms the shape into one that is ever more like the round three-sphere. In the long run, it becomes almost indistinguishable from this perfect, ideal shape. When the curvature is not strictly positive, however, solutions of the Ricci flow equation behave in a much more complicated way. This is because the equation is nonlinear. While parts of the shape may evolve towards a smoother, more regular state, other parts might develop singularities. This richer behavior posed serious difficulties. But it also held promise: it was conceivable that the formation of singularities could reveal Thurston's decomposition of a shape into its constituent geometric atoms.

Richard Hamilton

Hamilton was the driving force in developing the theory of Ricci flow in mathematics, both conceptually and technically. Among his many notable results is his 1999 paper [**5**], which showed that in a Ricci flow, the curvature is pushed towards the positive near a singularity. In that paper Hamilton also made use of the collapsing theory [**1**] mentioned earlier. Another result [**6**], which played a crucial role in Perelman's proof, was the Hamilton Harnack inequality, which generalized to positive Ricci flows a result of Peter Li and Shing-Tung Yau for positive solutions of Fourier's heat equation.

Hamilton had established the Ricci flow equation as a tool with the potential to resolve both conjectures as well as other geometric problems. Nevertheless, serious obstacles barred the way to a proof of the Poincaré conjecture. Notable among these obstacles was lack of an adequate understanding of the formation of singularities in Ricci flow, akin to the formation of black holes in the evolution of the cosmos. Indeed, it was not at all clear how or if formation of singularities could be understood. Despite the new front opened by Hamilton, and despite continued work by others using traditional topological tools for either a proof or a disproof, progress on the conjectures came to a standstill.

Such was the state of affairs in 2000, when John Milnor wrote an article describing the Poincaré conjecture and the many attempts to solve it. At that writing, it was not clear whether the conjecture was true or false, and it was not clear which method might decide the issue. Analytic methods (differential equations) were mentioned in a later version (2004). See [**8**] and [**9**].

Perelman announces a solution of the Poincaré conjecture

It was thus a huge surprise when Grigoriy Perelman announced, in a series of preprints posted on ArXiv.org in 2002 and 2003, a solution not only of the Poincaré conjecture, but also of Thurston's geometrization conjecture [**10**], [**11**], [**12**].

The core of Perelman's method of proof is the theory of Ricci flow. To its applications in topology he brought not only great technical virtuosity, but also new ideas. One was to combine collapsing theory in Riemannian geometry with Ricci flow to give an understanding of the parts of the shape that were collapsing onto a lower-dimensional space. Another was the introduction of a new quantity, the entropy, which instead of measuring disorder at the atomic level, as in the classical theory of heat exchange, measures disorder in the global geometry of the space. Perelman's entropy, like the thermodynamic entropy, is increasing in time: there is no turning back. Using his entropy function and a related local version (the L-length functional), Perelman was able to understand the nature of the singularities that formed under Ricci flow. There were just a few kinds, and one could write down simple models of their formation. This was a breakthrough of first importance.

Once the simple models of singularities were understood, it was clear how to cut out the parts of the shape near them as to continue the Ricci flow past the times at which they would otherwise form. With these results in hand, Perelman showed that the formation times of the singularities could not run into Zeno's wall: imagine a singularity that occurs after one second, then after half a second more, then after a quarter of a second more, and so on. If this were to occur, the "wall," which one would reach two seconds after departure, would correspond to a time at which the mathematics of Ricci flow would cease to hold. The proof would be unattainable. But with this new mathematics in hand, attainable it was.

The posting of Perelman's preprints and his subsequent talks at MIT, SUNY-Stony Brook, Princeton, and the University of Pennsylvania set off a worldwide effort to understand and verify his groundbreaking work. In the US, Bruce Kleiner and John Lott wrote a set of detailed notes on Perelman's work. These were posted online as the verification effort proceeded. A final version was posted to ArXiv.org in May 2006, and the refereed article appeared in *Geometry and Topology* in 2008. This was the first time that work on a problem of such importance was facilitated via a public website. John Morgan and Gang Tian wrote a book-long exposition of Perelman's proof, posted on ArXiv.org in July of 2006, and published by the American Mathematical Society in CMI's monograph series (August 2007). These expositions, those by other teams, and, importantly, the multi-year scrutiny of the mathematical community, provided the needed verification. Perelman had solved the Poincaré conjecture. After a century's wait, it was settled!

Among other articles that appeared following Perelman's work is a paper in the *Asian Journal of Mathematics*, posted on ArXiv.org in June of 2006 by the American-Chinese team, Huai-Dong Cao (Lehigh University) and Xi-Ping Zhu (Zhongshan University). Another is a paper by the European group of Bessieres, Besson, Boileau, Maillot, and Porti, posted on ArXiv.org in June of 2007. It was accepted for publication by *Inventiones Mathematicae* in October of 2009. It gives an alternative approach to the last step in Perelman's proof of the geometrization conjecture. Perelman's proof of the Poincaré and geometrization conjectures is a major mathematical advance. His ideas and methods have already found new applications in analysis and geometry; surely the future will bring many more.

JC, March 18, 2010

Bibliography

1. J. Cheeger and M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded. I and II, J. Differential Geom. Volume 23, Number 3 (1986); Volume 32, Number 1 (1990), 269–298. MR1064875 (92a:53066)
2. S. K. Donaldson. An application of gauge theory to four-dimensional topology. J. Differential Geom., 18, (1983), 279–315. MR710056 (85c:57015)
3. D. Friedan, Nonlinear Models in 2 + epsilon Dimensions, Annals of Physics 163, 318-419 (1985) MR811072 (87f:81130)
4. R. Hamilton, Three-manifolds with positive Ricci curvature, Journal of Differential Geometry, vol. 17, 255–306 (1982) MR664497 (84a:53050)
5. R. Hamilton, Non-singular solutions of the Ricci flow on three-manifolds, Comm. Anal. Geom. 7(4), 695–729 (1999) MR1714939 (2000g:53034)
6. R. Hamilton, The Harnack estimate for Ricci flow, Journal of Differential Geometry, vol. 37, 225–243 (1993) MR1198607 (93k:58052)
7. J. Honerkamp, (CERN), Chiral multiloops, Nucl. Phys. B36, 130–140 (1972)
8. J. Milnor, The Poincaré Conjecture (2000) www.claymath.org/sites/default/files/poincare.pdf
9. J. Milnor, The Poincaré Conjecture, in *The Millennium Prize Problems*, J. Carlson, A. Jaffe, A. Wiles, eds, AMS (2004) www.claymath.org/publications/special-editions/millennium-prize-problems MR2238275
10. G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv.org, November 11, 2002
11. G. Perelman, Ricci flow with surgery on three-manifolds, arXiv.org, March 10, 2003
12. G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, arXiv.org, July 17, 2003

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Geometry in 2, 3 and 4 Dimensions

Michael Atiyah

1. Introduction

Ten years ago, when the Millennium challenge of the Clay Mathematical Institute was launched here in Paris, I was one of the two speakers tasked with presenting the problems. The other was John Tate and 7 problems were divided between us, broadly based on a geometry/algebra division.

So it is appropriate that I introduce today's session devoted to the first of the seven problems to be solved. The whole world now knows that the century old conjecture made by Henri Poincaré, the leading French mathematician of his time, has been conclusively settled by the young Russian mathematician Grigoriy Perelman. It is without question a great event to be celebrated and the ten years we have had to wait is a short period for a problem of this importance. Time will tell how many decades will pass before the remaining six millennium problems succumb to the skill and efforts of the young mathematicians of the 21st century. I myself am optimistic that we will not have to wait too long for the next occasion, though I expect another presenter will be required.

In the subsequent lectures there will be more specialized presentations of the mathematical aspects of Perelman's proof, both about its achievement and about directions that it opens up for the future. My aim is to put things into a historical context by reviewing in broad terms the history of geometry over the past two centuries. As we know the Poincaré conjecture is about characterizing the 3-dimensional sphere in topological terms and its resolution by Perelman, combined with the earlier brilliant work of William Thurston, provides an essentially complete understanding of compact 3-dimensional manifolds. As such it sits on the cusp between the classical geometry of surfaces and the still emerging geometry of 4 dimensions, which may occupy mathematicians (and physicists) for many years to come.

After my historical review I will move on to discuss relations between geometry and physics which have enjoyed a remarkable renaissance in recent years. I will conclude with a speculative peep into the future, indicating some of the problems that lie ahead.

2. Historical context: dimensions 2 and 3

The single most important idea in differential geometry is that of curvature, pioneered by Gauss and then by Riemann. It is remarkable how far-reaching this

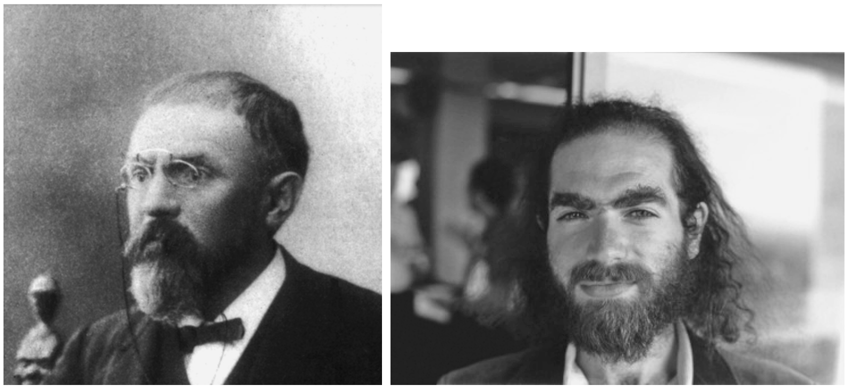

FIGURE 1. Henri Poincaré (1854–1912) and Grigoriy Perelman (1966–)

has proved to be, and we can trace its evolution through the increase in dimensions. Roughly we can divide the history of geometry into three eras.

19th century: dealing with 2 dimensions and the scalar cuvature R
20th century: dealing with 3 dimensions and the Ricci curvature R_{ij}
21st century: dealing with 4 dimensions and the Riemann curvature R_{ijkl}

Of course this is a great oversimplification and the boundaries between centuries are fluid. Moreover the Riemann curvature remains a basic object in higher-dimensional differential geometry. Nevertheless the work of Simon Donaldson has clearly shown the unique properties of 4-dimensions, and this poses the current challenge. The theory of (compact oriented) surfaces bridged the gaps between topology, differential geometry and algebraic geometry, with the seminal ideas being those of Niels Henrik Abel. The outcome was the classification of surfaces into 3 types depending on the genus g.

$g = 0$	sphere	positive curvature
$g = 1$	torus	zero curvature
$g \geq 2$	general case	negative curvature

It was Poincaré who laid the foundation of topology with the notion of homology as the "counting of holes" of different dimensions and the introduction of the fundamental group. The Poincaré conjecture originated when Poincaré realized that there was a 3-manifold with no homology other than the 3-sphere. This was the famous "fake" 3-sphere, arising from the icosahedron, whose symmetries appear in the fundamental group. This led Poincaré to formulate his famous conjecture:

A compact simply connected 3-manifold is topologically a sphere.

In the 20th century topology became a central topic and in 3 dimensions William Thurston outlined a comprehensive programme in which all 3-manifolds were included. Again, as in 2 dimensions, the classification involved the curvature. The 3- sphere typified positive curvature and hyperbolic 3-manifolds typified negative curvature, with a total of 8 different types in all as building bricks for general 3-manifolds. Perelman's proof, for the 3-sphere, extends naturally to the whole

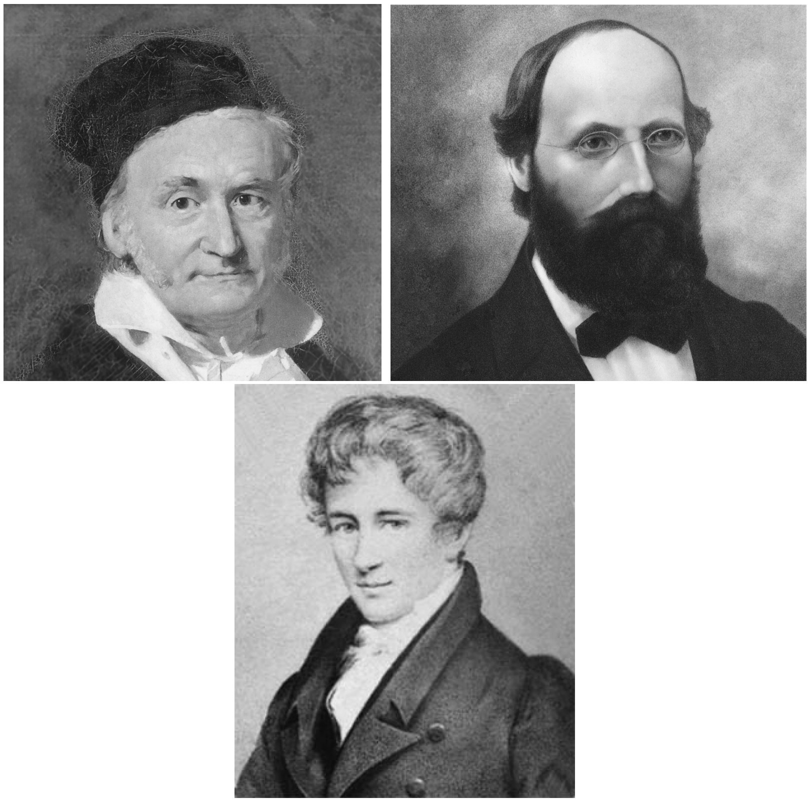

Figure 2. Gauss (1777–1855), Abel (1802–1829) and Riemann (1826–1866)

Figure 3. William Thurston (1946–2012)

Thurston programme and this brings to a close a century's work on the geometry of 3 dimensions. It is definitely the end of an era.

3. Complex algebraic geometry

While the move from dimension 2 to dimension 3 appears to be the obvious step there is a sense in which one should move from 2 to 4. This comes from the consideration of complex algebraic geometry. For complex dimension 1 this theory was started by Abel and continued by Riemann. For algebraic varieties of complex dimension n the real dimension is $2n$, so the case $n = 2$ leads to 4-dimensional real manifolds.

The key figures in the topology of higher-dimensional algebraic varieties were Lefschetz, Hodge, Cartan and Serre.

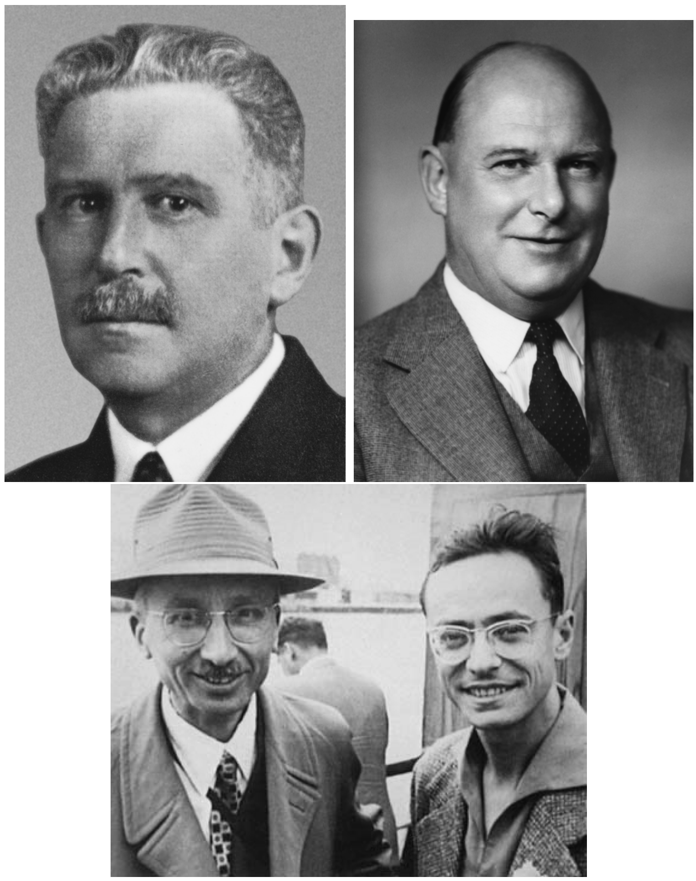

FIGURE 4. Solomon Lefschetz (1884–1972); William Hodge (1903–1975); Henri Cartan (1904–2008) and Jean-Pierre Serre (1926–)

While general algebraic geometry was one of the major developments of the second half of the 20th century, the topology of real 4-manifolds had a great surprise in store when Simon Donaldson made spectacular discoveries opening up an entirely new area.
This work of Donaldson emerged as a by-product of new ideas in physics, another example of which were the new knot invariants discovered by Vaughan Jones. This led to extensive developments linking geometry (particularly in low dimensions) to quantum physics. Much of this was due to Edward Witten and his colleagues.

So the 21st century has begun with the end of a chapter on 3-dimensions and with new problems emerging in 3 and 4 dimensions for the future.

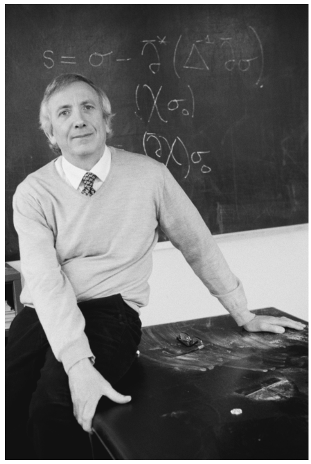

Figure 5. Simon Donaldson (1957–)

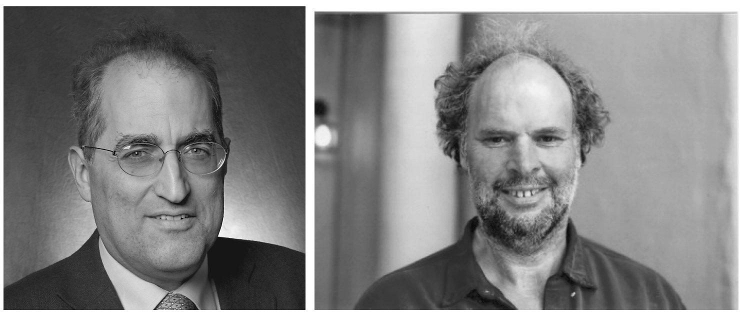

Figure 6. Edward Witten (1951–) and Vaughan Jones (1952–)

Future Problems

Relate Jones quantum invariants to Perelman-Thurston.
Understand the structure of simply-connected 4-manifolds and the relation to physics.

4. Speculation

Let me end with some personal speculations on the relation between geometry and physics. As we know Einstein extended the 3 dimensions of space to a 4-dimensional space-time where curvature embodies gravitational force. An idea due essentially to Hermann Weyl shows how an extra 5th dimension incorporates the Maxwell electro-magnetic field. While the 5-dimensional space has an indefinite metric of signature $(4,1)$ we can ignore time and get a 4-dimensional Riemannian manifold. Here Donaldson's theory comes naturally into its own and I am attracted by the idea that the phenomena he unearthed should play a key role in physics. I am exploring the possible role of such Riemannian 4-manifolds as models of nuclear matter, in which topology will relate directly to physics. These ideas are related to, but different from, the way Donaldson's theory is currently related to physical theory. Speculation is risky but essential for progress. But ideas evolve, in a

Darwinian process, with successful ones taking off and unsuccessful ones quietly withering. My speculation may or may not survive the competition. The future will tell.

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100 Years of Topology: Work Stimulated by Poincaré's Approach to Classifying Manifolds

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1. Introduction

Since its formulation in 1904, the Poincaré Conjecture has stood as a signal problem in topology. As such, it has attracted the attention of the leading topologists of each generation. As I will explain in this lecture, while the purely topological methods used to attack this question did not succeed, they have proved extremely fruitful in resolving closely related questions about manifolds. The Poincaré Conjecture continued to stand unresolved but the progress it generated made topology one of the most exciting and vibrant subjects during the twentieth century. The final irony of this story is that the method of solution comes not from the purely topological approach that Poincaré originally suggested but rather from more geometric and analytic approaches that have their foundations in other aspects of Poincaré's work. While it is impossible to know for sure what Poincaré would have thought of the history of his conjecture and the nature of the solution, it is natural and pleasing to speculate that he would have completely approved of the method.

This presentation is different from the others in this conference, which will be concerned either with details of the proof of the Poincaré Conjecture or the closely related Geometrization Conjecture or an exposition of related subjects. By and large those presentations will cover more geometric and analytic topics. My presentation mostly covers purely topological material. My aim is to show the background of Poincaré's work leading up to his conjecture as he grappled with how to understand the topology of manifolds. Then I will explain his direct approach to his conjecture about the 3-sphere and why the direct approach has been so tantalizing to generation after generation of topologists. I will then discuss how the study on manifolds evolved since Poincaré's time, and what successes successive generations of topologists did have with techniques that can be traced back to Poincaré. Lastly, I will sketch the modern developments where ideas from physics, geometry, and analysis have been brought to bear on the difficult questions about 3- and 4-dimensional manifolds.

It is clear from reading *l'Analysis Situs* and its complements that Poincaré's goal was to understand higher dimensional manifolds (higher in the sense of greater than 2, surfaces being much studied and well understood by then). He introduces various ways to present, or define, these spaces; he is concerned with explicit examples and with contexts in which the techniques of *l'Analysis Situs* can shed light on more geometric and dynamic questions, for example the study of algebraic surfaces. A recurring theme is the role of the Betti numbers and their generalization to include what Poincaré calls torsion coefficients and the fundamental group, which are closely related algebraic invariants. Throughout this series of papers, Poincaré is wondering and conjecturing, often incorrectly, whether these algebraic invariants are enough to determine the manifold up to isomorphism. He begins to understand the depth and difficulty of this question as he focuses in on 3-manifolds in the fifth and last complement to *l'Analysis Situs*. It is then that he formulates his famous conjecture: This conjecture concerns the lowest mysterious dimension, 3, and the simplest manifold in that dimension, S^3, and proposes a characterization of that manifold in terms of these algebraic invariants, namely a characterization in terms of the fundamental group.

Let me present a brief survey of what was to follow. The theory of characteristic classes flowing from the work of Chern, Weil, and Whitney, cobordism theory as introduced by Thom, Smale's proof of the h-cobordism theorem and the resolution of the high dimensional Poincaré Conjecture, exotic smooth structures on the spheres introduced by Milnor in 1956 and studied by Kervaire-Milnor in the early 1960s, leading to Browder-Novikov surgery theory in the late 1960s and Kirby-Siebenmann triangulation theory around 1970 gave answers for dimensions ≥ 5 to Poincaré's general search for a set of algebraic invariants to classify manifolds and what properties their algebraic invariants must have. This was the easy part and followed, in spirit at least, the path that Poincaré indicated. It was a purely topological discussion using differentiable techniques, combinatorial techniques (triangulations) and robust algebraic topology to arrive at the answers. The outcome is as good a classification scheme as possible—we cannot answer all the questions that Poincaré would have hoped for, but we know exactly what we can know. One thing that came out of this analysis that was completely unsuspected by Poincaré is that differentiable classification on the one hand and topological or combinatorial classification on the other hand are different. Thus, any smooth manifold of dimension at least 5 with trivial fundamental group and the homology of a sphere is homeomorphic to the sphere (this is Smale's theorem), but Milnor produced smooth manifolds with these properties starting in dimension 7 that are not diffeomorphic to the sphere. This divergence of smooth and topological manifolds is a high dimensional phenomenon; in dimension 3 the classifications agree.

The remaining dimensions, 3 and 4, have proven more difficult and have not been susceptible (to date) to purely topological reasoning. Geometry, analysis, and physics have played a large role in unravelling the mysteries of these dimensions. But before we get to that there is one part of the story in these exceptional dimensions where purely topological techniques have been shown to be powerful enough. This is when the 3-manifold has boundary of genus ≥ 1 or when the 3-manifold admits an embedded surface of genus ≥ 1 whose fundamental group injects into the fundamental group of the manifold. In these cases the work of Papakyriakopoulos in the 1950s on Dehn's lemma and the loop theorem led to work of Haken and Waldhausen in the 1960s to completely resolve the analogue of the Poincaré Conjecture for these manifolds.

This completes a review of the purely topological advances. Let us turn now to the more geometric and analytic work. The complete understanding of 3-manifolds requires the analytic method of Ricci flow introduced by Hamilton in the 1980s, developed by Hamilton thorugh the 1980s and 1990s, and extended masterfully by Perelman in 2002–2003. These developments will be explained in detail by others.

Manifolds of dimension 4 are even more of a mystery. In 1980 Freedman managed to push down the high dimensional techniques to prove the 4-dimensional analogue of the Poincaré Conjecture for topological 4-manifolds. At about the same time, Donaldson using the Anti-Self-Dual equations from physics introduced non-classical (i.e., not homotopy-theoretic) invariants for 4-manifolds. These have been used to show that differentiable 4-manifolds up to diffeomorphism are very complicated and their classification is unlike the topological classification and is also very different from the classifications one finds in other dimensions. This part of the story is still a mystery: we know that it is far more complicated than we currently understand but we have no idea how to understand completely these manifolds.

I hope this brief survey makes clear that, overall, Poincaré was prescient. The general approach he took, the types of questions he was asking, and the methods he was using form the basis of all of the developments of topology of manifolds and the applications of this field to geometric problems that followed. But history has shown that in focusing as he did on S^3, Poincaré was misguided on two fronts. First, even though 3 is the lowest mysterious dimension, it and 4 turn out to be the two hardest dimensions to deal with. Second, even though S^3 is in some sense the simplest 3-manifold it is the most difficult to deal with—'larger' 3-manifolds are easier to understand. In the end he formulated one of the most difficult questions as the next one to study. This was fortunate. It meant that no matter what startling progress was made, it was clear to all workers that that progress was not enough, Poincaré's original question remained there unanswered as a beacon to spur further work and as a test of new ideas in the study of the topology of manifolds.

I wish to thank Peter Shalen and Cameron Gordon for their help in preparing this account. Also, I recommend to the interested reader Cameron's excellent history [**14**] of 3-manifold topology up to 1960 which I drew on in preparing this account.

2. l'Analysis Situs, and its five complements

In 1892 Poincaré [**33**] published a short note entitled *Sur l'Analysis Situs.* This was followed in 1895 by the much longer *l'Analysis Situs* [**34**] and its complements, one through five, [**35, 36, 37, 38, 39**], published in 1899, 1900, 1902, 1902, and 1905. In 1901, he wrote an analysis of his scientific works (published in 1921) where he said " A method which lets us understand the qualitative relations in spaces of dimensions more than 3 could, to a certain extent, render service analogous to that rendered by figures ... In spite of everything, until now this branch of science has not been developed much....As far as I am concerned, every one of the diverse paths that I have followed, one after the other, have led me to analysis situs."

I think it is fair to say that this series of articles represents the founding of Topology, which is the modern name of what Poincaré called *Analysis Situs*, as an independent branch of mathematics. Many, if not most, of the themes that dominated the development of topology from 1900 until at least the 1960s are either explicitly introduced or at least foreshadowed in this series of articles. I will briefly describe the high points of these seven articles.

2.1. Sur l'Analysis Situs (1892) [**33**]**.** The paper is a short one. In it Poincaré gives an example of two manifolds with the same Betti numbers but with different fundamental group, thus answering a question in the negative that he had pondered: namely whether two closed manifolds with the same Betti numbers were topologically equivalent. Poincaré presents the examples as quotients of the unit cube

$$\{(x,y,z)\|0\leq x,y,z\leq 1\}$$

by identifications of opposite faces: The pair of faces $\{x=0\}$ and $\{x=1\}$ are identified by the map $x\mapsto x+1$; similarly the faces $\{y=0\}$ and $\{y=1\}$ are identified by $y\mapsto y+1$. These identifications produce a 3-manifold with boundary which is the product of a two-torus with the unit interval. The torus boundary components are then identified by

$$(x,y,z)\mapsto(\alpha x+\beta y,\gamma x+\delta y,z+1),$$

where

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

is an element of $SL(2,\mathbb{Z})$. This last identification produces a torus bundle over the circle with gluing (i.e., monodromy) A. Poincaré observes that two such manifolds constructed with monodromies A and A' have the same fundamental group if and only A and A' are conjugate in $SL(2,\mathbb{Z})$, and on the other hand the first (and hence second) Betti number for a general element of $SL(2,\mathbb{Z})$ is one.

2.2. l'Analysis Situs (1895) [**34**]**.** This is a long (121 pages), foundational paper. Poincaré begins by defending the study he is about to undertake by saying "Geometry in n-dimensions has a real goal; no one doubts this today. Objects in hyperspace are susceptible to precise definition like those in ordinary space, and even if we can't represent them to ourselves we can conceive of them and study them." There then follows a discursive introduction to the study of the topology of manifolds. Many of the approaches and techniques that came to dominate 20^{th} century topology are introduced in this paper. It truly is the beginning of Topology as an independent branch of mathematics. To give you a sense of the scope of this paper, I will briefly review its highlights.

Poincaré begins by defining a manifold (of dimension $n-p$) as a subspace of n-dimensional space given by p equalities:

$$F_1(x_1,\ldots,x_n)=0,...,F_p(x_1,\ldots,x_n)=0$$

and some inequalities:

$$\varphi_1(x_1,\ldots,x_n)>0,\ldots,\varphi_q(x_1,\ldots,x_n)>0,$$

subject to the condition that the F_j and φ_k are C^1 and the rank of the Jacobian differential of the system of $F_1,\ldots,F_p$ is p at every solution. He also considers manifolds defined by locally closed, one-one immersions from open subsets of Euclidean $n-p$-space. He goes on to consider manifolds covered by overlapping subsets of either type (though he is considering the real analytic situation where the extensions are given by analytic continuation). Having defined manifolds, he considers orientability, orientations, and homology. For him cycles are embedded closed submanifolds and the relation of homology is given by embedded compact submanifolds with boundaries. He defines the Betti numbers as the number of linearly independent cycles of each dimension[1].

[1]Actually, Poincaré's definition is one more than the number of linearly independent cycles.

Poincaré then turns to integration of what today are called differential forms over compact submanifolds. Namely, he writes down the condition that a form be closed and then he states (in modern language) that any closed form integrated over a cycle that is homologous to zero gives the result of zero. What Poincaré formulates is a special case of a general theorem, which goes under the name of (the higher dimensional) Stokes' Theorem. The first results along these lines date to the early 19^{th}-century and are due to Cauchy, Green, Stokes and Gauss for ordinary curves and surfaces in 3-space. The modern formulation required (one could argue forced) the notion of differential forms and exterior differentiation, both of which were slowly emerging at Poincaré's time and which were first written down in more or less modern form by Cartan, [**5**]. For a history of Stokes' Theorem see [**19**].

Next, Poincaré takes up the intersection number of closed, oriented manifolds of complementary dimension and shows that if one of the manifolds is homologous to zero then the intersection number is zero. He examines in more detail the case when one of the manifolds is of dimension 1 and the other is of codimension 1. He shows that if the manifold of codimension 1 is not homologous to zero then there is a closed 1-manifold (i.e., a circle) that has a non-trivial intersection number with it. Indeed, by separation arguments he finds a circle that has intersection number 1 with the given codimension-1 submanifold. He then generalizes this idea to higher dimensions arriving at a form of Poincaré duality for closed, orientable manifolds and concludes for example that the middle Betti number of a closed, oriented manifold of dimension $4k+2$ is always even.

Next, Poincaré passes to a form of combinatorial topology, giving another way to construct manifolds. He considers spaces made by identifying the codimension-1 faces in pairs of one or more polyhedra. He shows that in dimension 3 in order to get a manifold it is necessary and sufficient that the link of each vertex in the resulting space have Euler characteristic 2. A related construction of manifolds is to take a free, properly discontinuous group action on, say, Euclidean space. The relation with the previous example comes from taking a fundamental domain for the action and using the group to subdivide the boundary into faces which are then identified in pairs.

Poincaré then introduces the fundamental group. He does so by considering multi-valued functions and the action of the group of homotopy classes of based loops on these functions (hence, giving an explicit representation of the fundamental group). He describes a presentation of the fundamental group of a manifold obtained by gluing together in pairs the codimension-1 faces of a polyhedron: a generator for each pair of codimension-1 faces, a relation around each codimension-2 face. He also describes how to compute the first Betti number of such a manifold: namely, abelianize the relations associated with each codimension-2 face. Using this analysis Poincaré gives the examples from *Sur l'analysis situs* of 2-torus bundles over the circle with the same Betti numbers but with different fundamental groups, showing that the Betti numbers are not enough to determine the manifold up to homeomorphism. He then asks three questions:

(1) Given a presentation of a group is there a manifold with this as its fundamental group?
(2) How does one construct this manifold?
(3) If two manifolds have the same dimensions and the same fundamental group, are they homeomorphic?

Poincaré then turns to other ways to construct new manifolds from old. He considers (free) group actions on a manifold and constructs the quotient. His first example is the projective plane as a quotient of the usual sphere in 3-space by the antipodal action of $\mathbb{Z}/2\mathbb{Z}$. He also considers products of spheres $S^n \times S^n$ with the $\mathbb{Z}/2\mathbb{Z}$-action that interchanges the factors and makes the wrong claim that for all $n > 1$ the quotient of this action is a closed manifold. (What he actually establishes is that for $n > 1$ the quotient has no codimension-1 singularities.)

Lastly, Poincaré introduces the Euler number for a manifold that is presented as divided into pieces (cells), the interior of each being diffeomorphic to a ball of some dimension, e.g., a triangulation. He introduces the idea of subdivision for these presentations and, by taking a common subdivision, shows that the Euler number is independent of subdivision. He then computes the Euler number of a closed manifold and shows that it is zero if the dimension is odd and that it depends on the Betti numbers when the dimension is even.

So in this foundational paper we have the definition of manifolds and various ways of producing them: as solutions to equations and inequalities, as covered by images of open subsets of ordinary space, as quotients of polyhedra by gluing together faces, as quotients of free, properly discontinuous group actions. On these manifolds we have differential forms with exterior differentiation and integration. We have Betti numbers, Poincaré duality relating complementary dimensional Betti numbers, and we have the Euler characteristic. This is an excellent start for the new subject of Topology, but Poincaré was not finished with this subject. He went on to write 5 complements to this article where he develops these themes and considers applications of Topology to Algebraic Geometry.

2.3. The first complement (1899) [35]. This complement is concerned with the Poincaré duality result given in *l'Analysis Situs*. Poincaré states that Heegaard objected to his result that the Betti numbers in complementary degrees are equal and gave as an example of a 3-manifold with first Betti number 1 and second Betti number 0. (Here, the Betti number is taken to be the minimal number of cycles needed to generate the homology.) Poincaré points out that the discrepancy between Heegaard's definition and his own is that Heegaard is working over the integers and Poincaré is working over the rational numbers. Thus, in modern language Heegaard's example has torsion first homology and no second homology.

Poincaré goes on to say that Heegaard's objection is well-founded in one way; namely, Poincaré's proof of duality works in both cases, and therefore must not be correct. Poincaré's purpose in this complement is to rectify the proof of duality.

This time he approaches the proof from the combinatorial point of view. He revisits the notion from *l'Analysis Situs* of a polyhedral decomposition of a manifold into cells, each cell being a closed submanifold whose boundary is a union of cells of one lower dimension. Furthermore, the relative interiors of the cells are disjoint. Unless otherwise specified, Poincaré (and we also) restrict to decompositions where the interior of each cell is differeomorphic to an open ball of some dimension. He introduces the simplicial homology associated with such a decomposition: This is defined starting with a chain complex with free abelian chain groups, one $\mathbb{Z}$ in the k^{th} chain group for each k-dimensional cell in the decomposition. A generator of this free abelian factor is given by chosing an orientation for the cell in question; the opposite orientation is the opposite generator. The boundary map ∂ of the chain complex comes from the decomposition of the boundary of a cell as a union of cells of one lower dimension. Poincaré shows that $\partial \circ \partial = 0$. By definition the homology of this chain complex is the homology associated with the given decomposition.

Next Poincaré introduces the notion of a polyhedral subdivision, where each of the cells is divided into smaller cells (again keeping the condition that the interior of each cell is diffeomorphic to an open ball). Poincaré then argues that the associated homology is unchanged if one passes from such a polyhedral decomposition to a polyhedral subdivision. Then he argues that any two such polyhedral decompositions (which should be deformed slightly to be in general position) have a common polyhedral subdivision and hence the homology associated with a decomposition is in fact an invariant of the manifold independent of the polyhedral decomposition. He makes a brief argument (that he returns to in more detail in the next complement) to show that this homology is the same as the one computed from closed submanifolds modulo boundaries in the ambient manifold. (Of course, with hindsight we know that Poincaré was missing the crucial distinction between cycles, which in general have to be singular, and submanifolds. This point was not understood until Thom's work [**47**].) The basic idea in Poincaré's discussion is that since any closed submanifold is the union of cells of some polyhedral decomposition that cycle is accounted for in the homology computed using that decomposition, which after all is independent of the decomposition. Thus, every closed manifold is accounted for in the homology computed via a polyhedral decomposition. (Poincaré has more to say about this later.)

At this point he is prepared for his proof of the duality theorem, which as he states it says that the Betti numbers (computed using rational coefficients) of a closed, orientable manifold in complementary degrees are equal. He argues as follows: Begin with a polyhedral decomposition and take (what is now called) a barycentric subdivision. One does this by adding a vertex v_a in the relative interior of each cell a, and for any string $a_0 < a_1 < \cdots a_k$ of cells each strictly included in the boundary of the next, one constructs a k-dimensional ball inside a_k with the $v_{a_0}, \ldots, v_{a_k}$ as vertices. These various balls are required to fit together in the naturally consistent way so that they make a polyhedral decomposition which subdivides the original decomposition. Once this barycentric subdivision is created, one constructs the dual polyhedral decomposition to the original one. For each cell a of the original decomposition one takes the union of all the cells of the barycentric subdivision that meet a exactly in its vertex v_a. The relative interior of this union is diffeomorphic to a Euclidean space of dimension $n - k$ if a is k-dimensional and the ambient manifold is n-dimensional. These submanifolds construct the dual polyhedral cell decomposition. From this picture it is easy to conclude the duality of the Betti numbers: The $(n-k)^{th}$ Betti number of the dual polyhedral decomposition is identified with the k^{th}-Betti number of the original decomposition. But by the invariance result, the $(n-k)^{th}$ Betti number of the dual polyhedral decomposition is identified with the $(n-k)^{th}$ Betti number of the original decomposition.

Lastly, Poincaré remarks that this argument depends on being able to find a polyhedral decomposition of any manifold into cells with interiors diffeomorphic to open balls. He then gives a brief argument to establish this fact. In fact, he argues that every smooth manifold has a decomposition into simplices (i.e., has a smooth triangulation).

2.4. Second Complement (1900) [**36**]. Poincaré returns once again to the homology associated with a polyhedral decomposition of a manifold (as always into cells with interiors diffeomorphic to open balls); in particular he is concerned with the relationship of the decomposition and the dual decomposition, which recall is constructed by taking the barycentric subdivision of the original decomposition and then associating to each cell a of the original decomposition the union of the cells meeting a exactly in its barycenter.

Poincaré then presents the matrices giving the boundary operator of the chain complex associated with the polyhedral decomposition. Using standard row and column operations he diagonalizes this operator. From the matrices for the boundary operator from $q+1$ chains to q chains and that of the boundary operator from q chains to $q-1$ chains he computes the q^{th} Betti number as the difference of the rank of the kernel of the latter and the rank of the image of the former. From the matrix for the boundary operator from q-chains to $(q-1)$-chains he introduces the torsion coefficients associated to the $(q-1)$-homology group. These are the diagonal entries > 1 of the diagonalized matrix for the boundary operator. He then remarks that the difference between Heegaard's definition of the Betti numbers (using the integral coefficients) and his definition (using rational coefficients) is explained by these torsion coefficients. In particular, the two definitions agree when there are no torsion coefficients.

After giving some examples and analysing (only partially correctly) what accounts for torsion in homology, Poincaré states the false result that an n-dimensional closed, orientable manifold with all Betti numbers (except the 0^{th} and the top one) equal to zero and all torsion coefficients equal to zero is diffeomorphic to the n-sphere. He announces the result and says that the proof requires further developments. It is not clear if he means that he knows how to do it and simply needs to write down the details or whether it is still a proof in-progress. In any event he will show in the fifth complement that this statement is false.

2.5. Third Complement (1902) [37]. This complement and the fourth concern applications of the techniques and ideas of *Analysis Situs* to algebraic surfaces. This complement takes up the study of the fundamental group of a complex algebraic surface S given by an equation of the form

$$z^2 = F(x, y),$$

where $F(x, y)$ is a polynomial. Poincaré shows that if $F(x, y)$ describes a smooth curve then the resulting surface is simply connected. He does this by considering the surface S as fibered over the y-plane with hyperelliptic curves as fibers with a certain number of singular fibers. Removing the preimage of small disks in the y-plane centered at each singular point, Poincaré arrives at a 4-manifold with boundary, S_0 which is fibered by hyperelliptic curves. The boundary components of this manifold are 3-manifolds fibered over circles of the type he studied in an earlier complement. He expresses the fundamental group of S_0 as an extension of a free fuchsian group (the group of the punctured sphere given by y) by the fundamental group of the fuchsian group corresponding to the hyperelliptic curve which is the fiber over the base point. He then shows that putting back in the preimages of the disks around the singular points in the y-plane produces a quotient of this group that is trivial.

2.6. Fourth Complement (1902) [38]. In the fourth complement Poincaré continues his study of the topology of algebraic surfaces, concentrating in particular on their homology. He considers the surfaces as displayed over the projective line with the fibers being algebraic curves. Generically, these are smooth algebraic curves but the curves above a finite set of points, $A_1, \ldots, A_n$, in the projective line are singular. There is then the monodromy action (introduced earlier by Picard) of the fundamental group of $\mathbb{C}P^1 \setminus \{A_1, \ldots, A_n\}$ on H_1 of the generic fiber. Poincaré reduces the study of cycles of dimensions 1, 2, and 3 to a study of this action. In this he foreshadows the analysis that Lefschetz carried out in the 1920s on the topology of smooth algebraic varieties.

2.7. Fifth Complement (1905) [39]. The fifth complement is the one directly relevant to this conference. It is at the end of this long article that Poincaré formulates the question that soon became known as the Poincaré Conjecture. But before getting to that let me lay the groundwork by discussing what else one finds in this article dedicated to the study of the topology of 3-manifolds. The main result of the article is to give a counter-example to the statement he formulated in the second complement; namely that a n-manifold with the homology of the n-sphere is diffeomorphic to the n-sphere. The counter-example is a 3-manifold with the homology of the 3-sphere but whose fundamental group is non-trivial, so it is not diffeomorphic to S^3. Immediately after giving this example, Poincaré says: "There remains one question to treat: Is it possible that the fundamental group of [the 3-manifold] V is trivial yet V is not diffeomorphic to the sphere?" He goes on to say, "But this question would take us too far afield."

Poincaré studies the 3-manifold V by placing it in a Euclidean space and cutting it by a family of equations of the form $\varphi(x_1, \ldots, x_n) = t$, creating a family (depending on t) of codimension-1 submanifolds with certain singular values. Poincaré shows that generically the singular values are what today are called regular singular points, and he introduces the index of each such singular point—introducing what later become known as Morse theory. He studies the effect on homology of passing a critical point of index 1 or 2 and shows exactly how the first homology of the surface changes: If the index is two and if the vanishing cycle E (the circle in the level surface just below the critical level that contracts to a point at the critical level) is not homologous to zero then the cycles that persist are those that have zero intersection number with E and there is one additional homology, namely $E \cong 0$, so that as we pass this critical value the rank of the first homology of the level surface decreases by 2.

In preparation for his study of 3-manifolds using a Morse decomposition, Poincaré first turns to an analysis of curves on a Riemann surface. He shows that given two systems $\{C_1, \ldots, C_{2p}\}$ and $\{C'_1, \ldots, C'_{2p}\}$ of simple closed curves on a surface of genus p, each family generating the first homology, there is an equivalence (self-diffeomorphism) of the surface carrying each C_i to a curve homologous to C'_i for every $1 \leq i \leq 2p$ if and only if the intersection matrices satisfy

$$\left(C_i \cdot C_j\right)_{i,j=1}^{2p} = \left(C'_i \cdot C'_j\right)_{i,j=1}^{2p}.$$

Given this it is easy to see when a cycle, written as a linear combination of a basis $\sum_{i=1}^{2p} a_i C_i$ is homologous to a simple closed curve. It is if and only if the cycle is not divisible (i.e., if and only if the gcd of the coefficients a_i is 1).

Poincaré then turns to the question of when a based cycle C is homotopic (without preserving the basepoint) to a simple closed curve. This question he studies using 2-dimensional hyperbolic geometry, the Poincaré disk. The cycle C determines an element in the fundamental group of the surface and , after endowing the surface with a hyperbolic structure, this element is represented by a hyperbolic transformation of the Poincaré disk. This hyperbolic motion has two fixed points α and β on the circle at infinity. The condition that C be homotopic to an embedded curve is that for no element T of the fundamental group are the points $T\alpha$ and $T\beta$ on the circle at infinity separated by α and β. Poincaré derives an analogous condition for two simple closed curves on the surface to be homotopic to disjoint simple closed curves. Again the answer is in terms of the fixed points of the hyperbolic motions associated with the two curves. Notice that Poincaré's approach to this purely topological question about surfaces uses hyperbolic geometry rather than purely topological arguments.

Poincaré then turns to the study of the solid handlebody V' of genus p: this is a compact 3-manifold with boundary, say Σ, having a Morse function with a single critical point of index 0 and p critical points of index 1 and no higher index critical points. He shows that the critical points contribute p simple closed curves on Σ that bound p disjoint, properly embedded disks in V'. Cutting V' open along these p disks results in a 3-ball and the boundary is the 2-sphere made up of the union of Σ cut open along p simple closed curves and p disks. He then asks how many different ways can we present the given handlebody V'. Let the p boundary circles in the original presentation be $K_1, \ldots, K_p$ and let the boundary circles in the second presentation be $K_1', \ldots, K_p'$. The conditions that the K_i' must satisfy is (i) they are simple closed curves, (ii) they are disjoint, (iii) they are homotopically trivial in V', and (iv) the cycles K_i' generate the kernel of the map $H_1(\Sigma) \to H_1(V')$.

Now Poincaré turns to the study of a closed 3-manifold V. Using an appropriate Morse function, he writes V as the union of two solid handlebodies V' and V'' of genus, say, p. From the Morse function he produces two families of p curves $K_1', \ldots, K_p'$ and $K_1'', \ldots, K_p''$ on the separating surface Σ which is the boundary of V' and V''. Each of the K_i' bounds a disk in V' and analogously for the K_i'' in V''. This means that there is a homeomorphism of Σ to the boundary of a handlebody carrying the K_i' to a standard family of simple closed curves and similarly for the family K_i''. The first thing he does with this presentation is to express the fundamental group of V in terms of the fundamental group of the splitting surface Σ. He first shows that any element of the fundamental group of V is represented by a loop on Σ. Then he concludes that the fundamental group of V is the quotient of the fundamental group of Σ by the relations $[K_1'] = \cdots = [K_p'] = [K_1''] = \cdots = [K_p''] =$ identity. (Here, $[K]$ means the conjugacy class in the fundamental group represented by the simple closed curve K.) Of course, from this one can immediately deduce the homology of the manifold: it is the group obtained by abelianizing the quotient, or equivalently, beginning with the free abelian group which is the first homology of Σ and adding the $2p$ relations above to this group. Poincaré then reformulates the computation of the homology as the cokernel of the intersection matrix between the K_i' and the K_j''. Varying the bases we can arrange that this matrix is diagonal. The first Betti number is then the co-rank of the matrix (p-rank) and the torsion coefficients of the first homology are read off from the diagonal entries that are greater than one. In particular, the manifold has trivial first homology if and only if the determinant of this intersection matrix is ± 1 and the first Betti number is non-zero if and only if the determinant of the matrix is 0.

Poincaré is now ready to compute in an explicit example V. The genus of the splitting surface Σ for V is 2 so that we are concerned with two families of two curves $\{K_1', K_2'\}$ and $\{K_1'', K_2''\}$ on a surface Σ of genus 2. These generate the two handlebodies whose union is V. We can suppose that $\{K_1', K_2'\}$ is the standard family so that cutting Σ open along these simple closed curves gives a region which can be identified with a thrice punctured disk D_0 in the plane. Then each of the the curves K_1'' and K_2'' is given by drawing families of disjoint arcs on D_0 whose boundary points match in pairs. Furthermore, the two families of arcs are disjoint from each other. (There is the extra condition that when we cut Σ open along K_1'' and K_2'' we also get a surface diffeomorphic to the thrice punctured disk.) Figure 1 below is the picture that Poincaré drew in his text. Figure 2 (see next page) shows the resulting curves K_1'', K_2'' on the surface of genus 2.

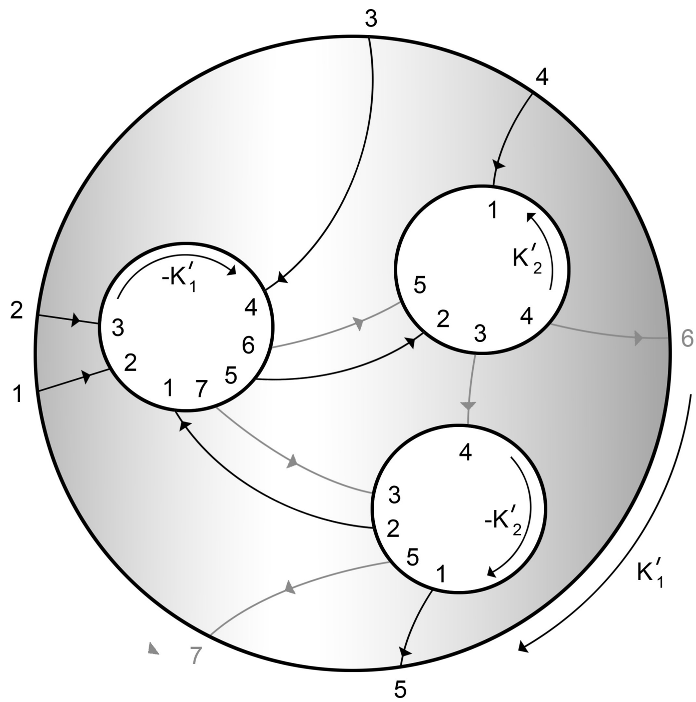

FIGURE 1. Poincaré's figure

The intersection matrix between the two sets of curves is:

$$\begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}.$$

Since the determinant of this matrix is 1, it follows that the first homology is trivial; i.e., the first Betti number is zero and there are no torsion coefficients.

How about the fundamental group? Poincaré computes this by a similar argument in non-abelian group theory. In modern language, we begin with the free group on two generators α and β (dual respectively to K_1' and K_2'). The curves K_1'' and K_2'' then give two elements in this free group and the quotient when these elements are set equal to the trivial element is the fundamental group of V. The word that K_i'' gives is read off by taking the intersections in order (and with signs as powers) of

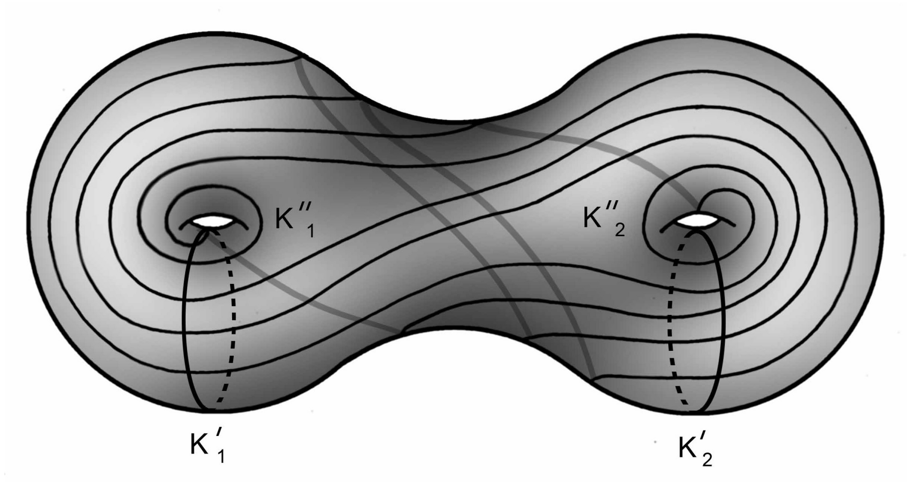

FIGURE 2. Curves drawn on the surface

K_i'' with K_1' and K_2'. Thus, the curve K_1'' gives the word $\alpha\alpha\alpha\alpha\beta\alpha^{-1}\beta = \alpha^4\beta\alpha^{-1}\beta$ and K_2'' gives the word $\alpha^{-1}\beta\alpha^{-1}\beta^{-1}\beta^{-1} = \alpha^{-1}\beta\alpha^{-1}\beta^{-2}$. (These words are determined only up to cyclic order depending on where on K_i'' we choose to start. But since we are setting the elements equal to the trivial element all choices lead to the same quotient group.) We then have a presentation of the fundamental group of V as

$$\langle \alpha, \beta | \alpha^4\beta\alpha^{-1}\beta = \alpha^{-1}\beta\alpha^{-1}\beta^{-2} = 1\rangle.$$

Adding the relation $(\alpha^{-1}\beta)^2 = 1$ we deduce that in the quotient group we have

$$\langle \alpha, \beta | (\alpha^{-1}\beta)^2 = (\beta^{-1})^3 = \alpha^5 = 1\rangle.$$

Setting $a = \alpha^{-1}\beta, b = \beta^{-1}$ and $c = \alpha$ we get the standard presentation of the $(2, 3, 5)$ triangle group

$$\langle a, b, c | a^2 = b^3 = c^5 = abc = 1\rangle$$

inside $SO(3)$. This group is the icosahedral group of order 60, denoted Γ_{60}. Hence, the fundamental group of the manifold V has a quotient which is this non-trivial group and thus the fundamental group of V is non-trivial and consequently V is not diffeomorphic to the 3-sphere even though it has the same homology as the 3-sphere. (It is in fact easy to see that the fundamental group of V is the binary icosahedral group, which is the pre-image of Γ_{60} in the double covering $SU(2) \to SO(3)$.) This example is called the *Poincaré homology sphere*.

Having constructed this example, Poincaré asks the natural follow-up question, What if the fundamental group is trivial? Is that enough to force the manifold to be diffeomorphic to the 3-sphere?

Actually, Poincaré asks this question and then goes on to say, "In other words, if V is simply connected can we change the families of curves generating the two handlebodies until they meet in the standard way (so that for each i, K_i'' meets only K_i' and meets it only once)?" Arranging this will certainly prove that the manifold V is diffeomorphic to S^3, but the converse is not obvious and was not established until the 1960s by Waldhausen [**50**].

To prepare the way for the higher dimensional analogues of the Poincaré Conjecture, let us reformulate it. Suppose that V has trivial fundamental group. Since the first homology of V is the abelianization of the fundamental group, V has trivial first Betti number and no torsion coefficients for the first homology. By Poincaré duality this implies that the second Betti number is trivial. Consequently, V automatically has the homology of S^3. In fact, as we know now, a much stronger statement is true: the manifold is homotopy equivalent to the sphere in the sense that there are maps from the manifold to S^3 and from S^3 to the manifold so that each composition is homotopic to the identity of the appropriate manifold. Such manifolds are called *homotopy* 3*-spheres*.

2.8. Why the Poincaré Conjecture has been so tantalizing. Poincaré's approach makes clear one reason that his conjecture is so tantalizing. It can be reformulated purely in terms of curves on a surface. One begins with two families $\mathcal{F}$ and $\mathcal{F}'$ of p disjoint simple curves on a surface of genus p, each family standard in its own right up to self-diffeomorphism of the surface. One is allowed to replace say $\mathcal{F}'$ by another family normally generating the same subgroup as $\mathcal{F}'$. Can one find such a replacement with the property that the resulting two families are dual in the sense that the i^{th} curve from $\mathcal{F}'$ meets only the i^{th} curve from $\mathcal{F}$ and meets that curve in a single point, if and only if the quotient of the fundamental group of the surface by the normal subgroup generated by the collection of curves in $\mathcal{F} \cup \mathcal{F}'$ is the trivial group? One feels that this is a problem one can attack without any sophisticated theory; surely one can understand curves on a surface and their intersection patterns well enough to decide the question.

There is a purely group theoretic reformulation of the question: Suppose we have a surface Σ of genus $p \geq 2$ and a homomorphism φ from the fundamental group of Σ onto $F_p \times F_p$ where F_p is the free group on p generators. Then φ factors through a surjection onto a non-trivial free product $A * B$. This purely group-theoretic question also seems tantalizingly approachable, but no one has succeeded in giving a direct, group theoretic, proof.

3. Work during the period 1904–1950

3.1. Dehn's work. The next major attack on the topology of 3-manifolds came from Max Dehn. In 1910 he published a paper [**8**] where he showed that taking two non-trivial knot complements (the complements of a solid-torus neighborhoods of two non-trivial knots in the 3-sphere) and sewing them together so that the meridian of one matches the longitude of the other and vice-versa produces a 3-manifold with the homology of S^3. Dehn also claimed that these manifolds were not diffeomorphic to S^3 since they contain a torus that does not bound a solid torus. But it was not until 1924 that Alexander ([**2**]) established that every torus in the 3-sphere bounds a solid torus. On the other hand, it is not too difficult to see (using Van Kampen's theorem which was formulated later but more or less understood even in Poincaré's day) that the fundamental groups of the manifolds that Dehn constructed in this way are non-trivial, so that these manifolds are indeed distinct from S^3. This produces a plethora of homology 3-spheres (manifolds with the homology of the 3-sphere). Dehn also introduced a notion that has proven to be of central importance in the theory of 3-manifolds; namely, the notion of *Dehn surgery*. Given a 3-manifold M and a knot $K \subset M$, one removes a solid torus neighborhood of K from M and sews this solid torus back into the resulting complement using some self-diffeomorphism of the boundary 2-torus. These self-diffeomorphsims up to isotopy are identified with $SL(2,\mathbb{Z})$ via the action of the diffeomorphism on the first homology of the two-torus. Dehn then constructed manifolds by this method, starting with a $(2,q)$-torus knot in S^3—these are knots lying on the surface of a standard torus and wrapping that torus linearly twice in one direction and q times in the other (q must be odd). Dehn identified which of the non-identity Dehn surgeries on $(2,q)$-torus knots produce manifolds with with the homology of the 3-sphere and showed that they all have non-trivial fundamental groups (usually infinite). When the knot is the $(2,3)$-torus knot there is a surgery that produces a manifold with the same group as Poincaré's example (later proved to be diffeomorphic to Poincaré's example).

The examples of Dehn surgery on $(2,q)$-torus knots were much better understood after the work of Seifert [**42**] and Seifert and Threlfall [**43**]. They considered 3-manifolds that admit locally free circle actions (now called Seifert fibrations). They showed that all of the examples coming from Dehn surgery on $(2,q)$-torus knots were such manifolds and they showed how to compute the fundamental group of these manifolds. In particular, Dehn's exceptional example was shown to have the same fundamental group as Poincaré's original example—it is the pre-image in $SU(2)$ of the symmetries of the regular icosahedron (Dehn had earlier considered this fundamental group but slightly miscomputed it.)

3.2. Work of Kneser. In [**24**] Kneser had also constructed a 3-manifold with the same fundamental group as Poincaré's example (and Dehn's exceptional example). He described the manifold as the space of regular icosahedra of volume 1 centered at the origin in $\mathbb{R}^3$; or said another way the quotient $SO(3)/\Gamma_{60}$, where recall that Γ_{60} is the group of symmetries of a regular icosahedron centered at the origin in 3-space. This description implies that this manifold has a spherical geometry: it is the quotient of the 3-sphere with its standard metric by a finite group of isometries acting freely. Thus, the quotient has a Riemannian metric of constant sectional curvature $+1$. Later it was shown that all these manifolds—Poincaré's original example, Dehn's example and the geometric quotient introduced by Kneser are diffeomorphic. It is now known, thanks to the work of Perelman, that this is the only homology 3-sphere (besides S^3) with finite fundamental group.

Kneser made another, even more important contribution to the study of the topology of 3-manifolds. He studied essential families of disjointly embedded 2-spheres in compact 3-manifolds. *Essential* means that each member is non-trivial in the sense that it does not bound a 3-ball and no two members of the family are parallel in the sense that they form the boundary of a region diffeomorphic to $S^2 \times I$ in the 3-manifold. He showed that for every compact 3-manifold V there is a finite upper bound to the number of two spheres in any essential family. This leads us to three definitions:

Definition 3.1.

(1) A 3-manifold is *prime* if it is not diffeomorphic to S^3 and if every separating 2-sphere in the manifold bounds a 3-ball.
(2) A 3-manifold is *irreducible* if every 2-sphere in the 3-manifold bounds a 3-ball.
(3) A 3-manifold V is a *connected sum* of X and Y, with X and Y being the *summands*, if there is a separating 2-sphere in V such that the two sides of this sphere are diffeomorphic to $X \setminus B^3$ and $Y \setminus B^3$. The connected

sum is *non-trivial* unless one of the two summands is diffeomorphic to S^3. In that case V is diffeomorphic to the other summand.

It is easy to see that the only (orientable) prime 3-manifold that is not irreducible is $S^2 \times S^1$. It is also easy to see that a 3-manifold is a non-trivial connected sum unless it is prime or diffeomorphic to S^3. The import in these terms of Kneser's result is that every compact 3-manifold is a connected sum of (finitely many) prime 3-manifolds. Such a decomposition is called a *prime decomposition* of the 3-manifold, and the prime manifolds that appear are the *prime factors* of the decomposition. Later, Milnor ([**27**]) showed that the prime factors that appear in any prime decomposition of a given 3-manifold are the same (up to order) no matter what prime decomposition is chosen. Thus, from a structural point of view 3-manifolds up to diffeomorphism are just like the natural numbers: there is a commutative product—connected sum—with a unit—S^3—and an infinite collection of primes. Every 3-manifold is a product (i.e., a connected sum) of a finite number of prime factors and the prime factors are unique up to order. This result reduces all questions about 3-manifolds to questions about prime 3-manifolds. For example, because the fundamental group of a connected sum of 3-manifolds is the free product of the fundamental groups of the factors, there is a counter-example to Poincaré's Conjecture if and only if there is a prime counter-example. Indeed, the set of homotopy 3-spheres (up to diffeomorphism) is the free monoid on the prime homotopy 3-spheres. Of course, we now know, thanks to Perelman, that this monoid has only one element, namely S^3. The unique connected sum decomposition result does not hold in dimensions greater than 3.

4. Work from 1950–1970

4.1. Work of Papakyriakopoulos. The next advances in the topology of 3-manifolds occurred in the 1950s and were due to Christos Papakyriakopoulos (universally known as Papa). He showed ([**32**]) two remarkable and very important results about surfaces in 3-manifolds. The first goes under the name of *Dehn's Lemma and the Loop Theorem.* It says that if M is a compact 3-manifold with boundary and if Σ is a boundary component of M with the property that the map on the fundamental groups induced by inclusion of Σ into M has a non-trivial kernel then (i) there is a simple closed curve in Σ that is not homotopically trivial in Σ but is homotopically trivial in M, and (ii) given any simple closed curve homotopically non-trivial in Σ but homotopically trivial in M, this curve is the boundary of a disk embedded into M. The second of Papa's results is called the *Sphere Theorem.* It says that if M has non-trivial second homotopy then there is either an embedded 2-sphere in M or an embedded projective plane in M so that the map induced by the inclusion of this surface into M is non-trivial on the second homotopy group.

Let us point out some consequences of Dehn's Lemma and the Loop Theorem. The first verifies a claim of Dehn's from 1910:

Corollary 4.1. *Any embedded 2-torus in S^3 bounds a solid torus.*

Proof. Let $T \subset S^3$ be an embedded 2-torus. The first remark is that by the uniqueness of the prime decomposition, any 2-sphere in S^3 bounds a 3-ball on each side. The surface T separates the 3-sphere. According to van Kampen's theorem the fundamental group of T cannot inject into the fundamental group of both sides since the fundamental group of the 3-sphere is trivial. Thus, by Dehn's Lemma and the Loop Theorem there is an embedded disk in the 3-sphere meeting T exactly in its boundary and this simple closed curve is homotopically non-trivial on T. The union N of a thickening of T and a neighborhood of the disk has two boundary components: a 2-torus parallel to T and a 2-sphere. The 2-sphere bounds a 3-ball whose interior is disjoint from N, and the union of N and this 3-ball is a solid torus whose boundary is a 2-torus parallel to T. Hence, T also bounds a solid torus. □

Another consequence is an analogue of the Poincaré Conjecture for knots.

COROLLARY 4.2. *A knot $K \subset S^3$ is the trivial knot (i.e., deforms as a knot in S^3 to a planar circle) if and only if the fundamental group of the complement $S^3 \setminus K$ is isomorphic to $\mathbb{Z}$.*

PROOF. First notice that the trivial knot has complement that is an open solid torus so the fundamental group of its complement is indeed isomorphic to $\mathbb{Z}$. Conversely, suppose that the fundamental group of the complement is isomorphic to $\mathbb{Z}$. Let ν be a solid torus neighborhood of K and let T be its boundary. Then the fundamental group of $(S^3 \setminus \operatorname{int} \nu)$ is also isomorphic to $\mathbb{Z}$. Hence, by Dehn's lemma and the Loop Theorem there is an embedded disk in $S^3 \setminus \operatorname{int} \nu$ whose boundary is a homotopically non-trivial simple closed curve on T. As before, this means that $S^3 \setminus \operatorname{int} \nu$ is a solid torus. From this it is easy to see that K bounds a, smoothly embedded disk in S^3, meaning that K continuously deforms as a knot to a planar circle. □

I believe that the feeling was that when Papa proved these results the Poincaré Conjecture and the complete classification of 3-manifolds would not be far behind. While Papa's results spurred tremendous progress in the subject, this was not to be the case in spite of a plethora of claimed proofs of the Poincaré Conjecture around this time.

4.2. Haken and Waldhausen. The work of Papa led in the 1960s to a much deeper understanding of 3-dimensional manifolds. Haken ([**16**]) generalized Kneser's argument for 2-spheres in a 3-manifold to surfaces of higher genus whose fundamental groups inject. He showed that in any given compact irreducible 3-manifold the number of non-parallel such surfaces is bounded. He also introduced hierarchies for compact, irreducible 3-manifolds admitting at least one embedded surface of genus ≥ 1 whose fundamental group injects into the manifold. In [**51**] Waldhausen used these ideas to study such manifolds, which he called *sufficiently large*. He was able to show the analogue of the Poincaré Conjecture: Two sufficiently large 3-manifolds with isomorphic fundamental groups are diffeomorphic. (It had been known since the work of Alexander, [**1**], in 1919 that there are lens spaces, which are quotients of S^3 by cyclic groups, with isomorphic fundamental groups that are not homeomorphic.) Thus, the theory of sufficiently large 3-manifolds was well understood by 1968. Of course, the sphere and any homotopy sphere are not sufficiently large, so the theory of sufficiently large 3-manifolds is orthogonal to the Poincaré Conjecture. What was then coming into focus is that the 'small' three manifolds, e.g., those with finite fundamental group are much harder to study than the sufficiently large ones.

5. Passage to Higher Dimensions

While the work on 3-manifolds was continuing, a broadening of perspective was taking place in topology. During the last half of the 1950s and then with a vengeance after Smale's work in 1961, attention of the topologist largely shifted from dimension 3 to higher dimensions, i.e., dimensions ≥ 5. Milnor ([**26**]) produced exotic smooth structures on the spheres starting in dimension 7. This was a surprise. It brought into focus something that had never been explicitly considered before: There were different categories of manifolds and classification results depend on the category. The three categories one considers are smooth (i.e., differentiable), piecewise linear (triangulated so that the link of every simplex is combinatorially equivalent to a standard triangulation of a sphere of the appropriate dimension and denoted pl) and topological. The issues had not arisen for Poincaré since, as we now know, every topological manifold of dimension ≤ 3 has a pl structure and a smooth structure, these refined structures being unique up to pl isomorphism and diffeomorphism, respectively. More precisely, the three categories of manifolds are equivalent in dimensions ≤ 3. It also follows from elementary arguments that every piecewise linear 4-manifold has a smooth structure unique up to diffeomorphism, i.e., that the pl category and the smooth category are equivalent in dimension 4.

In the differentiable category one might imagine, $C^1, C^2, ...C^\infty$ and real analytic manifolds but these categories are all equivalent, so one usually works with C^∞ manifolds. As Poincaré had already discovered in the first complement to *Analysis Situs* any smooth manifold has a smooth triangulation and hence carries a piecewise linear structure. Of course, any smooth manifold, where the overlaps between the coordinate charts are required to be smooth, is *a fortiori* a topological manifold, where the overlaps between the coordinate charts are required simply to be continuous. In a similar way any piecewise linear manifold is a topological manifold. What Milnor's examples showed is that the topological (and even piecewise linear) structure on the 7-sphere supports at least 7 different (i.e., non-diffeomorphic) smooth structures. In particular, this shows that the smooth version of the Poincaré Conjecture is not true in dimension 7. Not too long after that Kervaire ([**20**]) showed that there is a piecewise linear manifold of dimension 10 that admits no smooth structure. Later, manifolds with Lipschitz structures were introduced (see [**46**]) and it was shown that, except in dimension 4, every topological manifold has a unique Lipschitz structure up to bi-Lipschitz homeomorphism. In this category of manifolds some forms of analysis are possible that are not feasible with only a topological structure.

These results left open the question for the topological and piecewise linear versions of the analogue of the Poincaré Conjecture in dimensions ≥ 5. In 1961 Smale ([**44**]) showed, by using Morse theory in a way that can be viewed as a generalization of Poincaré's approach to the question in dimension 3, that a smooth manifold of dimension at least 5 that has trivial fundamental group and has the homology of the sphere is homeomorphic to the sphere. Shortly thereafter, John Stallings ([**45**]) showed that a piecewise linear manifold of dimension at least 7 that has trivial fundamental group and has the homology of the sphere has the property that after removing a point it becomes piecewise linearly equivalent to Euclidean space. Later, ([**7**], [**29**]) it was shown that topological manifolds of these dimensions with trivial fundamental group and the homology of the sphere are homeomorphic to the sphere.

Smale's method of proof established a much more general result, called the h-cobordism theorem. It says that any compact manifold with two boundary components $(W, \partial_- W, \partial_+ W)$ that is homotopy equivalent as a triple to $(N \times I, N \times \{0\}, N \times \{1\})$ where N is a simply connected manifold of dimension at least 5 is in fact diffeomorphic to $N \times I$. This theorem, together with cobordism theory introduced by René Thom ([**48**]) in the 1950s and 1960s and the related theory of characteristic classes, led to the Browder-Novikov surgery theory ([**4**]) which gives a highly successful way of describing simply connected manifolds of dimension ≥ 5.

At about the same time the relationship between the various categories of manifolds, at least in dimensions ≥ 5, was clarified. Kervaire and Milnor ([**21**]) gave a classification up to diffeomorphism of differentiable structures on the topological sphere or piecewise linear sphere. These form a finite abelian group under connected sum whose structure is reduced to number theoretic questions. These groups explain the difference between the piecewise linear theory and the smooth theory. As to topological manifolds, in 1969 Kirby and Siebenmann ([**23**]), showed that except for one minor twist related ironically enough to Poincaré's homology sphere, the classification of topological and piecewise linear manifolds of dimensions ≥ 5 coincide.

Thus, by 1970 the theory of manifolds of dimensions ≥ 5 was well understood but the questions about the 'low dimensions'—3 and 4—including the original Poincaré Conjecture remained a mystery. I think it is fair to say that all these results used the sort of purely topological, including differential topological and combinatorial topological, arguments along the lines that Poincaré foreshadowed and developed in *l'Analysis Situs* and its complements. But as the attention returned to the lower dimensions there was only one more result to come from the purely topological techniques. This was Michael Freedman's proof [**12**]) of the topological version of the h-cobordism theorem and Poincaré Conjecture in dimension 4. His technique was to find a way to extend Smale's argument to 4-dimensional manifolds. Although I have just characterized this as a purely topological argument, as it is, it falls outside the type of things that Poincaré considered. Poincaré used smooth and piecewise smooth techniques whereas Freedman's technique is purely topological, the surfaces he constructs in the course of his argument are topologically quite intricate and they cannot be made smooth or piecewise smooth.

Indeed at the same time Freedman was establishing the topological version of the h-cobordism theorem for 4 manifolds, Donaldson, in [**9**] using more geometric and analytic techniques inspired by physics (see below) was showing that the h-cobordism theorem does not extend to smooth 4-manifolds. This led quickly to examples of topological 4-manifolds, indeed smooth complex algebraic surfaces, with infinitely many non-diffeomorphic smooth structures—something that happens only in dimension 4. For every $n \neq 4$ Euclidean space $\mathbb{R}^n$ has a unique smooth structure up to diffeomorphism; there are uncountably many differentiable distinct smooth structures on $\mathbb{R}^4$.

5.1. Connections, characteristic classes, and cobordism theory. The advances in the understanding of high dimensional manifolds relied on other developments in topology and geometry. These involved looking not directly at manifolds but rather at auxiliary objects over manifolds. The most important auxiliary objects are principal bundles and associated fiber bundles and vector bundles. Of course, the frame bundle of the tangent bundle of the manifold is an example of a principal bundle and the tangent bundle itself is an associated vector bundle. These are important examples but not the only ones. Other bundles not directly derived from the manifold were also considered. These bundles can be equipped with an important geometric object, a connection. Connections have curvature and through the resulting Chern-Weil theory ([**6**]) curvature gives rise to characteristic classes, which can also be defined purely topologically, for example from the cohomology of appropriate classifying spaces. The relationship between geometry and topology in this theory is made clear by the Atiyah-Singer index theorem, [**3**] which relates the index of elliptic operators between sections of vector bundles in terms of the symbol of the operator and characteristic classes on the manifold.

These ideas are intricately woven into the study of high dimensional manifolds in many ways. Milnor's original examples of exotic smooth structures on S^7 were described as fiber bundles with fiber S^3 over S^4 associated to a principal $SO(4)$-bundle, and his argument used the classification of these bundles. The classification of the group of exotic structures on high dimensional spheres by Kervaire-Milnor relied in an essential way on Thom's cobordism theory ([**48**]) and the Hirzebruch index theorem ([**17**]) (a special case of the Aityah-Singer index theorem that was proved earlier) relating a topological invariant of a smooth manifold (its signature, or index which is computed from the intersection pairing on the middle dimensional homology) to a characteristic number computed using the Pontrjagin characteristic classes of the manifold. These ideas play an even more important role in the Browder-Novikov surgery theory ([**4**]) for understanding all high dimensional manifolds.

6. Geometry and physics and low dimensional topology

In the late 1970s and through the 1980s the focus of topology shifted back to the low dimensional manifolds—those of dimensions 3 and 4. The techniques were no longer purely topological—ideas and results from geometry and physics began to play a role.

6.1. Physics and 4-manifolds. In one of the great convergences of ideas from different fields, it turns out that Yang and Mills, [**55**], were trying to understand a physical theory analogous to electro-magnetism (EM) where the fields transformed under $SU(2)$ (instead of $U(1)$ for EM). They were led to write down a Lagrangian in terms of a connection on the principal $SU(2)$-bundle and matter fields in the associated two-dimensional complex vector bundle. The term involving only the connection that appears in the Lagrangian is the usual term computed from the partial derivatives of the connection, the same term that one finds in EM, together with a quadratic expression from the connection. What they had arrived at with the sum of these two terms is exactly the mathematical definition of the curvature of a connection. The term that appears in the Lagrangian related only to the connection is exactly the norm square of the curvature of the connection. (There are other terms involving the connection and the matter fields.) The classical equations of motion are the first order Euler-Lagrange equations for a critical point of the resulting action functional, called the Yang-Mills action. Thus, physically-inspired reasoning had led Yang and Mills to develop a crucially important physical theory that used the mathematics from some 40 years earlier. The resulting physical theories are called *gauge theories* and they have turned out to be central in all modern developments of high energy theoretical physics. For example, the standard model of high energy theoretical physics is written in terms of gauge theories for various gauge groups ($U(1)$ for EM, $SU(2)$ for the weak force, and $SU(3)$ for the strong force for QCD, the theory governing quarks and gluons). The greatest triumph of the gauge theories is in their accurate description of the interactions of the elementary particles and in particular the peculiar energy dependence of non-abelian gauge theories by Gross and Wilzcek ([**15**]) and Politzer ([**40**]) which implies asymptotic freedom for QCD.

In the study of 4-dimensional manifolds the impact of ideas from physics were being felt. Donaldson showed how to use the moduli space of solutions of the anti-self-dual equations for connections on an auxiliary principal $SU(2)$-bundle over a 4-manifold, equipped with a Riemannian metric, to produce non-classical (i.e., non-homotopy theoretic) invariants of 4-manifolds. By definition an anti-self-dual connection on a principal bundle over a 4-manifold is a connection whose curvature form is anti-self-dual. These connections are the minima for the pure Yang-Mills action functional, pure in the sense that there are no matter fields. Hence, anti-self-dual connections are special cases of Yang-Mills connections, which, recall, are critical points of the action functional. Later, Seiberg and Witten [**41**] constructed other invariants from a different physical theory, a gauge theory with structure group $U(1)$ and matter fields. By physics arguments Witten ([**53**]) established the relationship of these invariants to the invariants constructed by Donaldson. While this relationship has not been proven mathematically, the evidence for it is overwhelming and surely it is just a matter of time until this result is mathematically established. Since the Seiberg-Witten invariants are technically much easier to work with than Donaldson's original invariants, they have replaced them as the primary means of constructing 4-manifold invariants. These invariants were used to show that many naturally occurring 4-manifolds were not diffeomorphic, [**9, 13, 28**]. While they are not complete invariants, they do show that the classification of simply connected 4-manifolds is quite complicated. For example, it seems likely that there is an injective map from isotopy classes of knots in S^3 to diffeomorphism classes of 4-manifolds homeomorphic to a smooth hypersurface in $\mathbb{C}P^3$ of degree 4 (a $K3$ surface), see [**10**]. Unlike the situation with 3-manifolds where there was a clear guess as to the nature of the theory, the theory of smooth 4-manifolds is completely unknown and we have no good guesses as to the structure of this theory. This is the remaining mystery in the program that Poincaré outlined over one hundred years ago of understanding manifolds in terms of their algebraic invariants.

6.2. Geometrization of 3-manifolds. Thurston ([**49**]) introduced the idea of equipping (many) 3-manifolds with hyperbolic structures—these structures come from the Kleinian groups which Poincaré introduced and are the 3-dimensional analogue of surfaces that are the quotient of the Poincaré disk by fuchsian groups. Thurston was able to construct hyperbolic structures on prime 3-manifolds whose fundamental groups satisfy the obvious necessary conditions, provided that the 3-manifolds are sufficiently large. He also produced other examples of hyperbolic structures on non-sufficiently large 3-manifolds. The latter he produced by doing Dehn surgeries on other hyperbolic manifolds. This led him to conjecture that the obvious necessary fundamental group conditions are sufficient for a prime 3-manifold to admit a hyperbolic structure. Pursuing this line further, he asked himself what might be true for the other prime 3-manifolds. Pondering this led him to formulate the Geometrization Conjecture, which says roughly that every prime 3-manifold can be cut open along incompressible tori into pieces that admit complete homogeneous geometries of finite volume. A direct and immediate consequence of this conjecture is the Poincaré Conjecture. There are eight possibilities for the type of geometry with the most plentiful and the most interesting being hyperbolic geometry. This conjecture, which is what Perelman established, is explained in more detail by both McMullen and Thurston in their presentations at this conference. This reduces the problem of classifying closed 3-manifolds to the problem of classifying discrete, torsion-free subgroups of $SL(2,\mathbb{C})$ of finite covolume up to conjugacy. It gives a completely satisfying conceptual picture of the nature of all 3-manifolds, and it shows how closely related these topological objects are to homogeneous geometric ones. All closed 3-manifolds are made from homogeneous geometric ones by two simple operations—gluing along incompressible tori and connected sum. As a sidelight, this resolves in the affirmative the Poincaré Conjecture.

6.3. Invariants of 3-manifolds. There has been much other work in 3-manifold topology independent of the Geometrization Conjecture. This began with the Jones polynomial invariant for knots in S^3 ([**18**]), which was generalized by Witten ([**52**]) using a physical theory to invariants of all 3-manifolds. There are combinatorial definitions of the Jones polynomial, basically skein relations that say how the invariant is related before and after changing a crossing on the knot and doing surgery at the crossing. These types of relations led to other combinatorial notions of knot invariants and sometimes to 3-manifold invariants. One of the most powerful seems to be the Khovanov homology of knots in S^3 ([**22**]). These homology groups should be viewed as a categorification of, i.e., enrichment of, the Jones polynomial. Like the original Jones polynomial, these invariants are defined starting with a braid presentation of the knot. There is now a proposal due to Witten ([**54**]), coming from physics, for how to extend Khovanov theory for knots to give 3-manifold invariants but no mathematical treatment yet exists.

From a more geometric prospective, one can define 3-dimensional invariants using the anti-self-dual equations and the Seiberg-Witten equations. The idea, which essentially goes back to Floer ([**11**]), is to consider finite energy solutions to these equations on the 4-manifold obtained by crossing the given 3-manifold with $\mathbb{R}$. This leads to invariants of 3-manifolds called Donaldson-Floer invariants and Seiberg-Witten Floer invariants. There is a related set of invariants coming from Floer's original definition of Floer homology in the symplectic context. These are due to Ozsváth-Szabo̧ [**30**]. They defined an invariant called Heegaard-Floer homology. It is believed to be equivalent to the Seiberg-Witten Floer homology (see [**25**]) and hence closely related to the Donaldson-Floer homology, and on the other hand it has a purely combinatorial definition, [**31**]. It brings us full circle because it is defined from the intersection pattern of the two families of simple closed curves on the splitting surface of a Heegaard decomposition of the 3-manifold.

How all of these invariants of a 3-manifold, inspired by combinatorics and physics, are related to the geometric decomposition of the 3-manifold is another tantalizing mystery. An understanding of that could conceivably lead to a purely topological proof of the Poincaré Conjecture since one would understand the relationship of invariants made from the intersection pattern of curves on a Heegaard surface to the geometric decomposition of the manifold. This would complete the 3-dimensional program in the manner that Poincaré laid it out.

References

1. J. Alexander. Note on two 3-dimensional manifolds with the same group. *Trans. Amer. Math. Soc.*, 20:339–342, 1919. MR1501131
2. J. Alexander. On the subdivision of 3-space by a polyhedron. *Proceedings of the National Academy of Sciences*, 10:6–8, 1924.
3. M. Atiyah and I. Singer. The index of elliptic operators. I. *Ann. of Math. (2)*, 87:484–530, 1968. MR0236950 (38:5243)
4. W. Browder. *Surgery on simply-connected manifolds.* Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65. MR0358813 (50:11272)
5. E. Cartan. *Les Systemes Differentiels Exterieurs et leurs Applications Geometriques.* Hermann, 1945.
6. S-S. Chern. On the characteristic ring of a differentiable manifold. *Acad. Sinica Science Record*, 2:1–5, 1947. MR0023053 (9:297i)
7. E. Connell. A topological H-cobordism theorem for $n \geq 5$. *Illinois J. Math.*, 11:300–309, 1967. MR0212808 (35:3673)
8. M Dehn. Uber die topologie des dreidimensionalen Raumes. *Math. Ann.*, 69:137 – 168, 1910. MR1511580
9. S. Donaldson. Irrationality and the h-cobordism conjecture. *J. Differential Geom.*, 26(1):141–168, 1987. MR892034 (88j:57035)
10. R. Fintushel and R. Stern. Knots, links, and 4-manifolds. *Invent. Math.*, 134(2):363–400, 1998. MR1650308 (99j:57033)
11. A. Floer. An instanton-invariant for 3-manifolds. *Comm. Math. Phys.*, 118(2):215–240, 1988. MR956166 (89k:57028)
12. M. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982. MR679066 (84b:57006)
13. R. Friedman and J. Morgan. Algebraic surfaces and Seiberg-Witten invariants. *J. Algebraic Geom.*, 6(3):445–479, 1997. MR1487223 (99b:32045)
14. C. McA. Gordon. 3-dimensional Topology up to 1960. In *History of Topology*, pages 449 – 489. Elsevier Science B. V., 1999. MR1674921 (2000h:57003)
15. D. Gross and F. Wilzcek. Ultraviolet behavior of nonabelian gauge theories. *Phys.Rev.Lett.*, 30:1343–1346, 1973.
16. W. Haken. Theorie der Normalflächen. *Acta Math.*, 105:245 – 375, 1961. MR0141106 (25:4519a)
17. F. Hirzebruch. Neue topologische Methoden in der algebraischen Geometrie. pages viii+165, 1956. MR0082174 (18:509b)
18. V. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):103–111, 1985. MR766964 (86e:57006)
19. V. Katz. A history of Stokes' Theorem. *Mathematics Magazine*, 52:146–156, 1979. MR533433 (80j:01018)
20. M. Kervaire. A manifold which does not admit any differentiable structure. *Comment. Math. Helv.*, 34:257–270, 1960. MR0139172 (25:2608)
21. M. Kervaire and J.. Milnor. Groups of homotopy spheres. I. *Ann. of Math. (2)*, 77:504–537, 1963. MR0148075 (26:5584)
22. M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000. MR1740682 (2002j:57025)
23. R. Kirby and L. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations.* Princeton University Press, Princeton, N.J., 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.
24. H. Kneser. Geschlossene flächen in dreidimensionale Mannifgaltigkeiten. *Nachr. Ges. Wiss. Göttingen*, pages 128 – 130, 1929.
25. C. Kutluhan, Y-J. Lee, and C. Taubes. HF=HM I: Heegaard Floer homology and Seiberg–Witten Floer homology. *ArXiv.MathGT:1007.1979.*
26. J. Milnor. On manifolds homeomorphic to the 7-sphere. *Ann. of Math. (2)*, 64:399–405, 1956. MR0082103 (18:498d)
27. J. Milnor. A unique factorization theorem for 3-manifolds. *Amer. J. Math.*, 84:1 – 7, 1962. MR0142125 (25:5518)

28. J. Morgan and T. Mrowka. On the diffeomorphism classification of regular elliptic surfaces. *Internat. Math. Res. Notices*, (6):183–184, 1993. MR1224116 (94g:57035)
29. M. H. A. Newman. The engulfing theorem for topological manifolds. *Ann. of Math. (2)*, 84:555–571, 1966. MR0203708 (34:3557)
30. P. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)*, 159(3):1027–1158, 2004. MR2113019 (2006b:57016)
31. P. Ozsváth, Z. Szabó, and D. Thurston. Legendrian knots, transverse knots and combinatorial Floer homology. *Geom. Topol.*, 12(2):941–980, 2008. MR2403802 (2009f:57051)
32. C. Papakyriakopoulos. On Dehn's lemma and the asphericity of knots. *Ann. of Math.*, 66:1 –26, 1957. MR0090053 (19:761a)
33. H. Poincaré. Sur l'analysis situs. *Comptes rendus de l'Académie des Science*, 115:633 –636, 1892.
34. H. Poincaré. Analysis Situs. *Journal de l'Ecole Polytechnique*, 1:1 –121, 1895.
35. H. Poincaré. Complément a l'analysis situs. *Rendiconti del Circolo Matematico di Palermo*, 13:285 –343, 1899.
36. H. Poincaré. Second complément a l'analysis situs. *Proceedings of the London Mathematical Society*, 32:277 – 308, 1900.
37. H. Poincaré. Certaines surfaces algrébriques; troisième complément a l'analysis situs. *Bulletin de la Societé Mathématique de France*, 30:49 – 70, 1902. MR1504408
38. H. Poincaré. Les cycles des surfaces algébriques; quatrième complément a l'analysis situs. *Journal de Mathématiques*, 8:169 – 214, 1902.
39. H. Poincaré. Cinquième complément a l'analysis situs. *Rendiconti del Circolo matematico di Palermo*, 18:45 –110, 1905.
40. H. Politzer. Reliable perturbative results for strong interactions? *Phys.Rev.Lett.*, 30:1346–1349, 1973.
41. N. Seiberg and E. Witten. Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory. *Nuclear Phys. B*, 426(1):19–52, 1994. MR1293681 (95m:81202a)
42. H. Seifert. Topologie dreidimensionaler gefaserter Räume. *Acta math.*, 60:147 – 238, 1932.
43. H. Seifert and W. Threlfall. *A Textbook of Topology.* Academic Press, 1980. MR575168 (82b:55001)
44. S. Smale. Generalized Poincaré's conjecture in dimensions greater than four. *Ann. of Math. (2)*, 74:391–406, 1961. MR0137124 (25:580)
45. J. Stallings. The piecewise-linear structure of Euclidean space. *Proc. Cambridge Philos. Soc.*, 58:481–488, 1962. MR0149457 (26:6945)
46. D. Sullivan. Hyperbolic geometry and homeomorphisms. In *Geometric topology: Proceedings of the 1977 Georgia Topology Conference held in Athens, Georgia, August 1-12, 1977*, pages 543 – 555. Academic Press, 1979. MR537749 (81m:57012)
47. R. Thom. Sous-variétés et classes d'homologie des variétés différentiables. I. Le théorème général. *C. R. Acad. Sci. Paris*, 236:453–454, 1953. MR0054961 (14:1005a)
48. R. Thom. Variétés différentiables cobordantes. *C. R. Acad. Sci. Paris*, 236:1733–1735, 1953. MR0055689 (14:1112b)
49. W. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982. MR648524 (83h:57019)
50. F. Waldhausen. Heegaard-Zerlegungen der 3-Sphäre. *Topology*, 7:195–203, 1968. MR0227992 (37:3576)
51. F. Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Ann. of Math*, 87:56 – 88, 1968. MR0224099 (36:7146)
52. E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989. MR990772 (90h:57009)
53. E. Witten. Monopoles and four-manifolds. *Math. Res. Lett.*, 1(6):769–796, 1994. MR1306021 (96d:57035)
54. E. Witten. Fivebranes and knots, 2011. preprint, arXiv:1101.3216 [hep-th]. MR2852941
55. C. N. Yang and F. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Phys. Rev.*, 96:191 –195, 1954. MR0065437 (16:432j)

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The Evolution of Geometric Structures on 3-Manifolds

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Abstract. This paper gives an overview of the geometrization conjecture and approaches to its proof.

1. Introduction

In 1300, Dante described a universe in which the concentric terraces of hell—nesting down to the center of the earth—are mirrored by concentric celestial spheres, rising and converging to a single luminous point. Topologically, this finite yet unbounded space would today be described as a three-dimensional sphere.

In 1904, Poincaré asked if the 3-sphere is the only closed 3-manifold in which every loop can be shrunk to a point; a positive answer became known as the Poincaré conjecture. Although the theory of manifolds developed rapidly in the following generations, this conjecture remained open.

In the 1980's, Thurston showed that a large class of 3-manifolds are hyperbolic—they admit rigid metrics of constant negative curvature. At the same time he proposed a geometric description of all 3-dimensional manifolds, subsuming the Poincaré conjecture as a special case.

Both the Poincaré conjecture and Thurston's geometrization conjecture have now been established through the work of Perelman. The confirmation of this achievement was recognized by a conference at the Institut Henri Poincaré in 2010. This article—based on a lecture at that conference—aims to give a brief and impressionistic introduction to the geometrization conjecture: its historical precedents, the approaches to its resolution, and some of the remaining open questions. Additional notes, and references to some of the many works treating these topics in detail, are collected at the end.

2. Surfaces and tilings

We begin by recalling the geometrization theorem in dimension two.

Theorem 2.1. *Any closed, orientable topological surface S can be presented as a quotient of S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$ by a discrete group of isometries.*

Concretely, this means S can be tiled by spherical, Euclidean or hyperbolic polygons.

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Figure 1. Illustration of Dante's cosmology by Gustave Doré (1867).

Proof. By the classification of surfaces, we may assume S is a sphere, a torus, or a surface of genus $g \geq 2$. The theorem is immediate in the first two cases. In terms of tilings, one can assemble S^2 out of 8 spherical triangles with all angles $90°$; and a torus can be tiled by 8 Euclidean squares, which unfold to give the checkerboard tiling of $\mathbb{E}^2$.

Next we observe that a surface of genus $g = 2$ can be assembled from 8 regular pentagons (see Figure 2). The right-angled pentagons needed for this tiling do not exist in spherical or Euclidean geometry, but they do exist in the hyperbolic plane. Passing to the universal cover $\widetilde{S}$, we obtain a periodic tiling of $\mathbb{H}^2$ and an isometric action of the deck group $\Gamma \cong \pi_1(S)$ on $\mathbb{H}^2$ yielding S as its quotient. Any surface of genus $g \geq 3$ covers a surface of genus 2, so it too can be tiled by pentagons—one just needs more of them. □

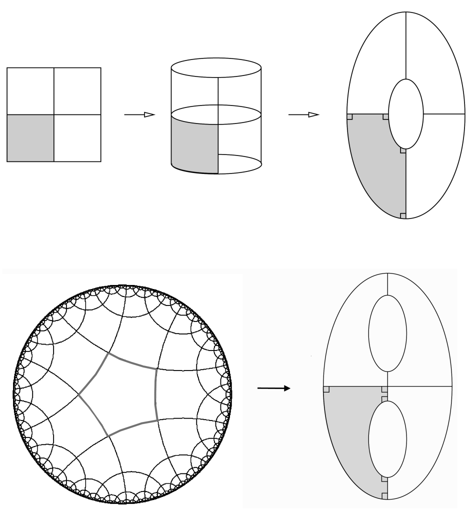

Figure 2. Tilings of surfaces of genus 1 and 2.

Uniformization. The geometrization theorem for surfaces was essentially known to Klein and his contemporaries in the 1870's, although the classification of abstract surfaces according to genus, by Dehn and Heegaard, was not proved until 1907. It is definitely more elementary than the *uniformization theorem*, proved in the same era, which asserts that every algebraic curve can be analytically parameterized by $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}^2$.

The converse is also true, as was shown by Poincaré; in particular:

THEOREM 2.2. *Every compact Riemann surface of the form $X = \mathbb{H}^2/\Gamma$ is isomorphic to an algebraic curve.*

For the proof, we need to construct meromorphic functions on X. A natural approach is to start with any rational function $f(z)$ on the unit disk $\Delta \cong \mathbb{H}^2$, and then make it invariant by forming the *Poincaré series*

$$\Theta(f) = \sum_{\gamma \in \Gamma} \gamma^*(f) = \sum_{\gamma \in \Gamma} f(\gamma(z)).$$

The result can then be regarded as a meromorphic function on X.

Unfortunately this series has no chance of converging: the orbit $\gamma(z)$ accumulates on points in $\partial\Delta$ where $f(z) \neq 0$, so the terms in the sum do not even tend to zero.

However, the sum does converge if we replace the function $f(z)$ with the quadratic differential $q = f(z)\,dz^2$, since then $|q|$ behaves like an area form, and the total area near the boundary of the disk is finite. This makes $\Theta(q)$ into a meromorphic form on X; and ratios of these forms, $\Theta(q_1)/\Theta(q_2)$, then give enough meromorphic functions to map X to an algebraic curve.

Thus algebra, geometry and topology are mutually compatible in dimension two.

Poincaré's Θ-operator also plays an unexpected role in the theory of 3-manifolds; see §5.

3. The geometrization conjecture for 3-manifolds

We now turn to the 3-dimensional case.

In contrast to the case of surfaces, which are ordered by genus, the world of 3-manifolds resembles an evolutionary tree, with phyla and species whose intricate variations admit, at best, a partial ordering by various measures of complexity.

An organizing principle for 3-manifolds seemed elusive until, in the 1980's, Thurston proposed:

CONJECTURE 3.1 (The geometrization conjecture). *All compact 3-manifolds can be built using just 8 types of geometry.*

The 8 geometries featured in this conjecture come from the following simply-connected homogeneous spaces:

(1) The spaces of constant curvature, S^3, $\mathbb{E}^3$ and $\mathbb{H}^3$;
(2) The product spaces $\mathbb{R} \times S^2$ and $\mathbb{R} \times \mathbb{H}^2$; and
(3) The 3-dimensional Lie groups Nil, Sol and $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

A 3-manifold M is *geometric* if it can be presented as the quotient $M = H/\Gamma$ of one of these homogeneous spaces by a discrete group of isometries. By gluing together geometric 3-manifolds along suitable spheres or tori, one obtains *composite* manifolds.

The geometrization conjecture states that any compact 3-manifold is either geometric or composite. In other words, any 3-manifold can be factored into 'geometric primes'.

Hyperbolic manifolds. Most of the 8 geometries are only required to describe fairly simple 3-manifolds: products or twisted products of circles and surfaces (spaces of 'dimension $2\frac{1}{2}$'). The manifolds covered by S^3 and $\mathbb{E}^3$ are also special – they are just finite quotients of the sphere or the 3-torus.

The only remaining case is that of *hyperbolic manifolds*—those of the form $M = \mathbb{H}^3/\Gamma$. Thus a principal corollary of the geometrization conjecture is that *most 3-manifolds are hyperbolic.*[1]

On the other hand, since S^3 is the only closed, simply-connected geometric 3-manifold, the geometrization conjecture also implies the Poincaré conjecture.

Spherical and hyperbolic dodecahedra. As in the case of dimension two, geometric 3-manifolds correspond to periodic tilings. Two such are shown in Figure 3.

The first tiling comes from the *Poincaré homology sphere* $M = S^3/\Gamma$, where $\Gamma \cong \widetilde{A_5}$ is a $\mathbb{Z}/2$ extension of the alternating group on 5 symbols. A fundamental domain for the action of Γ on S^3 is given by a spherical dodecahedron D. The faces of D are regular pentagons meeting in angles of 120°; this allows 3 copies of D to fit together neatly along an edge. (In Euclidean space, the angle would be about 116°.)

Poincaré originally speculated that the condition $H_1(M, \mathbb{Z}) = 0$ would be sufficient to characterize the 3-sphere. The space $M = S^3/\Gamma$ just constructed (also discovered by Poincaré) provides a counterexample, since $\pi_1(M) \cong \widetilde{A_5}$ abelianizes to the trivial group.

The second image in Figure 3 shows a tiling of $\mathbb{H}^3$ by infinitely many negatively-curved dodecahedra. Now the dodecahedra meet four to an edge; their faces are the same right-angled pentagons that appeared in Figure 2. This pattern provides a hyperbolic metric on a closed 3-manifold M that can be obtained as a 4-fold cover of S^3 branched over the Borromean rings.

Arithmetic groups. Additional examples of hyperbolic 3-manifolds are provided by *arithmetic subgroups* of $\mathrm{SL}_2(\mathbb{C})$. The simplest of these are the *Bianchi groups* $\mathrm{SL}_2(\mathcal{O})$, where $\mathcal{O} = \mathbb{Z}[\sqrt{-d}]$ or $\mathbb{Z}[(1+\sqrt{-d})/2]$ is the ring of integers in a complex quadratic field.

The action of a Bianchi group by Möbius transformations on $\widehat{\mathbb{C}}$ extends to an isometric action on $\mathbb{H}^3$. Passing to a subgroup of finite index, we can ensure that $\Gamma \subset \mathrm{SL}_2(\mathcal{O})$ is torsion-free, and hence $M = \mathbb{H}^3/\Gamma$ is a manifold. General principles insure that M has finite volume, but it is never closed; instead, it is homeomorphic to the complement of a knot or link in some closed 3-manifold $\overline{M}$.

An important example is provided by the Eisenstein integers, $\mathcal{O} = \mathbb{Z}[\omega]$ where $\omega^3 = 1$. In this case $\Gamma \subset \mathrm{SL}_2(\mathbb{Z}[\omega])$ can be chosen so that $M = \mathbb{H}^3/\Gamma \cong S^3 - K$ is the complement of the figure-eight knot in the sphere. Similarly, using $\mathcal{O} = \mathbb{Z}[i]$, one

[1]A spatial universe of constant negative curvature (as would be consistent with a uniform distribution of matter and energy) can therefore have almost any global topological form.

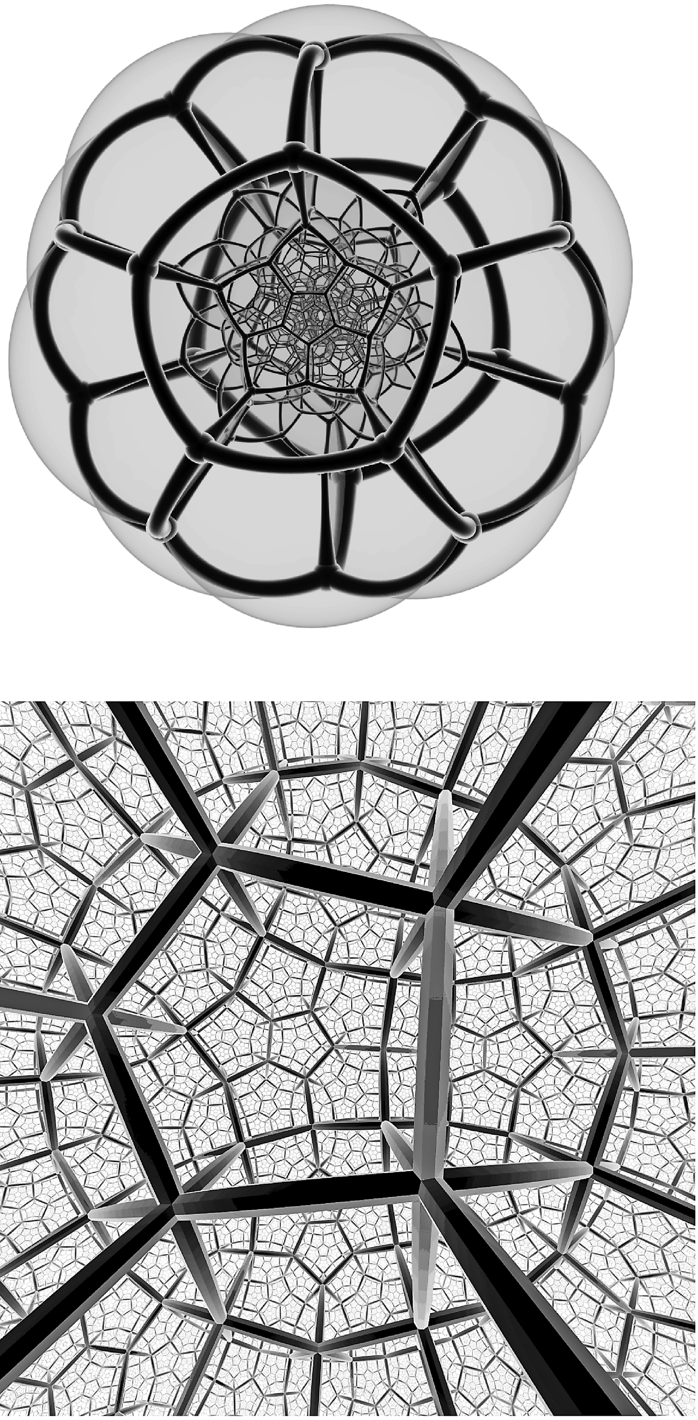

FIGURE 3. Tilings of S^3 and $\mathbb{H}^3$

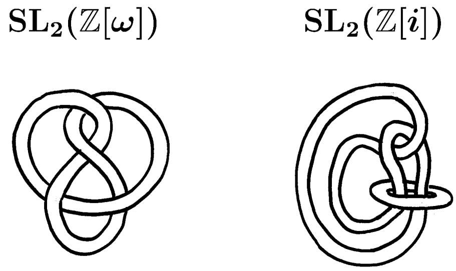

FIGURE 4. Arithmetic knots and links

obtains an arithmetics hyperbolic structure on the complement of the Whitehead link (Figure 4).

These examples are related to tilings of $\mathbb{H}^3$ by regular ideal tetrahedra and octahedra respectively.

Rigidity. The architectural integrity of the frameworks shown in Figure 3, and the arithmeticity of the preceding examples, reflect an important feature of the passage from 2 to 3 dimensions: while topology becomes more flexible in higher dimensions, geometry becomes more rigid. A precise statement is furnished by:

THEOREM 3.2 (Mostow Rigidity). *The geometry of a finite-volume hyperbolic 3-manifold is uniquely determined by its fundamental group.*

Because of this uniqueness, geometric quantities such as the hyperbolic volume of M^3 or the length of its shortest geodesic are actually *topological invariants.* For example, the figure-eight knot satisfies

$$\mathrm{vol}(S^3 - K) = 6\text{Л}(\pi/3) = 6\int_0^{\pi/3} \log \frac{1}{2\sin\theta}\, d\theta = 2.0298832\ldots$$

The geometrization conjecture becomes even more striking when seen in light of this rigidity.

A comparison to number theory. The influence of the Poincaré conjecture on low-dimensional topology can be compared to the influence of Fermat's last theorem on number theory. Both conjectures have been driving forces in mathematics, but both their formulations are essentially negative.

The geometrization conjecture placed the Poincaré conjecture in the context of a comprehensive picture of 3-dimensional topology that could be tested and developed in many new directions. Similarly, work of Frey, Ribet and Serre in the 1980's showed that Fermat's last theorem would follow from the modularity conjecture, which states:

Every elliptic curve E defined over $\mathbb{Q}$ is dominated by a modular curve of the form $X_0(N) = \mathbb{H}^2/\Gamma_0(N)$.

Like the geometrization conjecture, the modularity conjecture is constructive and testable. For example, in 1993 Cremona calculated all the modular elliptic curves with conductor $N \leq 999$, lending support to the conjecture and furnishing important arithmetic invariants of these elliptic curves.

Some of the experimental work carried out for 3-manifolds will be discussed in the sections that follow.

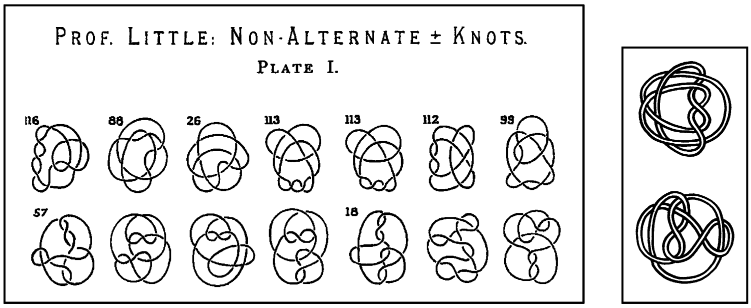

FIGURE 5. Tait and Little's knot tables (excerpt), 1899; the Perko pair, 1974.

4. Knots

What has become of all the simpler vortex atoms?

—P. G. Tait, 1876.

Knots and links provide a glimpse of the full complexity of 3-dimensional topology. In this section we discuss Thurston's results on hyperbolic 3-manifolds, and their impact on knot theory.

Hyperbolic knots. A *knot* is a smoothly embedded circle in S^3. The union of finitely many disjoint knots is a *link*. By removing a thickened link in S^3 (a union of solid tori) and gluing it back in with a twist, one obtains a new 3-manifold M. Lickorish showed that all orientable 3-manifolds can be obtained by surgery on links in S^3 [**Li**].

In the early 1980's Thurston established several major cases of the geometrization conjecture, including the following unexpected results:

(1) Almost all *knots* are hyperbolic;
(2) Almost all *surgeries* of S^3 along hyperbolic knots and links yield hyperbolic manifolds; and
(3) The result M of *gluing together* two hyperbolic 3-manifolds is hyperbolic, unless $\pi_1(M)$ contains a copy of $\mathbb{Z}^2$.

Here a knot or link L is *hyperbolic* if $S^3 - L$ is homeomorphic to a finite volume hyperbolic manifold $\mathbb{H}^3/\Gamma$. In the first statement, just torus knots and satellite knots must be avoided; in the second, finitely many surgeries must be excluded on each component of the link. The third statement is the key to proving the first two; it will be taken up in §5. All three results make precise, in various ways, the statement that most 3-manifolds are hyperbolic.

Tabulating knots. To put these results in context, we recount some history.

A knot can be conveniently described by a crossing diagram, showing its projection to a plane. Motivated by Lord Kelvin's theory of atoms as vortex rings, whose different shapes would account for the different chemical elements, in the period 1876–1899 Tait and Little (aided by Kirkman) assembled a census of all 249 (prime) knots with 10 or fewer crossings (see e.g. Figure 5). It is a demanding but straightforward task to enumerate all such knot diagrams; the challenge is to tell when two different diagrams actually represent the same knot.

In the 1960's J. H. Conway invented a more efficient combinatorial notation for knots, based on tangles. This notation allowed him to replicate the work by Tait and Little in a matter of days, and to extend the existing tables to include all 552 knots with 11 crossings.

Both tables, however, contained a duplication: in 1974, the lawyer K. Perko discovered that the two diagrams shown at the right in Figure 5 actually represent the same knot.

Algorithms and geometry. With such pitfalls in mind, the prospect of proceeding further seemed daunting. Nevertheless, in 1998, Hoste, Thistlethwaite and Weeks succeeded in tabulating all knots up to 16 crossings—all 1,701,936 of them [**HTW**].

How was such a tabulation possible? Its cornerstone was a computer program, developed by Weeks, to find the hyperbolic structure on $M = S^3 - K$. Although based on a heuristic algorithm, in practice this program almost always succeeds. The hyperbolic structure, in turn, yields a host of numerical invariants for K, such as the volume of $M = S^3 - K$; and it also provides a canonical triangulation of M (dual to a fundamental domain 'centered' on K). This triangulation is a complete invariant of K, so it suffices to eliminate all duplicate hyperbolic knots. (In particular, the algorithm immediately recognizes the Perko pair as two diagrams for the same knot.)

We remark that the practical computation of hyperbolic structures for knots, while motivated by Thurston's results, does not logically rely upon them; nor do the existing proofs of the existence of hyperbolic structures yet explain why such computations are so robust.

5. Evolving geometric structures

We now turn to the proof of the geometrization conjecture. We will discuss two important processes for deforming a topological 3-manifold towards its optimal geometric shape: conformal iteration and the Ricci flow.

1. Haken manifolds. We begin with some terminology. Let M be a compact orientable 3-manifold, possibly with boundary. A connected orientable surface $S \subset M^3$ is *incompressible* if $S \neq S^2$ and $\pi_1(S)$ maps injectively into $\pi_1(M)$.

A 3-manifold is *Haken* if it can be built up, starting from 3-balls, by successively gluing along incompressible submanifolds of the boundary. Any knot or link complement is Haken, as is any irreducible 3-manifold with boundary. Thus most of the results stated in §4 for knots are consequences of:

THEOREM 5.1 (Thurston). *The geometrization conjecture holds for Haken 3-manifolds.*

Iteration on Teichmüller space. Since the seven simpler geometries are understood for Haken manifolds, the main point in the proof of Theorem 5.1 is to treat the hyperbolic case. At the critical inductive step, one has an open hyperbolic 3-manifold M with incompressible boundary, and a gluing involution $\tau : \partial M \to \partial M$ (see Figure 6). The task is to produce a hyperbolic metric on the closed manifold M/τ.

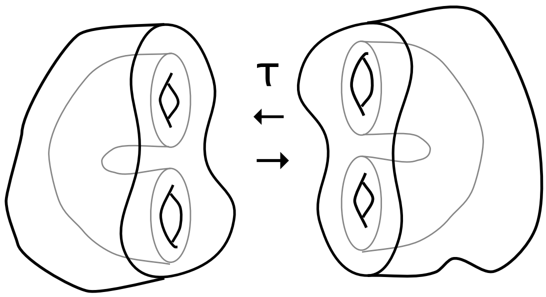

FIGURE 6. Gluing succeeds unless it is obstructed by a torus

A generalization of Mostow rigidity shows that hyperbolic structures on the interior of M correspond to conformal structures on ∂M. They are therefore parameterized by Teichmüller space, a finite-dimensional complex manifold homeomorphic to a ball. Thurston showed a solution to the gluing problem corresponds to a fixed point for a topologically-defined holomorphic map

$$\sigma \circ \tau : \mathrm{Teich}(\partial M) \to \mathrm{Teich}(\partial M).$$

By iterating this map, we obtain an evolving sequence of hyperbolic structures on M. If the sequence converges, then M/τ is hyperbolic.

One obstruction to convergence comes from $\pi_1(M/\tau)$: the fundamental group of a closed, negatively curved manifold never contains a copy of $\mathbb{Z}^2$. In fact, as Thurston showed, this is the only obstruction.

THEOREM 5.2. *M/τ is hyperbolic $\iff$ $\pi_1(M/\tau)$ does not contain $\mathbb{Z}^2$.*

SKETCH OF THE PROOF. We describe an approach based on complex analysis developed in [**Mc1**]. At a given point $X \in \mathrm{Teich}(\partial M)$, the Poincaré series operator introduced in §2 provides a map $\Theta_X : Q(\Delta) \to Q(X)$ from L^1 holomorphic quadratic differentials on the disk to those on X. It turns out the norm of the operator depends only on the location of X in moduli space, and satisfies $\|\Theta_X\| < 1$. Using the fact that $Q(X)$ forms the cotangent space to Teichmüller space at X, one can also show that $\sigma \circ \tau$ is a contraction in the Teichmüller metric with the bound

$$|(\sigma \circ \tau)'(X)| \le \|\Theta_X\| < 1.$$

Now start with an arbitrary Riemann surface $X_0 \in \mathrm{Teich}(\partial M)$ and form the sequence

$$X_n = (\sigma \circ \tau)^n(X_0).$$

Then the bound above shows we have uniform contraction—and hence convergence to a fixed point—*unless* $[X_n]$ tends to infinity in moduli space. But in this case $X_n \cong \partial M$ develops short geodesics, which bound cylinders in M that are joined together by τ to yield a torus in M/τ as shown in Figure 6. Thus $\pi_1(M/\tau)$ contains the obstruction $\mathbb{Z}^2$. □

This gluing construction is the pivotal step in Thurston's proof of Theorem 5.1. It also resonates with similar approaches to the topology of rational maps, the classification of surface diffeomorphisms and Mordell's conjecture in the function field case; see e.g. [**DH**], [**Mc2**], [**Mc5**].

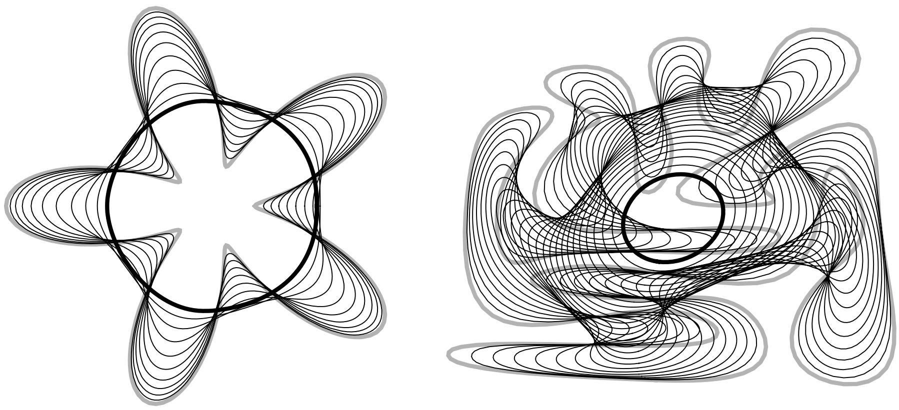

FIGURE 7. Curvature flow

For simplicity we have assumed M is not a virtual product, $S\times[0,1]$. The gluing theorem remains true in this case, but a somewhat different proof is required.

2. Evolution by curvature. We now turn to the second approach, used by Perelman to complete the proof of the geometrization conjecture.

> *Darwin recognized that his weak and negative force... could only play [a] creative role if variation met three crucial requirements: copious in extent, small in range of departure from the mean, and isotropic.*
>
> — S. J. Gould.

In 1982, Hamilton introduced the Ricci flow

$$\frac{dg_{ij}}{dt} = -2R_{ij}$$

for an evolving Riemannian metric g_{ij} on a manifold M. This is a (nonlinear) heat-equation type flow driven by the Ricci tensor, a contraction of the Riemann curvature form that also plays a central role in general relativity.

The idea of the Ricci flow is shown in Figure 7: here, an initial space of variable curvature evolves continuously until it becomes recognizable as a round circle. The manifold bends in response to its own shape, continuously adapting so that as $t \to \infty$ a metric of constant curvature may emerge.

Hamilton made several pioneering contributions to geometrization using the Ricci flow, including a proof of the Poincaré conjecture for manifolds with positive Ricci curvature [**Ham1**]. In this case the manifold shrinks to a point in finite time, but under rescaling it converges to a round unit sphere.

We remark that the Ricci flow, like natural selection itself, satisfies the 3 principles enunciated by Gould: as a differential equation on the whole manifold, it is copious in extent but small in departure from the mean; and it is *isotropic*, since the Ricci curvature is an intrinsic invariant of the metric.

Perelman's work. There are two main obstacles to long-term evolution under the Ricci flow: singularities may develop, which rapidly pinch off and break the manifold into pieces; and the manifold may *collapse*: it may become filled with short loops, even though its curvature remains bounded.

Perelman's work addresses both of these obstacles, and indeed turns them into the cornerstones of a successful proof of the geometrization conjecture. In brief, he shows that in dimension three:

(1) Singularities of the Ricci flow always occur along shrinking 2-spheres, which split M into a connected sum of smaller pieces. These singularities can be sidestepped by an explicit surgery operation.
(2) Curvature evolution with surgery defines a flow which continues for all time.
(3) In the limit as $t \to \infty$, a geometric structure on the pieces of M becomes visible, either through convergence to a metric of constant curvature or through collapsing.

As a consequence we have:

THEOREM 5.3 (Perelman). *Both the Poincaré conjecture and the geometrization conjecture are true.*

Comparison. Many detailed accounts of Perelman's work are now available in the literature. Here we will only add a few comparisons between these two different evolutionary processes.

(1) The discrete dynamics of conformal iteration takes place on the finite-dimensional space of hyperbolic manifolds. It proceeds through a sequence of classical, finite-sided hyperbolic polyhedra with varying shapes, converging to a form suitable for gluing.

    The continuous Ricci flow, on the other hand, takes place in the infinite-dimensional space of smooth metrics. Constant curvature and homogeneous geometry emerge only in the limit.
(2) Iteration on Teichmüller space is a contraction, and hence guaranteed to converge if a fixed point exists. In this way it leverages Mostow rigidity (which implies the fixed point is unique).

    The analysis of the Ricci flow, on the other hand, pivots on monotonicity. Various entropy-like quantities increase under the flow, allowing one to obtain compactness results and to rule out *breathers* (oscillating solutions to the flow which cannot possibly converge).

    Rigidity or uniqueness of the limiting geometry is not apparent from this perspective.
(3) The approach for Haken manifolds is bottom-up: the geometry of M is assembled inductively from smaller geometric pieces, by cutting along a hierarchy of surfaces.

    The Ricci flow approach is top-down; the metric evolves on the manifold as a whole, splitting it into pieces as singularities develop. Thus it can be applied to 3-manifolds which are too tightly wound (or too homotopically simple) to contain an incompressible surface.

Because of these features, the evolutionary approach based on the Ricci flow is able to treat the geometrization conjecture in full.

Remark: The cone-manifold approach. By Thurston's Theorem 5.1, any 3-manifold contains a knot such that $M^3 - K$ is hyperbolic. One can then try to increase the cone angle along the knot from 0 to 360°, to obtain a geometric structure on M. This cone-manifold approach to geometrization works well for constructing orbifolds (see e.g. [**CHK**]), but it runs into difficulties, still unresolved, when the strands of the knot collide.

The Ricci flow, on the other hand, smooths out such conical singularities, diffusing the knot so it can freely pass through itself.

6. Open problems

To conclude, we mention two of the many remaining open problems in the theory of 3-manifolds.

1. Surfaces in 3-manifolds. As we have seen, a useful approach to simplifying a 3-manifold involves cutting it open along an incompressible surface. A central problem, still open, is to understand how often such surfaces exist; in particular, to establish:

CONJECTURE 6.1 (Waldhausen, 1968). Every closed, irreducible 3-manifold M with infinite fundamental group has a finite cover which contains an incompressible surface.

Part of the impact of the proof of the geometrization conjecture is that it allows topological problems to be studied by geometric means. For example, in the conjecture above, one may now assume M is hyperbolic.

In 2003, Dunfield and Thurston verified Conjecture 6.1 for the more than ten thousand different hyperbolic 3-manifolds appearing in the Hodgson–Weeks census [**DT**] . Further progress includes the following result from 2010:

THEOREM 6.2 (Kahn–Markovic). *If M is a closed hyperbolic 3-manifold, then $\pi_1(M)$ contains a surface group.*

The proof uses ergodic theory on the frame bundle of M to analyze statistical properties of the pairs of pants it contains, which are then pieced together (in enormous numbers) to form a closed, immersed surface [**KM**].

It remains a challenge to find a finite cover where this surface becomes embedded.[2]

2. Quantum topology. The curved space of general relativity becomes a sea of virtual particles when viewed through the lens of quantum mechanics. Similarly, quantum topology gives a new perspective on 3-manifolds.

An example is provided by the knot polynomial $V(K,t)$ discovered by Jones in 1984. The Jones polynomial can be computed from a knot diagram by a simple inductive procedure, but it proves difficult to say what $V(K,t)$ measures in terms of classical topology.

On the other hand, Witten found a useful description in terms of physics: in 1988 he proposed that for each integer $k \geq 0$, the value of $V(K,t)$ at the root of unity $q = \exp(2\pi i/(2+k))$ should satisfy the relation:

$$\langle K \rangle = \int \mathrm{Tr}\left(\textstyle\int_K A\right) e^{2\pi i k\, \mathrm{CS}(A)}\, DA = (q^{1/2} + q^{-1/2}) V(K, q^{-1}).$$

Here A is an SU(2)-connection on the trivial $\mathbb{C}^2$ bundle over S^3. The factor $e^{2\pi i k\, \mathrm{CS}(A)}\, DA$ represents a formal probability measure on the space of all connections, coming from quantum field theory and the Chern-Simons action. Finally

[2]The proof of Conjecture 6.1 has recently been completed by Agol and Wise.

$\langle K\rangle$ is the expected value of the random variable $\mathrm{Tr}(\int_K A)$, which measures the twisting of the connection along the knot.

Many additional invariants of low-dimensional manifolds have emerged from the perspective of quantum field theory in recent decades, and been made rigorous using combinatorial methods and gauge theory.

How might these developments relate to the geometrization conjecture? A possible connection is provided by:

CONJECTURE 6.3 (Kashaev, Murakami–Murakami). *The hyperbolic volume of $S^3 - K$ can be calculated using the Jones polynomials of the cables of K; in fact, we have*

$$\mathrm{vol}(S^3 - K) = \lim_{n\to\infty} \frac{2\pi \log |V_n(K, e^{2\pi i/n})|}{n}.$$

Here $V_n(K,t) = \sum_{j=0}^{n/2} \binom{n-j}{j} V(K^{n-2j}, t)$, where K^i is the cabled link formed by i parallel copies of K.

The idea behind this conjecture is that the SU(2) connections on $S^3 - K$ should be sensitive to the flat $\mathrm{SL}_2(\mathbb{C})$ connection defining its hyperbolic structure.

At present, the volume conjecture above has been verified for only a handful of knots, including the figure-eight knot. It hints, however, at a deeper connection between geometric and quantum topology, mediated perhaps by the multitude of fluctuating combinatorial descriptions that a single geometric manifold can admit.

7. Notes and references

§1. For a historical perspective on the Poincaré conjecture, see [**Mil**].

§2. Poincaré's works on Fuchsian groups and Θ-series are collected in [**Po**].

§3. The geometrization conjecture is formulated in [**Th1**]. For more on the eight 3-dimensional geometries, see e.g. [**Sc**] and [**Th5**].

A variant of the hyperbolic tiling shown in Figure 3, in which five dodecahedra meet along an edge, was discovered by Seifert and Weber in 1933 [**SW**]. Seifert and Weber also related their example to the Poincaré sphere and to a covering of S^3 branched over the Whitehead link. The graphics in Figure 3 were produced by Fritz Obermeyer and by the Geometry Center.

It is known that the figure-eight knot is the only arithmetic knot [**Re**]. Additional arithmetic links are described in [**Hat**].

The original proof of Mostow rigidity (generalized to manifolds of finite volume by Prasad) was based on ergodic theory and quasiconformal mappings [**Mos**]. For a more geometric proof, due to Gromov, see e.g. [**Rat**, §11].

A discussion of the modularity conjecture and Fermat's last theorem can be found in Mazur's article [**Maz**]. The proof of the modularity conjecture was completed in 2001, through work of Breuil, Conrad, Diamond, Taylor and Wiles.

§4. Thurston's results are presented in [**Th1**]. In the 1990's, Casson–Jungreis and Gabai made important progress on the seven non-hyperbolic geometries by characterizing Seifert fiber spaces [**CJ**], [**Ga**].

The work of Tait appears in [**Ta**] and its sequels. Figure 5 is taken from Little's paper [**Lit**]. These authors had no rigorous methods even to distinguish the trefoil knot from the unknot. The success of their tabulations, especially for alternating knots, is due in part to the validity of the Tait conjectures, which were finally proved using the Jones polynomial (see e.g. [**MeT**]). One of these conjectures asserts that the writhe of a reduced alternating diagram is an invariant of the knot; the Perko pair shows this is false for non-alternating knots.

Conway's work on knots up to 11 crossings appears in [**Con**].

§5. Thurston's proof of the geometrization conjecture for Haken manifolds is outlined in [**Mor1**]. Large portions appear in [**Th2**], [**Th3**] and [**Th4**]. An orbifold construction allows one to reduce to the case where gluing is along the full boundary of M; some additional work is required to keep track of the parabolic locus. A complete proof is presented by Kapovich in [**Kap**]. The case of 3-manifolds that fiber over the circle, which requires a different gluing argument, is treated in detail in [**Th3**] and [**Ot**]; see also [**Mc4**]. The analytic proof of the gluing step presented here appears in [**Mc1**]; see also [**Mc3**].

Gould's statement on natural selection is taken from [**Go**, p.60]. Figure 7 actually depicts two examples of the mean curvature flow for hypersurfaces, a variant of the Ricci flow studied in dimension two by Gage and Hamilton [**GH**]; see also [**Gr**]. For Hamilton's work on the Ricci flow for 3-manifolds, see [**Ham1**], [**Ham2**] and [**Ham3**].

Perelman's proof, which appeared in [**Per1**], [**Per2**] and [**Per3**], is surveyed in [**And**], [**Mor2**] and [**Be**], and presented in detail in [**KL**], [**MT1**], [**MT2**], and [**CZ**]; see also the forthcoming book [**BMP**].

§6. For more on the Jones polynomial and quantum topology, see e.g. [**J1**], [**J2**], [**Wit**] and Atiyah's book [**At**]. The volume conjecture is formulated in [**Ka**] and [**MM**]; see also the survey [**Mur**].

References

[And] M. T. Anderson. Geometrization of 3-manifolds via the Ricci flow. *Notices Amer. Math. Soc.* **51**(2004), 184–193. MR2026939 (2004i:53095)

[At] M. Atiyah. *The Geometry and Physics of Knots*. Cambridge University Press, 1990. MR1078014 (92b:57008)

[BMP] L. Bessières, G. Besson, Michel Boileau, S. Maillot, and J. Porti. *Geometrisation of 3-Manifolds*. Eur. Math. Soc., 2010. MR2683385 (2012d:57027)

[Be] G. Besson. Preuve de la conjecture de Poincaré en déformant la métrique par la courbure de Ricci (d'après G. Perel'man). In *Séminaire Bourbaki 2004/2005*, pages 309–347. Astérisque, vol. 307, 2006. MR2296423 (2008b:53088)

[CZ] H.-D. Cao and X.-P. Zhu. A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow. *Asian J. Math.* **10**(2006), 165–492. MR2233789 (2008d:53090)

[CJ] A. Casson and D. Jungreis. Convergence groups and Seifert fibered 3-manifolds. *Invent. math.* **118**(1994), 441–456. MR1296353 (96f:57011)

[Con] J. H. Conway. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pages 329–358. Pergamon, 1970. MR0258014 (41:2661)

[CHK] D. Cooper, C. D. Hodgson, and S. P. Kerckhoff. *Three-Dimensional Orbifolds and Cone-Manifolds*. Mathematical Society of Japan, 2000. MR1778789 (2002c:57027)

[DH] A. Douady and J. Hubbard. A proof of Thurston's topological characterization of rational maps. *Acta Math.* **171**(1993), 263–297. MR1251582 (94j:58143)

[DT] N. Dunfield and W. Thurston. The virtual Haken conjecture: experiments and examples. *Geometry and Topology* **7**(2003), 399–441. MR1988291 (2004i:57024)

[Ga] D. Gabai. Convergence groups are Fuchsian groups. *Annals of Math.* **136**(1992), 447–510. MR1189862 (93m:20065)

[GH] M. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *J. Differential Geom.* **23**(1986), 69–96. MR840401 (87m:53003)

[Go] S. J. Gould. *The Structure of Evolutionary Theory.* Harvard University Press, 2002.

[Gr] M. A. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differential Geom.* **26**(1987), 85–314. MR906392 (89b:53005)

[Ham1] R. S. Hamilton. Three manifolds with positive Ricci curvature. *J. Diff. Geom.* **17**(1982), 255–306. MR664497 (84a:53050)

[Ham2] R. S. Hamilton. The formation of singularities in the Ricci flow. In *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, pages 7–136. Int. Press, 1995. MR1375255 (97e:53075)

[Ham3] R. S. Hamilton. Non-singular solutions of the Ricci flow on three-manifolds. *Comm. Analysis and Geometry* **7**(1999), 695–729. MR1714939 (2000g:53034)

[Hat] A. Hatcher. Hyperbolic structures of arithmetic type on some link complements. *J. London Math. Soc.* **27**(1983), 345–355. MR692540 (84m:57005)

[HTW] J. Hoste, M. Thistlethwaite, and J. Weeks. The first 1,701,936 knots. *Math. Intelligencer* **20**(1998), 33–48. MR1646740 (99i:57015)

[J1] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math.* **126**(1987), 335–388. MR908150 (89c:46092)

[J2] V. F. R. Jones. On the origin and development of subfactors and quantum topology. *Bull. Amer. Math. Soc.* **46**(2009), 309–326. MR2476415 (2009k:46113)

[KM] J. Kahn and V. Markovic. Immersing almost geodesic surfaces in a closed hyperbolic 3-manifold. *Preprint, 2010.*

[Kap] M. Kapovich. *Hyperbolic Manifolds and Discrete Groups.* Birkhäuser Boston, 2001. MR1792613 (2002m:57018)

[Ka] R. M. Kashaev. The hyperbolic volume of knots from the quantum dilogarithm. *Lett. Math. Phys.* **39**(1997), 269–275. MR1434238 (98b:57012)

[KL] B. Kleiner and J. Lott. Notes on Perelman's papers. *Geom. Topol.* **12**(2008), 2587–2855. MR2460872 (2010h:53098)

[Li] W.B.R. Lickorish. Representation of orientable combinatorial 3-manifolds. *Annals of Mathematics* **76**(1962), 531–540. MR0151948 (27:1929)

[Lit] C. N. Little. Non-alternate $+/-$ knots. *Trans. Royal. Soc. Edinburgh* **39**(1898-9), 771–778.

[Maz] B. Mazur. Number theory as gadfly. *Amer. Math. Monthly* **98**(1991), 593–610. MR1121312 (92f:11077)

[Mc1] C. McMullen. Iteration on Teichmüller space. *Invent. math.* **99**(1990), 425–454. MR1031909 (91a:57008)

[Mc2] C. McMullen. Rational maps and Kleinian groups. In *Proceedings of the International Congress of Mathematicians (Kyoto, 1990)*, pages 889–900. Springer-Verlag, 1991. MR1159274 (93h:57024)

[Mc3] C. McMullen. Riemann surfaces and the geometrization of 3-manifolds. *Bull. Amer. Math. Soc.* **27**(1992), 207–216. MR1153266 (93b:57013)

[Mc4] C. McMullen. *Renormalization and 3-Manifolds which Fiber over the Circle*, volume 142 of *Annals of Math. Studies.* Princeton University Press, 1996. MR1401347 (97f:57022)

[Mc5] C. McMullen. From dynamics on surfaces to rational points on curves. *Bull. Amer. Math. Soc.* **37**(2000), 119–140. MR1713286 (2000j:11093)

[MeT] W. Menasco and M. Thistlethwaite. The Tait flyping conjecture. *Bull. Amer. Math. Soc.* **25**(1991), 403–412. MR1098346 (92b:57017)

[Mil] J. Milnor. Towards the Poincaré conjecture and the classification of 3-manifolds. *Notices Amer. Math. Soc.* **50**(2003), 1226–1233. MR2009455 (2004h:57022)

[Mor1] J. Morgan. On Thurston's uniformization theorem for three dimensional manifolds. In *The Smith Conjecture*, pages 37–125. Academic Press, 1984. MR758464

[Mor2] J. Morgan. Recent progress on the Poincaré conjecture and the classification of 3-manifolds. *Bull. Amer. Math. Soc.* **42**(2005), 57–78. MR2115067 (2005i:57015)

[MT1] J. Morgan and G. Tian. *Ricci Flow and the Poincaré Conjecture.* Amer. Math. Soc., 2007.

[MT2] J. Morgan and G. Tian. Completion of the proof of the geometrization conjecture. *Preprint, 2008.*

[Mos] D. Mostow. *Strong Rigidity of Locally Symmetric Spaces*, volume 78 of *Annals of Math. Studies.* Princeton University Press, 1972.

[Mur] H. Murakami. A quantum introduction to knot theory. In *Primes and knots*, volume 416 of *Contemp. Math.*, pages 137–165. Amer. Math. Soc., 2006. MR2276140 (2008f:57018)

[MM] H. Murakami and J. Murakami. The colored Jones polynomials and the simplicial volume of a knot. *Acta Math.* **186**(2001), 85–104. MR1828373 (2002b:57005)

[Ot] J.-P. Otal. *Le théorème d'hyperbolisation pour les variétés fibrées de dimension trois.* Astérisque, vol. 235, 1996.

[Per1] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. `arXiv:math/0211159`.

[Per2] G. Perelman. Ricci flow with surgery on three-manifolds. `arXiv:math/0303109`.

[Per3] G. Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. `arXiv:math/0307245`.

[Po] H. Poincaré. *Oeuvres. Tome II: Fonctions fuchsiennes.* Éditions Jacques Gabay, 1995. MR1401791 (97c:01053)

[Rat] J. G. Ratcliffe. *Foundations of Hyperbolic Manifolds.* Springer-Verlag, 1994. MR1299730 (95j:57011)

[Re] A. W. Reid. Arithmeticity of knot complements. *J. London Math. Soc.* **43**(1991), 171–184. MR1099096 (92a:57011)

[Sc] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.* **15**(1983), 401–487. MR705527 (84m:57009)

[SW] H. Seifert and C. Weber. Die beiden Dodekaederräume. *Math. Z.* **37**(1933), 237–253. MR1545392

[Ta] P. G. Tait. On knots I. *Trans. Roy. Soc. Edinburgh* **28**(1876-7), 145–190.

[Th1] W. P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc.* **6**(1982), 357–381. MR648524 (83h:57019)

[Th2] W. P. Thurston. Hyperbolic structures on 3-manifolds I: Deformations of acylindrical manifolds. *Annals of Math.* **124**(1986), 203–246. MR855294 (88g:57014)

[Th3] W. P. Thurston. Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds which fiber over the circle. *Preprint, 1986.* MR855294 (88g:57014)

[Th4] W. P. Thurston. Hyperbolic structures on 3-manifolds III: Deformation of 3-manifolds with incompressible boundary. *Preprint, 1986.* MR855294 (88g:57014)

[Th5] W. P. Thurston. *Three-Dimensional Geometry and Topology*, volume 1. Princeton University Press, 1997. MR1435975 (97m:57016)

[Wit] E. Witten. Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* **121**(1989), 351–399. MR990772 (90h:57009)

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Invariants of Manifolds and the Classification Problem

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1. Introduction

This article follows the structure of the author's lecture at the meeting in Paris closely, but the opportunity has been taken to add more detail, particularly in the last section.

It is a truism that the problem of classifying manifolds, like any other mathematical objects, has two aspects.

- If M_1, M_2 are manifolds which are not, in fact, equivalent, prove that $M_1 \neq M_2$. This is the problem of finding suitable *invariants*.
- If M_1, M_2 are manifolds which are, in fact, equivalent, prove that $M_1 = M_2$. This is the problem of *constructing equivalences*.

Of course there are different flavours of this problem depending on the notion of "equivalence" in question. We usually consider the C^∞ classification problem, so our manifolds have smooth structures and the equivalences we seek are diffeomorphisms. Another obvious but salient point is that one can distinguish between *general classification theorems*, dealing with all manifolds in some large class, and *specific problems*, where we have some pair of manifolds M_1, M_2 in our hands and we want to decide if they are diffeomorphic.

The solutions of the Poincaré conjecture, and the Geometrisation conjecture, are of course giant advances in this general classification problem in manifold topology. The purpose of this lecture is to discuss various other developments and questions, partly in the light of the solution of the Poincaré conjecture. One main theme is that there has been huge progress in the past 25 years in the first aspect above, the construction of new invariants, but rather little in the second aspect. Another theme is the interaction between these questions of manifold topology and ideas from geometry and analysis.

2. The role of geometry: contrast with high dimensions

A paradigm for the application of geometry and analysis to topology is the Riemann Mapping Theorem. This tells us that if U is a proper open subset of $\mathbf{C}$ which is connected and simply connected then there is a holomorphic diffeomorphism f mapping the unit disc D onto U. In particular we derive the "topological" corollary that U is diffeomorphic to the disc. While this is intuitively plausible, and can be proved by purely "topological" methods, the statement is far from trivial, particularly when we think of the exotic possibilities for U—for example U might be the interior of a closed fractal curve. One way of thinking about the contribution of geometry here is that in the topological formulation the sought-for map is not at all unique; we can compose any one choice with a diffeomorphism of the disc to get another. It is often easier to prove the existence of an object which is unique than to prove the existence of *some* object in a large class. (The Riemann map is of course unique if we prescribe $f(0)$ and the argument of $f'(0)$.)

We can formulate essentially the same argument in the language of differential geometry rather than complex analysis. In the situation above, the set U admits a complete Riemannian metric of constant curvature -1. This follows because we can transport the standard Poincaré metric on the disc using the map f. On the other hand it is quite conceivable that we might be able to prove this differential geometric statement directly. In general, suppose that a manifold M of dimension n admits a complete Riemannian metric of constant sectional curvature κ. Then we know that M is diffeomorphic to a quotient X/Γ where X is the simply connected model and Γ is a discrete subgroup of the isometry group of X, isomorphic to $\pi_1(M)$ and acting freely on X. So in particular if M is simply connected (as in the case of U above), we know that M is diffeomorphic to X (the Cartan-Hadamard theorem). Explicitly, the diffeomorphism is constructed using geodesics on M and the exponential map.

These general remarks lead to some insight into to the special place of geometrization in 3-dimensions. For this we need to recall some facts of Riemannian geometry. Any Riemannian n-manifold has a Riemann curvature tensor R_{ijkl} (working in an orthonormal basis for the tangent space at a point). This is skew symmetric in the indices (ij) and (kl) and satisfies the Bianchi identity

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

In more invariant terms the curvature tensor lies in the kernel of an $O(n)$-invariant linear map from $\Lambda^2 \otimes \Lambda^2$ to $\Lambda^1 \otimes \Lambda^3$. This map is surjective so its kernel has dimension

$$d(n) = \left(\frac{n(n-1)}{2}\right)^2 - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n^2-1)}{12}.$$

A metric has constant sectional curvature κ if $R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{jk}g_{il})$. So the condition of (prescribed) constant sectional curvature can be regarded locally as $d(n)$ equations for the $n(n+1)/2$ variables making up the entries g_{ij} of the metric tensor. Now it is obvious that the quartic polynomial $d(n)$ becomes much larger than $n(n+1)/2$ for large n. In fact we have

n	$n(n+1)/2$	$d(n)$
2	1	1
3	6	6
4	10	20
5	15	50

So just in dimensions $n = 2, 3$ is $d(n)$ equal to $n(n+1)/2$ which is to say that just in those dimensions is the constant sectional curvature condition a “determined” PDE, with the same as number of equations as variables. These simple considerations go a long way to explain the special role of “geometrisation”—in the sense of spaces of constant sectional curvature (and other homogeneous geometrical structures which can be thought of as degenerations of these)—in dimensions 2 and 3. In higher dimensions it is quite unreasonable to expect that a "typical" manifold will admit such a structure.

Another point of view on this is given by the *Ricci tensor*. In any dimension this is a symmetric 2-tensor given by the contraction

$$R_{jl} = \sum_i R_{ijil}.$$

The coincidence $d(n) = n(n+1)/2$ when $n = 2, 3$ reflects the fact that in these dimensions the full curvature tensor can be recovered from the Ricci tensor. In higher dimensions there is another irreducible component of the curvature: the *Weyl tensor*. Thus the constant sectional curvature condition, in dimensions 2,3, is equivalent to the *Einstein equation*

$$R_{jl} = (n-1)\kappa g_{jl}.$$

In any dimension, this Einstein equation is a determined equation, indeed the Ricci tensor has the same type as the metric tensor. Thus it is reasonable to hope that, in contrast to constant sectional curvature, can find such Einstein metrics in a fairly general context. In fact the Einstein equation is a determined elliptic equation for the metric, when proper account is taken of the invariance under the diffeomorphism group and the second Bianchi identity. One way in which one may hope to find solutions is from the asymptotics of the corresponding parabolic equation—Hamilton's Ricci flow:

$$\frac{\partial g_{jl}}{\partial t} = -2R_{jl}.$$

Of course, in dimension 3 this is exactly what Perelman achieved. The simple point we want to make here is that even if one were able to develop the analysis sufficiently, one would only expect the Ricci flow in higher dimensions to prove the existence of Einstein metrics in some degree of generality and this would not have any immediate bearing on topology because we do not have a complete description of Einstein metrics similar to the description X/Γ in the constant sectional curvature case.

What we *do* have in higher dimensions $n \geq 5$ is Smale's h-cobordism theorem. This gives a very general criterion for constructing diffeomorphism given "homotopy information". More precisely, we say that n-manifolds M_1, M_2 are h-cobordant if there is an $(n+1)$ manifold W with boundary

$$\partial W = M_1 \sqcup M_2,$$

such that the inclusions $M_i \to W$ are homotopy equivalences. The h-cobordism theorem asserts that if $n \geq 5$ and if M_1, M_2 are simply connected, h-cobordant, n-manifolds then they are diffeomorphic. The diffeomorphism can be constructed by integrating a suitable ODE. That is, the proof goes by constructing a nowhere-vanishing vector field v on W such that the flow lines of v go from M_1 to M_2. For each point $x \in M_1$ we let $f(x)$ be the terminus in M_2 of the flow line starting from x and then $f : M_1 \to M_2$ is the desired diffeomorphism. (There is a more sophisticated version, the s-cobordism theorem, which takes account of the fundamental group.)

The h-cobordism theorem (in the smooth category) definitely fails in dimension 4 [**10**], so four dimensional differential topology lies outside both the truly "low-dimensional" range $n = 2, 3$ in which topology is closely related to locally homogeneous geometric structures and outside the truly "high dimensional" range $n \geq 5$ where the h-cobordism theorem applies, and where we rather suspect that there may be an abundance of Einstein metrics, but without any particular topological significance. In the context of Ricci flow with surgery, Hamilton gave in [**22**], Section 3, a fascinating insight into this kind of low-dimension/high-dimension division.

3. Some geometric structures in four dimensions

It is natural to hope that there may be some kinds of "geometric structures" on four-dimensional manifolds which will play a role in the topological classification problem, although as we have explained above one certainly has to allow something more flexible than locally homogenous geometries in the sense of Thurston. (That said, the study of the locally homogeneous theory in 4-dimensions is still an interesting topic [**45**].) Of course this may not turn out to be a fruitful direction at all, but in any case there is a wide variety of geometric structures which have been studied in four dimensions and which are interesting in their own right, regardless of any possible application to topology. In this section we will give a very brief sketch of the some of these ideas.

One organising principle is the trinity of structures

$$g, \omega, J$$

in multilinear algebra. Here we have in mind a real vector space V of dimension 4 on which we can consider a Euclidean structure $g \in s^2V^*$, a symplectic form $\omega \in \Lambda^2V^*$ or a complex structure $I : V \to V, I^2 = -1$. Given any two of these we can write down a hermitian structure, whereby V becomes a complex vector space with a hermitian inner product. We may also consider a quaternionic structure, in which we have $I, J, K : V \to V$ satisfying the quaternion relations $I^2 = J^2 = K^2 = -1, IJ = K$. Of course in our situation these algebraic structures will be considered on the tangent spaces $V = TM_p$ of a 4-dimensional manifold M. Now we have, at least, the following differential-geometric structures which can be considered on four-dimensional manifolds.

- Einstein, Ricci-soliton
- Anti-self dual.
- symplectic, almost-complex
- Kähler, complex algebraic
- complex
- hypercomplex
- hyperkähler
- complex symplectic

(This is presented as a list but it would really be better to think of some kind of graph, indicating the diverse connections between the concepts.) First we have the Riemannian theories. Alongside Einstein metrics one considers *Ricci solitons*, which are fixed points of the Ricci flow up to diffeomorphism. There is another class one can consider within Riemannian geometry which is special to the 4-dimensional situation. On an oriented Riemannian 4-manifold the Weyl tensor decomposes into two pieces W^+, W^- (*self-dual* and *anti-self dual*), and the metric is called anti-self-dual if $W^+ = 0$. This equation is conformally invariant and is an elliptic equation for the conformal structure, modulo diffeomorphism. It is related to complex geometry via the "twistor construction" [**4**]. Next we have the symplectic and complex theories and Kähler structures which are a natural intersection of Riemannian, symplectic and complex geometry and which, on compact manifolds of complex dimension two, can always be deformed to complex projective surfaces. A different kind of mix of complex and symplectic geometry is furnished by complex-symplectic structures, where one has a non-degenerate *holomorphic* 2-form. It is equivalent to say that one has a pair ω_1, ω_2 of closed real 2-forms satisfying the conditions

$$\omega_1^2 = \omega_2^2 > 0 \quad \omega_1 \wedge \omega_2 = 0.$$

The *hyperkähler* case is in a sense the intersection of all these theories. A hyperkähler manifold is a manifold with a Riemannian metric and three complex structures, obeying the quaternion relations and such that the metric is Kähler with respect to each structure. (Hypercomplex structures are similar, but without the requirement of a metric). In four dimensions this is the same at least up to a covering, as saying that both the Ricci tensor and the self-dual Weyl tensor vanish. They are also complex symplectic manifolds: in fact a hyperkähler structure is equivalent to a triple of closed forms $\omega_1, \omega_2, \omega_3$ satisfying the relations

$$\omega_1^2 = \omega_2^2 = \omega_3^2 > 0 \quad \omega_i \wedge \omega_j = 0 \quad i \neq j.$$

There is a complete classification of compact hyperkähler 4-manifolds: the only non-flat examples are "K3 surfaces". These are all diffeomorphic, one model is given by a smooth surface of degree 4 in $\mathbf{CP}^3$. From some points of view one finds that $K3$ surfaces are the "simplest" compact 4-manifolds— they have the metrics whose curvature tensor is just the small piece W^- and they have the simplest possible non-trivial Seiberg–Witten invariants. (Of course they are also the prototypes of *Calabi-Yau* manifolds in general complex geometry). They furnish a special case example where we *do* have some hold on the Einstein condition. A compact oriented 4-manifold M has two characteristic numbers, the Euler characteristic $\chi(M)$ and the signature $\sigma(M)$ (the latter is the signature of the intersection form on $H_2(M, \mathbf{R})$). The *Hitchin–Thorpe inequality* states that if M has an Einstein metric g then

$$|\sigma(M)| \leq \frac{2}{3}\chi(M).$$

If equality holds (in the simply-conected case, say) then M is a $K3$ surface and the metric g is a hyperkähler. The proof goes by Chern-Weil theory which gives an identity, for an Einstein metric,

$$8\pi^2(\frac{2}{3}\chi(M) \pm \sigma(M)) = \int_M S^2 + |W^\pm|^2,$$

where S is the scalar curvature and W^+, W^- are the components of the Weyl tensor, as above. So if $\sigma(M) + \frac{2}{3}\chi(M) = 0$ we deduce that the scalar curvature and W^+ both vanish which implies that the metric is hyperkähler (and the case $\sigma(M) - \frac{2}{3}\chi(M) = 0$ follows by switching orientation). Thus if we consider a smooth compact simply-connected, oriented 4-manifold M with $\sigma(M) + \frac{2}{3}\chi(M) = 0$ and if we have some way (for example by Ricci flow) of proving the existence of an Einstein metric on M, then we could make the topological deduction that M is diffeomorphic to a K3 surface. Conversely, as we will discuss further in Section 5 below, there are examples of manifolds M which are homotopy equivalent, but not diffeomorphic, to $K3$ surfaces. We see then that these manifolds cannot support

Einstein metrics and the differences in the smooth structures must be reflected in some way in the behaviour of the Ricci flow. Related matters are discussed in detail in Tian's talk in this meeting. Let us just mention here that there are intriguing questions about the Ricci flow in 4-dimensions even in simple model problems. One recent example is provided by the work of Chen, LeBrun and Weber [**9**] who prove the existence of an Einstein metric on the connected sum $\mathbf{CP}^2\sharp\overline{\mathbf{CP}^2}\sharp\overline{\mathbf{CP}^2}$. Although the metric is not Kähler it is conformal to a Kähler metric and in fact to an "extremal metric". The striking thing here is that there is also a (Kähler) Ricci soliton metric on this same manifold, so we have a case where there are two distinct fixed points of the Ricci flow, modulo diffeomorphism. In fact the same is true, by older work of Calabi and Page, in the even simpler case of $\mathbf{CP}^2\sharp\overline{\mathbf{CP}^2}$ and the metrics here both have $U(2)$ symmetry.

Of course as we have said before, there is no certainty that any of these structures will lead to great insights into the topological classification problem. Equally one can consider still further permutations of these ideas. For example one can relax the hyperkähler condition to consider a triple of symplectic forms $\omega_1, \omega_2, \omega_3$ such that the matrix $\omega_i \wedge \omega_j$ is positive definite at each point [**11**].

4. Invariants in low-dimensional topology

Since the early 1980's many "new" invariants of three and four dimensional manifolds have been discovered. What we mean by "new" here is that they go beyond, and have a different character to, those arising from classical algebraic topology. These developments intertwine a variety of different fields

- 3 and 4 dimensional topology, knot theory
- Symplectic and contact topolology, foliations of 3-manifolds.

Further, they are intimately connected with developments in Theoretical Physics. There is now a constellation of differ ent but related theories, to the extent that there are probably few people, and certainly not this author, who are familiar with all the latest developments. As some headings we can mention

- Casson invariants
- Yang–Mills instanton invariants
- Seiberg–Witten invariants
- Floer homology
- Gromov–Witten invariants
- Contact homology
- Symplectic Field Theory
- Heegard Floer Theory
- Fukaya categories
- Jones–Witten invariants of knots and 3-manifolds
- Khovanov homology

We will not try to say anything systematic about all these developments but just note some fundamental ideas which appear in many of them, under three slogans.

- *Integration.* By which we mean functional integrals of the kind developed in Quantum Field theory which are, generally speaking, not mathematically rigorous. These are the basis of Witten's approach to the Jones invariants (and, more recently, to Khovanov homology [**48**]). They are also give the background, at least historically, for the definition of the Seiberg–Witten invariants of 4-manifolds.

- *Counting.* By which we mean invariants defined by "counting" the solutions of suitable elliptic partial differential equations. These can be used to extend ideas from ordinary differential topology to infinite dimensional situations, for example in the manner in which one defined the degree of a smooth map $f : S^p \to S^p$ by counting the points in the generic fibre $f^{-1}(y)$. "Counting" is a slogan here, since in reality one has to take account of signs, transversality and other technical issues. Examples include the Casson invariant which counts flat connections over a 3-manifold (or equivalently conjugacy classes of representations of the fundamental group), and Gromov–Witten invariants which count holomorphic curves and are related to classical enumerative questions in algebraic geometry.
- $\partial^2 = 0$. By which we mean the idea of Floer producing a chain complex from geometric data whose homology yields "topological invariants". These constructions can often be interpreted formally as computing the "middle dimensional" homology of an infinite dimensional space. Examples include Floer's original theories for 3-manifolds and Lagrangian intersections in symplectic manifolds, and many later developments growing from his idea.

Note that the geometry and topology of infinite dimensional spaces is a theme running through all these three items, and makes the pervasive interaction with Quantum Field Theory very natural. The infinite dimensional spaces in question are generally either spaces of G-connections over a manifold, for a Lie group G, or spaces of maps from one manifold—often a circle or Riemann surface—to another. One unifying concept is that of a "topological field theory" [**3**] by which one expects a theory that yields numerical invariants of manifolds in some dimension n may generate structures which assign *vector spaces* (such as Floer groups) to manifolds in dimension $(n-1)$, *categories* to manifolds of dimension $(n-2)$ and yet more esoteric structures in still lower dimensions [**33**].

Bringing order into this profusion, and understanding the precise connections between all these developments is an outstanding problem in mathematics today. Some connections are well-established and almost obvious; for example the Casson invariant of a 3-manifold can be viewed as the Euler characteristic of the Floer homology. Other connections are well-established but much deeper: for example Taubes' relation SW $\Leftrightarrow$ Gr [**42**] between the Seiberg–Witten invariants of a symplectic 4-manifold and the Gromov–Witten invariants defined by counting holomorphic curves using a choice of compatible almost-complex structure. (Note that the Seiberg–Witten equation depends on a Riemannian metric g while the holomorphic curve equations use an almost complex structure I: the passage between them makes essential use of a symplectic form ω, although this form does not appear explicitly in either equation.) In other cases there are well-established conjectures which are not completely proved, for example in the case of the Seiberg–Witten and Yang–Mills instanton invariants. In other cases there are clear hints of some relationship although the picture is still mysterious: for example in the case of Seidel and Smith's work [**41**] relating Khovanov homology to the the symplectic version of Floer theory. In other cases still it seems very unclear what the final picture will be—for example in relating the Floer theories of 3-manifolds based on Yang–Mills instantons and on the Seiberg–Witten equations.

5. A selection of notable results, questions and developments

Lacking the space for a systematic overview, we will present in this section a number of topics, many—but not all— involving very recent work, hoping to give some picture of the field. The author is particularly grateful to Ivan Smith for tutorials on these more recent developments.

5.1. Successes of invariants in 4 dimensions. Since 1984 the new invariants have been used to distinguish many smooth 4-manifolds which appear identical from the point of view of classical algebraic topology. We will restrict attention to simply connected, oriented 4-manifolds. In this case there are just three "classical" invariants: the integers b_2^+, b_2^- giving the dimensions of positive and negative subspaces for the intersection form and the "parity" w which is 0 or 1 according as the intersection form is an even form or an odd form. (Thus the sum $b_2^+ + b_2^-$ is equal to the second number $b_2 = \dim H_2$ and the parity is 0 if and only if the Stiefel-Whitney class w_2 vanishes.) The examples we refer to above are of pairs M_1, M_2 with these same values of b_2^+, b_2^- and parity which can be shown not to be diffeomorphic. Such a wealth of examples is now known that there is little point in collecting more without some special feature. One subject of investigation has been the "geography problem" of which values of (b_2^+, b_2^-, w) are realised by distinct 4-manifolds. In particular there is interest in searching for "small" examples, where "small" means with a small second Betti number. For 15 years the record was held by Kotschick [**26**], who gave examples with $b_2^+ = 1, b_2^- = 8$. In fact Kotschick's examples are complex algebraic surfaces and the distinction between them, as smooth 4-manifolds, is related to the fact that they have different Kodaira dimension, as complex surfaces. In 2005, J. Park [**40**] constructed examples with $b_2^+ = 1, b_2^- = 7$. Although it is not known whether these manifolds are complex surfaces, techniques from complex geometry play a major role in his construction. He started with a complex surface containing a special configuration of rational curves (i.e., 2-spheres) and then applied a "rational blow-down" construction, due to Fintushel and Stern [**13**], to remove the homology classes represented by these curves. In some situations the rational blow-down has a complex geometry interpretation in terms of smoothings of a singular surface, but the construction makes sense at the topological level even when this smoothing is obstructed. Since Park's breakthrough there has been rapid progress in the construction of steadily smaller examples. Some of these give new examples of complex algebraic surfaces [**31**], and have independent interest for algebraic geometers. Akmedov and Park [**2**] and independently Fintushel and Stern [**15**] have constructed examples with $b_+^2 = 1, b_-^2 = 2$. Further progress seems likely, but the verification that the manifolds constructed are simply connected is often a very delicate issue.

5.1.1. *Questions which appear out of reach.* Despite the abundance of cases indicated above in which we can distinguish smooth 4-manifolds, there are also many pairs of manifolds M_1, M_2 which we suspect to be distinct but which escape the current methods. One well-known case is that of *Horikawa surfaces*. Here we start with the complex quadric surface which is the product $S^2 \times S^2$. For each l there is a Hirzebruch surface Σ_{2l} which is a ruled surface fibering over S^2 with fibre S^2, diffeomorphic but not biholomorphic to $S^2 \times S^2$. There is a section

Δ of $\Sigma_{2l} \to S^2$ which is an embedded sphere of self-intersection $-2l$. We take M_1 to be a double cover of $S^2 \times S^2$ branched over a curve of bi-degree $(6, 12)$ and take M_2 to be the double cover of Σ_6 branched over the union of Δ and another, disjoint, curve C in Σ_6. If the homology class of C is chosen correctly, then M_1 and M_2 are homeomorphic but it is known from algebraic geometry that they are not "deformation equivalent"–i.e., they cannot be placed in the same connected moduli space of complex structures. The question is whether M_1, M_2 are diffeomorphic. In this situation general facts from Seiberg–Witten theory show that those invariants cannot distinguish the manifolds and at present there are no other tools to apply. We simply do not know the answer. It is quite possible that these manifolds are diffeomorphic, and perhaps this will be established by the application of sufficient insight and ingenuity. A related situation arises in the work of Catanese and Wajnryb [**7**] who show that certain (simply connected) iterated branched covers of $S^2 \times S^2$ which are not deformation equivalent are, in fact, diffeomorphic. Their proof uses an analysis of deformations of a singular surface. There is a variant of the question in which one asks whether M_1, M_2 are symplectomorphic (with their natural symplectic structures), and, while there are some techniques that can in principle be applied here, the problem seems far out of reach [**5**]. Again there is a parallel question for iterated covers which has been studied in detail by Catanese, Lönne and Wajnryb [**8**] but the complete answer is not yet clear.

Other well-known instances where we are unable to distinguish 4-manifolds are provided by work of Fintushel and Stern [**14**]. They start from a knot K in the 3-sphere and a K3 surface M containing a 2-torus T with a trivial tubular neighbourhood. Then they construct another manifold M_K by cutting out the torus neighbourhood and gluing in the product $S^1 \times (S^3 \setminus K)$. They show that the Seiberg–Witten invariants of M_K capture precisely the Alexander polynomial of K, while M_K is always homeomorphic to M. Thus, since we can easily write down knots with different Alexander polynomials, we get hordes of mutually distinct smooth 4-manifolds homeomorphic to the K3 surface. One the other hand we can write down non-trivial knots whose Alexander polynomials vanish. Then we get manifolds with the same Seiberg–Witten invariants as the $K3$ surface but we rather suspect that they are not diffeomorphic. But again we have no tools to establish this, and perhaps little reason to believe the answer should go one way or the other.

The best-known open problem in 4-manifold theory is probably the *four-dimensional smooth Poincaré conjecture*: which, after Freedman, is the question whether there is a manifold M homeomorphic but not diffeomorphic to S^4. For some time this seemed inaccessible since the "new" 4-manifold invariants rely on non-trivial second homology. But there is interesting recent work of Freedman, Gompf, Morrison and Walker who show that a certain invariant defined by Rasmussen, arising from Khovanov homology, could potentially detect counterexamples (i.e., exotic spheres). Freedman et al. report on extensive computer calculations, but some promising candidates for counterexamples were shown to be standard by Akbulut. Rather than speculate about the truth of the smooth Poincare conjecture let us just note two facts that seem vaguely relevant.

- The work of Bauer and Furuta [**6**] shows that more subtle information can sometimes be squeezed, with great profit, from the Seiberg–Witten equations.

- At one time there appeared to be the possibility of *disproving* the 3-dimensional Poincaré conjecture using the Rohlin invariant which, from its definition, did not obviously vanish on homotopy spheres. Showing that, in fact, this scheme would not work was the original motivation for Casson's introduction of his invariant. Casson proved that his invariant reduces modulo 2 to the Rohlin invariant but it clearly vanishes on homotopy spheres.

5.2. A small corner where there is a complete classification. The, possibly gloomy, conclusion in the previous subsection is that any kind of systematic understanding of smooth 4-manifolds seems a long way off. Thus it is very satisfying to have complete results even under limiting hypotheses. The only such result known to the author occurs in the discussion of compact symplectic 4-manifolds. Given a symplectic form ω on M we have two basic topological invariants

- The de Rham cohomology class $[\omega] \in H^2(M, \mathbf{R})$.
- The first Chern class $c_1 \in H^2(M; \mathbf{Z})$ which can be defined by choosing any compatible almost-complex structure on M.

Thus if M is a compact 4-manifold we get a numerical invariant from the cup product $c_1.\omega = \langle c_1 \cup [\omega], [M] \rangle$ and in particular we have a "sign" $+, 0, -$ depending on whether this number is positive, zero or negative. This number is loosely connected with the sign of curvature in Riemannian geometry: in the Kähler case it is 2π times the integral of the scalar curvature. Now we have

THEOREM 5.1. *If (M, ω) is a compact symplectic 4-manifold and $c_1.[\omega] > 0$ then M is diffeomorphic to a blow-up of M_0, where M_0 is either the complex projective plane $\mathbf{CP}^2$ of an S^2 bundle over a surface.*

This result is due to Li and Liu [**32**], building on earlier work of Gromov, Taubes, McDuff and others. A more precise version of the theorem gives a complete description of the possible symplectic forms, up to symplectomorphism. For simplicity let us consider the case when we know that M is homotopy equivalent to $\mathbf{CP}^2$, which was considered by Gromov in his renowned paper [**20**]. We have a generator h of $H_2(M, \mathbf{Z})$ which we choose so that $h.\omega > 0$. Standard algebraic topology shows that in this situation $c_1.h = \pm 3$ and the positivity hypothesis means that $c_1.h = 3$. Now Gromov chooses an almost-complex structure compatible with ω. Suppose we know that h is represented by an embedded holomorphic curve Σ. Then standard algebraic topology (the adjunction formula) shows that Σ is a 2-sphere. Fix a point p on Σ and consider the deformations of Σ among holomorphic curves passing through p. The fact that the self-intersection number of Σ is 1 shows that there is a 2 (real) parameter family of such curves, and further that there is a unique curve with prescribed complex tangent line at p. The latter deduction uses Gromov's compactness theorem for holomorphic curves: since Σ represents a generator of homology there is no way that curves in the family can break up into unions of curves, or develop singularities. In a similar way, it follows that for each point $q \neq p$ there is a *unique* holomorphic curve Σ_q in the family passing through q. So we get a map $\pi : \mathbf{CP}^2 \setminus \{p\} \to S^2$ which takes a point q to the tangent line at p of Σ_q. Of course in the model case where we have the standard complex structure on $\mathbf{CP}^2$ the curves are just the projective lines through the fixed point p. Now given the map π, which has a standard model in a punctured neighbourhod of p, it is easy to deduce that M is diffeomorphic to $\mathbf{CP}^2$, and in fact that the symplectic form is equivalent to a multiple of the standard one.

The general case is handled in a similar fashion, assuming that one can find a holomorphic sphere in M with non-negative self-intersection. Thus the essential problem is how to find such holomorphic spheres. This is where Taubes' relation [**43**] between the Seiberg–Witten equations and holomorphic curves enters in a crucial way. One of Taubes' main results is that for a compact symplectic 4-manifold (N, ω) with $b^+(N) > 1$ the Poincaré dual of $-c_1(N)$ is represented by a holomorphic curve. This is obtained by deforming the Seiberg–Witten equations in a 1-parameter family, from an equation which has a trivial solution to an equation whose solution is localised in a certain sense around a holomorphic curve. This result does not immediately apply to the case at hand—for example a homotopy $\mathbf{CP}^2$ has $b^+ = 1$—so a more sophisticated version is needed, involving "wall-crossing" formulae. But the upshot is that Taubes' theory provides the holomorphic spheres required to "sweep out" the 4-manifold. (There is a slightly different approach, using less analysis, to some of Taubes' results in [**12**])

The general scheme of the proof of Theorem 5.1 can be summarised by:

Symplectic hypothesis	$\Rightarrow$	Seiberg–Witten information	$\Rightarrow$	holomorphic curves	$\Rightarrow$	Classification (ruling)

One can perhaps think (loosely) of the construction of the diffeomorphisms—sweeping out the manifold by a family of 2-spheres through a point—as a mixture of the two constructions discussed in Section 2: the Riemannian geometry construction, sweeping out a manifold by geodesics through a point, and the Riemann mapping construction (since the holomorphic curve condition is a version of the Cauchy-Riemann equations).

In the case of Kähler surfaces the the division into three cases $c_1.\omega = +, 0, -$ is closely related to the division by Kodaira dimension, with $c_1.\omega > 0$ corresponding roughly to Kodaira dimension $-\infty$: *i.e.,* rational and ruled surfaces. Many of the arguments above can be viewed as an extension to the symplectic case of the algebro-geometric classification theory for surfaces of Kodaira dimension $-\infty$. Of course one would like to go further. An interesting question is to ask what are the simply-connected compact symplectic 4-manifolds M with $c_1(M) = 0$. The natural conjecture is that they should all be equivalent to $K3$ surfaces (with standard symplectic forms). Morgan and Szabo showed that M must be homeomorphic to a $K3$ surface [**35**], and it is known that it must have the same Seiberg–Witten invariants but it seems hard to go further.

An issue to ponder is: what are the good questions in 4-manifold theory? It is well-known that it is not reasonable to ask for a complete classification of compact 4-manifold, because that would involve a complete classification of finitely presented groups (appearing as the fundamental groups). The abundance of examples suggests that it may not be reasonable to try to classify all simply connected smooth 4-manifolds. The classification of symplectic 4-manifolds with $c_1.\omega > 0$ is an example of a "good question", with a complete solution. The classification of symplectic 4-manifolds with $c_1 = 0$ may perhaps be a "good question" in this sense, even though the problem seems out of reach at the moment.

5.3. Three and Four dimensions. There has been some decisive progress in the last 5 years on the borderline between 3 and 4 dimensional topology. A central result is

THEOREM 5.2. *Let N be a closed oriented 3-manifold. The product $N \times S^1$ admits a symplectic structure if and only if N fibres over the circle.*

In one direction—the construction of a symplectic form given a fibration—the argument is elementary and goes back to a note of Thurston [**44**]. The hard part is the converse and different, but related, proofs have been found by different groups of workers. The first proof was given by Friedl and Vidussi [**18**] and uses a considerable amount of 3-manifold topology. Let $f : N \to S^1$ be a smooth map and $\Sigma = f^{-1}(\theta) \subset N$ for some generic θ, so Σ is a smooth surface. Let N_0 be the complement of a tubular neighbourhood of Σ: this is a 3-manifold with two boundary components $i_+(\Sigma), i_-(\Sigma)$. A 1961 theorem of Stallings gives a criterion in terms of fundamental groups under which f can be deformed to a fibration: under suitable hypotheses on Σ we need $i_\pm$ to induce isomorphisms from $\pi_1(\Sigma)$ to $\pi_1(N_0)$. Roughly speaking, the argument of Friedl and Vidussi is to obtain this kind of *homotopy* information from *homology* data which in turn is supplied, in the case when $N \times S^1$ has a symplectic structure, from Seiberg–Witten theory. For each homotopy class of maps from N to S^1 there is a classical invariant, the Alexander polynomial Δ. Friedl and Vidussi consider a generalisation Δ^α of this depending on a homomorphism α from $\pi_1(N)$ to a finite group. The central result in their argument is that the homotopy class is represented by a fibration if and only if for every α the degree of Δ^α is given by a certain explicit formula. When $N \times S^1$ is symplectic this degree formula follows from a Seiberg–Witten argument of Kronheimer, depending on Taubes' work. The other half of the proof, showing that this degree formula is a sufficient condition for the existence of a fibration is pure 3-manifold theory. The strategy is to reduce to the case when $\pi_1(N)$ is "residually finite soluble"; a strategy related to a result of Thurston that the fundamental group of any 3-manifold is residually finite. At a crucial point the argument uses deep and recent results of Agol [**1**] in 3-manifold theory. An important part is played by the "Thurston norm" on the homology of a 3-manifold, which is well-known to be related to the Alexander polynomial.

Another proof was given soon after by a combination of work of Kutluhan and Taubes, Kronheimer and Mrowka and Ni. This goes back to fundamental advances of Ghiggini[**19**], Ni [**37**] and Juhasz [**23**] who gave necessary and sufficient conditions for the complement of a knot in S^3 to be fibred in terms of Ozsvath and Szabo's "knot Floer homology". In [**28**], Kronheimer and Mrowka prove an analogous result for general fibred 3-manifolds in terms of a version of the Seiberg–Witten Floer theory attached to the manifold. A crucial ingredient in these arguments is Gabai's theory of sutured manifolds and taught foliations. Comparing with the first proof; the Alexander polynomial is related to the simplest Seiberg–Witten invariant of a 3-manifolds which is, roughly speaking, the Euler characterstic of the more refined Seiberg–Witten Floer theory. Thus the second proof builds in more refined Seiberg–Witten information, but the problem is to show that if $N \times S^1$ admits a symplectic structure then the Seiberg–Witten Floer homology of N does have the appropriate structure. This was accomplished by Kutluhan and Taubes in [**30**]. The techniques are related to those developed by Taubes in his proof of the Weinstein conjecture for 3-manifolds.

The general line of these arguments can be indicated by

Symplectic hypothesis	$\Rightarrow$	Seiberg–Witten information	$\Rightarrow$	Alexander polynomial Thurston norm sutured/foliated structures	$\Rightarrow$	Construction of fibration

An open problem in a similar vein is whether the Fintushel-Stern homotopy K3 surface M_K, obtained from a knot K, is symplectic if and only if the knot K is fibred.

5.4. Khovanov homology and instanton Floer theory. In recent work Kronheimer and Mrowka [**29**] prove that "Khovanov homology detects the unknot"– that is a knot $K \subset S^3$ is trivial if and only if it has the same Khovanov homology as the trivial knot. We recall the bare outline of the definition of Khovanov homology. This starts from a generic plane projection of a knot K with l crossings. At each crossing we can consider two local modifications, patterned on the hyperbolae $xy = \pm\epsilon$ for small ϵ. Such a modification will in general define a link. So we get 2^l links, indexed by the subsets of $\{1, \dots, l\}$ (after fixing some choices). If π is such a subset the corresponding link K_π is a trivial link with some number $N(\pi)$ of components. Now Khovanov defines a vector space V_π (over some fixed field) with one basis element for each component and sets

$$C_* = \bigoplus_\pi \Lambda^* V_\pi.$$

There is an elementary, combinatorial, way to define a differential $\partial : C_* \to C_*$ making (C_*, ∂) a chain complex and the Khovanov homology is defined to be its homology. In fact C_* has a bi-grading, so the Khovanov homology groups are bi-graded. Of course the striking thing is that this homology is actually independent of the plane projection but, given a choice of projection, the definition is in principle completely straightforward–for example to implement on a computer (although in practice the calculations soon become very large, as in [**17**]). So a consequence of these results is a linear algebra algorithms for detecting, from a knot projection, whether a knot is trivial. Another, earlier, route to the same end is provided by knot Floer homology and the result of Oszvath and Szabo [**38**] showing that knot Floer homology detects the unknot, together with work of Manoescu, Oszvath and Szabo [**34**] establishing that knot Floer homology can be computed algorithmically.

Parts of the Kronheimer and Mrowka argument follow similar arguments of Ozsvath and Szabo [**39**] for the Heegard Floer homology of a branched cover. In these arguments the Khovanov homology appears as a kind of universal construction for theories which obey a "skein relation". Suppose we have some theory which assigns to each link L in the 3-sphere a graded vector space $I_*(L)$. Represent the link by a plane projection and make two other links L_0, L_1 by changing a crossing as above. Then by a skein relation we mean an exact triangle

$$\dots \to I_*(L_0) \to I_*(L) \to I_*(L_1) \to I_*(L_0) \to \dots,$$

(where we ignore grading shift) with suitable naturality properties. To give an indication of the main idea, imagine (very imprecisely) that the knowledge of two terms in an exact triangle determines the third. Then for any knot K we could successively undo all the crossing and conclude that $I_*(K)$ is determined by the $I_*(K_\pi)$ for the 2^l choices of π. Suppose further that the theory obeys a "Künneth formula"; that if L, L' are widely-separated links then

$$I_*(L \cup L') = I_*(L) \otimes I_*(L').$$

Each K_π is a trivial link, determined solely by the number $N(\pi)$ of its components and repeated application of the Künneth formula shows that $I_*(K_\pi)$ is the tensor product of $N(\pi)$ copies of $I_*(U)$ where U is the trivial knot. Suppose finally that $I_*(U)$ is 2-dimensional with a preferred basis $(1, e)$. Then $I_*(K_\pi)$ can be identified with V_π: that is to say it has a basis made up of expressions $e_{i_1} e_{i_2} \dots e_{i_p}$ where $i_1, \dots i_p$ label components of K_π. We conclude that under these hypotheses $\bigoplus_\pi I_*(K_\pi)$ can be identified with the Kohovanov chain group C_*. Of course in reality two terms of an exact triangle do not determine the third: the homomomorphisms of the triangle contain extra information and keeping track of all this data one expects that the true statement is that *there is a spectral sequence with E_2 term the Khovanov homology of K and converging to $I_*(K)$*. A more familiar analogy is the Atiyah-Hirzebruch spectral sequence for generalised cohomology theories. If $\mathcal{K}^*$ is such a theory and X_ν are the skeleta of a CW-complex X then we have exact triangles

$$\dots \to \mathcal{K}^*(X_\nu) \to \mathcal{K}^*(X_{\nu-1}) \to \mathcal{K}^*(X_\nu, X_{\nu-1}) \dots .$$

Our first, very imprecise, approximation would say that $\mathcal{K}^*(X_\nu)$ is determined by $\mathcal{K}^*(X_{\nu-1})$ and the relative cohomology $\mathcal{K}^*(X_\nu, X_{\nu-1})$. By excision and suspension the latter is the cellular cochain group for ordinary cohomology with co-efficients $\mathcal{K}^*(S^0)$ and of course the precise statement is that there is a spectral sequence with E_2 term $H^*(X, \mathcal{K}^*(S^0))$ converging to $\mathcal{K}^*(X)$. Set in this analogy, the Khovanov homology for link invariants plays the role of ordinary cohomology of spaces.

The theory $I_*(L)$ to which Kronheimer and Mrowka apply these ideas is a variant of the instanton Floer homology for 3-manifolds utilising connections with singularities along the link. This is closely related to a theory considered by Floer in work—never properly written up—from around about 1990, which one can now see was far ahead of its time [**16**]. Many of these ideas: the skein relation, the Künneth formula and the "resolution" of a knot by undoing all crossings can be found in Floer's work. The ideas are also related to the work of Kronhimer and Mrowka in proving that all knots have "Property P" [**27**]. In the case at hand, their strategy is to show that there is a spectral sequence from the Khovanov homology to their instanton theory $I_*(K)$ and then to use independent arguments to show that for a non-trivial knot K, $I_*(K)$ is in a certain sense non-trivial.

5.5. The Volume conjecture and complexification. In 2001 H. and J. Murakami [**36**] (building on work of Kashaev [**24**]) proposed a "volume conjecture" which connects the theory of the Jones polynomial with hyperbolic structures and Thurston's Geometrisation programme. In its simplest form the conjecture considers a knot $K \subset S^3$ whose complement admits a complete, finite-volume, hyperbolic structure and predicts that as $N \to \infty$

$$\left| \frac{J_N(K, e^{2\pi i/N})}{J_N(K_0, e^{2\pi i/N})} \right| \sim \exp\left(\frac{N}{2\pi} \mathrm{Vol}(S^3 \setminus K) \right).$$

Here J_N are versions of the Jones polynomial and K_0 is the trivial knot. In fact the numerator and denominator on the left hand side both vanish so the expression is understood by taking a limit of $J_N(\ , q)$ as q tends to $e^{2\pi i/N}$. This equation has been verified in many cases. In Witten's Quantum Field Theory interpretation of the Jones theory

$$J_N(K, e^{2\pi i/k}) = \int e^{ikCS(A)} \mathrm{Tr}_V \mathrm{Hol}(K, A)\, \mathcal{D}A.$$

This is a functional integral over the space of connections A on an $SU(2)$ bundle over S^3. The quantity $CS(A)$ is the Chern-Simons invariant of a connection A; $\mathrm{Hol}(K, A)$ denotes the holonomy of a connection around K and Tr_V denotes the trace in the N-dimensional irreducible representation V of $SU(2)$. Inserting the holonomy term has the effect that we are really considering connections with a singularity along K, just as in the work of Kronheimer and Mrowka in the previous subsection. Of course the Jones polynomial has alternative rigorous mathematical definitions, so the volume conjecture is a precise mathematical statement. However, as explained by Gukov [**21**] the Quantum Field Theory interpretation gives a lot of insight into why such a formula should hold. In Witten's original paper [**46**] he explained why certain asymptotics as $k \to \infty$ of these invariants could be derived in terms of flat $SU(2)$ connections, applying the principle of stationary phase to the oscillatory integral. The case at hand is different but the hyperbolic structure can be regarded as a flat $SL(2, \mathbf{C})$ connection over the knot complement and the hyperbolic volume naturally appears as 2π times the imaginary part of the Chern-Simons invariant of this connection. Thus, when the theory is complexified to consider the space of all $SL(2, \mathbf{C})$ connections, the conjecture appears as a relation between the asymptotics of integrals over the real locus and in a neighbourhood of a particular complex critical point. Witten has recently given a very comprehensive discussion of the issues which arise [**47**].

The complexification appearing here, replacing the compact gauge group $SU(2)$ by its complexification $SL(2, \mathbf{C})$ and studying Chern-Simons Theory on a fixed 3-manifold, is related to another kind of complexification in which the 3-manifold is replaced by a Calabi-Yau 3-fold. An example of the latter arises in Thomas' theory of "holomorphic Casson invariants", which "count" holomorphic bundles over Calabi-Yau 3-folds [**43**]. This has been a very active topic in string theory, and it is quite possible that relations with these ideas around the volume conjecture will emerge in the future.

References

1. I. Agol *Criteria for virtual fibering* Jour. Topology 1 (2008) 269-284 MR2399130 (2009b:57033)
2. A. Akhmedov and B. D. Park *Exotic smooth structures on small 4-manifolds with odd signatures* Inventiones Math. 181 (2010) 577-603 MR2660453 (2012b:57048)
3. M. F. Atiyah *Topological Field Theories* I.H.E.S Sci. Publ. Math. 68 (1988) 175-186 MR1001453 (90e:57059)
4. M. F. Atiyah, N. J. Hitchin and I. M. Singer *Self-duality in four-dimensional Riemannian geometry* Proc. Roy. Soc. London A 362 (1978) 425-461 MR506229 (80d:53023)
5. D. Auroux *The canonical pencils on Horikawa surfaces* Geometry and Topology 10 (2006) 2173-2217 MR2284054 (2007m:14065)
6. S. Bauer and M. Furuta *A stable cohomotopy refinement of Seiberg–Witten invariants, I* Inventiones Math. 155 (2004) No.1. 1-19 MR2025298 (2005c:57040)
7. F. Catanese and B. Wajnryb *Diffeomorphisms of simply connected algebraic surfaces* Jour. Diff. Geometry 76 (2007) No. 2 177-213 MR2330412 (2008c:14057)

8. F. Catanese , M. Lönne and B. Wajnryb *Moduli spaces of surfaces and monodromy invariants* Proc. Gokova Geometry and Topology Conf. 2009 58-98 Int. Press 2010 MR2655304 (2011g:14085)
9. X-X Chen, C. LeBrun and B. Weber *On conformally Kähler Einstein manifolds* Jour. Amer. Math. Soc. 21 (2008) No. 4 1137-1168 MR2425183 (2010h:53054)
10. S. Donaldson *Irrationality and the h-cobordism conjecture* Jour. Diff. Geom. 26 (1987) No. 1 141-168 MR892034 (88j:57035)
11. S. Donaldson *Two-forms on four-manifolds and elliptic equations* In: inspired by S-S Chern Nankai Tracts in Mathemetics 11 153-172 World Scientific 2006 MR2313334 (2008c:53090)
12. S. Donaldson and I. Smith *Lefschetz pencils and the canonical class for symplectic four-manifolds* Topology 42 (2003) 743-785 MR1958528 (2004c:53142)
13. R. Fintushel and R. Stern *Rational blow-downs of smooth 4-manifolds* Jour. Diff. Geom. 46 (1997) 181-235 MR1484044 (98j:57047)
14. R. Fintushel and R. Stern *Knots, links and 4-manifolds* Inventiones Math. (1998) No. 2 363-400 MR1650308 (99j:57033)
15. R. Fintushel and R. Stern *Pinwheels and nullhomologous surgery on 4-manifolds with* $b^+ = 1$ Algebraic and Geometric Topology 11 (2011) 1649-1699 MR2821436 (2012i:57043)
16. A. Floer *Instanton homology, surgery and knots* In: Geometry of low dimensional manifolds Vol 1, Cambridge UP 1990 97-114 MR1171893 (93h:57049)
17. M. Freedman, R. Gompf, S. Morrison and K. Walker *Man and machine thinking about the smooth four-dimensional Poincaré conjecture* Quantum Topology 1 (2010) No. 2 171-208 MR2657647 (2011f:57061)
18. S. Friedl and S. Vidussi *Twisted Alexander polynomials and fibred 3-manifolds* arxiv 1001.0132 MR2768657 (2012j:57028)
19. P. Ghiggini *Knot Floer homology detects genus-one fibred knots* Amer. J. Math. 130 (2008) No. 5 1151-1169 MR2450204 (2010f:57013)
20. M. Gromov *Pseudoholomorphic curves in symplectic manifolds* Inventiones Math. 82 (1985) No. 2 307-347 MR809718 (87j:53053)
21. S. Gukov *Three-dimensional quantum gravity, Chern-Simons Theory and the A-polynomial* Commun. Math. Phys. 255 (2005) 577-627 MR2134725 (2006f:58029)
22. R. Hamilton *The formation of singularities in the Ricci flow* Surveys in Differential Geometry II International Press 1995 7-136 MR1375255 (97e:53075)
23. A. Juhasz *Floer homology and surface decompositions* Geometry and Topology 12 No.1 (2008) 299-350 MR2390347 (2009a:57021)
24. R. Kashaev *The hyperbolic volume of knots from quantum dilogarithms* Letters in Math. Phys. 39 1997 No. 3 269-275 MR1434238 (98b:57012)
25. M. Khovanov *A categorification of the Jones polynomial* Duke Math. J. 101 (2000) no. 3 359-426 MR1740682 (2002j:57025)
26. D. Kotschick *On manifolds homeomorphic to* $\mathbf{CP}^2\sharp 8\overline{\mathbf{CP}^2}$ Inventiones Math. 95 (1989) No. 3 591-600 MR979367 (90a:57047)
27. P. B. Kronheimer and T. S. Mrowka *Witten's conjecture and Property P* Geometry and Topology 8 (2004) 295-310 MR2023280 (2004m:57023)
28. P. B. Kronheimer and T. S. Mrowka *Knots, sutures and excision* Jour. Differential Geometry 84 (2010) 301-364 MR2652464 (2012m:57059)
29. P. B. Kronheimer and T. S. Mrowka *Khovanov homology is an unknot detector* Pub. Math. Inst. Hautes Etudes Sc. 113 (2011) 97-208 MR2805599
30. C. Kutluhan and C. H. Taubes *Seiberg–Witten Floer homology and symplectic forms on* $S^1 \times M^3$ Geometry and Topology 13 2009 493-525 MR2469523 (2010h:57044)
31. Y. Lee and J. Park *A simply connected surface of general type with* $p_g = 0, K^2 = 2$ Inventiones Math. 170 (2007) No. 3 483-505 MR2357500 (2008m:14076)
32. T J Li and A Liu *General wall-crossing formula* Math. Res. Letters 2 1995 797-810 MR1362971 (96m:57053)
33. J. Lurie *On the classification of topological field theories* Current developments in mathematics 2008 128-280 Int. Press. 2009 MR2555928 (2010k:57064)
34. C. Manolescu, P. Oszvath, S. Sarkar *A combinatorial description of knot Floer homology* Annals of Math. Ser. 2 169 (2008) 633-660 MR2480614 (2009k:57047)
35. J. Morgan and Z. Szabo *Homotopy K3 surfaces and mod 2 Seiberg–Witten invariants* math. Res. Letters 4 (1997) No. 1 17-21 MR1432806 (97k:57041)

36. H. Murakami and J. Murakami *The coloured Jones polynomial and the simplicial volume of a knot* Acta Math. 186 (2001) No. 1 85-104 MR1828373 (2002b:57005)
37. Y. Ni *Knot Floer homology detects fibred knots* Inventiones Math. 170 (2007) No. 3 577-608 MR2357503 (2008j:57053)
38. P. Oszvath and Z. Szabo *Holomorphic disks and genus bounds* Geometry and Topology 8 (2004) 311-344 MR2023281 (2004m:57024)
39. P. Ozsvath and Z. Szabo *On the Heegard Floer homology of branched double covers* Advances in Math. 194 (2005) 1-33 MR2141852 (2006e:57041)
40. J. Park *Simply connected symplectic 4-manifolds with* $b_2^+ = 1, c_1^2 = 2$ Inventiones Math. 159 (2005) 657-667 MR2125736 (2006c:57024)
41. P. Seidel and I. Smith *A link invariant from the symplectic geometry of the nilpotent slice* Duke Math. Jour. 134 (2006) No. 3 453-514 MR2254624 (2007f:53118)
42. C. H. Taubes *SW ⇒ Gr: from the Seiberg–Witten equations to pseudoholomorphic curves* Jour. Amer. Math. Soc. 9 (1996) 845-918 MR1362874 (97a:57033)
43. R. Thomas *A holomorphic Casson invariant for Calabi-Yau 3-folds and bundles on K3 fibrations* Jour. Diff. Geom. 54 (2000) No. 2 367-438 MR1818182 (2002b:14049)
44. W. Thurston *Some simple examples of symplectic manifolds* Proc. Amer. That. Soc. 55 (1976) No. 2 467-468 MR0402764 (53:6578)
45. C. Wall *Geometric structures on compact complex analytic surfaces* Topology 25 (1986) No. 2 119-153 MR837617 (88d:32038)
46. E. Witten *Quantum Field Theory and the Jones polynomial* Commun Math Phys. 121 (1989) 351-399 MR990772 (90h:57009)
47. E. Witten *Analytic continuation of Chern-Simons theory* In: Chern-Simons Gauge theories: 20 years after 347-446 AMS/IP Studies Adv. Math. 50 Amer. Math Soc 2011 MR2809462 (2012i:81252)
48. E. Witten *Fivebranes and knots* Quantum Topology 3 (2012) 1-137 MR2852941

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Volumes of Hyperbolic 3-Manifolds

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Abstract. We discuss the recent proof that the Weeks manifold is the unique lowest volume closed hyperbolic 3-manifold; in particular the role of Perelman's work (after Agol, Dunfield, Storm & Thurston) in the argument.

Introduction

This paper is based on the first author's lecture at the 2010 Clay Research Conference held in Paris. That conference celebrated the proof by Grisha Perelman of both the Poincaré conjecture and Thurston's Geometrization conjecture. Not coincidentally, Poincaré, Thurston and Perelman play major roles in the story this paper tells. Poincaré made major contributions to the foundations of hyperbolic geometry, topology and 3-manifold theory; e.g., he introduced the fundamental group and showed that $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{Isom}(\mathbb{H}^3)$. A byproduct of Thurston's work on geometrization was his seminal work on volumes that made possible the problem addressed in this paper of finding the lowest volume closed hyperbolic 3-manifold. Perelman's work on Ricci flow (which forms the core of his proof of geometrization) plays a crucial role in the work of Agol–Dunfield that in turn is needed in the resolution of the low-volume problem.

Background on cusped hyperbolic surfaces and 3-manifolds is given in sections 1–2. Section 3 recalls Mostow's rigidity theorem and Thurston's fundamental result on volumes. Section 4 states several of the natural problems arising from Thurston's theorem, most of which are still open, and the hyperbolic complexity conjecture of Thurston, Hodgson–Weeks, Matveev–Fomenko. Section 5 states our main result and some of its history. Section 6 states the $\log(3)/2$ theorem of [**GMT**] that plays a crucial role in the proof. Section 7 explains Perelman's role, after Agol, Dunfield, Storm and Thurston [**ADST**]. Sections 8–9 give some of the intuition behind the main result and some of the formalism needed to make it a proof.

The resolution of the main result was the culmination of a long line of research spanning 30 years by many authors, much of which is not discussed in this paper. For a much more detailed outline of the proof, the reader should consult the expository paper *Mom technology and hyperbolic 3-manifolds* [**GMM3**]. That paper also

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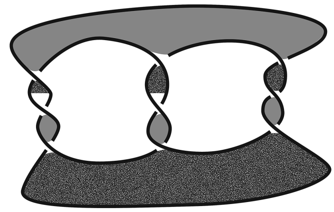

FIGURE 1. What surface is this?

provides more information about partial results on the problems stated in section 4, a long list of other problems, and directions for future research.

Unless otherwise stated all manifolds in this paper are orientable and connected.

1. Hyperbolic surfaces

The well-known classification of surfaces asserts that any closed surface is homeomorphic to a surface of genus ≥ 0. If S is a closed surface of genus g, then $\chi(S) = 2 - 2g$, thus the topology of a closed surface is determined by its Euler characteristic. For compact surfaces we have the following classification theorem.

THEOREM 1.1. *Let S be a compact connected surface, then S is homeomorphic to $S_{g,p}$, the surface of genus g with p open discs [with disjoint closures] removed. Furthermore two compact connected surfaces S and T are homeomorphic if and only if*
1) $\chi(S) = \chi(T)$ and
2) $|\partial S| = |\partial T|$, where $|X|$ denotes the number of components of X.

Thus the topology of a compact surface is determined by two easily computed invariants. Part of the beauty of topology lies in the fact that homeomorphic spaces are not always obviously homeomorphic. See Figure 1.

The interior of a compact surface S supports a complete finite-volume *hyperbolic* metric, i.e. a metric of constant -1 curvature, if and only if $\chi(S) < 0$. The remarkable Gauss–Bonnet theorem asserts that for complete finite-volume hyperbolic surfaces: Euler characteristic, a combinatorial invariant, is a linear function of area, a geometric invariant.

THEOREM 1.2 (Gauss–Bonnet). *If S is a complete finite-volume hyperbolic surface, then* $\text{area}(S) = -2\pi\chi(S)$.

This immediately follows from the general Gauss–Bonnet theorem

$$\int_S K dA = 2\pi\chi(S)$$

after taking $K = -1$.

Thus area is an excellent, though not complete, measure of topological complexity of a hyperbolic surface. At most finitely many such surfaces have the same area.

Figure 2 shows how to explicitly construct a finite-volume hyperbolic metric on the punctured torus S. Start with the regular ideal 4-gon in the hyperbolic plane and then glue opposite edges by an isometry. For each pair of edges there is an $\mathbb{R}$-parameter of choices that produces an orientable surface. Choose the unique

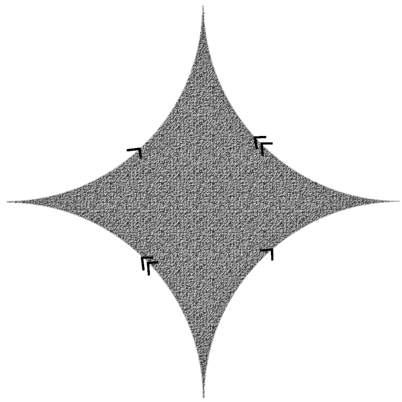

FIGURE 2. Symmetrically identify opposite edges to obtain the cusped punctured torus S

gluing that takes the pair of nearest points to each other. The resulting complete finite-volume hyperbolic surface will have a *cusp*, that is, a subset homeomorphic to $S^1 \times [1, \infty)$ and isometric to $\mathbb{R} \times [1, \infty)/f$, where $\mathbb{R} \times [1, \infty) \subset \mathbb{H}^2$, $\mathbb{H}^2$ denotes the upper-half-space model of hyperbolic 2-space, and $f(x, y) = (x + d, y)$ for some $d > 0$.

We can choose the cusp to be *maximal*, i.e., so that its interior is embedded and its boundary is tangent to itself. The preimage of a cusp in $\mathbb{H}^2$ is a union of *horoballs*. Figure 3 shows some of the preimage horoballs of the maximal cusp of S, viewed in the upper-half-space model. It also shows 4 fundamental domains and all the horoballs that intersect them. A calculation shows that a fundamental domain for the maximal cusp is $[0, 4\sqrt{2}] \times [1, \infty)$ and that area(maximal cusp) $=$ $4\sqrt{2}$. Also Gauss–Bonnet implies that area$(S) = 2\pi$. It follows that the ratio area(maximal cusp)/area$(S) = 0.90...$, which is at first glance a strikingly large ratio. By Boroczky [**B**], an optimal 2-dimensional horosphere packing has density $3/\pi$.

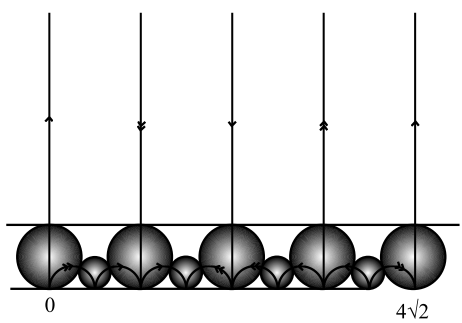

FIGURE 3. A horoball diagram for T showing horoballs at hyperbolic distance $\log(2)$ from the horoball at infinity. Also shown are four fundamental domains for T and one fundamental domain for the cusp.

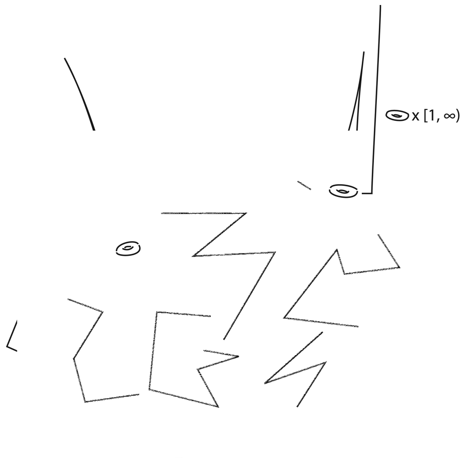

FIGURE 4. Schematic view of a 2-cusped hyperbolic 3-manifold

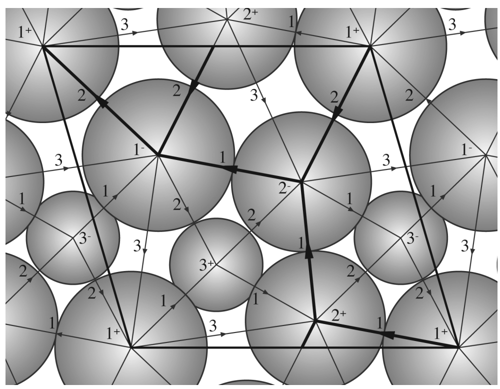

FIGURE 5. m011 horoball diagram showing horoballs at hyperbolic distance 0.50 from the horoball at infinity and an internal Mom-2 structure. Vol(m011) = 2.718... Vol(maximal cusp) = 2.134...

2. Hyperbolic 3-manifolds

A schematic picture of a 2-cusped hyperbolic 3-manifold appears in figure 4. It emphasizes that each cusp of a complete finite-volume hyperbolic 3-manifold is topologically a torus$\times[1,\infty)$. Analogous to the situation in dimension two, the preimage of a cusp is a 3-dimensional horoball. If a manifold has a unique cusp, then it can be expanded to a maximal one. Figure 5 shows a *horoball diagram* for the maximal cusp of the 1-cusped hyperbolic 3-manifold m011(in Weeks' SnapPea notation [**We**]) in the upper-half-space model of $\mathbb{H}^3$. The diagram shows the projection of various horoballs to the $(x, y)-$plane. The *horoball at infinity* is not shown in this figure. It consists of the horoball $\mathbb{R}\times\mathbb{R}\times[1,\infty)$. The largest horoballs have Euclidean diameter 1 and are tangent to the horoball at infinity. Note that $\pi_1(\text{cusp}) = \mathbb{Z}\oplus\mathbb{Z}$ and acts on the upper-half-space by Euclidean translations. The parallelogram is a fundamental domain for this action restricted to the (x,y)-plane.

Note that the cusp takes up over 75% of the volume of the manifold. The various edges, thick edges and numbers have important significance and will be explained in future sections.

3. Foundational results on volumes

The following theorem was first proved by Mostow [**Mo**] in 1968 for closed manifolds and extended independently by Marden [**Ma**] ($n = 3$) and Prasad [**Pra**] ($n \geq 3$) for complete finite-volume manifolds in 1972.

THEOREM 3.1 (Mostow Rigidity). *If ρ_0, ρ_1 are complete finite-volume hyperbolic metrics on the n-manifold N, $n \geq 3$, then there exists an isometry $f : N_{\rho_0} \to N_{\rho_1}$ such that f is homotopic to the identity. Here N_ρ denotes N with the ρ metric.*

The following is an immediate consequence of the Mostow Rigidity and Gauss–Bonnet theorems.

COROLLARY 3.2. Volume is a topological invariant of complete finite-volume hyperbolic manifolds.

REMARKS 3.3. Mostow Rigidity does not assert that f is isotopic to the identity. That f is isotopic to the identity is classical for $n = 2$, proved by Gabai–Meyerhoff–N. Thurston for $n = 3$ [**GMT**] and false for $n > 10$ and N closed [**FJ**]. (Using Igusa's stability theorem, rather than a truncated version, that result can be improved, using the same proof, to $n > 8$ [**F**].) An assertion related to and stronger than the Mostow Rigidity for $n = 3$, the space of hyperbolic metrics is contractible [**G**].

The following seminal result of Thurston [**Th1**], generalizing work of Jorgensen and Gromov opened the door to many interesting volume problems.

THEOREM 3.4 (Thurston 1977). *Volumes of complete hyperbolic 3-manifolds are a well-ordered closed subset of $\mathbb{R}$ of order-type ω^ω. Only finitely many manifolds can have the same volume.*

Order type ω^ω means that there is a smallest volume, a next smallest volume, $\cdots$, then a first limit volume, then a next volume, then a next volume, $\cdots$, then a second limit volume, $\cdots$. Eventually, there is a first limit of limit volumes, then a next volume etc. This is schematically depicted in Figure 6.

FIGURE 6

In contrast to dimension-3, by Gauss–Bonnet ($n = 2$) and Wang (1972) [**Wa**] ($n \geq 4$), the volumes of hyperbolic n-manifolds form a discrete subset of $\mathbb{R}$.

4. Problems on volumes

The following is an immediate corollary of Theorem 3.4.

THEOREM 4.1 (Thurston). *Any set of hyperbolic 3-manifolds has a minimal volume element.*

This result immediately gave rise to a host of interesting problems, most of which are still open. See [**GMM3**] for more information about these and other such problems.

PROBLEM 4.2. What is/are the hyperbolic 3-manifold(s) of least volume?

ANSWER 4.3. The Weeks manifold is the unique closed hyperbolic 3-manifold of least volume [**GMM2**].

PROBLEM 4.4. What is/are the least volume n-cusped 3-manifold(s)?

ANSWER 4.5. When $n = 1$, Chris Cao and Rob Meyerhoff [**CM**] showed in 2001 that the figure-8 knot complement and its sister are the two least volume manifolds.

When $n = 2$, Ian Agol [**Ag2**] showed in 2010 that the Whitehead link and its sister are the two least volume manifolds.

The problem is open for $n \geq 3$.

PROBLEM 4.6. What is/are the least volume fibered hyperbolic 3-manifold(s)?

PROBLEM 4.7. What is/are the least volume Haken hyperbolic 3-manifolds(s)?

PROBLEM 4.8. What is/are the least volume non-orientable 3-manifold(s)?

The following related problem is a special case of a question of Siegel that predates Thurston.

PROBLEM 4.9. What is the least volume 3-orbifold?

ANSWER 4.10. This was solved by Gehring–Martin [**GM**] and Marshall–Martin [**MM**] in two papers culminating a long line of research.

During the period 1978-1987, starting with Thurston, various mathematicians coming from different points of view made conjectures relating volume and topological complexity. We packaged them together [**GMM1**] to offer the following open-ended conjecture.

CONJECTURE 4.11. (Hyperbolic Complexity Conjecture) (Thurston, Hodgson–Weeks, Matveev–Fomenko) Low volume hyperbolic 3-manifolds are obtained by filling low topological complexity cusped hyperbolic 3-manifolds.

REMARK 4.12. Part of the challenge is to find a good measure of topological complexity and find reasonable notions of low. We believe that the *Mom number* [**GMM1**] recalled here in Definition 8.1 offers a measure of topological complexity amenable to this conjecture.

5. The Quest for the lowest volume manifold

The Weeks manifold (see Figure 7), also known as the Matveev–Fomenko manifold, is the closed manifold obtained by (5/1, 5/2) surgery on the Whitehead link. It is known that the Weeks manifold is arithmetic [**MR**] and that vol(Weeks)=0.9427... . It was independently conjectured to be the smallest volume closed manifold around 1984 by Josef Przytcki and Jeffrey Weeks. The former through his study of punctured torus bundles and the latter through computer experimentation that evolved into his SnapPea program.

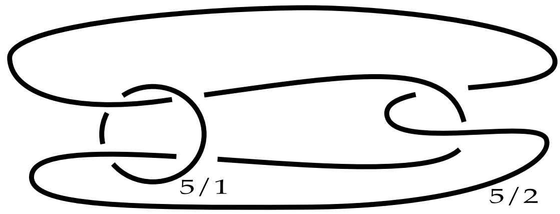

FIGURE 7. The Weeks Manifold. Volume(Weeks) = 0.9427....

THEOREM 5.1 (Gabai–Meyerhoff–Milley [**GMM2**]). *The Weeks manifold is the unique lowest volume closed hyperbolic 3-manifold.*

The quest for the lowest volume closed 3-manifold has a long history. Part of that history is captured in Table 1, which lists successive improvements in lower bounds for vol(smallest). Interestingly the proof of Theorem 5.1 used most of the techniques needed to establish the partial results of Table 1.

1979	Meyerhoff	0.0006
1986	Meyerhoff	0.0008
1991	Gehring–Martin	0.0010
1996	Gabai–Meyerhoff–N. Thurston	0.16668
1999	Przeworski	0.2766
7/2000	Przeworski	0.2814
10/2000	Marshall–Martin	0.2855
10/2000	Marshall–Martin + Przeworski	0.2903
2001	Agol	0.32
2002	Przeworski	0.33
2005	Agol–Dunfield	0.67
2007	Gabai–Meyerhoff–Milley	0.9427...

TABLE 1. Historic Lower Bounds for Vol(Smallest)

6. The log(3)/2 theorem

A crucial ingredient in the proof of Theorem 5.1 is the following.

THEOREM 6.1 (Gabai–R. Meyerhoff–N. Thurston[**GMT**]). *If γ is a shortest geodesic in the closed orientable hyperbolic 3-manifold N, then either*
1) TubeRadius(γ) $\geq$ log(3)/2 *or*
2) vol(N) $\geq$ 1.10....

REMARKS 6.2. The proof was done with rigorous computer assistance. It sufficed to analyze a compact 3-complex-dimensional *rectangle* in $\mathbb{C}^3$. (The proof of compactness used a lemma of Meyerhoff's [**Mey**] that enabled him to obtain the first explicit lower bound of 0.0006 for vol(smallest).) The rectangle was chopped up into about 500,000,000 subboxes. All but 7 of the boxes were eliminated by one of about 32,000 reasons. The seven boxes contained parameters for covering spaces of all the thin-tubed manifolds.

The argument relies on the fact that $\mathrm{Isom}(\mathbb{H}^3) = PSL(2,\mathbb{C})$ (Poincaré 1883 [**Po**]) and that the representation of $\pi_1(N) \to PSL(2,\mathbb{C})$ lifts to $SL(2,\mathbb{C})$ (Thurston). For this reason, the geometry of hyperbolic 3-manifolds is amenable to computer calculation.

A few years later Champanerkar, Lewis, Lipyanskiy and Meltzer [**CLLMR**] showed that each box contains a unique manifold and two of them are isometric. Thus any thin-tubed manifold is covered by one of six manifolds that are denoted $N_0, \cdots, N_5$.

Jones and Reid [**JR**] showed that N_0 (also known as vol3) nontrivially covers no manifold and Reid [**CLLMR**] extended this to N_1 and N_5. Very recently Maria Trnkova and the first author [**GT**] have shown that N_2 and N_4 each nontrivially 2-fold cover a manifold and the shortest geodesics of these quotient manifolds have $\log(3)/2$ tubes. Furthermore, N_2, N_3 and N_4 nontrivially cover no other manifolds and vol3 is the unique closed hyperbolic 3-manifold that does not have *some* geodesic with a $\log(3)/2$ tube.

7. The role of Perelman: After Agol, Dunfield, Storm & Thurston

Using Perelman's work on Ricci flow, Ian Agol and Nathan Dunfield proved the following result.

Theorem 7.1 (Agol–Dunfield). [**ADST**] *Let N be a closed hyperbolic 3-manifold, and γ a simple closed geodesic in N of length L and tube radius R. Let V denote a tube of radius R about γ. Let N_γ denote $N \setminus \gamma$ with a complete hyperbolic metric. Then*

$$\begin{aligned}\mathrm{vol}(N_\gamma) &\le (\coth(2R))^3\big(\mathrm{vol}(N) + (\pi/2)(L\tanh(R)\tanh(2R))\big)\\ &= \coth(2R)^3\big(\mathrm{vol}(N) + \mathrm{vol}(V)\mathrm{sech}(2R)\big)\end{aligned}$$

Corollary 7.2. [**ADST**], [**ACS**] If N is a minimal volume closed hyperbolic 3-manifold, then N is obtained by filling a 1-cusped hyperbolic 3-manifold of volume at most 2.848.

Proof. Let γ be a shortest geodesic in N. Let R denote its tube radius. As in [**ADST**] apply Theorem 6.1 to assume that $R \ge \log(3)/2$ and apply Andrew Przeworski's tube-packing estimate [**Prez**] to assume that $\mathrm{vol}(V) \le .91\,\mathrm{vol}(N)$. Since $\mathrm{vol(Weeks)} \le 0.9428$ it follows that

$$0.9428 > \mathrm{vol}(N) \ge \mathrm{vol}(N_\gamma)/\big(\coth(\log(3))^3(1 + .91\mathrm{sech}(\log(3))\big)$$

and hence $\mathrm{vol}(N_\gamma) < 2.848$. □

Idea of proof of Theorem 7.1. We outline the argument of [**ADST**]. Figure 8 shows how to construct a C^0 metric g on N_γ such that the scalar curvature of g is ≥ -6. The 3-manifold N_γ with the g metric has volume equal to the right-hand side of the first inequality of Theorem 7.1. The metric g can be approximated by a smooth metric also with scalar curvature ≥ -6, along the lines of Bray and Miao [**Br**], [**Mi**]. Apply Perelman's Ricci flow with surgery to this metric to obtain the hyperbolic metric on N_γ. Perelman's monotonicity formula implies that volume is monotonically decreasing under the flow, thus Theorem 7.1 follows.

Actually, Perelman's results require that N_γ be compact. However, by Thurston's filling theorem, N_γ is the Gromov–Hausdorff limit of the manifolds $\{N_\gamma(p_i, q_i)\}$ where $N_\gamma(p_i, q_i)$ is obtained by filling N_γ. One can put a metric g_i on $N_\gamma(p_i, q_i)$

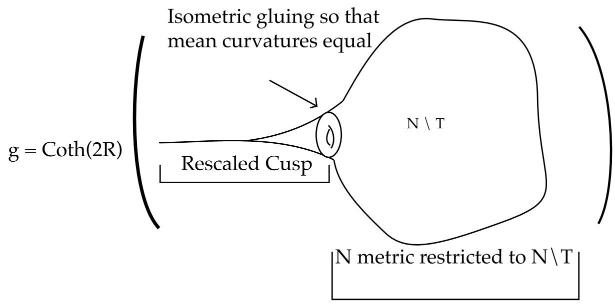

FIGURE 8

analogous to the metric g, thereby obtaining estimates for $\mathrm{vol}(N_\gamma(p_i, q_i))$ as in Theorem 7.1. Since $\lim_{i\to\infty} \mathrm{vol}(N_\gamma(p_i, q_i)) = \mathrm{vol}(N_\gamma)$ the result follows. □

Historical Note The above result is a generalization of an earlier work of Agol [**Ag1**], that obtained the lower bound of 0.32 for vol(smallest) in 2001. Instead of using Perelman, Agol invoked the main result of Boland–Connell–Souto [**BCS**] that in turn is the finite-volume version of a fundamental result of Besson–Courtois–Gallot [**BCG**]. Besson–Courtois–Gallot showed that the volume of a closed hyperbolic 3-manifold is minimized by the hyperbolic metric among all metrics satisfying a certain condition on Ricci curvature.

8. Mom technology I

The next two sections discuss our work towards finding the lowest-volume closed hyperbolic 3-manifold, see [**GMM1**], [**GMM2**], and [**Mill**].

The motivating idea to address Conjecture 4.11 and begin to address Problem 4.2 is as follows. Given a hyperbolic manifold N (cusped or not) of low volume, we expect to find an embedded compact submanifold $M \subset N$ of low topological complexity such that ∂M is a union of at least *two* tori and N is obtained by filling in some of the tori with solid tori and attaching cusps to the others. Such a manifold is called a *Mom* manifold to N and may arise as follows. (The reader might want to stare at Figure 5 before proceeding further.) Suppose that N has exactly one cusp. Let T denote the boundary of the maximal cusp. Push T slightly into the cusp so that it is embedded. Expand T to the inside of N. By Morse theory, for generic times t the expanded T will be a manifold M_t diffeomorphic to $T \times I$ with handles attached to the $T \times 1$ side. (As described by Morgan in his lecture, Poincaré understood, pre-Morse, the rudiments of this Morse theory. We were motivated by Smale's [**Sm**] spectacular use of this type of idea to prove the Poincaré conjecture in dimensions ≥ 5.) In our setting, since $\mathrm{vol}(N)$ is small, the pushed T must rapidly and repeatedly bump into itself so M_t should have several 1- and 2-handles for t small. Experiments with SnapPea suggest that by judiciously choosing a subset of the 1- and 2-handles we can find a submanifold M with an equal number of 1- and 2-handles. This suggests that ∂M is a union of tori that in turn cuts off cusps and solid tori to the outside. Such an M is our desired *Mom* manifold. Section 9 explains how to find the desired Mom submanifold in the manifold m011.

A major step in the proof of Theorem 5.1 is in carrying out a more sophisticated version of this procedure for 1-cusped hyperbolic manifolds of volume ≤ 2.848. We conjecture that a variant of this program can be carried out for closed hyperbolic 3-manifolds to give a much deeper understanding of low volume closed hyperbolic 3-manifolds as well as a Ricci flow free proof of Theorem 5.1. In the closed case we expect that T can be taken to be the boundary of a (nearly) maximal tube W about a shortest geodesic and we expect the Mom manifold to live in $N \setminus \text{int}(W)$.

Note that in the horoball diagram of m011 (Figure 5) the parallelogram is fairly small and the big horoballs (i.e., those at hyperbolic distance ≤ 0.5 from the one at infinity) are packed fairly tightly. Our intuition is that these two phenomena hold in general for 1-cusped hyperbolic 3-manifolds with $\text{vol}(N)$ small, though with a somewhat smaller value than 0.5. Indeed, failure of the first property implies that the volume of the cusp cut off by T is too large and failure of the second implies that too much volume lies outside (and sometimes inside) the cusp. This relies on the basic fact that the volume of a cusp V is equal to $\text{area}(\partial V)/2$. We also expect that since $\text{vol}(N)$ is small, most of the volume of N will lie in the maximal cusp. These three properties are key to finding the Mom manifold within N. Note that three closely packed horoballs will give rise, using the above construction, to a valence-3 2-handle. Valence-3 means that each 2-handle runs over the 1-handles exactly 3 times, counted with multiplicity.

We now formally define our measure of topological complexity. In §9 more formalism is introduced and we use it to explain how to find a Mom-2 manifold within m011.

Definition 8.1. Let M be a compact 3-manifold whose boundary is a union of at least two tori. The *Mom-complexity* of M is the least n such that M can be constructed by starting with $T \times I$, where T is a torus, then attaching n 1-handles and then n valence-3 2-handles to the $T \times 1$ side. A compact 3-manifold with boundary a union of at least two tori is a *Mom-n* manifold if it has Mom-complexity $\leq n$. A Mom-n manifold is said to be *hyperbolic* if its interior supports a complete hyperbolic structure.

The following is the main technical result of [**GMM2**]. It makes precise an earlier informal statement.

Theorem 8.2 (Gabai–Meyerhoff–Milley). *If N is a complete 1-cusped hyperbolic manifold and $\text{vol}(N) \leq 2.848$, then N is obtained by filling a hyperbolic Mom-3 manifold and adding one cusp.*

The following result, proven in [**GMM1**], classifies the hyperbolic Mom-3 manifolds.

Theorem 8.3 (Gabai–Meyerhoff–Milley). *There are exactly 3 hyperbolic Mom-2 manifolds. They are m125, m129 and m203. There are exactly 18 hyperbolic Mom-3 manifolds that are not Mom-2's. They are m202, m292, m295, m328, m329, m359, m366, m367, m391, m412, s596, s647, s774, s776, s780, s785, s898, and s959.*

In [**GMM1**] we conjectured that there are exactly 117 hyperbolic Mom-4 manifolds that are not Mom-$n, n \leq 3$ and we explicitly listed these manifolds.

Around 1978, in his seminal work on hyperbolic structures, Thurston showed how to explicitly analyze the hyperbolic fillings of the figure-8 knot complement. Using the more recent filling analysis of Futer–Kalfagianni–Purcell [**FKP**], the computer programs SnapPea [**We**] and Snap [**G**] and rigorization results of Harriet Moser [**Mos**], Peter Milley proved the following result.

THEOREM 8.4 (Milley [**Mill**]). *If N is a 1-cusped hyperbolic 3-manifold, $vol(N) \leq 2.848$ and N is obtained by filling and adding a cusp to a Mom-3 manifold, then N is one of m003, m004, m006, m007, m009, m010, m011, m015, m016, m017. If N is a closed hyperbolic 3-manifold, $vol(N) \leq 0.943$, and N is obtained by filling a Mom-3 manifold, then N is the Weeks manifold.*

Putting these results together we obtain.

Theorem 5.1 (Gabai–Meyerhoff–Milley [**GMM2**]) The Weeks manifold is the unique closed hyperbolic 3-manifold of least volume.

THEOREM 8.5 (Gabai–Meyerhoff–Milley [**GMM2**]). *The ten 1-cusped manifolds of volume at most 2.848 are exactly m003, m004, m006, m007, m009, m010, m011, m015, m016, m017.*

The two smallest 1-cusped manifolds, m003 and m004, were earlier determined by Cao and Meyerhoff [**CM**]. They introduced the *lessvol* method for finding a lower bound on volume outside the maximal cusp. Lessvol plays a crucial role in the proof of Theorem 8.2.

9. Mom technology II

This section gives a very brief introduction to some of the ideas that go into the proof of Theorem 8.2. A much more detailed outline of the proofs of that result and Theorem 5.1 can be found in [**GMM3**].

DEFINITION 9.1. Consider a maximal cusp V in the 1-cusped hyperbolic 3-manifold N. Its preimage in $\mathbb{H}^3$ is a collection of horoballs $\mathcal{H} \subset \mathbb{H}^3$. The action of $\pi_1(N)$ on $\mathbb{H}^3$ permutes the horoballs $\mathcal{H}$. This action induces an equivalence relation on the sets of unordered pairs and unordered triples of horoballs. Call an equivalence class of pairs an *orthoclass*. Order the orthoclasses $\theta(1), \theta(2), \cdots$ so that $i \leq j$ implies that $o(i) \leq o(j)$, where $o(k)$ denotes the distance between the two balls in any representative of $\theta(k)$. A minimal length geodesic arc connecting two horoballs is called an *orthocurve*.

An unordered triple (H_i, H_j, H_k) of horoballs give rise to an unordered triple $(\theta(p), \theta(q), \theta(r))$ of orthoclasses by restricting to pairs of horoballs. We call such a triple of horoballs a (p, q, r)-triple.

REMARK 9.2. Two distinct classes of triples of horoballs may give rise to the same (p, q, r)-triple of orthoclasses and not all (p, q, r)-triples occur as triples of horoballs in a given 1-cusped manifold.

EXAMPLE 9.3. Examine carefully Figure 5. The number next to each vertex or edge corresponds to the orthoclass of a pair of horoballs. The number next to a vertex is the orthoclass of the pair of horoballs consisting of the ball lying below the vertex and the horoball at infinity. The number next to an edge corresponds to the pair of balls lying below the endpoints of the edge. Each edge is labeled with an arrow and each vertex is labeled with a sign. These arrows and signs orient the corresponding orthocurve. A minus (resp. plus) sign indicates that the orthocurve is oriented from the horoball below the vertex (resp. horoball at infinity) to the horoball at infinity (resp. the horoball below the vertex). The action of $\pi_1(N)$ preserves the oriented orthocurves.

A triangle corresponds to a triple of horoballs. The numbers p, q, and r along the edges of a triangle give rise to the corresponding (p,q,r)-triple. Note that there are (1,1,2) and (1,2,2) triples for m011. An edge also gives rise to a triple, where the third ball is the horoball at infinity. In that case the edge label together with the vertex labels give the (p,q,r)-triple. Construction 9.6 below shows how the triples of horoballs of type (1,1,2) and (1,2,2) give rise to an embedded Mom-2 manifold inside of N.

Definition 9.4. A 1-cusped hyperbolic 3-manifold has a *geometric Mom-n structure* if there exist n distinct classes of triples of horoballs whose pairs of horoballs involve only n different orthoclasses, i.e., only n different values appear in the corresponding (p,q,r)-triples.

Example 9.5. For example, if there are three classes of triples of horoballs for N of type (1,1,2), (1,2,3), (1,2,3), then N has a geometric Mom-3 structure.

Much of [**GMM2**] is involved with showing that a 1-cusped manifold of volume ≤ 2.848 has a geometric Mom-n structure $n \leq 3$, involving only the first four orthoclasses and satisfying other geometric conditions. An outline of the argument can be found in [**GMM3**].

The following construction describes how a geometric Mom-n structure on N potentially can give rise to a genuine Mom-n embedded in N.

Construction 9.6. An orthocurve θ_j projects to a geodesic arc α_j in N with endpoints in the maximal torus T and triples of orthocurves give rise to totally geodesic hexagons with edges alternating on horoballs and orthocurves, hence lie on horocycles or geodesics. See Figure 9. Thus a geometric Mom-n structure *potentially* gives rise to a Mom-n submanifold of N as follows. Shrink the maximal cusp V slightly to be embedded and let T be the resulting embedded boundary torus. Thicken slightly to an embedded $T \times I$. Suppose the geometric Mom-n structure involves orthoclasses $\theta(i_1), \cdots, \theta(i_n)$. The 1-handles correspond to the slightly thickened arcs $\alpha_1, \cdots, \alpha_n$. The 2-handles correspond to the projections of the hexagons into N, slightly thickened.

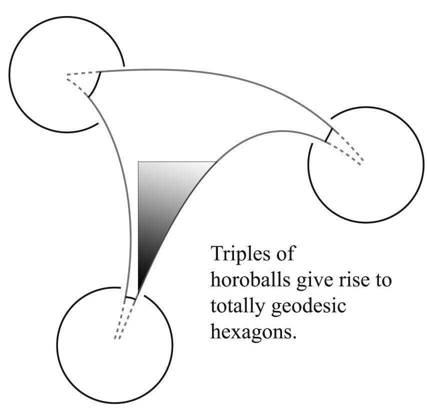

Figure 9

The problem with this construction is that the 2-handles may not be embedded or may pop into the cusp side of T. Said another way, viewed in $\mathbb{H}^3$, the 2-handles

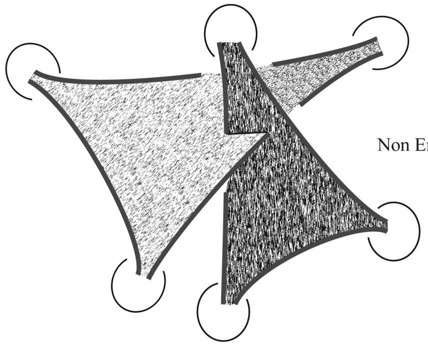

FIGURE 10

may have undesirable pairwise intersections, one possibility being as in Figure 10, or a hexagon corresponding to a triple may penetrate into a fourth horoball. Another issue is that even if everything is embedded, the resulting submanifold $M \subset N$ might not have ∂M being a union of tori and even if they are tori, they might not bound solid tori. It turns out that ∂M is a union of tori when the geometric Mom-n structure has $n \leq 3$ and satisfies a technical condition called *torus friendly*.

We show in [**GMM2**] that if $\mathrm{vol}(N) \leq 2.848$, then N has a torus friendly geometric Mom-n structure $n \leq 3$ satisfying certain geometric conditions. Furthermore, if there exist non-desirable intersections, then N has a torus-friendly geometric Mom-≤ 3 structure of *smaller* complexity satisfying the same geometric conditions. An induction argument shows that eventually we obtain a geometric Mom-k structure that defines an embedded Mom-k manifold $M \subset N$ with $k \leq 3$. By construction, the image of $\pi_1(M)$ in $\pi_1(N)$ is non-abelian, so in the terminology of [**GMM2**] M is *nonelementarily* embedded in N.

There is the final technical issue that M is possibly not hyperbolic and/or all its boundary components do not cut off solid tori and cusps to the outside. But it is shown in [**GMM1**] that if N has a nonelementarily embedded Mom-n submanifold $n \leq 4$, then it has one M_1 that is hyperbolic of non-greater Mom complexity and each component of ∂M_1 cuts off either a solid torus or a cusp to the outside.

Example 9.3 *continued.* Each dark edge of Figure 5 corresponds to a triple of horoballs. While there are six dark edges, they involve only two classes of triples of horoballs, and these are of type (1,1,2) and (1,2,2). Thus this pair of triples of horoballs is a geometric Mom-2 structure. Construction 9.6 produces the desired embedded Mom-2 manifold inside of manifold m011.

References

[Ag2] I. Agol, The minimal volume orientable hyperbolic 2-cusped 3-manifolds, *Proc. AMS* **138** (2010), 3723–3732. MR2661571 (2011k:57023)

[Ag1] I. Agol, Volume change under drilling, *Geom. Topol.* **6** (2002) 905–916. MR1943385 (2004e:57021)

[ADST] I. Agol, P. Storm, and W. Thurston, appendix by N. Dunfield, Lower bounds on volumes of hyperbolic Haken 3-manifolds, *J. Amer. Math. Soc.* **20** (2007), no. 4, 1053–1077. MR2328715 (2008i:53086)

[ACS] I. Agol, M. Culler, and P. Shalen, Dehn surgery, homology and hyperbolic volume, *Algebr. Geom. Topol.* **6** (2006), 2297–2312. MR2286027 (2008f:57024)

[BCG] G. Besson, G. Courtois and S. Gallot, Entropies et rigidites des espaces localement symmetric de courbure strictement negative, *Geom. Funct. Anal.* **5** (1995), 731-799. MR1354289 (96i:58136)

[BCS] J. Boland, C. Connell and J. Souto, Volume rigidity for finite-volume manifolds, *Amer. J. of Math.* **127** (2005), 535–550. MR2141643 (2006b:53052)

[B] K. Boroczky, Packing of spheres in spaces of constant curvature, *Acta Mathematica Academiae Scientiarum Hungaricae* **32** (1978), 243–261. MR512399 (80h:52014)

[Br] H. Bray, Proof of the Riemannian Penrose inequality using the positive mass theorem, *J. Diff. Geom.* **59** (2001), 177–267. MR1908823 (2004j:53046)

[CG] A. Casson and C. Gordon, Reducing Heegaard splittings, *Topology Appl.* **27** (1987), 275–283. MR918537 (89c:57020)

[CM] C. Cao and R. Meyerhoff, The orientable cusped hyperbolic 3-manifolds of minimum volume, *Invent. math.* **146** (2001), 451–478. MR1869847 (2002i:57016)

[CLLMR] A. Champanerkar, J. Lewis, M. Lipyanskiy, S. Meltzer, Exceptional regions and associated exceptional hyperbolic 3-manifolds, with an appendix by Alan W. Reid, *Experiment. Math.* **16** (2007), 106–118. MR2312981 (2008c:57030)

[F] F. Farrell, personal communication.

[FJ] F. Farrell and L. Jones, A topological analogue of Mostow's rigidity theorem, *J. AMS* **2** (1989), 257-370. MR973309 (90h:57023a)

[FKP] D. Futer, E. Kalfagianni, and J. Purcell, Dehn filling, volume, and the Jones polynomial, *J. Diff. Geom.* **78** (2008), 429–464. MR2396249 (2009c:57010)

[G] D. Gabai, The Smale conjecture for hyperbolic 3-manifolds: Isom(M^3)$\simeq$ Diff(M^3), *J. Diff. Geom.* **58** (2001), 113–149. MR1895350 (2003c:57016)

[GMM1] D. Gabai, R. Meyerhoff and P. Milley, Mom-technology and volumes of hyperbolic 3-manifolds, *Comment. Math. Helv.* **86** (2011), 145-188. MR2745279 (2012d:57031)

[GMM2] D. Gabai, R. Meyerhoff and P. Milley, Minimum volume cusped hyperbolic three-manifolds, *J. AMS* **22** (2009), 1157–1215. MR2525782 (2011a:57031)

[GMM3] D. Gabai, R. Meyerhoff and P. Milley, Mom technology and hyperbolic 3-manifolds, *Cont. Math.* **510** (2010), 81–107. MR2581832 (2011f:57026)

[GMT] D. Gabai, R. Meyerhoff and N. Thurston, Homotopy hyperbolic 3-manifolds are hyperbolic, *Annals of Math.* **157** (2003), 335–431. MR1973051 (2004d:57020)

[GM] F. Gehring and G. Martin, Minimal co-volume hyperbolic lattices, I: the spherical points of a Kleinian group, *Annals of Math.* **170** (2009), 123–161. MR2521113 (2010h:57029)

[GT] D. Gabai and M. Trnkova, Exceptional hyperbolic 3-manifolds, Comment. Math. Helv., to appear.

[La] M. Lackenby, Heegaard splittings, the virtually Haken conjecture and property (τ), *Invent. math.* **164** (2006) 317–359. MR2218779 (2007c:57030)

[JR] K. Jones and A. Reid, Vol 3 and other exceptional hyperbolic 3-manifolds, *Proc. Amer. Math. Soc.* **129** (2001), 2175–2185. MR1825931 (2002e:57024)

[Ma] A. Marden, The geometry of finitely generated Kleinian groups, *Annals of Math.* **99** (1974), 383–462. MR0349992 (50:2485)

[MM] T. Marshall and G. Martin, Minimal co-volume hyperbolic lattices, II: Simple torsion in Kleinian groups, preprint, 2008. MR2925384

[MF] S. Matveev and A. Fomenko, Isoenergetic surfaces of Hamiltonian systems, the enumeration of three-dimensional manifolds in order of growth of their complexity, and the calculation of the volumes of closed hyperbolic 3-manifolds, translated in *Russian Math. Surveys* **43** (1988), 3–24. MR937017 (90a:58052)

[Mey] R. Meyerhoff, A lower bound for the volume of hyperbolic 3-manifolds, *Canad. J. Math.* **39** (1987), 1038–1056. MR918586 (88k:57049)

[MR] C. Maclachlan and A. Reid, The Arithmetic of Hyperbolic 3-Manifolds, GTM **219** 2003, Springer–Verlag, New York. MR1937957 (2004i:57021)

[Mi] P. Miao, Positive mass theorem on manifolds admitting corners along a hypersurface, *Adv. Theor. Math. Phys.* **6** (2002), 1163–1182. MR1982695 (2005a:53065)

[Mill] P. Milley, Minimum-volume hyperbolic 3-manifolds, *J. Topol.* **2** (2009), no. 2, 181–192. MR2499442 (2010d:57018)

[Mos] H. Moser, Proving a manifold to be hyperbolic once it has been approximated to be so, *Algebr. Geom. Topol.* **9** (2009), 103–133. MR2471132 (2009k:57028)

[Mo] G. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, *Inst. Hautes Etudes Sci. Publ. Math.* **34** (1968), 53–104. MR0236383 (38:4679)

[Snap] O. Goodman, *Snap*, http://www.ms.unimelb.edu.au/ snap/.

[Po] H. Poincaré, Memoires sur les groupes kleineens, *Acta Math.* **3** (1883), 49–92. MR1554613

[P1] G. Perelman, Ricci flow with surgery on three-manifolds, arXiv:math.DG/0303109.

[P2] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159.

[Pra] G. Prasad, Strong rigidity of $\mathbb{Q}$-rank 1 lattices, *Invent. math.* **21** (1973), 255-286. MR0385005 (52:5875)

[Prez] A. Przeworski, A universal upper bound on density of tube packings in hyperbolic space, *J. Diff. Geom.* **72** (2006), 113–127. MR2215457 (2007b:57030)

[Ra] T. Rado, Uber den Begriff der Riemann Flache, *Acta Univ. Szeged* **2**, (1924–1926), 101–121.

[Sm] S. Smale, Generalized Poincaré's conjecture in dimensions greater than four, *Ann. of Math.* (2) **74** (1961) 391–406. MR0137124 (25:580)

[Th1] W. Thurston, The geometry and topology of 3-manifolds, Princeton University lecture notes, 1978.

[Th2] W. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, *B. AMS* **6** (1982), 357–381. MR648524 (83h:57019)

[Wa] H. Wang, Topics in totally discontinuous groups, *Symmetric Spaces*, Boothby–Weiss, New York (1972), 460–485.

[We] J. Weeks, *SnapPea*, available from the author at www.geometrygames.org.

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Manifolds: Where Do We Come From? What Are We? Where Are We Going?

Mikhail Gromov

Abstract. Descendants of algebraic kingdoms of high dimensions, enchanted by the magic of Thurston and Donaldson, lost in the whirlpools of the Ricci flow, topologists dream of an ideal land of manifolds—perfect crystals of mathematical structure which would capture our vague mental images of geometric spaces. We browse through the ideas inherited from the past hoping to penetrate through the fog which conceals the future.

1. Ideas and Definitions

We are fascinated by knots and links. Where does this feeling of beauty and mystery come from? To get a glimpse at the answer let us move by 25 million years in time, which is, roughly, what separates us from orangutans: 12 million years to our common ancestor on the phylogenetic tree and then 12 million years back by another branch of the tree to the present day orangutans.

But are there topologists among orangutans? Yes, there definitely are: many orangutans are good at "proving" the triviality of elaborate knots, e.g. they fast master the art of untying boats from their mooring when they fancy taking rides downstream in a river, much to the annoyance of people making these knots with a different purpose in mind.

A more amazing observation was made by a zoo-psychologist Anne Russon in mid 90's at Wanariset Orangutan Reintroduction Project (see p. 114 in [**67**]).

> "...Kinoi [a juvenile male orangutan], when he was in a possession of a hose, invested every second in making giant hoops, carefully inserting one end of his hose into the other and jamming it in tight. Once he'd made his hoop, he passed various parts of himself back and forth through it—an arm, his head, his feet, his whole torso—as if completely fascinated with idea of going through the hole."

Playing with hoops and knots, where there is no visible goal or any practical gain—be it an ape or a $3D$-topologist—appears fully "non-intelligent" to a practically minded observer. But we, geometers, feel thrilled at seeing an animal whose space perception is so similar to ours.

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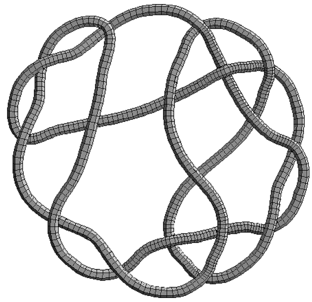


It is unlikely, however, that Kinoi would formulate his ideas the way we do and that, unlike our students, he could be easily intimidated into accepting "equivalence classes of atlases" and "ringed spaces" as appropriate definitions of his topological playground. (Despite such display of disobedience, we would enjoy the company of young orangutans; they are charmingly playful creatures, unlike the aggressive and reckless chimpanzees—our nearest evolutionary neighbors.) Apart from topology, orangutans do not rush to accept another human definition, namely that of "tools", as of

> "external *detached* objects (to exclude a branch used for climbing a tree) employed for reaching specific goals".

(The use of tools is often taken by zoo-psychologists for a measure of "intelligence" of an animal.)

Being imaginative arboreal creatures, orangutans prefer a broader definition: For example (see [**67**]):

- they bunch up leaves to make wipers to clean their bodies without detaching the leaves from a tree;
- they often break branches but deliberately leave them attached to trees when it suits their purposes—these could not have been achieved if orangutans were bound by the "detached" definition.

Moral. Our best definitions, e.g. that of a manifold, tower as prominent landmarks over our former insights. Yet, we should not be hypnotized by definitions. After all, they are remnants of the past and tend to misguide us when we try to probe the future.

Remark. There is a non-trivial similarity between the neurological structures underlying the behaviour of playful animals and that of working mathematicians (see [**31**]).

2. Homotopies and Obstructions

For more than half a century, starting from Poincaré, topologists have been laboriously stripping their beloved science of its geometric garments.

"Naked topology", reinforced by homological algebra, reached its to-day breathtakingly high plateau with the following

Theorem (Serre $[S^{n+N} \to S^N]$-Finiteness Theorem (1951)). *There are at most finitely many homotopy classes of maps between spheres $S^{n+N} \to S^N$ but for the two exceptions:*

- *equivi-dimensional case where* $n = 0$ $\pi_N(S^N) = \mathbb{Z}$; *the homotopy class of a map* $S^N \to S^N$ *in this case is determined by an integer that is the degree of a map. (Brouwer 1912, Hopf 1926. We define degree in Section* 4.*) This is expressed in the standard notation by writing*

$$\pi_N(S^N) = \mathbb{Z}.$$

- *Hopf case, where* N *is even and* $n = 2N - 1$. *In this case* $\pi_{2N-1}(S^N)$ *contains a subgroup of finite index isomorphic to* $\mathbb{Z}$.

It follows that

the homotopy groups $\pi_{n+N}(S^N)$ are finite for $N >> n$,

where, by the *Freudenthal suspension theorem* of 1928 (this is easy),

the groups $\pi_{n+N}(S^N)$ for $N \geq n$ do not depend on N.

These are called *the stable homotopy groups of spheres* and are denoted π_n^{st}.

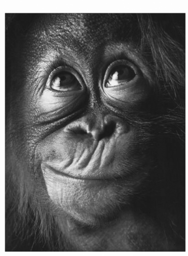
H. Hopf proved in 1931 that the map $f : S^3 \to S^2 = S^3/\mathbb{T}$, for the group $\mathbb{T} \subset \mathbb{C}$ of the complex numbers with norm one which act on $S^3 \subset \mathbb{C}^2$ by $(z_1, z_2) \mapsto (tz_1, tz_2)$, is *non-contractible.*

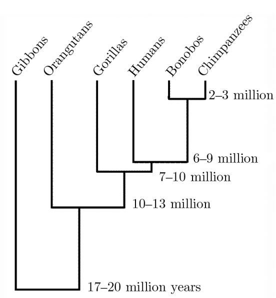
In general, the unit tangent bundle $X = UT(S^{2k}) \to S^{2k}$ has finite homology $H_i(X)$ for $0 < i < 4k - 1$. By Serre's theorem, there exists a map $S^{4k-1} \to X$ of positive degree and the composed map $S^{4k-1} \to X \to S^{2k}$ generates an *infinite* cyclic group of *finite index* in $\pi_{4k-1}(S^{2k})$.

The proof by Serre—a geometer's nightmare—consists in tracking a multitude of linear-algebraic relations between the homology and homotopy groups of infinite dimensional spaces of maps between spheres and it tells you next to nothing about the geometry of these maps. (See [**57**] for a "semi-geometric" proof of the finiteness of the stable homotopy groups of spheres and Section 5 of this article for a related discussion. Also, the construction in [**23**] may be relevant.)

Recall that the set of the homotopy classes of maps of a sphere S^M to a connected space X makes *a group* denoted $\pi_M(X)$, (π is for Poincaré who defined *the fundamental group* π_1) where the definition of the group structure depends on distinguished points $x_0 \in X$ and $s_0 \in S^M$. The groups π_M defined with different x_0 are mutually isomorphic, and if X is *simply connected*, i.e. $\pi_1(X) = 1$, then they are *canonically* isomorphic.

This point in S^M may be chosen with the representation of S^M as the one point compactification of the Euclidean space $\mathbb{R}^M$, denoted $\mathbb{R}^M_\bullet$, where this infinity point $\bullet$ is taken for s_0. It is convenient, instead of maps $S^m = \mathbb{R}^m_\bullet \to (X, x_0)$, to deal with maps $f : \mathbb{R}^M \to X$ "with compact supports", where the *support* of an f





is the closure of the (open) subset $\mathrm{supp}(f) = \mathrm{supp}_{x_0}(f) \subset \mathbb{R}^m$ which consists of the points $s \in \mathbb{R}^m$ such that $f(s) \neq x_0$.

A pair of maps $f_1, f_2 : \mathbb{R}^M \to X$ with *disjoint* compact supports obviously defines "the joint map" $f : \mathbb{R}^M \to X$, where the homotopy class of f (obviously) depends only on those of f_1, f_2, provided $\mathrm{supp}(f_1)$ lies in the left half space $\{s_1 < 0\} \subset \mathbb{R}^m$ and $\mathrm{supp}(f_2) \subset \{s_1 > 0\} \subset \mathbb{R}^M$, where s_1 is a non-zero linear function (coordinate) on $\mathbb{R}^M$.

The composition of the homotopy classes of two maps, denoted $[f_1] \cdot [f_2]$, is defined as the homotopy class of the joint of f_1 moved far to the left with f_2 moved far to the right.

Geometry is sacrificed here for the sake of algebraic convenience: first, we break the symmetry of the sphere S^M by choosing a base point, and then we destroy the symmetry of $\mathbb{R}^M$ by the choice of s_1. If $M = 1$, then there are essentially two choices: s_1 and $-s_1$, which correspond to interchanging f_1 with f_2—nothing wrong with this as the composition is, in general, non-commutative.

In general $M \geq 2$, these $s_1 \neq 0$ are, homotopically speaking, parametrized by the unit sphere $S^{M-1} \subset \mathbb{R}^M$. Since S^{M-1} is connected for $M \geq 2$, the composition is commutative and, accordingly, the composition in π_i for $i \geq 2$ is denoted $[f_1]+[f_2]$. Good for algebra, but the $O(M+1)$-ambiguity seems too great a price for this. (An algebraist would respond to this by pointing out that the ambiguity is resolved in the language of *operads* or something else of this kind.)

But this is, probably, unavoidable. For example, the best you can do for maps $S^M \to S^M$ in a given non-trivial homotopy class is to make them symmetric (i.e. equivariant) under the action of the maximal torus $\mathbb{T}^k$ in the orthogonal group $O(M+1)$, where $k = M/2$ for even M and $k = (M+1)/2$ for M odd.

And if $n \geq 1$, then, with a few exceptions, there are no apparent symmetric representatives in the homotopy classes of maps $S^{n+N} \to S^N$; yet Serre's theorem does carry a geometric message.

> If $n \neq 0, N-1$, then every continuous map $f_0 : S^{n+N} \to S^N$ is homotopic to a map $f_1 : S^{n+N} \to S^N$ of dilation bounded by a constant,

$$\mathrm{dil}(f_1) =_{\mathrm{def}} \sup_{s_1 \neq s_2 \in S^{n+N}} \frac{\mathrm{dist}(f(s_1), f(s_2))}{\mathrm{dist}(s_1, s_2)} \leq \mathrm{const}(n, N).$$

Dilation Questions. (1) What is the asymptotic behaviour of $\mathrm{const}(n, N)$ for $n, N \to \infty$?

For all we know the *Serre dilation constant* $\mathrm{const}_S(n, N)$ may be bounded for $n \to \infty$ and, say, for $1 \leq N \leq n-2$, but a bound one can see offhand is that by an exponential tower $(1+c)^{(1+c)^{(1+c)^{\cdots}}}$, of height N, since each geometric implementation of the homotopy lifting property in a Serre fibrations may bring along an exponential dilation. Probably, the (questionably) geometric approach to the Serre theorem via "singular bordisms" (see [**74**], [**23**],[**1**] and Section 5) delivers a better estimate.

(2) Let $f : S^{n+N} \to S^N$ be a contractible map of dilation d, e.g. f equals the m-multiple of another map where m is divisible by the order of $\pi_{n+N}(S^N)$.

What is, roughly, the minimum $D_{\min} = D(d, n, N)$ of dilations of maps F of the unit ball $B^{n+N+1} \to S^N$ which are equal to f on $\partial(B^{n+N+1}) = S^{n+N}$?

Of course, this dilation is the most naive invariant measuring the "geometric size of a map". Possibly, an interesting answer to these questions needs a more imaginative definition of "geometric size/shape" of a map, e.g. in the spirit of the minimal degrees of polynomials representing such a map.

Serre's theorem and its descendants underlie most of the topology of the high dimensional manifolds. Below are frequently used corollaries which relate homotopy problems concerning general spaces X to the homology groups $H_i(X)$ (see Section 4 for definitions) which are much easier to handle.

THEOREM ($[S^{n+N} \to X]$-Theorems). *Let X be a compact connected triangulated or cellular space, (defined below) or, more generally, a connected space with finitely generated homology groups $H_i(X)$, $i = 1, 2, \dots$. If the space X is simply connected, i.e. $\pi_1(X) = 1$, then its homotopy groups have the following properties.*

(1) *Finite Generation. The groups $\pi_m(X)$ are (Abelian!) finitely generated for all $m = 2, 3, \dots$.*

(2) *Sphericity. If $\pi_i(X) = 0$ for $i = 1, 2, N-1$, then the (obvious) Hurewicz homomorphism*

$$\pi_N(X) \to H_N(X),$$

which assigns, to a map $S^N \to X$, the N-cycle represented by this N-sphere in X, is an isomorphism. (This is elementary, Hurewicz 1935.)

(3) *$\mathbb{Q}$-Sphericity. If the groups $\pi_i(X)$ are finite for $i = 2, N-1$ (recall that we assume $\pi_1(X) = 1$), then the Hurewicz homomorphism tensored with rational numbers,*

$$\pi_{N+n}(X) \otimes \mathbb{Q} \to H_{N+n}(X) \otimes \mathbb{Q},$$

is an isomorphism for $n = 1, \dots, N-2$.

Because of the finite generation property, The $\mathbb{Q}$-sphericity is equivalent to the following.

THEOREM (Serre m-Sphericity Theorem). *(3′) Let the groups $\pi_i(X)$ be finite (e.g. trivial) for $i = 1, 2, \dots, N-1$ and $n \leq N-2$. Then an m-multiple of every $(N+n)$-cycle in X for some $m \neq 0$ is homologous to an $(N+n)$-sphere continuously mapped to X; every two homologous spheres $S^{N+n} \to X$ become homotopic when composed with a non-contractible i.e. of degree $m \neq 0$, self-mapping $S^{n+N} \to S^{n+N}$. In more algebraic terms, the elements $s_1, s_2 \in \pi_{n+N}(X)$ represented by these spheres satisfy $ms_1 - ms_2 = 0$.*

The following is the dual of the m-Sphericity.

THEOREM (Serre $[\to S^N]_{\mathbb{Q}}$- Theorem). *Let X be a compact triangulated space of dimension $n+N$, where either N is odd or $n < N-1$. Then a non-zero multiple of every homomorphism $H_N(X) \to H_N(S^N)$ can be realized by a continuous map $X \to S^N$.*

If two continuous maps are $f, g : X \to S^N$ are homologous, i.e. if the homology homomorphisms $f_*, g_* : H_N(X) \to H_N(S^N) = \mathbb{Z}$ are equal, then there exists a continuous self-mapping $\sigma : S^N \to S^N$ of non-zero degree such that the composed maps $\sigma \circ f$ and $\sigma \circ f : X \to S^N$ are homotopic.

These $\mathbb{Q}$-theorems follow from the Serre finiteness theorem for maps between spheres by an elementary argument of *induction by skeletons* and rudimentary *obstruction theory* which run, roughly, as follows.

Cellular and Triangulated Spaces. Recall that a *cellular space* is a topological space X with an ascending (finite or infinite) sequence of closed subspaces $X_0 \subset X_1 \subset \dots \subset X_i \subset \dots$ called he *i-skeleta* of X, such that $\bigcup_i (X_i) = X$ and such that X_0 is a discrete finite or countable subset. Every X_i, $i > 0$, is obtained by attaching a countably (or finitely) many i-balls B^i to X_{i-1} by continuous maps of the boundaries $S^{i-1} = \partial(B^i)$ of these balls to X_{i-1}.

For example, if X is a *triangulated* space then it comes with homeomorphic *embeddings* of the i-simplices $\Delta^i \to X_i$ extending their boundary maps, $\partial(\Delta^i) \to X_{i-1} \subset X_i$ where one additionally requires (here the word "simplex", which is, topologically speaking, is indistinguishable from B^i, becomes relevant) that the intersection of two such simplices Δ^i and Δ^j imbedded into X is a simplex Δ^k which is a *face* simplex in $\Delta^i \supset \Delta^k$ and in $\Delta^j \supset \Delta^k$.

If X is a non-simplicial cellular space, we also have continuous maps $B^i \to X_i$ but they are, in general, embeddings *only* on the interiors $B^i \setminus \partial(B^i)$, since the attaching maps $\partial(B^i) \to X_{i-1}$ are not necessarily injective. Nevertheless, the images of B^i in X are called *closed cells*, and denoted $B_i \subset X_i$, where the union of all these i-cells equals X_i.

Observe that the homotopy equivalence class of X_i is determined by that of X_{i-1} and by the homotopy classes of maps from the spheres $S^{i-1} = \partial(B^i)$ to X_{i-1}. We are free to take any maps $S^{i-1} \to X_{i-1}$ we wish in assembling a cellular X which make cells more efficient building blocks of general spaces than simplices. For example, the sphere S^n can be made of a 0-cell and a single n-cell.

If $X_{i-1} = S^l$ for some $l \leq i-1$ (one has $l < i-1$ if there is no cells of dimensions between l and $i-1$) then the homotopy equivalence classes of X_i with a single i-cell one-to-one correspond to the homotopy group $\pi_{i-1}(S^l)$.

On the other hand, every cellular space can be approximated by a homotopy equivalent simplicial one, which is done by induction on skeletons X_i with an approximation of *continuous* attaching maps by *simplicial* maps from $(i-1)$-spheres to X_{i-1}.

Recall that a *homotopy equivalence* between X_1 and X_2 is given by a pair of maps $f_{12} : X_1 \to X_2$ and $f_{21} : X_2 \to X_1$, such that both composed maps $f_{12} \circ f_{21} : X_1 \to X_1$ and $f_{21} \circ f_{12} : X_2 \to X_2$ are homotopic to the identity.

Obstructions and Cohomology. Let Y be a connected space such that $\pi_i(Y) = 0$ for $i = 1, \dots, n-1 \geq 1$, let $f : X \to Y$ be a continuous map and let us construct, by induction on $i = 0, 1, \dots, n-1$, a map $f_{\text{new}} : X \to Y$ which is homotopic to f and which sends X_{n-1} to a point $y_0 \in Y$ as follows.

Assume $f(X_{i-1}) = y_0$. Then the resulting map $B^i \overset{f}{\to} Y$, for each i-cell B^i from X_i, makes an i-sphere in Y, because the boundary $\partial B^i \subset X_{i-1}$ goes to a single point—our to y_0 in Y.

Since $\pi_i(Y) = 0$, this B^i in Y can be contracted to y_0 without disturbing its boundary. We do it all i-cells from X_i and, thus, contract X_i to y_0. One cannot, in general, extend a continuous map from a closed subset $X' \subset X$ to X, but one always can extend a continuous *homotopy* $f'_t : X' \to Y$, $t \in [0,1]$, of a given map $f_0 : X \to Y$, $f_0|X' = f'_0$, to a homotopy $f_t : X \to Y$ for all closed subsets $X' \subset X$, similarly to how one extends $\mathbb{R}$-valued functions from $X' \subset X$ to X.

The contraction of X to a point in Y can be obstructed on the n-th step, where $\pi_n(Y) \neq 0$, and where each *oriented* n-cell $B^n \subset X$ mapped to Y with $\partial(B^n) \to y_0$

represents an element $c \in \pi_n(Y)$ which may be non-zero. (When we switch an orientation in B^n, then $c \mapsto -c$.)

We assume at this point, that our space X is a triangulated one, switch from B^n to Δ^n and observe that the function $c(\Delta^n)$ is (obviously) an *n-cocycle* in X with values in the group $\pi_n(Y)$, which means (this is what is longer to explain for general cell spaces) that the sum of $c(\Delta^n)$ over the $n+2$ face-simplices $\Delta^n \subset \partial\Delta^{n+1}$ *equals zero*, for all Δ^{n+1} in the triangulation (if we canonically/correctly choose orientations in all Δ^n).

The cohomology class $[c] \in H^n(X; \pi_n(X))$ of this cocycle does not depend (by an easy argument) on how the $(n-1)$-skeleton was contracted. Moreover, every cocycle c' in the class of $[c]$ can be obtained by a homotopy of the map on X_n which is kept constant on X_{n-2}. (Two A-valued n-cocycles c and c', for an abelian group A, are *in the same cohomology class* if there exists an A-valued function $d(\Delta^{n-1})$ on the oriented simplices $\Delta^{n-1} \subset X_{n-1}$, such that $\sum_{\Delta^{n-1} \subset \Delta^n} d(\Delta^{n-1}) = c(\Delta^n) - c'(\Delta^n)$ for all Δ^n. The set of the cohomology classes of n-cocycles with a natural additive structure is called the *cohomology group* $H^n(X; A)$. It can be shown that $H^n(X; A)$ depends only on X but not an a particular choice of a triangulation of X. See Section 4 for a lighter geometric definitions of homology and cohomology.)

In particular, if $\dim(X) = n$ we, thus, equate the set $[X \to Y]$ of the homotopy classes of maps $X \to Y$ with the cohomology group $H^n(X; \pi_n(X))$. Furthermore, this argument applied to $X = S^n$ shows that $\pi_n(X) = H_n(X)$ and, in general, that the following is true.

> The set of the homotopy classes of maps $X \to Y$ equals the set of homomorphisms $H_n(X) \to H_n(Y)$, provided $\pi_i(Y) = 0$ for $0 < i < \dim(X)$.

Finally, when we use this construction for proving the above $\mathbb{Q}$-theorems where one of the spaces is a sphere, we keep composing our maps with self-mappings of this sphere of suitable degree $m \neq 0$ that kills the obstructions by the Serre finiteness theorem.

For example, if X is a finite cellular space without 1-cells, one can define the *homotopy multiple* l^*X, for every integer l, by replacing the attaching maps of all $(i+1)$-cells, $S^i \to X_i$, by l^{k_i}-multiples of these maps in $\pi_i(X_i)$ for $k_2 << k_3 << \ldots$, where this l^*X comes along with a map $l^*X \to X$ which induces isomorphisms on all homotopy groups tensored with $\mathbb{Q}$.

The obstruction theory, developed by Eilenberg in 1940 following Pontryagin's 1938 paper, well displays the logic of algebraic topology: the geometric symmetry of X (if there was any) is broken by an arbitrary triangulation or a cell decomposition and then another kind of symmetry, an Abelian algebraic one, emerges on the (co)homology level.

Serre's idea is that the homotopy types of finite simply connected cell complexes as well as of finite diagrams of continuous maps between these are *finitary arithmetic objects* which can be encoded by finitely many polynomial equations and non-equalities with integer coefficients, and where the structural organization of the homotopy theory depends on *non-finitary objects* which are *inductive limits of finitary ones*, such as the homotopy types of spaces of continuous maps between finite cell spaces.

3. Generic Pullbacks

A common zero set of N smooth (i.e. infinitely differentiable) functions $f_i : \mathbb{R}^{n+N} \to \mathbb{R}$, $i = 1, \ldots, N$, may be very nasty even for $N = 1$—every closed subset in $\mathbb{R}^{n+1}$ can be represented as a zero of a smooth function. However, if the functions f_i are taken *in general position*, then the common zero set is a smooth n-submanifold in $\mathbb{R}^{n+N}$.

Here and below, "f in general position" or "generic f", where f is an element of a topological space F, e.g. of the space of C^∞-maps with the C^∞-topology, means that what we say about f applies to all f in an *open and dense* subset in F. Sometimes, one allows not only open dense sets in the definition of genericity but also their *countable* intersections.

Generic smooth (unlike continuous) objects are as nice as we expect them to be; the proofs of this "niceness" are local-analytic and elementary (at least in the cases we need); everything trivially follows from *Sard's theorem + the implicit function theorem.*

The representation of manifolds with functions generalizes as follows.

Generic Pullback Construction. (Pontryagin 1938, Thom 1954). Start with a smooth N-manifold V, e.g. $V = \mathbb{R}^N$ or $V = S^N$, and let $X_0 \subset V$ be a smooth submanifold, e.g. $0 \in \mathbb{R}^N$ or a point $x_0 \in S^N$. Let W be a smooth manifold of dimension M, e.g. $M = n + N$.

Theorem. *If $f : W \to V$ is a generic smooth map, then the pullback $X = f^{-1}(X_0) \subset W$ is a smooth submanifold in W with* $\operatorname{codim}_W(X) = \operatorname{codim}_V(X_0)$*, i.e. $M - \dim(X) = N - \dim(X_0)$. Moreover, if the manifolds W, V and X_0 are oriented, then X comes with a natural orientation. Furthermore, if W has a boundary then X is a smooth submanifold in W with a boundary $\partial(X) \subset \partial(W)$.*

Example (a). Let $f : W \subset V \supset X_0$ be a smooth, possibly non-generic, embedding of W into V. Then a small generic perturbation $f' : W \to V$ of f remains an embedding, such that image $W' = f'(W) \subset V$ in V becomes *transversal* (i.e. nowhere tangent) to X_0. One sees with the full geometric clarity (with a picture of two planes in the 3-space which intersect at a line) that the intersection $X = W' \cap X_0 (= (f')^{-1}(X_0))$ is a submanifold in V with $codim_V(X) = codim_V(W) + codim_V(X_0)$.

Example (b). Let $f : S^3 \to S^2$ be a smooth map and $S_1, S_2 \in S^3$ be the pullbacks of two generic points $s_1, s_2 \in S^2$. These S_i are smooth closed curves; they are naturally oriented, granted orientations in S^2 and in S^3.

Let $D_i \subset B^4 = \partial(S^3)$, $i = 1, 2$, be generic smooth oriented surfaces in the ball $B^4 \supset S^3 = \partial(B^4)$ with their oriented boundaries equal S_i and let $h(f)$ denote the *intersection index* (defined in the next section) between D_i.

Suppose, the map f is homotopic to zero, extend it to a smooth generic map $\varphi : B^4 \to S^2$ and take the φ-pullbacks $D_i^\varphi = \varphi^{-1}(s_i) \subset B^4$ of s_i.

Let S^4 be the 4-sphere obtained from the two copies of B^4 by identifying the boundaries of the balls and let $C_i = D_i \cup D_i^\varphi \subset S^4$.

Since $\partial(D_i) = \partial(D_i^\varphi) = S_i$, these C_i are *closed* surfaces; hence, the intersection index between them equals zero (because they are homologous to zero in S^4, see the next section), and since D_i^φ do not intersect, the intersection index $h(f)$ between D_i is also zero.

It follows that *non-vanishing* of the *Hopf invariant* $h(f)$ implies that f is *non-homotopic to zero.*

For instance, the Hopf map $S^3 \to S^2$ is non-contractible, since every two transversal flat dicks $D_i \subset B^4 \subset \mathbb{C}^2$ bounding equatorial circles $S_i \subset S^3$ intersect at a single point.

The essential point of the seemingly trivial pull-back construction, is that starting from "simple manifolds" $X_0 \subset V$ and W, we produce complicated and more interesting ones by means of "complicated maps" $W \to V$. (It is next to impossible to make an interesting manifold with the "equivalence class of atlases" definition.)

For example, if $V = \mathbb{R}$, and our maps are functions on W, we can generate lots of them by using algebraic and analytic manipulations with functions and then we obtain maps to $\mathbb{R}^N$ by taking N-tuples of functions.

And less obvious (smooth generic) maps, for all kind of V and W, come as smooth generic approximations of continuous maps $W \to V$ delivered by the algebraic topology.

Following Thom (1954) one applies the above to maps into one point compactifications $V_\bullet$ of open manifolds V where one still can speak of generic pullbacks of smooth submanifolds X_0 in $V \subset V_\bullet$ under maps $W \to V_\bullet$

Thom spaces. The Thom space of an N*-vector bundle* $V \to X_0$ over a compact space X_0 (where the pullbacks of all points $x \in X_0$ are Euclidean spaces $\mathbb{R}^N_x = \mathbb{R}^N$) is the one point compactifications $V_\bullet$ of V, where X_0 is canonically embedded into $V \subset V_\bullet$ as the zero section of the bundle (i.e. $x \mapsto 0 \in \mathbb{R}^N_x$).

If $X = X^n \subset W = W^{n+N}$ is a smooth submanifold, then the total space of its *normal bundle* denoted $U^\perp \to X$ is (almost canonically) diffeomorphic to a small (normal) ε-neighbourhood $U(\varepsilon) \subset W$ of X, where every ε-ball $B^N(\varepsilon) = B^N_x(\varepsilon)$ normal to X at $x \in X$ is radially mapped to the fiber $\mathbb{R}^N = \mathbb{R}^N_x$ of $U^\perp \to X$ at x.

Thus the Thom space $U^\perp_\bullet$ is identified with $U(\varepsilon)_\bullet$ and the tautological map $W_\bullet \to U(\varepsilon)_\bullet$, that equals the identity on $U(\varepsilon) \subset W$ and sends the complement $W \setminus U(\varepsilon)$ to $\bullet \in U(\varepsilon)_\bullet$, defines the *Atiyah–Thom map* for all closed smooth submanifold $X \subset W$,

$$A^\perp_\bullet : W_\bullet \to U^\perp_\bullet.$$

Recall that every $\mathbb{R}^N$-bundle over an n-dimensional space with $n < N$, can be induced from the tautological bundle V over the Grassmann manifold $X_0 = \mathrm{Gr}_N(\mathbb{R}^{n+N})$ of N-planes (i.e. linear N-subspaces in $\mathbb{R}^{n+N}$) by a continuous map, say $G : X \to X_0 = \mathrm{Gr}_N(\mathbb{R}^{n+N})$.

For example, if $X \subset \mathbb{R}^{n+N}$, one can take *the normal Gauss map* for G that sends $x \in X$ to the N-plane $G(x) \in \mathrm{Gr}_N(\mathbb{R}^{n+N}) = X_0$ which is parallel to the normal space of X at x.

Since the Thom space construction is, obviously, functorial, every $U^\perp$-bundle inducing map $X \to X_0 = \mathrm{Gr}_N(\mathbb{R}^{n+N})$ for $X = X^n \subset W = W^{n+N}$, defines a map $U^\perp_\bullet \to V_\bullet$ and this, composed with $A^\perp_\bullet$, gives us the *Thom map*

$$T_\bullet : W_\bullet \to V_\bullet \text{ for the tautological } N\text{-bundle } V \to X_0 = \mathrm{Gr}_N(\mathbb{R}^{n+N}).$$

Since all n-manifolds can be (obviously) embedded (by generic smooth maps) into Euclidean spaces $\mathbb{R}^{n+N}$, $N >> n$, every *closed*, i.e. compact without boundary, n-manifold X comes from the generic pullback construction applied to maps f from

$S^{n+N} = \mathbb{R}^{n+N}_\bullet$ to the Thom space $V_\bullet$ of the canonical N-vector bundle $V \to X_0 = \mathrm{Gr}_N(\mathbb{R}^{n+N})$,

$$X = f^{-1}(X_0) \text{ for generic } f : S^{n+N} \to V_\bullet \supset X_0 = \mathrm{Gr}_N(\mathbb{R}^{n+N}).$$

In a way, Thom has discovered the source of all manifolds in the world and responded to the question "Where are manifolds coming from?" with the following.

1954 ANSWER. *All closed smooth n-manifolds X come as pullbacks of the Grassmannians $X_0 = \mathrm{Gr}_N(\mathbb{R}^{n+N})$ in the ambient Thom spaces $V_\bullet \supset X_0$ under generic smooth maps $S^{n+N} \to V_\bullet$.*

The manifolds X obtained with the generic pull-back construction come with a grain of salt: generic maps are abundant but it is hard to put your finger on any one of them—we cannot say much about topology and geometry of an individual X. (It seems, one cannot put *all* manifolds in one basket without some "random string" attached to it.)

But, empowered with Serre's theorem, this construction unravels an amazing structure in the "space of all manifolds" (Before Serre, Pontryagin and following him Rokhlin proceeded in the reverse direction by applying smooth manifolds to the homotopy theory via the Pontryagin construction.)

Selecting an object X, e.g. a submanifold, from a given collection $\mathcal{X}$ of similar objects, where there is *no distinguished* member $X^\star$ among them, is a notoriously difficult problem which had been known since antiquity and can be traced to De Cael of Aristotle. It reappeared in 14th century as *Buridan's ass problem* and as *Zermelo's choice problem* at the beginning of 20th century.

A geometer/analyst tries to select an X by first finding/constructing a "value function" on $\mathcal{X}$ and then by taking the "optimal" X. For example, one may go for n-submanifolds X of *minimal volumes* in an $(n+N)$-manifold W endowed with a *Riemannian metric*. However, minimal manifolds X are usually singular except for hypersurfaces $X^n \subset W^{n+1}$ where $n \leq 6$ (Simons, 1968).

Picking up a "generic" or a "random" X from $\mathcal{X}$ is a geometer's last resort when all "deterministic" options have failed. This is aggravated in topology, since

- there is no known construction delivering *all* manifolds X in a reasonably controlled manner besides generic pullbacks and their close relatives;
- on the other hand, geometrically interesting manifolds X are not anybody's pullbacks. Often, they are "complicated quotients of simple manifolds", e.g. $X = S/\Gamma$, where S is a symmetric space, e.g. the hyperbolic n-space, and Γ is a discrete isometry group acting on S, possibly, with fixed points.

(It is obvious that every surface X is homeomorphic to such a quotient, and this is also so for compact 3-manifolds by a theorem of Thurston. But if $n \geq 4$, one does not know if every closed smooth manifold X is homeomorphic to such an S/Γ. It is hard to imagine that there are infinitely many non-diffeomorphic but mutually homeomorphic S/Γ for the hyperbolic 4-space S, but this may be a problem with our imagination.)

Starting from another end, one has *ramified covers* $X \to X_0$ of "simple" manifolds X_0, where one wants the ramification locus $\Sigma_0 \subset X_0$ to be a subvariety with "mild singularities" and with an "interesting" fundamental group of the complement $X_0 \setminus \Sigma_0$, but finding such Σ_0 is difficult (see the discussion following (3) in Section 7).

And even for simple $\Sigma_0 \subset X_0$, the description of ramified coverings $X \to X_0$ where X are *manifolds* may be hard. For example, this is non-trivial for ramified coverings over the flat n-torus $X_0 = \mathbb{T}^n$ where Σ_0 is the union of several flat $(n-2)$-subtori in general position where these subtori may intersect one another.

4. Duality and the Signature

Cycles and Homology. If X is a smooth n-manifold X one is inclined to define "geometric i-cycles" C in X, which represent homology classes $[C] \in H_i(X)$, as "compact oriented i-submanifolds $C \subset X$ with singularities of *codimension two*".

This, however, is too restrictive, as it rules out, for example, closed self-intersecting curves in surfaces, and/or the double covering map $S^1 \to S^1$.

Thus, we allow $C \subset X$ which may have singularities of codimension one, and, besides orientation, a locally constant integer valued function on the non-singular locus of C.

First, we define dimension on all closed subsets in smooth manifolds with the usual properties of monotonicity, locality and max-additivity, i.e. $\dim(A \cup B) = \max(\dim(A), \dim(B))$.

Besides we want our dimension to be monotone under *generic* smooth maps of compact subsets A, i.e. $\dim(f(A)) \leq \dim(A)$ and if $f : X^{m+n} \to Y^n$ is a *generic* map, then $f^{-1}(A) \leq \dim(A) + m$.

Then we define the "generic dimension" as the *minimal* function with these properties which coincides with the ordinary dimension on smooth compact submanifolds. This depends, of course, on specifying "generic" at each step, but this never causes any problem in-so-far as we do not start taking limits of maps.

An i-cycle $C \subset X$ is a closed subset in X of dimension i with a *$\mathbb{Z}$-multiplicity function* on C defined below, and with the following set decomposition of C.

$$C = C_{\text{reg}} \cup C_{\times} \cup C_{\text{sing}},$$

such that

- C_{sing} is a closed subset of dimension$\leq i-2$.
- C_{reg} is an open and dense subset in C and it is a smooth i-submanifold in X.

$C_{\times} \cup C_{\text{sing}}$ is a closed subset of dimension $\leq i-1$. Locally, at every point, $x \in C_{\times}$ the union $C_{\text{reg}} \cup C_{\times}$ is diffeomorphic to a collection of smooth copies of $\mathbb{R}^i_+$ in X, called *branches*, meeting along their $\mathbb{R}^{i-1}$-boundaries where the basic example is the union of hypersurfaces in general position.

- The *$\mathbb{Z}$-multiplicity structure*, is given by an orientation of C_{reg} and a locally constant *multiplicity/weight* $\mathbb{Z}$-function on C_{reg}, (where for $i = 0$ there is only this function and no orientation) such that the sum of these oriented multiplicities over the branches of C at each point $x \in C_{\times}$ equals zero.

Every C can be modified to C' with empty $C'_{\times}$ and if $\operatorname{codim}(C) \geq 1$, i.e. $\dim(X) > \dim(C)$, also with weights $= \pm 1$.

For example, if $2l$ oriented branches of C_{reg} with multiplicities 1 meet at $C_{\times}$, divide them into l pairs with the partners having *opposite* orientations, keep these partners attached as they meet along $C_{\times}$ and separate them from the other pairs.

No matter how simple, this separation of branches is, say with the total weight $2l$, it can be performed in $l!$ different ways. Poor C' burdened with this ambiguity becomes rather non-efficient.

If X is a closed oriented n-manifold, then it itself makes an n-cycle which represents what is called the *fundamental class* $[X] \in H_n(X)$. Other n-cycles are integer combinations of the oriented connected components of X.

It is convenient to have singular counterparts to manifolds with boundaries. Since "chains" were appropriated by algebraic topologists, we use the word "plaque", where an $(i+1)$*-plaque* D with a boundary $\partial(D) \subset D$ is the same as a cycle, except that there is a subset $\partial(D)_\times \subset D_\times$, where the sums of oriented weights do not cancel, where the closure of $\partial(D)_\times$ equals $\partial(D) \subset D$ and where $\dim(\partial(D) \setminus \partial(D)_\times) \leq i-2$.

Geometrically, we impose the local conditions on $D \setminus \partial(D)$ as on $(i+1)$-cycles and add the local i-cycle conditions on (the closed set) $\partial(D)$, where this $\partial(D)$ comes with the canonical weighted orientation induced from D.

(There are *two* opposite canonical induced orientations on the boundary $C = \partial D$, e.g. on the circular boundary of the 2-disc, with no apparent rational for preferring one of the two. We choose the orientation in $\partial(D)$ defined by the frames of the tangent vectors $\tau_1, ..., \tau_i$ such that the orientation given to D by the $(i+1)$-frames $\nu, \tau_1, ..., \tau_i$ agrees with the original orientation, where ν is the *inward* looking normal vector.)

Every plaque can be "subdivided" by enlarging the set $D_\times$ (and/or, less essentially, D_{sing}). We do not care distinguishing such plaques and, more generally, the equality $D_1 = D_2$ means that the two plaques have a common subdivision.

We go further and write $D = 0$ if the weight function on D_{reg} equals zero.

We denote by $-D$ the plaque with the either minus weight function or with the opposite orientation.

We define $D_1 + D_2$ if there is a plaque D containing both D_1 and D_2 as its sub-plaques with the obvious addition rule of the weight functions.

Accordingly, we agree that $D_1 = D_2$ if $D_1 - D_2 = 0$.

On genericity. We have not used any genericity so far except for the definition of dimension. But from now on we assume all our object to be generic. This is needed, for example, to define $D_1 + D_2$, since the sum of arbitrary plaques *is not* a plaque, but the sum of generic plaques, obviously, *is.*

Also if you are used to genericity, it is obvious to you that

> if $D \subset X$ is an i-plaque (i-cycle) then the image $f(D) \subset Y$ under a generic map $f: X \to Y$ is an i-plaque (i-cycle).

Notice that for $\dim(Y) = i+1$ the self-intersection locus of the image $f(D)$ becomes a part of $f(D)_\times$ and if $\dim(Y) = i+1$, then the new part the $\times$-singularity comes from $f(\partial(D))$.

It is even more obvious that

> the pullback $f^{-1}(D)$ of an i-plaque $D \subset Y^n$ under a generic map $f: X^{m+n} \to Y^n$ is an $(i+m)$-plaque in X^{m+n}; if D is a cycle and X^{m+n} is a closed manifold (or the map f is proper), then $f^{-1}(D)$ is cycle.

As the last technicality, we extend the above definitions to arbitrary triangulated spaces X, with "smooth generic" substituted by "piecewise smooth generic" or by piecewise linear maps.

Homology. Two i-cycles C_1 and C_2 in X are called *homologous*, written $C_1 \sim C_2$, if there is an $(i+1)$-plaque D in $X \times [0,1]$, such that $\partial(D) = C_1 \times 0 - C_2 \times 1$.

For example *every contractible cycle $C \subset X$ is homologous to zero*, since the cone over C in $Y = X \times [0,1]$ corresponding to a smooth generic homotopy makes a plaque with its boundary equal to C.

Since small subsets in X are contractible, a cycle $C \subset X$ is homologous to zero if and only if it admits a decomposition into a sum of "arbitrarily small cycles", i.e. if, for every locally finite covering $X = \bigcup_i U_i$, there exist cycles $C_i \subset U_i$, such that $C = \sum_i C_i$.

The *homology group* $H_i(X)$ is defined as the Abelian group with generators $[C]$ for all i-cycles C in X and with the relations $[C_1] - [C_2] = 0$ whenever $C_1 \sim C_2$.

Similarly one defines $H_i(X;\mathbb{Q})$, for the field $\mathbb{Q}$ of rational numbers, by allowing C and D with fractional weights.

Example. Every closed orientable n-manifold X with k connected components has $H_n(X) = \mathbb{Z}^k$, where $H_n(X)$ is generated by the fundamental classes of its components.

This is obvious with our definitions since the only plaques D in $X \times [0,1]$ with $\partial(D) \subset \partial(X \times [0,1]) = X \times 0 \cup X \times 1$ are combination of the connected components of $X \times [0,1]$ and so $H_n(X)$ equals the group of n-cycles in X. Consequently,

every closed orientable manifold X is non-contractible.

The above argument may look suspiciously easy, since it is even hard to prove *non-contractibility of S^n* and issuing from this the *Brouwer fixed point theorem* within the world of *continuous* maps without using generic smooth or combinatorial ones, except for $n = 1$ with the covering map $\mathbb{R} \to S^1$ and for S^2 with the Hopf fibration $S^3 \to S^2$.

The catch is that the difficulty is hidden in the fact that a *generic* image of an $(n+1)$-plaque (e.g. a cone over X) in $X \times [0,1]$ is again an $(n+1)$-plaque. What is obvious, however, without any appeal to genericity is that $H_0(X) = \mathbb{Z}^k$ for every manifold or a triangulated space with k components.

The spheres S^n have $H_i(S^n) = 0$ for $0 < i < n$, since the complement to a point $s_0 \in S^n$ is homeomorphic to $\mathbb{R}^n$ and a generic cycles of dimension $< n$ misses s_0, while $\mathbb{R}^n$, being contractible, has zero homologies in positive dimensions.

It is clear that continuous maps $f : X \to Y$, when generically perturbed, define homomorphisms $f_{*i} : H_i(X) \to H_i(Y)$ for $C \mapsto f(C)$ and that

homotopic maps $f_1, f_2 : X \to Y$ induce equal homomorphisms $H_i(X) \to H_i(Y)$.

Indeed, the cylinders $C \times [0,1]$ generically mapped to $Y \times [0,1]$ by homotopies f_t, $t \in [0,1]$, are plaque D in our sense with $\partial(D) = f_1(C) - f_2(C)$. It follows, that

homology is invariant under homotopy equivalences $X \leftrightarrow Y$ for manifolds X, Y as well as for triangulated spaces.

Similarly, if $f : X^{m+n} \to Y^n$ is a proper (pullbacks of compact sets are compact) smooth generic map between *manifolds* where Y has no boundary, then the pullbacks of cycles define homomorphism, denoted, $f^! : H_i(Y) \to H_{i+m}(X)$, which is invariant under proper homotopies of maps.

The homology groups are much easier do deal with than the homotopy groups, since the definition of an i-cycle in X is purely local, while "spheres in X" cannot be recognized by looking at them point by point. (Holistic philosophers must feel triumphant upon learning this.)

Homologically speaking, a space *is* the sum of its parts: the locality allows an effective computation of homology of spaces X assembled of simpler pieces, such as cells, for example.

The locality+additivity is satisfied by the *generalized homology functors* that are defined, following Sullivan, by limiting possible singularities of cycles and plaques [**6**]. Some of these, e.g. *bordisms* we meet in the next section.

Definition. Degree of a map. Let $f : X \to Y$ be a smooth (or piece-wise smooth) generic map between closed connected oriented equidimensional manifolds Then the degree $\deg(f)$ can be (obviously) equivalently defined either as the image $f_*[X] \in \mathbb{Z} = H_n(Y)$ or as the $f^!$-image of the generator $[\bullet] \in H_0(Y) \in \mathbb{Z} = H_0(X)$.

For, example, l-sheeted covering maps $X \to Y$ have degrees l. Similarly, one sees that

> finite covering maps between arbitrary spaces are surjective on the rational homology groups.

To understand the local geometry behind the definition of degree, look closer at our f where X (still assumed compact) is allowed a non-empty boundary and observe that the f-pullback $\tilde{U}_y \subset X$ of some (small) open neighbourhood $U_y \subset Y$ of a generic point $y \in Y$ consists of finitely many connected components $\tilde{U}_i \subset \tilde{U}$, such that the map $f : \tilde{U}_i \to U_y$ is a diffeomorphism for all $\tilde{U}_i$.

Thus, every $\tilde{U}_i$ carries two orientations: one induced from X and the second from Y via f. The sum of +1 assigned to $\tilde{U}_i$ where the two orientation agree and of −1 when they disagree is called the *local degree* $\deg_y(f)$.

If two generic points $y_1, y_2 \in Y$ can be joined by a path in Y which does not cross the f-image $f(\partial(X)) \subset Y$ of the boundary of X, then $\deg_{y_1}(f) = \deg_{y_2}(f)$ since the f-pullback of this path, (which can be assumed generic) consists, besides possible closed curves, of several segments in Y, joining ±1-degree points in $f^{-1}(y_1) \subset \tilde{U}_{y_1} \subset X$ with ∓1-points in $f^{-1}(y_2) \subset \tilde{U}_{y_2}$.

Consequently, the local degree does not depend on y if X has no boundary. Then, clearly, it coincides with the homologically defined degree.

Similarly, one sees in this picture (without any reference to homology) that the local degree is invariant under generic homotopies $F : X \times [0,1] \to Y$, where the smooth (typically disconnected) pull-back curve $F^{-1}(y) \subset X \times [0,1]$ joins ±1-points in $F(x,0)^{-1}(y) \subset X = X \times 0$ with ∓1-points in $F(x,1)^{-1}(y) \subset X = X \times 1$.

Geometric versus algebraic cycles. Let us explain how the geometric definition matches the algebraic one for triangulated spaces X.

Recall that the homology of a triangulated space is algebraically defined with $\mathbb{Z}$-*cycles* which are $\mathbb{Z}$-chains, i.e. formal linear combinations $C_{\mathrm{alg}} = \sum_s k_s \Delta_s^i$ of oriented i-simplices Δ_s^i with integer coefficients k_s, where, by the definition of "algebraic cycle" , these sums have zero algebraic boundaries, which is equivalent to $c(C_{\mathrm{alg}}) = 0$ for every $\mathbb{Z}$-cocycle c cohomologous to zero (see Section 2).

But this is exactly the same as our generic cycles C_{geo} in the i-skeleton X_i of X and, tautologically, $C_{\mathrm{alg}} \overset{\mathrm{taut}}{\mapsto} C_{\mathrm{geo}}$ gives us a homomorphism from the algebraic homology to our geometric one.

On the other hand, an $(i+j)$-simplex minus its center can be radially homotoped to its boundary. Then the obvious reverse induction on skeleta of the triangulation shows that the space X minus a subset $\Sigma \subset X$ of codimension $i+1$ can be homotoped to the i-skeleton $X_i \subset X$.

Since every generic i-cycle C misses Σ it can be homotoped to X_i where the resulting map, say $f : C \to X_i$, sends C to an algebraic cycle.

At this point, the equivalence of the two definitions becomes apparent, where, observe, the argument applies to all cellular spaces X with piece-wise linear attaching maps.

The usual definition of homology of such an X amounts to working with all i-cycles contained in X_i and with $(i+1)$-plaques in X_{i+1}. In this case the group of i-cycles becomes a subspace of the group spanned by the i-cells, which shows, for example, that the rank of $H_i(X)$ does not exceed the number of i-cells in X_i.

We return to generic geometric cycles and observe that if X is a *non-compact* manifold, one may drop "compact" in the definition of these cycles. The resulting group is denoted $H_1(X, \partial_\infty)$. If X is compact with boundary, then this group of the interior of X is called the *relative homology group* $H_i(X, \partial(X))$. (The ordinary homology groups of this interior are canonically isomorphic to those of X.)

Intersection Ring. The intersection of cycles in general position in a smooth manifold X defines a multiplicative structure on the homology of an n-manifold X, denoted

$$[C_1] \cdot [C_2] = [C_1] \cap [C_2] = [C_1 \cap C_2] \in H_{n-(i+j)}(X)$$
$$\text{for } [C_1] \in H_{n-i}(X) \text{ and } [C_2] \in H_{n-j}(X),$$

where $[C] \cap [C]$ is defined by intersecting $C \subset X$ with its small generic perturbation $C' \subset X$.

(Here genericity is most useful: intersection is painful for simplicial cycles confined to their respective skeleta of a triangulation. On the other hand, if X is a *not* a manifold one may adjust the definition of cycles to the local topology of the singular part of X and arrive at what is called the *intersection homology.*)

It is obvious that the intersection is respected by $f^!$ for proper maps f, but not for f_*. The former implies. in particular, that this product is invariant under oriented (i.e. of degrees +1) homotopy equivalences between *closed equidimensional* manifolds. (But $X \times \mathbb{R}$, which is homotopy equivalent to X has trivial intersection ring, whichever is the ring of X.)

Also notice that the intersection of cycles of *odd* codimensions is *anti-commutative* and if one of the two has *even* codimension it is *commutative.*

The intersection of two cycles of complementary dimensions is a 0-cycle, the total $\mathbb{Z}$-weight of which makes sense if X is oriented; it is called *the intersection index of the cycles.*

Also observe that the intersection between C_1 and C_2 equals the intersection of $C_1 \times C_2$ with the diagonal $X_{\text{diag}} \subset X \times X$.

Example (a). The intersection ring of the complex projective space $\mathbb{C}P^k$ is multiplicatively generated by the homology class of the hyperplane, $[\mathbb{C}P^{k-1}] \in H_{2k-2}(\mathbb{C}P^k)$, with the only relation $[\mathbb{C}P^{k-1}]^{k+1} = 0$ and where, obviously, $[\mathbb{C}P^{k-i}] \cdot [\mathbb{C}P^{k-j}] = [\mathbb{C}P^{k-(i+j)}]$.

The only point which needs checking here is that the homology class $[\mathbb{C}P^i]$ *(additively) generates* $H_i(\mathbb{C}P^k)$, which is seen by observing that $\mathbb{C}P^{i+1} \setminus \mathbb{C}P^i$, $i = 0, 1, ..., k-1$, is an open $(2i+2)$-cell, i.e. the open topological ball B^{2i+2}_{op} (where the cell attaching map $\partial(B^{2i+2}) = S^{2i+1} \to \mathbb{C}P^i$ is the quotient map $S^{2i+1} \to S^{2i+1}/\mathbb{T} = \mathbb{C}P^{i+1}$ for the obvious action of the multiplicative group $\mathbb{T}$ of the complex numbers with norm 1 on $S^{2i+1} \subset \mathbb{C}^{2i+1}$).

Example (b). The intersection ring of the n-torus is isomorphic to the exterior algebra on n-generators, i.e. the only relations between the multiplicative generators $h_i \in H_{n-1}(\mathbb{T}^n)$ are $h_i h_j = -h_j h_i$, where h_i are the homology classes of the n coordinate subtori $\mathbb{T}^{n-1}_i \subset \mathbb{T}^n$.

This follows from the Ḱunneth formula below, but can be also proved directly with the obvious cell decomposition of $\mathbb{T}^n$ into 2^n cells.

The intersection ring structure immensely enriches homology. Additively, $H_* = \oplus_i H_i$ is just a graded Abelian group – the most primitive algebraic object (if finitely generated) – fully characterized by simple numerical invariants: the rank and the orders of their cyclic factors.

But the ring structure, say on H_{n-2} of an n-manifold X, for $n = 2d$ defines a symmetric d-form, on $H_{n-2} = H_{n-2}(X)$ which is, a polynomial of degree d in r variables with integer coefficients for $r = \mathrm{rank}(H_{n-2})$. All number theory in the world cannot classify these for $d \geq 3$ (to be certain, for $d \geq 4$).

One can also intersect non-compact cycles, where an intersection of a compact C_1 with a non-compact C_2 is compact; this defines the *intersection pairing*

$$H_{n-i}(X) \otimes H_{n-j}(X, \partial_\infty) \overset{\cap}{\to} H_{n-(i+j)}(X).$$

Finally notice that generic 0 cycles C in X are finite sets of points $x \in X$ with the "orientation" signs ± 1 attached to each x in C, where the sum of these ± 1 is called the *index of* C. If X is *connected*, then $ind(C) = 0$ if and only if $[C] = 0$.

Thom Isomorphism. Let $p : V \to X$ be a *fiber-wise oriented* smooth (which is unnecessary) $\mathbb{R}^N$-bundle over X, where $X \subset V$ is embedded as the zero section and let $V_\bullet$ be Thom space of V. Then there are two natural homology homomorphisms.

> *Intersection.* $\cap : H_{i+N}(V_\bullet) \to H_i(X)$. This is defined by intersecting generic $(i+N)$-cycles in $V_\bullet$ with X.
> *Thom Suspension.* $S_\bullet : H_i(X) \to H_i(V_\bullet)$, where every cycle $C \subset X$ goes to the Thom space of the restriction of V to C, i.e. $C \mapsto (p^{-1}(C))_\bullet \subset V_\bullet$.

These $\cap$ and $S_\bullet$ are mutually reciprocal. Indeed $(\cap \circ S_\bullet)(C) = C$ for all $C \subset X$ and also $(S_\bullet \circ \cap)(C') \sim C'$ for all cycles C' in $V_\bullet$ where the homology is established by the fiberwise radial homotopy of C' in $V_\bullet \supset V$, which fixes $\bullet$ and move each $v \in V$ by $v \mapsto tv$. Clearly, $tC' \to (S_\bullet \circ \cap)(C')$ as $t \to \infty$ for all generic cycles C' in $V_\bullet$.

Thus we arrive at the Thom isomorphism

$$H_i(X) \leftrightarrow H_{i+N}(V_\bullet).$$

Similarly we see that

> The Thom space of every $\mathbb{R}^N$-bundle $V \to X$ is $(N-1)$-connected, i.e. $\pi_j(V_\bullet) = 0$ for $j = 1, 2, ...N-1$.

Indeed, a generic j-sphere $S^j \to V_\bullet$ with $j < N$ does not intersect $X \subset V$, where X is embedded into V by the zero section. Therefore, this sphere radially (in the fibers of V) contracts to $\bullet \in V_\bullet$.

Euler Class. Let $f : X \to B$ be a fibration with $\mathbb{R}^{2k}$-fibers over a smooth closed oriented manifold B. Then the intersection indices of $2k$-cycles in B with $B \subset X$, embedded as the zero section, defines an *integer cohomology class*, i.e. a *homomorphism* (additive map) $e : H_{2k}(B) \to \mathbb{Z} \subset \mathbb{Q}$, called the *Euler class* of the fibration. (In fact, one does not need B to be a manifold for this definition.)

Observe that the Euler number *vanishes* if and only if the homology projection homomorphism ${}_0f_{*2k} : H_{2k}(V \smallsetminus B; \mathbb{Q}) \to H_{2k}(B; \mathbb{Q})$ is *surjective*, where $B \subset X$ is embedded by the zero section $b \mapsto 0_b \in \mathbb{R}^k_b$ and ${}_0f : V \smallsetminus B \to B$ is the restriction of the map (projection) f to $V \smallsetminus B$.

Moreover, it is easy to see that the *ideal* in $H^*(B)$ generated by the Euler class (for the $\smile$*-ring structure on cohomology* defined later in this section) equals the kernel of the cohomology homomorphism ${}_0f^* : H^*(B) \to H^*(V \smallsetminus B)$.

If B is a closed connected oriented manifold, then $e[B]$ is called the *Euler number* of $X \to B$ also denoted e.

In other words, the number e equals the self-intersection index of $B \subset X$. Since the intersection pairing is symmetric on H_{2k} the sign of the Euler number does not depend on the orientation of B, but it does depend on the orientation of X.

Also notice that if X is embedded into a larger $4k$-manifold $X' \supset X$ then the self-intersection index of B in X' equals that in X.

If X equals the tangent bundle $T(B)$ then X is canonically oriented (even if B is non-orientable) and the Euler number is non-ambiguously defined and it equals the self-intersection number of the diagonal $X_{diag} \subset X \times X$.

Theorem (Poincaré-Hopf Formula). *The Euler number e of the tangent bundle $T(B)$ of every closed oriented $2k$-manifold B satisfies*

$$e = \chi(B) = \sum_{i=0,1,\ldots 2k} rank(H_i(B; \mathbb{Q})).$$

(If $n = \dim(B)$ is odd, then $\sum_{i=0,1,\ldots n} rank(H_i(B;\mathbb{Q})) = 0$ by the Poincaré duality.)

It is hard the believe this may be true! A single cycle (let it be the *fundamental* one) knows something about all of the homology of B.

The most transparent proof of this formula is, probably, via the Morse theory (known to Poincaré) and it hardly can be called "trivial".

A more algebraic proof follows from the K'unneth formula (see below) and an expression of the class $[X_{\text{diag}}] \in H_{2k}(X \times X)$ in terms of the intersection ring structure in $H_*(X)$.

The Euler number can be also defined for connected *non-orientable* B as follows. Take the canonical oriented double covering $\tilde{B} \to B$, where each point $\tilde{b} \in \tilde{B}$ over $b \in B$ is represented as b + an orientation of B near b. Let the bundle $\tilde{X} \to \tilde{B}$ be induced from X by the covering map $\tilde{B} \to B$, i.e. this $\tilde{X}$ is the obvious double covering of X corresponding to $\tilde{B} \to B$. Finally, set $e(X) = e(\tilde{X})/2$.

The Poincaré-Hopf formula for non-orientable $2k$-manifolds B follows from the orientable case by the *multiplicativity* of the Euler characteristic χ which is valid for all compact triangulated spaces B,

an l-sheeted covering $\tilde{B} \to B$ has $\chi(\tilde{B}) = l \cdot \chi(B)$.

If the homology is defined via a triangulation of B, then $\chi(B)$ equals the alternating sum $\sum_i (-1)^i N(\Delta^i)$ of the numbers of i-simplices by straightforward linear algebra and the multiplicativity follows. But this is not so easy with our geometric cycles. (If B is a closed manifold, this also follows from the Poincaré-Hopf formula and the obvious multiplicativity of the Euler number for covering maps.)

Theorem (Ḱunneth Theorem). *The rational homology of the Cartesian product of two spaces equals the graded tensor product of the homologies of the factors. In fact, the natural homomorphism*

$$\bigoplus_{i+j=k} H_i(X_1;\mathbb{Q}) \otimes H_j(X_2;\mathbb{Q}) \to H_k(X_1 \times X_2;\mathbb{Q}),\ \ k = 0,1,2,...$$

is an isomorphism. Moreover, if X_1 and X_2 are closed oriented manifolds, this homomorphism is compatible (if you say it right) with the intersection product.

This is obvious if X_1 and X_2 have cell decompositions such that the numbers of i-cells in each of them equals the ranks of their respective H_i. In the general case, the proof is cumbersome unless you pass to the language of chain complexes where the difficulty dissolves in linear algebra. (Yet, keeping track of geometric cycles may be sometimes necessary, e.g. in the algebraic geometry, in the geometry of foliated cycles and in evaluating the so called *filling profiles* of products of Riemannian manifolds.)

Theorem (Poincaré $\mathbb{Q}$-Duality). *Let X be a connected oriented n-manifold. The intersection index establishes a linear duality between homologies of complementary dimensions:*

$$H_i(X;\mathbb{Q}) \textit{ equals the } \mathbb{Q}\textit{-linear dual of } H_{n-i}(X,\partial_\infty;\mathbb{Q}).$$

In other words, the intersection pairing

$$H_i(X) \otimes H_{n-i}(X,\partial_\infty) \overset{\cap}{\to} H_0(X) = \mathbb{Z}$$

is $\mathbb{Q}$-faithful: a multiple of a compact i-cycle C is homologous to zero if and only if its intersection index with every non-compact $(n-i)$-cycle in general position equals zero.

Furthermore, if X equals the interior of a compact manifolds with a boundary, then a multiple of a non-compact cycle is homologous to zero if and only if its intersection index with every compact generic cycle of the complementary dimension equals zero.

Proof of $(H_i \leftrightarrow H^{n-i})$ for Closed Manifolds X. We, regretfully, break the symmetry by choosing some *smooth* triangulation T of X which means this T is locally as good as a triangulation of $\mathbb{R}^n$ by affine simplices (see below).

Granted T, assign to each generic i-cycle $C \subset X$ the intersection index of C with every oriented Δ^{n-i} of T and observe that the resulting function $c^\perp : \Delta^{n-i} \mapsto ind(\Delta^{n-i} \cap C)$ is a $\mathbb{Z}$-valued cocycle (see section 2), since the intersection index of C with every $(n-i)$-sphere $\partial(\Delta^{n-i+1})$ equals zero, because these spheres are homologous to zero in X.

Conversely, given a $\mathbb{Z}$-cocycle $c(\Delta^{n-i})$ construct an i-cycle $C_\perp \subset X$ as follows. Start with $(n-i+1)$-simplices Δ^{n-i+1} and take in each of them a smooth oriented curve S with the boundary points located at the centers of the $(n-i)$-faces of Δ^{n-i+1}, where S is normal to a face Δ^{n-i} whenever it meets one and such that the intersection index of the (slightly extended across Δ^{n-i}) curve S with Δ^{n-i} equals $c(\Delta^{n-i})$. Such a curve, (obviously) exists because the function c is a *cocycle*. Observe, that the union of these S over all $(n-i+1)$-simplices in the boundary sphere $S^{n-i+1} = \partial\Delta^{n-i+2}$ of every $(n-i+2)$-simplex in T is a *closed* (disconnected) curve in S^{n-i+1}, the intersection index of which with every $(n-i)$-simplex $\Delta^{n-i} \subset S^{n-i+1}$ equals $c(\Delta^{n-i})$ (where this intersection index is evaluated in S^{n-i+1} but *not* in X).

Then construct by induction on j the (future) intersection $C_\perp^j$ of $C_\perp$ with the $(n-i+j)$-skeleton T_{n-i+j} of our triangulation by taking the cone from the center of each simplex $\Delta^{n-i+j} \subset T_{n-i+j}$ over the intersection of $C_\perp^j$ with the boundary sphere $\partial(\Delta^{n-i+j})$.

It is easy to see that the resulting $C_\perp$ is an i-cycle and that the composed maps $C \to c^\perp \to C_\perp$ and $c \to C_\perp \to c^\perp$ define identity homomorphisms $H_i(X) \to H_i(X)$ and $H^{n-i}(X;\mathbb{Z}) \to H^{n-i}(X;\mathbb{Z})$ correspondingly and we arrive at the Poincaré $\mathbb{Z}$-isomorphism,

$$H_i(X) \leftrightarrow H^{n-i}(X;\mathbb{Z}).$$

To complete the proof of the $\mathbb{Q}$-duality one needs to show that $H^j(X;\mathbb{Z}) \otimes \mathbb{Q}$ equals the $\mathbb{Q}$-linear dual of $H_j(X;\mathbb{Q})$. To do this we represent $H_i(X)$ by algebraic $\mathbb{Z}$-cycles $\sum_j k_j\Delta^i$ and now, in the realm of algebra, appeal to the linear duality between homologies of the chain and cochain complexes of T:

> the natural pairing between classes $h \in H_i(X)$ and $c \in H^i(X;\mathbb{Z})$, which we denote $(h,c) \mapsto c(h) \in \mathbb{Z}$, establishes, when tensored with $\mathbb{Q}$, an isomorphism between $H^i(X;\mathbb{Q})$ and the $\mathbb{Q}$-linear dual of $H_i(X;\mathbb{Q})$
>
> $$H^i(X;\mathbb{Q}) \leftrightarrow \mathrm{Hom}[H_i(X;\mathbb{Q})] \to Q]$$
>
> for all compact triangulated spaces X.

□

Corollaries. (a) The non-obvious part of the Poincaré duality is the claim that, for ever $\mathbb{Q}$-*homologically non-trivial* cycle C, there is a cycle C' of the complementary dimension, such that the intersection index between C and C' *does not vanish.*

But the easy part of the duality is also useful, as it allows one to give a *lower* bound on the homology by producing sufficiently many *non-trivially* intersecting cycles of complementary dimensions.

For example it shows that closed manifolds are non-contractible (where it reduces to the degree argument). Also it implies that the Künneth pairing $H_*(X;\mathbb{Q})\otimes H_*(Y;\mathbb{Q}) \to H_*(X \times Y;\mathbb{Q})$ is injective for closed orientable manifolds X.

(b) Let $f : X^{m+n} \to Y^n$ be a smooth map between closed orientable manifolds such the homology class of the pullback of a generic point is not homologous to zero, i.e. $0 \neq [f^{-1}(y_0)] \in H_m(X)$. Then the homomorphisms $f^{!i} : H_i(Y;\mathbb{Q}) \to H_{i+m}(X;\mathbb{Q})$ are *injective* for all i.

Indeed, every $h \in H_i(Y;\mathbb{Q})$ different from zero comes with an $h' \in H_{n-i}(Y)$ such that the intersection index d between the two is $\neq 0$. Since the intersection of $f^!(h)$ and $f^!(h')$ equals $d[f^{-1}(y_0)]$ none of $f^!(h)$ and $f^!(h')$ equals zero. Consequently/similarly *all* $f_{*j} : H_j(X) \to H_j(Y)$ *are surjective.* For example,

(b1) *Equidimensional maps f of positive degrees between closed oriented manifolds are surjective on rational homology.*

(b2) Let $f : X \to Y$ be a smooth fibration where the fiber is a closed oriented manifold with *non-zero Euler characteristic*, e.g. homeomorphic to S^{2k}. Then the fiber is *non-homologous to zero*, since the Euler class e of the fiberwise tangent bundle, which defined on *all* of X, does not vanish on $f^{-1}(y_0)$; hence, f_* *is surjective.*

Recall that the unit tangent bundle fibration $X = UT(S^{2k}) \to S^{2k} = Y$ with S^{2k-1}-fibers has $H_i(X;\mathbb{Q}) = 0$ for $1 \le i \le 4k-1$, since the Euler class of $T(S^{2k})$ *does not* vanish; hence; f_* *vanishes on all* $H_i(X;\mathbb{Q})$, $i > 0$.

Geometric Cocycles. We gave only a combinatorial definition of cohomology, but this can be defined more invariantly with geometric i-cocycles c being "generically locally constant" functions on oriented plaques D such that $c(D) = -c(-D)$ for reversing the orientation in D, where $c(D_1 + D_2) = c(D_1) + c(D_2)$ and where the final cocycle condition reads $c(C) = 0$ for all *i-cycles C* which are *homologous to zero.* Since every $C \sim 0$ decomposes into a sum of small cycles, the condition $c(C) = 0$ needs to be verified only for (arbitrarily) "small cycles" C.

Cocycles are as good as Poincaré's dual cycles for detecting non-triviality of geometric cycles C: if $c(C) \neq 0$, then, C is non-homologous to zero and also c is not cohomologous to zero.

If we work with $H^*(X;\mathbb{R})$, these cocycles $c(D)$ can be averaged over measures on the space of smooth self-mapping $X \to X$ homotopic to the identity. (The averaged cocycles are kind of duals of generic cycles.) Eventually, they can be reduced to differential forms invariant under a given compact connected automorphism group of X, that let cohomology return to geometry by the back door.

On Integrality of Cohomology. In view of the above, the rational cohomology classes $c \in H^i(X;\mathbb{Q})$ can be *defined* as homomorphisms $c : H_i(X) \to \mathbb{Q}$. Such a c is called *integer* if its image is contained in $\mathbb{Z} \subset \mathbb{Q}$. (Non-integrality of certain classes underlies the existence of *nonstandard smooth structures* on topological spheres discovered by Milnor, see Section 6.)

The $\mathbb{Q}$-duality does not tell you the whole story. For example, the following simple property of *closed n-manifolds X* depends on the full homological duality:

Connectedness/Contractibility. If X is a closed k-connected n-manifold, i.e. $\pi_i(X) = 0$ for $i = 1, ..., k$, then the complement to a point, $X \smallsetminus \{x_0\}$, is $(n-k-1)$-contractible, i.e. there is a homotopy f_t of the identity map $X \smallsetminus \{x_0\} \to X \smallsetminus \{x_0\}$ with $P = f_1(X \smallsetminus \{x_0\})$ being a smooth triangulated subspace $P \subset X \smallsetminus \{x_0\}$ with $\mathrm{codim}(P) \ge k+1$.

For example, if $\pi_i(X) = 0$ for $1 \le i \le n/2$, then X is homotopy equivalent to S^n.

Smooth triangulations. Recall, that "smoothness" of a triangulated subset in a smooth n-manifold, say $P \in X$, means that, for every closed i-simplex $\Delta \subset P$, there exist

- an open subset $U \subset X$ which contains Δ,
- an affine triangulation P' of $\mathbb{R}^n$, $n = dim(X)$,
- a diffeomorphism $U \to U' \subset \mathbb{R}^n$ which sends Δ onto an i-simplex Δ' in P'.

Accordingly, one defines the notion of a *smooth triangulation T* of a smooth manifold X, where one also says that the *smooth structure in X is compatible* with T. Every smooth manifold X can be given a smooth triangulation, e.g. as follows.

Let S be an affine (i.e. by affine simplices) triangulation of $\mathbb{R}^M$ which is invariant under the action of a lattice $\Gamma = \mathbb{Z}^M \subset \mathbb{R}^M$ (i.e. S is induced from a triangulation of the M-torus $\mathbb{R}^M/\Gamma$) and let $X \subset_f \mathbb{R}^M$ be a smoothly embedded (or immersed) closed n-submanifold. Then there (obviously) exist

- an arbitrarily small positive constant $\delta_0 = \delta_0(S) > 0$,
- an arbitrarily large constant $\lambda \geq \lambda_0(X, f, \delta_0) > 0$,
- δ-small moves of the vertices of S for $\delta \leq \delta_0$, where these moves themselves depend on the embedding f of X into $\mathbb{R}^M$ and on λ, such that the simplices of the correspondingly moved triangulation, say $S' = S'_\delta = S'(X, f)$ are δ'-*transversal* to the λ-scaled X, i.e. to $\lambda X = X \subset_{\lambda f} \mathbb{R}^M$.

The δ'-*transversality* of an affine simplex $\Delta' \subset \mathbb{R}^M$ to $\lambda X \subset \mathbb{R}^M$ means that the affine simplices Δ'' obtained from Δ' by arbitrary δ'-moves of the vertices of Δ' for some $\delta' = \delta'(S, \delta) > 0$ are transversal to λX. In particular, the intersection "angles" between λX and the i-simplices, $i = 0, 1, ..., M-1$, in S' are all $\geq \delta'$.

If λ is sufficiently large (and hence, $\lambda X \subset \mathbb{R}^M$ is nearly flat), then the δ'-transversality (obviously) implies that the intersection of λX with each simplex and its neighbours in S' in the vicinity of each point $x \in \lambda X \subset \mathbb{R}^M$ has the same combinatorial pattern as the intersection of the tangent space $T_x(\lambda X) \subset \mathbb{R}^M$ with these simplices. Hence, the (cell) partition $\Pi = \Pi_{f'}$ of λX induced from S' can be subdivided into a triangulation of $X = \lambda X$.

Almost all of what we have presented in this section so far was essentially understood by Poincaré, who switched at some point from geometric cycles to triangulations, apparently, in order to prove his duality. (See [**41**] for pursuing further the first Poincaré approach to homology.)

The language of geometic/generic cycles suggested by Poincaré is well suited for observing and proving the multitude of obvious little things one comes across every moment in topology. (I suspect, geometric, even worse, some algebraic topologists think of cycles while they draw commutative diagrams. Rephrasing J.B.S. Haldane's words: "Geometry is like a mistress to a topologist: he cannot live without her but he's unwilling to be seen with her in public".)

But if you are *far* away from manifolds in the homotopy theory it is easier to work with *cohomology* and use the *cohomology product* rather than intersection product.

The cohomology product is a bilinear pairing, often denoted $H^i \otimes H^j \overset{\smile}{\to} H^{i+j}$, which is the Poincaré dual of the intersection product $H_{n-i} \otimes H_{n-j} \overset{\cap}{\to} H_{n-i-j}$ in closed oriented n-manifolds X.

The $\smile$-product can be defined for all, say triangulated, X as the dual of the intersection product on the relative homology, $H_{M-i}(U; \infty) \otimes H_{M-j}(U; \infty) \to H_{M-i-j}(U; \infty)$, for a small regular neighbourhood $U \supset X$ of X embedded into some $\mathbb{R}^M$. The $\smile$ product, so defined, is invariant under continuous maps $f : X \to Y$:

$$f^*(h_1 \smile h_2) = f^*(h_1) \smile f^*(h_2) \text{ for all } h_1, h_2 \in H^*(Y).$$

It easy to see that the $\smile$-pairing equals the composition of the K'unneth homomorphism $H^*(X) \otimes H^*(X) \to H^*(X \times X)$ with the restriction to the diagonal $H^*(X \times X) \to H^*(X_{\text{diag}})$.

You can hardly expect to arrive at anything like Serre's finiteness theorem without a linearized (co)homology theory; yet, geometric constructions are of a great help on the way.

Topological and $\mathbb{Q}$-manifolds. The combinatorial proof of the Poincaré duality is the most transparent for open subsets $X \subset \mathbb{R}^n$ where the standard decomposition S of $\mathbb{R}^n$ into cubes is the combinatorial dual of its own translate by a generic vector.

Poincaré duality remains valid for all oriented *topological* manifolds X and also for all *rational homology* or $\mathbb{Q}$-*manifolds*, that are compact triangulated n-spaces where the *link* $L^{n-i-1} \subset X$ of every i-simplex Δ^i in X has the same rational homology as the sphere S^{n-i-1}, where it follows from the (special case of) *Alexander duality.*

> The rational homology of the complement to a topologically embedded k-sphere as well as of a rational homology sphere, into S^n (or into a $\mathbb{Q}$-manifold with the rational homology of S^n) equals that of the $(n-k-1)$-sphere.

(The link $L^{n-i-1}(\Delta^i)$ is the union of the simplices $\Delta^{n-i-1} \subset X$ which do not intersect Δ^i and for which there exists an simplex in X which contains Δ^i and Δ^{n-i-1}.)

Alternatively, an n-dimensional space X can be embedded into some $\mathbb{R}^M$ where the duality for X reduces to that for "suitably regular" neighbourhoods $U \subset \mathbb{R}^M$ of X which admit Thom isomorphisms $H_i(X) \leftrightarrow H_{i+M-n}(U_\bullet)$.

If X is a topological manifold, then "locally generic" cycles of complementary dimension intersect at a discrete set which allows one to define their geometric intersection index. Also one can define the intersection of several cycles C_j, $j = 1, ...k$, with $\sum_j \dim(C_i) = \dim(X)$ as the intersection index of $\times_j C_j \subset X^k$ with $X_{\text{diag}} \subset X^k$, but anything more then that can not be done so easily.

Possibly, there is a comprehensive formulation with an obvious invariant proof of the "functorial Poincaré duality" which would make transparent, for example, the multiplicativity of the signature (see below) and the topological nature of rational Pontryagin classes (see section 10) and which would apply to "cycles" of dimensions βN where $N = \infty$ and $0 \leq \beta \leq 1$ in spaces like these we shall meet in section 11.

Signature. The intersection of (compact) k-cycles in an oriented, possibly non-compact and/or disconnected, $2k$-manifold X defines *a bilinear* form on the homology $H_k(X)$. If k is odd, this form is antisymmetric and if k is even it is symmetric.

The signature of the latter, i.e. the number of positive minus the number of negative squares in the diagonalized form, is called $\text{sig}(X)$. This is well defined if $H_k(X)$ has finite rank, e.g. if X is compact, possibly with a boundary.

Geometrically, a diagonalization of the intersection form is achieved with a *maximal* set of *mutually disjoint* k-cycles in X where each of them has a *non-zero* (positive or negative) self-intersection index. (If the cycles are represented by smooth closed oriented k-*submanifolds*, then these indices equal the Euler numbers of the normal bundles of these submanifolds. In fact, such a maximal system of submanifolds always exists as it was derived by Thom from the Serre finiteness theorem.)

Example (a). $S^{2k} \times S^{2k}$ has zero signature, since the $2k$-homology is generated by the classes of the two coordinate spheres $[s_1 \times S^{2k}]$ and $[S^{2k} \times s_2]$, which both have zero self-intersections.

Example (b). The complex projective space $\mathbb{C}P^{2m}$ has signature one, since its middle homology is generated by the class of the complex projective subspace $\mathbb{C}P^m \subset \mathbb{C}P^{2m}$ with the self-intersection = 1.

Example (c). The tangent bundle $T(S^{2k})$ has signature 1, since $H_k(T(S^{2k}))$ is generated by $[S^{2k}]$ with the self-intersection equal the Euler characteristic $\chi(S^{2k}) = 2$.

It is obvious that $\mathrm{sig}(mX) = m \cdot \mathrm{sig}(X)$, where mX denotes the disjoint union of m copies of X, and that $\mathrm{sig}(-X) = -\mathrm{sig}(X)$, where "−" signifies reversion of orientation. Furthermore

Theorem (Rokhlin 1952). *The oriented boundary X of every compact oriented $(4k+1)$-manifold Y has zero signature.*

(Oriented boundaries of non-orientable manifolds may have non-zero signatures. For example the double covering $\tilde{X} \to X$ with $\mathrm{sig}(\tilde{X}) = 2\mathrm{sig}(X)$ non-orientably bounds the corresponding 1-ball bundle Y over X.)

Proof. If k-cycles C_i, $i = 1, 2$, bound relative $(k+1)$-cycles D_i in Y, then the (zero-dimensional) intersection C_1 with C_2 bounds a relative 1-cycle in Y which makes the index of the intersection zero. Hence,

> the intersection form vanishes on the kernel $\ker_k \subset H_k(X)$ of the inclusion homomorphism $H_k(X) \to H_k(Y)$.

On the other hand, the obvious identity

$$[C \cap D]_Y = [C \cap \partial D]_X$$

and the Poincaré duality in Y show that the orthogonal complement $\ker_k^\perp \subset H_k(X)$ with respect to the intersection form in X is contained in $\ker_k$. □

Observe that this argument depends entirely on the Poincaré duality and it equally applies to the topological and $\mathbb{Q}$-manifolds with boundaries.

Also notice that the K'unneth formula and the Poincaré duality (trivially) imply the Cartesian multiplicativity of the signature for closed manifolds,

$$\mathrm{sig}(X_1 \times X_2) = \mathrm{sig}(X_1) \cdot \mathrm{sig}(X_2).$$

For example, the products of the complex projective spaces $\times_i \mathbb{C}P^{2k_i}$ have signatures one. (The K'unneth formula is obvious here with the cell decompositions of $\times_i \mathbb{C}P^{2k_i}$ into $\times_i (2k_i + 1)$ cells.)

Amazingly, the *multiplicativity of the signature* of *closed* manifolds under covering maps can not be seen with comparable clarity.

Theorem (Multiplicativity Formula). *If $\tilde{X} \to X$ is an l-sheeted covering map, then*

$$\mathrm{sign}(\tilde{X}) = l \cdot \mathrm{sign}(X).$$

This can be sometimes proved by elementary, means, e.g. if the fundamental group of X is *free*. In this case, there obviously exist closed hypersurfaces $Y \subset X$ and $\tilde{Y} \subset \tilde{X}$ such that $\tilde{X} \setminus \tilde{Y}$ is diffeomorphic to the disjoint union of l copies of $X \setminus Y$. This implies multiplicativity, since *signature is additive*:

> removing a closed hypersurface from a manifold does not change the signature.

Therefore,

$$\mathrm{sig}(\tilde X) = \mathrm{sig}(\tilde X \smallsetminus \tilde Y) = l \cdot \mathrm{sig}(X \smallsetminus Y) = l \cdot \mathrm{sig}(X).$$

(This "additivity of the signature" easily follows from the Poincaré duality as observed by S. Novikov.)

In general, given a finite covering $\tilde X \to X$, there exists an *immersed* hypersurface $Y \subset X$ (with possible self-intersections) such that the covering trivializes over $X \smallsetminus Y$; hence, $\tilde X$ can be assembled from the pieces of $X \smallsetminus Y$ where each piece is taken l times. One still has an addition formula for some "stratified signature" but it is rather involved in the general case.

On the other hand, the multiplicativity of the signature can be derived in a couple of lines from the Serre finiteness theorem (see below).

5. The Signature and Bordisms

Let us prove the multiplicativity of the signature by constructing a compact oriented manifold Y with a boundary, such that the *oriented* boundary $\partial(Y)$ equals $m\tilde X - mlX$ for some integer $m \neq 0$.

Embed X into $\mathbb{R}^{n+N}$, $N >> n = 2k = \dim(X)$ let $\tilde X \subset \mathbb{R}^{n+N}$ be an embedding obtained by a small generic perturbation of the covering map $\tilde X \to X \subset \mathbb{R}^{n+N}$ and $\tilde X' \subset \mathbb{R}^{n+N}$ be the union of l generically perturbed copies of X.

Let $\tilde A_\bullet$ and $\tilde A'_\bullet$ be the Atiyah-Thom maps from $S^{n+N} = \mathbb{R}^{n+N}_\bullet$ to the Thom spaces $\tilde U_\bullet$ and $U'_\bullet$ of the normal bundles $\tilde U \to \tilde X$ and $\tilde U' \to \tilde X'$.

Let $\tilde P : \tilde X \to X$ and $\tilde P' : \tilde X' \to X$ be the normal projections. These projections, obviously, induce the normal bundles $\tilde U$ and $\tilde U'$ of $\tilde X$ and $\tilde X'$ from the normal bundle $U^\perp \to X$. Let

$$\tilde P_\bullet : \tilde U_\bullet \to U^\perp_\bullet \text{ and } \tilde P'_\bullet : \tilde U'_\bullet \to U^\perp_\bullet$$

be the corresponding maps between the Thom spaces and let us look at the two maps f and f' from the sphere $S^{n+N} = \mathbb{R}^{N+n}_\bullet$ to the Thom space $U^\perp_\bullet$,

$$f = \tilde P_\bullet \circ \tilde A_\bullet : S^{n+N} \to U^\perp_\bullet, \text{ and } f' = \tilde P'_\bullet \circ \tilde A'_\bullet : S^{n+N} \to U^\perp_\bullet.$$

Clearly

$$f^{-1}(X) = \tilde X \text{ and } (f')^{-1}(X) = \tilde X'. \qquad [\tilde\bullet\tilde\bullet']$$

On the other hand, the homology homomorphisms of the maps f and f' are related to those of $\tilde P$ and $\tilde P'$ via the Thom suspension homomorphism $S_\bullet : H_n(X) \to H_{n+N}(U^\perp_\bullet)$ as follows

$$f_*[S^{n+N}] = S_\bullet \circ \tilde P_*[\tilde X] \text{ and } f'_*[S^{n+N}] = S_\bullet \circ \tilde P'_*[\tilde X'].$$

Since $deg(\tilde P) = deg(\tilde P') = l$,

$$\tilde P_*[\tilde X] = \tilde P'_*[\tilde X'] = l \cdot [X] \text{ and } f'[S^{n+N}] = f[S^{n+N}] = l \cdot S_\bullet[X] \in H_{n+N}(U^\perp_\bullet);$$

hence,

> some non-zero m-multiples of the maps $f, f' : S^{n+N} \to U^\perp_\bullet$ can be joined by a (smooth generic) homotopy $F : S^{n+N} \times [0,1] \to U^\perp_\bullet$

by Serre's theorem, since $\pi_i(U^\perp_\bullet) = 0$, $i = 1, ...N-1$.

Then, because of $[\tilde{\bullet}\tilde{\bullet}']$, the pullback $F^{-1}(X) \subset S^{n+N} \times [0,1]$ *establishes a bordism between* $m\tilde{X} \subset S^{n+N} \times 0$ *and* $m\tilde{X}' = mlX \subset S^{n+N} \times 1$. This implies that $m \cdot \mathrm{sig}(\tilde{X}) = ml \cdot \mathrm{sig}(X)$ and since $m \neq 0$ we get $\mathrm{sig}(\tilde{X}) = l \cdot \mathrm{sig}(X)$. QED.

Bordisms and the Rokhlin-Thom-Hirzebruch Formula. Let us modify our definition of homology of a manifold X by allowing only *non-singular* i-cycles in X, i.e. smooth closed oriented i-submanifolds in X and denote the resulting Abelian group by $\mathcal{B}^o_i(X)$.

If $2i \geq n = \dim(X)$ one has a (minor) problem with taking sums of non-singular cycles, since generic i-submanifolds may intersect and their union is unavoidably singular. We assume below that $i < n/2$; otherwise, we replace X by $X \times \mathbb{R}^N$ for $N >> n$, where, observe, $\mathcal{B}^o_i(X \times \mathbb{R}^N)$ does not depend on N for $N >> i$.

Unlike homology, the *bordism groups* $\mathcal{B}^o_i(X)$ may be non-trivial even for a contractible space X, e.g. for $X = \mathbb{R}^{n+N}$. (Every cycle in $\mathbb{R}^n$ equals the boundary of any cone over it but this does not work with manifolds due to the *singularity* at the apex of the cone which is not allowed by the definition of a bordism.) In fact, we have the following.

Theorem (Thom, 1954). *if $N >> n$, then the bordism group $\mathcal{B}^o_n = \mathcal{B}^o_n(\mathbb{R}^{n+N})$ is canonically isomorphic to the homotopy group $\pi_{n+N}(V_\bullet)$, where $V_\bullet$ is the Thom space of the tautological oriented $\mathbb{R}^N$-bundle V over the Grassmann manifold $V = \mathrm{Gr}^{or}_N(\mathbb{R}^{n+N+1})$*

Proof. Let $X_0 = \mathrm{Gr}^{\mathrm{or}}_N(\mathbb{R}^{n+N})$ be the Grassmann manifold of *oriented* N-planes and $V \to X_0$ the tautological oriented $\mathbb{R}^N$ bundle over this X_0.

(The space $\mathrm{Gr}^{\mathrm{or}}_N(\mathbb{R}^{n+N})$ equals the double cover of the space $\mathrm{Gr}_N(\mathbb{R}^{n+N})$ of non-oriented N-planes. For example, $\mathrm{Gr}^{\mathrm{or}}_1(\mathbb{R}^{n+1})$ equals the sphere S^n, while $\mathrm{Gr}_1(\mathbb{R}^{n+1})$ is the projective space, that is S^n divided by the $\pm$-involution.)

Let $U^\perp \to X$ be the oriented normal bundle of X with the orientation induced by those of X and of $\mathbb{R}^N \supset X$ and let $G : X \to X_0$ be the oriented Gauss map which assigns to each $x \in X$ the oriented N-plane $G(x) \in X_0$ parallel to the oriented normal space to X at x.

Since G induces $U^\perp$ from V, it defines the Thom map $S^{n+N} = \mathbb{R}^{n+N}_\bullet \to V_\bullet$ and every bordism $Y \subset S^{n+N} \times [0,1]$ delivers a homotopy $S^{n+N} \times [0,1] \to V_\bullet$ between the Thom maps at the two ends $Y \cap S^{n+N} \times 0$ and $Y \cap S^{n+N} \times 1$.

This define a *homomorphism*

$$\tau_{b\pi} : \mathcal{B}^o_n \to \pi_{n+N}(V_\bullet)$$

since the additive structure in $\mathcal{B}^o_n(\mathbb{R}^{i+N})$ agrees with that in $\pi_{i+N}(V^o_\bullet)$. (Instead of checking this, which is trivial, one may appeal to the general principle: "two natural Abelian group structures on the same set must coincide.")

Also note that one needs the extra 1 in $\mathbb{R}^{n+N+1}$, since bordisms Y between manifolds in $\mathbb{R}^{n+N}$ lie in $\mathbb{R}^{n+N+1}$, or, equivalently, in $S^{n+N+1} \times [0,1]$.

On the other hand, the generic pullback construction

$$f \mapsto f^{-1}(X_0) \subset \mathbb{R}^{n+N} \supset \mathbb{R}^{n+N}_\bullet = S^{n+N}$$

defines a homomorphism $\tau_{\pi b} : [f] \to [f^{-1}(X_0)]$ from $\pi_{n+N}(V_\bullet)$ to $\mathcal{B}^o_n$, where, clearly $\tau_{\pi b} \circ \tau_{b\pi}$ and $\tau_{b\pi} \circ \tau_{\pi b}$ are the identity homomorphisms. □

Now Serre's $\mathbb{Q}$-sphericity theorem implies the following.

THEOREM (Thom Theorem). *The (Abelian) group $\mathcal{B}^o_i$ is finitely generated; $\mathcal{B}^o_n \otimes \mathbb{Q}$ is isomorphic to the rational homology group $H_i(X_0;\mathbb{Q}) = H_i(X_0)\otimes\mathbb{Q}$ for $X_0 = \mathrm{Gr}^{\mathrm{or}}_N(\mathbb{R}^{i+N+1})$.*

Indeed, $\pi_i(V^\bullet) = 0$ for $N >> n$, hence, by Serre,

$$\pi_{n+N}(V_\bullet)\otimes\mathbb{Q} = H_{n+N}(V_\bullet;\mathbb{Q}),$$

while

$$H_{n+N}(V_\bullet;\mathbb{Q}) = H_n(X_0;\mathbb{Q})$$

by the Thom isomorphism.

In order to apply this, one has to compute the homology $H_n(\mathrm{Gr}^{\mathrm{or}}_N(\mathbb{R}^{N+n+j}));\mathbb{Q})$, which, as it is clear from the above, is independent of $N \geq 2n+2$ and of $j > 1$; thus, we pass to

$$\mathrm{Gr}^{\mathrm{or}} =_{\mathrm{def}} \bigcup_{j,N\to\infty} \mathrm{Gr}^{\mathrm{or}}_N(\mathbb{R}^{N+j}).$$

Let us state the answer in the language of *cohomology*, with the advantage of the *multiplicative structure* (see section 4) where, recall, the cohomology product $H^i(X)\otimes H^j(X) \overset{\smile}{\to} H^{i+j}(X)$ for closed oriented n-manifold can be defined via the Poincaré duality $H^*(X)\leftrightarrow H_{n-*}(X)$ by the intersection product $H_{n-i}(X)\otimes H_{n-j}(X) \overset{\cap}{\to} H_{n-(i+j)}(X)$.

> The cohomology ring $H^*(\mathrm{Gr}^{\mathrm{or}};\mathbb{Q})$ is the polynomial ring in some distinguished *integer* classes, called Pontryagin classes $p_k \in H^{4k}(\mathrm{Gr}^{\mathrm{or}};\mathbb{Z})$, $k = 1,2,3,\ldots$ [**50**].

(It would be awkward to express this in the homology language when $N = \dim(X)\to\infty$, although the cohomology ring $H^*(X)$ is canonically isomorphic to $H_{N-*}(X)$ by Poincaré duality.)

If X is a smooth oriented n-manifold, its Pontryagin classes $p_k(X)\in H^{4k}(X;\mathbb{Z})$ are defined as the classes induced from p_k by the normal Gauss map

$$G \to \mathrm{Gr}^{\mathrm{or}}_N(\mathbb{R}^{N+n}) \subset \mathrm{Gr}^{\mathrm{or}}$$

for an embedding $X \to \mathbb{R}^{n+N}$, $N >> n$.

EXAMPLE (a). (see [**50**]). The the complex projective spaces have

$$p_k(\mathbb{C}P^n) = \binom{n+1}{k} h^{2k}$$

for the generator $h\in H^2(\mathbb{C}P^n)$ which is the Poincaré dual to the hyperplane $\mathbb{C}P^{n-1}\subset\mathbb{C}P^{n-1}$.

EXAMPLE (b). The rational Pontryagin classes of the Cartesian products $X_1\times X_2$ satisfy

$$p_k(X_1\times X_2) = \sum_{i+j=k} p_i(X_1)\otimes p_j(X_2).$$

If Q is a unitary (i.e. a product of powers) monomial in p_i of graded degree $n = 4k$, then the value $Q(p_i)[X]$ is called *the (Pontryagin) Q-number.* Equivalently, this is the value of $Q(p_i)\in H^{4i}(\mathrm{Gr}^{\mathrm{or}};\mathbb{Z})$ on the image of (the fundamental class) of X in $\mathrm{Gr}^{\mathrm{or}}$ under the Gauss map.

The Thom theorem now can be reformulated as follows. *Two closed oriented n-manifolds are $\mathbb{Q}$-bordant if and only if they have equal Q-numbers for all monomials Q. Thus, $\mathcal{B}^o_n\otimes\mathbb{Q} = 0$, unless n is divisible by 4 and the rank of $\mathcal{B}^o_n\otimes\mathbb{Q}$ for $n = 4k$*

equals the number of Q-monomials of graded degree n, that are $\prod_i p_i^{k_i}$ *with* $\sum_i k_i = k$. (We shall prove this later in this section, also see [**50**].)

For example, if $n = 4$, then there is a single such monomial, p_1; if n=8, there two of them: p_2 and p_1^2; if $n = 12$ there three monomials: p_3, p_1p_2 and p_1^3; if $n = 16$ there are five of them.

In general, the number of such monomials, say $\pi(k) = \mathrm{rank}(H^{4k}(\mathrm{Gr}^{\mathrm{or}};\mathbb{Q})) = \mathrm{rank}_{\mathbb{Q}}(\mathcal{B}^o_{4k})$ (obviously) equals the number of the conjugacy classes in the permutation group $\Pi(k)$ (which can be seen as a certain subgroup in the Weyl group in $SO(4k)$), where, by the Euler formula, the generating function $E(t) = 1 + \sum_{k=1,2,\dots} \pi(k)t^k$ satisfies

$$1/E(t) = \prod_{k=1,2,\dots} (1-t^k) = \sum_{-\infty<k<\infty} (-1)^k t^{(3k^2-k)/2},$$

Here the first equality is obvious, the second is tricky (Euler himself was not able to prove it) and where one knows now-a-days that

$$\pi(k) \sim \frac{\exp(\pi\sqrt{2k/3})}{4k\sqrt{3}} \text{ for } k \to \infty.$$

Since the top Pontryagin classes p_k of the complex projective spaces do not vanish, $p_k(\mathbb{C}P^{2k}) \neq 0$, the products of these spaces constitute a basis in $\mathcal{B}^o_n \otimes \mathbb{Q}$.

Finally, notice that the bordism groups together make a commutative ring under the Cartesian product of manifolds, denoted $\mathcal{B}^o_*$, and the Thom theorem says that

$\mathcal{B}^o_* \otimes \mathbb{Q}$ is the polynomial ring over $\mathbb{Q}$ in the variables $[\mathbb{C}P^{2k}]$, $k = 0, 2, 4, \dots$.

Instead of $\mathbb{C}P^{2k}$, one might take the compact quotients of the *complex hyperbolic spaces* $\mathbb{C}H^{2k}$ for the generators of $\mathcal{B}^o_* \otimes \mathbb{Q}$. The quotient spaces $\mathbb{C}H^{2k}/\Gamma$ have two closely related attractive features: their tangent bundles admit natural *flat connections* and their rational Pontryagin numbers are *homotopy invariant*, see section 10. It would be interesting to find "natural bordisms" between (linear combinations of) Cartesian products of $\mathbb{C}H^{2k}/\Gamma$ and of $\mathbb{C}P^{2k}$, e.g. associated to complex analytic ramified coverings $\mathbb{C}H^{2k}/\Gamma \to \mathbb{C}P^{2k}$.

Since the signature is additive and also multiplicative under this product, it defines a homomorphism $[\mathrm{sig}] : \mathcal{B}^o_* \to \mathbb{Z}$ which can be expressed in each degree $4k$ by means of a universal polynomial in the Pontryagin classes, denoted $L_k(p_i)$, by

$$\mathrm{sig}(X) = L_k(p_i)[X] \text{ for all closed oriented } 4k\text{-manifolds } X.$$

For example,

$$L_1 = \frac{1}{3}p_1,\ L_2 = \frac{1}{45}(7p_2 - p_1^2),\ L_3 = \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3).$$

Accordingly,

$$\mathrm{sig}(X^4) = \frac{1}{3}p_1[X^4], \qquad \text{(Rokhlin 1952)}$$

$$\mathrm{sig}(X^8) = \frac{1}{45}(7p_2(X^8) - p_1^2(X^8))[X^8], \qquad \text{(Thom 1954)}$$

and where a concise general formula (see blow) was derived by Hirzebruch who evaluated the coefficients of L_k using the above values of p_i for the products

$X = \times_j \mathbb{C}P^{2k_j}$ of the complex projective spaces, which all have $\mathrm{sig}(X) = 1$, and by substituting these products $\times_j \mathbb{C}P^{2k_j}$ with $\sum_j 4k_j = n = 4k$, for $X = X^n$ into the formula $\mathrm{sig}(X) = L_k[X]$. The outcome of this seemingly trivial computation is unexpectedly beautiful.

Hirzebruch Signature Theorem. Let

$$R(z) = \frac{\sqrt{z}}{\tanh(\sqrt{z})} = 1 + z/3 - z^2/45 + \dots = 1 + 2\sum_{l>0}(-1)^{l+1}\frac{\zeta(2l)z^l}{\pi^{2l}} = 1 + \sum_{l>0}\frac{2^{2l}B_{2l}z^l}{(2l)!},$$

where $\zeta(2l) = 1 + \frac{1}{2^{2l}} + \frac{1}{3^{2l}} + \frac{1}{4^{2l}} + \dots$ and let

$$B_{2l} = (-1)^l 2l\zeta(1-2l) = (-1)^{l+1}(2l)!\zeta(2l)/2^{2l-1}\pi^{2l}$$

be the *Bernoulli numbers* [**47**],

$$B_2 = 1/6, B_4 = -1/30, \dots, B_{12} = -691/2730, B_{14} = 7/6, \dots,$$
$$B_{30} = 8615841276005/14322, \dots .$$

Write

$$R(z_1) \cdot \dots \cdot R(z_k) = 1 + P_1(z_j) + \dots + P_k(z_j) + \dots$$

where P_j are homogeneous symmetric polynomials of degree j in $z_1, \dots, z_k$ and rewrite

$$P_k(z_j) = L_k(p_i)$$

where $p_i = p_i(z_1, \dots, z_k)$ are the elementary symmetric functions in z_j of degree i. The Hirzebruch theorem says the following.

> The above L_k is exactly the polynomial which makes the equality $L_k(p_i)[X] = \mathrm{sig}(X)$.

A significant aspect of this formula is that the Pontryagin numbers and the signature are *integers* while the Hirzebruch polynomials L_k have non-trivial denominators. This yields certain *universal divisibility* properties of the Pontryagin numbers (and sometimes of the signatures) for smooth closed orientable $4k$-manifolds.

But despite a heavy integer load carried by the signature formula, its derivation depends only on the *rational* bordism groups $\mathcal{B}_n^o \otimes \mathbb{Q}$. This point of elementary linear algebra was overlooked by Thom (isn't it incredible?) who derived the signature formula for 8-manifolds from his special and more difficult computation of the true bordism group $\mathcal{B}_8^o$. However, the shape given by Hirzebruch to this formula is something more than just linear algebra.

> *Question.* Is there an implementation of the analysis/arithmetic encoded in the Hirzebruch formula by some infinite dimensional manifolds?

Computation of the Cohomology of the Stable Grassmann Manifold. First, we show that the cohomology $H^*(\mathrm{Gr}^{\mathrm{or}}; \mathbb{Q})$ is multiplicatively generated by some classes $e_i \in H^*(\mathrm{Gr}^{\mathrm{or}}; \mathbb{Q})$ and then we prove that the L_i-classes are multiplicatively independent. (See [**50**] for computation of the integer cohomology of the Grassmann manifolds.)

Think of the unit tangent bundle $UT(S^n)$ as the space of orthonormal 2-frames in $\mathbb{R}^{n+1}$, and recall that $UT(S^{2k})$ is a rational homology $(4k-1)$-sphere.

Let $W_k = \mathrm{Gr}^{\mathrm{or}}_{2k+1}(\mathbb{R}^\infty)$ be the Grassmann manifold of oriented $(2k+1)$-planes in $\mathbb{R}^N$, $N \to \infty$, and let W''_k consist of the pairs (w, u) where $w \in W_k$ is an $(2k+1)$-plane $\mathbb{R}^{2k+1}_w \subset \mathbb{R}^\infty$, and u is an orthonormal frame (pair of orthonormal vectors) in $\mathbb{R}^{2k+1}_w$.

The map $p : W''_k \to W_{k-1} = \mathrm{Gr}^{\mathrm{or}}_{2k-1}(\mathbb{R}^\infty)$ which assigns, to every (w, u), the $(2k-1)$-plane $u^\perp_w \subset \mathbb{R}^{2k+1}_w \subset \mathbb{R}^\infty$ normal to u is a fibration with *contractible fibers* that are spaces of orthonormal 2-frames in $\mathbb{R}^\infty \ominus u^\perp_w = \mathbb{R}^{\infty-(2k-1)}$; hence, *$p$ is a homotopy equivalence.*

A more interesting fibration is $q : W_k \to W''_k$ for $(w, u) \mapsto w$ with the fibers $UT(S^{2k})$. Since $UT(S^{2k})$ is a rational $(4k-1)$-sphere, the kernel of the cohomology homomorphism $q^* : H^*(W''_k; \mathbb{Q}) \to H^*(W_k; \mathbb{Q})$ is generated, as a $\smile$-ideal, by the rational Euler class $e_k \in H^{4k}(W''_k; \mathbb{Q})$.

It follows by induction on k that the rational cohomology algebra of $W_k = \mathrm{Gr}^{\mathrm{or}}_{2k+1}(\mathbb{R}^\infty)$ is generated by certain $e_i \in H^{4i}(W_k; \mathbb{Q})$, $i = 0, 1, ..., k$, and since

$$\mathrm{Gr}^{\mathrm{or}} = \varprojlim_{k\to\infty} \mathrm{Gr}^{\mathrm{or}}_{2k+1},$$

these e_i also generate the cohomology of $\mathrm{Gr}^{\mathrm{or}}$.

Direct Computation of the L-Classes for the Complex Projective Spaces. Let $V \to X$ be an oriented vector bundle and, following Rokhlin-Schwartz and Thom, *define* L-classes of V, without any reference to Pontryagin classes, as follows. Assume that X is a manifold with a trivial tangent bundle; otherwise, embed X into some $\mathbb{R}^M$ with large M and take its small regular neighbourhood. By Serre's theorem, there exists, for every homology class $h \in H_{4k}(X) = H_{4k}(V)$, an $m = m(h) \neq 0$ such that the m-multiple of h is representable by a closed $4k$-submanifold $Z = Z_h \subset V$ that equals the pullback of a generic point in the sphere S^{M-4k} under a generic map $V \to S^{M-4k} = \mathbb{R}^{M-4k}_\bullet$ with "compact support", i.e. where all but a compact subset in V goes to $\bullet \in S^{M-4k}$. Observe that such a Z has trivial normal bundle in V.

Define $L(V) = 1 + L_1(V) + L_2(V) + ... \in H^{4*}(V; \mathbb{Q}) = \oplus_k H^{4k}(V; \mathbb{Q})$ by the equality $L(V)(h) = \mathrm{sig}(Z_h)/m(h)$ for all $h \in H_{4k}(V) = H_{4k}(X)$.

If the bundle V is induced from $W \to Y$ by an $f : X \to Y$ then $L(V) = f^*(L(W))$, since, for $\dim(W) > 2k$ (which we may assume), the generic image of our Z in W has *trivial* normal bundle.

It is also clear that the bundle $V_1 \times V_2 \to X_1 \times X_2$ has $L(V_1 \times V_2) = L(V_1) \otimes L(V_2)$ by the Cartesian multiplicativity of the signature.

Consequently the L-class of the *Whitney sum* $V_1 \oplus V_2 \to X$ of V_1 and V_2 over X, which is defined as the restriction of $V_1 \times V_2 \to X \times X$ to $X_{\mathrm{diag}} \subset X \times X$, satisfies

$$L(V_1 \oplus V_2) = L(V_1) \smile L(V_2).$$

Recall that the complex projective space $\mathbb{C}P^k$ – the space of $\mathbb{C}$-lines in $\mathbb{C}^{k+1}$ comes with the canonical $\mathbb{C}$-line bundle represented by these lines and denoted $U \to \mathbb{C}P^k$, while the same bundle with the reversed orientation is denoted U^-. (We always refer to the canonical orientations of $\mathbb{C}$-objects.)

Observe that $U^- = \mathrm{Hom}_{\mathbb{C}}(U \to \theta)$ for the trivial $\mathbb{C}$-bundle

$$\theta = \mathbb{C}P^k \times \mathbb{C} \to \mathbb{C}P^k = \mathrm{Hom}_{\mathbb{C}}(U \to U)$$

and that the Euler class $e(U^-) = -e(U)$ equals the generator in $H^2(\mathbb{C}P^k)$ that is the Poincaré dual of the hyperplane $\mathbb{C}P^{k-1} \subset \mathbb{C}P^k$, and so e^l is the dual of $\mathbb{C}P^{k-l} \subset \mathbb{C}P^k$.

The Whitney $(k+1)$-multiple bundle of U^-, denoted $(k+1)U^-$, equals the tangent bundle $T_k = T(\mathbb{C}P^k)$ plus θ. Indeed, let $U^\perp \to \mathbb{C}P^k$ be the $\mathbb{C}^k$ bundle of the normals to the lines representing the points in $\mathbb{C}P^k$. It is clear that $U^\perp \oplus U = (k+1)\theta$, i.e. $U^\perp \oplus U$ is the trivial $\mathbb{C}^{k+1}$-bundle, and that, tautologically,

$$T_k = \mathrm{Hom}_{\mathbb{C}}(U \to U^\perp).$$

It follows that

$$T_k \oplus \theta = \mathrm{Hom}_{\mathbb{C}}(U \to U^\perp \oplus U) = \mathrm{Hom}_{\mathbb{C}}(U \to (k+1)\theta) = (k+1)U^-.$$

Recall that

$$\mathrm{sig}(\mathbb{C}P^{2k}) = 1; \text{ hence, } L_k((k+1)U^-) = L_k(T_k) = e^{2k}.$$

Now we compute $L(U^-) = 1 + \sum_k L_k = 1 + \sum_k l_{2k} e^{2k}$, by equating e^{2k} and the $2k$-degree term in the $(k+1)$th power of this sum.

$$(1 + \sum_k l_{2k} e^{2k})^{k+1} = 1 + ... + e^{2k} + ...$$

Thus,

$$(1 + l_1 e^2)^3 = 1 + 3l_1 + ... = 1 + e^2 + ...,$$

$$\text{which makes } l_1 = 1/3 \text{ and } L_1(U^-) = e^2/3.$$

Then

$$(1 + l_1 e^2 + l_2 e^4)^5 = 1 + ... + (10l_1 + 5l_2)e^4 + ... = 1 + ... + e^4 + ...$$

which implies that $l_2 = 1/5 - 2l_1 = 1/5 - 2/3$ and $L_2(U^-) = (-7/15)e^4$, etc.

Finally, we compute all L-classes $L_j(T_{2k}) = (L(U^-))^{k+1}$ for $T_{2k} = T(\mathbb{C}P^{2k})$ and thus, all $L(\times_j \mathbb{C}P^{2k_j})$.

For example,

$$(L_1(\mathbb{C}P^8))^2[\mathbb{C}P^8] = 10/3 \text{ while } (L_1(\mathbb{C}P^4 \times \mathbb{C}P^4))^2[\mathbb{C}P^4 \times \mathbb{C}P^4] = 2/3$$

which implies that $\mathbb{C}P^4 \times \mathbb{C}P^4$ and $\mathbb{C}P^8$, which have equal signatures, are *not* rationally bordant, and similarly one sees that the products $\times_j \mathbb{C}P^{2k_j}$ are multiplicatively independent in the bordism ring $\mathcal{B}_*^o \otimes Q$ as we stated earlier.

Combinatorial Pontryagin Classes. Rokhlin-Schwartz and independently Thom applied their definition of L_k, and hence of the *rational* Pontryagin classes, to *triangulated (not necessarily smooth) topological manifolds* X by observing that the pullbacks of *generic* points $s \in S^{n-4k}$ under piece-wise linear map are $\mathbb{Q}$-manifolds and by pointing out that the signatures of $4k$-manifolds are invariant under bordisms by such $(4k+1)$-dimensional $\mathbb{Q}$-manifolds with boundaries (by the Poincaré duality issuing from the Alexander duality, see Section 4). Thus, they have shown, in particular, that the following holds.

> Rational Pontryagin classes of smooth manifolds are invariant under piece-wise smooth homeomorphisms between smooth manifolds.

The combinatorial pull-back argument breaks down in the topological category since there is no good notion of a generic *continuous* map. Yet, S. Novikov (1966) proved that the L-classes and, hence, the rational Pontryagin classes are invariant under *arbitrary homeomorphisms* (see Section 10).

The Thom-Rokhlin-Schwartz argument delivers a definition of rational Pontryagin classes for all $\mathbb{Q}$-manifolds which are by far more general objects than smooth (or combinatorial) manifolds due to possibly enormous (and beautiful) fundamental groups $\pi_1(L^{n-i-1})$ of their links.

Yet, the naturally defined bordism ring $\mathbb{Q}\mathcal{B}^o_n$ of oriented $\mathbb{Q}$-manifolds is only marginally different from $\mathcal{B}^o_*$ in the degrees $n \neq 4$ where the natural homomorphisms $\mathcal{B}^o_n \otimes \mathbb{Q} \to \mathbb{Q}\mathcal{B}^o_n \otimes \mathbb{Q}$ are isomorphisms. This can be easily derived by *surgery* (see section 9) from Serre's theorems. For example, if a $\mathbb{Q}$-manifold X has a single singularity – a cone over $\mathbb{Q}$-sphere Σ then a connected sum of several copies of Σ bounds a smooth $\mathbb{Q}$-ball which implies that a multiple of X is $\mathbb{Q}$-bordant to a smooth manifold.

On the contrary, the group $\mathbb{Q}\mathcal{B}^o_4 \otimes \mathbb{Q}$, is much bigger than $\mathcal{B}^o_4 \otimes \mathbb{Q} = \mathbb{Q}$ as $\mathrm{rank}_{\mathbb{Q}}(\mathbb{Q}\mathcal{B}^o_4) = \infty$ (see [**44**], [**22**], [**23**] and references therein).

(It would be interesting to have a notion of "refined bordisms" between $\mathbb{Q}$-manifold that would partially keep track of $\pi_1(L^{n-i-1})$ for $n > 4$ as well.)

The simplest examples of $\mathbb{Q}$-manifolds are one point compactifications $V^{4k}_\bullet$ of the tangent bundles of even dimensional spheres, $V^{4k} = T(S^{2k}) \to S^{2k}$, since the boundaries of the corresponding $2k$-ball bundles are $\mathbb{Q}$-homological $(2k-1)$-spheres – the unit tangent bundles $UT(S^{2k}) \to S^{2k}$.

Observe that the tangent bundles of spheres are *stably trivial* – they become trivial after adding trivial bundles to them, namely the tangent bundle of $S^{2k} \subset \mathbb{R}^{2k+1}$ stabilizes to the trivial bundle upon adding the (trivial) normal bundle of $S^{2k} \subset \mathbb{R}^{2k+1}$ to it. Consequently, the manifolds $V^{4k} = T(S^{2k})$ have all characteristic classes zero, and $V^{4k}_\bullet$ have all $\mathbb{Q}$-classes zero except for dimension $4k$.

On the other hand, $L_k(V^{4k}_\bullet) = \mathrm{sig}(V^{4k}_\bullet) = 1$, since the tangent bundle $V^{4k} = T(S^{2k}) \to S^{2k}$ has non-zero Euler number. Hence,

> the $\mathbb{Q}$-manifolds $V^{4k}_\bullet$ multiplicatively generate all of $\mathbb{Q}\mathcal{B}^o_* \otimes \mathbb{Q}$ except for $\mathbb{Q}\mathcal{B}^o_4$.

Local Formulae for Combinatorial Pontryagin Numbers. Let X be a closed oriented triangulated (smooth or combinatorial) $4k$-manifold and let

$$\{S^{4k-1}_x\}_{x \in X_0}$$

be the *disjoint* union of the oriented links S^{4k-1}_x of the vertices x in X. Then there exists, for each monomial Q of the total degree $4k$ in the Pontryagin classes, an assignment of rational numbers $Q[S_x]$ to all S^{4k-1}_x, where $Q[S^{4k-1}_x]$ depend only on the combinatorial types of the triangulations of S_x induced from X, such that the Pontryagin Q-number of X satisfies (Levitt-Rourke 1978),

$$Q(p_i)[X] = \sum_{S^{4k-1}_x \in [[X]]} Q[S^{4k-1}_x].$$

Moreover, there is a *canonical* assignment of *real numbers* to S^{4k-1}_x with this property which also applies to all $\mathbb{Q}$-manifolds (Cheeger 1983). There is no comparable *effective combinatorial* formulae with *a priori* rational numbers $Q[S_x]$ despite many efforts in this direction, see [**24**] and references therein. (Levitt-Rourke theorem is purely existential and Cheeger's definition depends on the L_2-analysis of differential forms on the non-singular locus of X away from the codimension 2 skeleton of X.)

Questions. Let $\{[S^{4k-1}]_\triangle\}$ be a finite collection of combinatorial isomorphism classes of oriented triangulated $(4k-1)$-spheres let $\mathbb{Q}^{\{[S^{4k-1}]_\triangle\}}$ be the $\mathbb{Q}$-vector space of functions $q:\{[S^{4k-1}]_\triangle\}\to\mathbb{Q}$ and let X be a closed oriented triangulated $4k$-manifold homeomorphic to the $4k$-torus (or any parallelizable manifold for this matter) with all its links in $\{[S^{4k-1}]_\triangle\}$.

Denote by $q(X)\in\mathbb{Q}^{\{[S^{4k-1}]_\triangle\}}$ the function, such that $q(X)([S^{4k-1}]_\triangle)$ equals the number of copies of $[S^{4k-1}]_\triangle$ in $\{S^{4k-1}_x\}_{x\in X_0}$ and let $\mathcal{L}\{[S]_\triangle\}\subset\mathbb{Q}^{\{[S^{4k-1}]_\triangle\}}$ be the linear span of $q(X)$ for all such X.

The above shows that the vectors $q(X)$ of "q-numbers" satisfy, besides $2k+1$ *Euler-Poincaré and Dehn-Somerville equations*, about $\frac{\exp(\pi\sqrt{2k/3})}{4k\sqrt{3}}$ linear "Pontryagin relations".

Observe that the Euler-Poincaré and Dehn-Somerville equations do not depend on the $\pm$-orientations of the links but the "Pontryagin relations" are anti-symmetric since $Q[-S^{4k-1}]_\triangle=-Q[S^{4k-1}]_\triangle$. Both kind of relations are valid for all $\mathbb{Q}$-manifolds.

What are the codimensions $\mathrm{codim}(\mathcal{L}\{[S^{4k-1}]_\triangle\}\subset\mathbb{Q}^{\{[S^{4k-1}]_\triangle\}}$, i.e. the numbers of independent relations between the "q-numbers", for "specific" collections $\{[S^{4k-1}]_\triangle\}$?

It is pointed out in [**23**] that

- The spaces $\mathbb{Q}_\pm^{\mathcal{S}^i}$ of antisymmetric $\mathbb{Q}$-linear combinations of *all* combinatorial spheres make a *chain complex* for the differential $q^i_\pm:\mathbb{Q}_\pm^{\mathcal{S}^i}\to\mathbb{Q}_\pm^{\mathcal{S}^{i-1}}$ defined by the linear extension of the operation of taking the oriented links of all vertices on the triangulated i-spheres $S^i_\triangle\in\mathcal{S}^i$.
- The operation $q^i_\pm$ with values in $\mathbb{Q}_\pm^{\mathcal{S}^{i-1}}$, which is obviously defined on all closed oriented combinatorial i-manifolds X as well as on combinatorial i-spheres, satisfies
$$q^{i-1}_\pm(q^i_\pm(X))=0\,,$$
i.e. i-manifolds represent *i-cycles* in this complex.

Furthermore, it is shown in [**23**] (as was pointed out to me by Jeff Cheeger) that all such anti-symmetric relations are generated/exhausted by the relations issuing from $q^{n-1}_\pm\circ q^n_\pm=0$, where this identity can be regarded as an "oriented (Pontryagin in place of Euler-Poincaré) counterpart" to the Dehn-Somerville equations.

The exhaustiveness of $q^{n-1}_\pm\circ q^n_\pm=0$ and its (easy, [**25**]) Dehn-Somerville counterpart, probably, imply that in most (all?) cases the Euler-Poincaré, Dehn-Somerville, Pontryagin and $q^{n-1}_\pm\circ q^n_\pm=0$ make the full set of affine (i.e. homogeneous and non-homogeneous linear) relations between the vectors $q(X)$, but it seems hard to effectively (even approximately) evaluate the number of independent relations issuing from for $q^{n-1}_\pm\circ q^n_\pm=0$ for particular collections $\{[S^{n-1}]_\triangle\}$ of allowable links of X^n.

Examples. Let $D_0=D_0(\Gamma)$ be a *Dirichlet-Voronoi (fundamental polyhedral) domain* of a generic lattice $\Gamma\subset\mathbb{R}^M$ and let $\{[S^{n-1}]_\triangle\}$ consist of the (isomorphism classes of naturally triangulated) boundaries of the intersections of D_0 with generic affine n-planes in $\mathbb{R}^M$.

What is $codim(\mathcal{L}\{[S^{n-1}]_\triangle\} \subset \mathbb{Q}\{[S^{n-1}]_\triangle\})$ in this case?

What are the (affine) relations between the "geometric q-numbers" i.e. the numbers of combinatorial types of intersections σ of λ-scaled submanifolds $X \subset_f \mathbb{R}^M$, $\lambda \to \infty$, (as in the triangulation construction in the previous section) with the Γ-translates of D_0?

Notice, that some of these σ are *not* convex-like, but these are negligible for $\lambda \to \infty$. On the other hand, if λ is sufficiently large all σ can be made convex-like by a small perturbation f' of f by an argument which is similar to but slightly more technical than the one used for the triangulation of manifolds in the previous section.

Is there anything special about the "geometric q-numbers" for "distinguished" X, e.g. for round n-spheres in $\mathbb{R}^N$?

Observe that the *ratios* of the "geometric q-numbers" are asymptotically defined for many *non-compact complete* submanifolds $X \subset \mathbb{R}^M$.

For example, if X is an affine subspace $A = A^n \subset \mathbb{R}^M$, these ratios are (obviously) expressible in terms of the volumes of the connected regions Ω in $D \subset \mathbb{R}^M$ obtained by cutting D along hypersurfaces made of the affine n-subspaces $A' \subset \mathbb{R}^M$ which are parallel to A and which *meet* the $(M - n - 1)$-skeleton of D.

What is the number of our kind of relations between these volumes?

There are similar relations/questions for intersection patterns of particular X with other fundamental domains of lattices Γ in Euclidean and some non-Euclidean spaces (where the finer asymptotic distributions of these patterns have a slight arithmetic flavour).

If $f : X^n \to \mathbb{R}^M$ is a generic map *with singularities* (which may happen if $M \leq 2n$) and $D \subset \mathbb{R}^M$ is a small convex polyhedron in $\mathbb{R}^M$ with its faces being δ-transversal to f (e.g. $D = \lambda^{-1}D_0$, $\lambda \to \infty$ as in the triangulations of the previous section), then the pullback $f^{-1}(D) \subset X$ is not necessarily a topological cell. However, some local/additive formulae for certain characteristic numbers may still be available in the corresponding "non-cell decompositions" of X.

For instance, one (obviously) has such a formula for the Euler characteristic for all kind of decompositions of X. Also, one has such a "local formula" for $sig(X)$ and $f : X \to \mathbb{R}$ (i.e for $M = 1$) by Novikov's signature additivity property mentioned at the end of the previous section.

It seems not hard to show that *all* Pontryagin numbers can be thus locally/additively expressed for $M \geq n$, but it is unclear what are precisely the Q-numbers which are combinatorially/locally/additively expressible for given $n = 4k$ and $M < n$.

(For example, if $M = 1$, then the Euler characteristic and the signature are, probably, *the only* "locally/additively expressible" invariants of X.)

Bordisms of Immersions. If the allowed singularities of oriented n-cycles in $\mathbb{R}^{n+k}$ are those of collections of n-planes in general position, then the resulting homologies are the bordism groups of *oriented immersed* manifolds $X^n \subset \mathbb{R}^{n+k}$ (R.Wells, 1966). For example if $k = 1$, this group is isomorphic to the stable homotopy group $\pi_n^{\mathrm{st}} = \pi_{n+N}(S^N)$, $N > n+1$, by the Pontryagin pullback construction, since a small generic perturbation of an oriented X^n in $\mathbb{R}^{n+N} \supset \mathbb{R}^{n+1} \supset X^n$ is *embedded* into $\mathbb{R}^{n+N}$ with a *trivial* normal bundle, and where *every* embedding

$X^n \to \mathbb{R}^{n+N}$ with the *trivial normal bundle* can be isotoped to such a perturbation of an immersion $X^n \to \mathbb{R}^{n+1} \subset \mathbb{R}^{n+N}$ by the *Smale-Hirsch immersion theorem.* (This is obvious for $n = 0$ and $n = 1$).

Since immersed oriented $X^n \subset \mathbb{R}^{n+1}$ have *trivial* stable normal bundles, they have, for $n = 4k$, *zero* signatures by the Serre finiteness theorem. Conversely, the finiteness of the stable groups $\pi_n^{\mathrm{st}} = \pi_{n+N}(S^N)$ can be (essentially) reduced by a (framed) surgery of X^n (see section 9) to the vanishing of these signatures.

The complexity of $\pi_{n+N}(S^N)$ shifts in this picture one dimension down to bordism invariants of the "decorated self-intersections" of immersed $X^n \subset \mathbb{R}^{n+1}$, which are partially reflected in the structure of the l-sheeted coverings of the loci of l-multiple points of X^n.

The Galois group of such a covering may be equal the full permutation group $\Pi(l)$ and the "decorated invariants" live in certain "decorated" bordism groups of the *classifying spaces* of $\Pi(l)$, where the "dimension shift" suggests an inductive computation of these groups that would imply, in particular, Serre's finiteness theorem of the stable homotopy groups of spheres. In fact, this can be implemented in terms of configuration spaces associated to the iterated loop spaces as was pointed out to me by Andras Szùcs, also see [**1**], [**74**].

The simplest bordism invariant of codimension one immersions is the parity of the number of $(n+1)$-multiple points of generically immersed $X^n \subset \mathbb{R}^{n+1}$. For example, the figure $\infty \subset \mathbb{R}^2$ with a single double point represents a *non-zero* element in $\pi_{n=1}^{\mathrm{st}} = \pi_{1+N}(S^N)$. The number of $(n+1)$-multiple points also can be *odd* for $n = 3$ (and, trivially, for $n = 0$) but it is always *even* for codimension one immersions of *orientable* n-manifolds with $n \neq 0, 1, 3$, while the non-orientable case is more involved [**16**], [**17**].

One knows, (see next section) that every element of the stable homotopy group $\pi_n^{\mathrm{st}} = \pi_{n+N}(S^N)$, $N >> n$, can be represented for $n \neq 2, 6, 14, 30, 62, 126$ by an immersion $X^n \to \mathbb{R}^{n+1}$, where X^n is a *homotopy sphere*; if $n = 2, 6, 14, 30, 62, 126$, one can make this with an X^n where $\mathrm{rank}(H_*(X_n)) = 4$.

What is the smallest possible size of the topology, e.g. homology, of the *image* $f(X^n) \subset \mathbb{R}^{n+1}$ and/or of the homologies of the (natural coverings of the) subsets of the l-multiple points of $f(X^n)$?

Geometric Questions about Bordisms. Let X be a closed oriented Riemannian n-manifold with locally bounded geometry, which means that every R-ball in X admits a λ-*bi-Lipschitz homeomorphism* onto the Euclidean R-ball.

Suppose X is bordant to zero and consider all compact Riemannian $(n+1)$-manifolds Y extending $X = \partial(Y)$ with its Riemannian tensor and such that the local geometries of Y are bounded by some *constants* $R' << R$ and $l' >> \lambda$ with the obvious precaution near the boundary.

One can show that the infimum of the volumes of these Y is bounded by

$$\inf_Y \mathrm{Vol}(Y) \leq F(\mathrm{Vol}(X)),$$

with the power exponent bound on the function $F = F(V)$. (F also depends on R, λ, R', λ', but this seems non-essential for $R' << R, \lambda' >> \lambda$.)

What is the true asymptotic behaviour of $F(V)$ for $V \to \infty$?

It may be linear for all we know and the above "dimension shift" picture and/or the construction from [**23**] may be useful here.

Is there a better setting of this question with some curvature integrals and/or spectral invariants rather than volumes?

The real cohomology of the Grassmann manifolds can be analytically represented by invariant differential forms. Is there a compatible analytic/geometric representation of $\mathcal{B}^o_n \otimes \mathbb{R}$? (One may think of a class of *measurable n-foliations*, see section 10, or, maybe, something more sophisticated than that.)

6. Exotic Spheres

In 1956, to everybody's amazement, Milnor found smooth manifolds Σ^7 which were *not diffeomorphic to* S^N; yet, each of them was decomposable into the union of two 7-balls $B_1^7, B_2^7 \subset \Sigma^7$ intersecting over their boundaries $\partial(B_1^7) = \partial(B_2^7) = S^6 \subset \Sigma^7$ like in the ordinary sphere S^7.

In fact, this decomposition does imply that Σ^7 is "ordinary" in the *topological* category: such a Σ^7 is (obviously) *homeomorphic* to S^7.

The subtlety resides in the "equality" $\partial(B_1^7) = \partial(B_2^7)$; this identification of the boundaries is far from being the identity map from the point of view of either of the two balls—it does not come from any diffeomorphisms $B_1^7 \leftrightarrow B_2^7$.

The equality $\partial(B_1^7) = \partial(B_2^7)$ can be regarded as a self-diffeomorphism f of the round sphere S^6 – the boundary of standard ball B^7, but this f *does not extend* to a diffeomorphism of B^7 in Milnor's example; otherwise, Σ^7 would be diffeomorphic to S^7. (Yet, f radially extends to a *piecewise smooth homeomorphism* of B^7 which yields a piecewise smooth homeomorphism between Σ^7 and S^7.)

It follows, that such an f *can not* be included into a family of diffeomorphisms bringing it to an isometric transformations of S^6. Thus, any geometric "energy minimizing" flow on the diffeomorphism group $\mathrm{diff}(S^6)$ either gets stuck or develops singularities. (It seems little, if anything at all, is known about such flows and their singularities.)

Milnor's spheres Σ^7 are rather innocuous spaces – the boundaries of (the total spaces of) 4-ball bundles $\Theta^8 \to S^4$ in some in some $\mathbb{R}^4$-bundles $V \to S^4$, i.e. $\Theta^8 \subset V$ and, thus, our Σ^7 are certain S^3-bundles over S^4.

All 4-ball bundles, or equivalently $\mathbb{R}^4$-bundles, over S^4 are easy to describe: each is determined by two numbers: the *Euler number* e, that is the self-intersection index of $S^4 \subset \Theta^8$, which assumes all integer values, and the Pontryagin number p_1 (i.e. the value of the Pontryagin class $p_1 \in H^4(S^4)$ on $[S^4] \in H_4(S^4)$) which may be an *arbitrary even integer.*

(Milnor explicitly construct his fibrations with maps of the 3-sphere into the group SO(4) of orientation preserving linear isometries of $\mathbb{R}^4$ as follows. Decompose S^4 into two round 4-balls, say $S^4 = B_+^4 \cup B_-^4$ with the common boundary $S_\partial^3 = B_+^4 \cap B_-^4$ and let $f : s_\partial \mapsto O_\partial \in SO(4)$ be a smooth map. Then glue the boundaries of $B_+^4 \times \mathbb{R}^4$ and $B_-^4 \times \mathbb{R}^4$ by the diffeomorphism $(s_\partial, s) \mapsto (s_\partial, O_\partial(s))$ and obtain $V^8 = B_+^4 \times \mathbb{R}^4 \cup_f B_-^4 \times \mathbb{R}^4$ which makes an $\mathbb{R}^4$-fibration over S^4.

To construct a specific f, identify $\mathbb{R}^4$ with the quaternion line $\mathbb{H}$ and S^3 with the multiplicative group of quaternions of norm 1. Let $f(s) = f_{ij}(s) \in SO(4)$ act by $x \mapsto s^i x s^j$ for $x \in \mathbb{H}$ and the left and right quaternion multiplication. Then Milnor computes: $e = i + j$ and $p_1 = \pm 2(i - j)$.)

Obviously, all Σ^7 are 2-connected, but $H_3(\Sigma^7)$ may be non-zero (e.g. for the trivial bundle). It is not hard to show that Σ^7 has the same homology as S^7, hence, homotopy equivalent to S^7, if and only if $e = \pm 1$ which means that the selfintersection index of the zero section sphere $S^4 \subset \Theta^8$ equals ± 1; we stick to $e = 1$ for our candidates for Σ^7.

The basic example of Σ^7 with $e = \pm 1$ (the sign depends on the choice of the orientation in Θ^8) is the ordinary 7-sphere which comes with the *Hopf fibration* $S^7 \to S^4$, where this S^7 is positioned as the unit sphere in the quaternion plane $\mathbb{H}^2 = \mathbb{R}^8$, where it is freely acted upon by the group $G = S^3$ of the unit quaternions and where S^7/G equals the sphere S^4 representing the quaternion projective line.

If Σ^7 is diffeomorphic to S^7 one can attach the 8-ball to Θ^8 along this S^7-boundary and obtain a *smooth* closed 8-manifold, say Θ^8_+ .

Milnor observes that the signature of Θ^8_+ equals ± 1, since the homology of Θ^8_+ is represented by a single cycle – the sphere $S^4 \subset \Theta^8 \subset \Theta^8_+$ the selfintersection number of which equals the Euler number.

Then Milnor invokes the *Thom signature theorem*

$$45\,\mathrm{sig}(X) + p_1^2[X] = 7p_2[X]$$

and concludes that the number $45 + p_1^2$ must be divisible by 7; therefore, the boundaries Σ^7 of those Θ^8 which fail this condition, say for $p_1 = 4$, must be exotic. (You do not have to know the definition of the Pontryagin classes, just remember they are *integer* cohomology classes.)

Finally, using quaternions, Milnor explicitly constructs a Morse function $\Sigma^7 \to \mathbb{R}$ with only two critical points – maximum and minimum on each Σ^7 with $e = 1$; this yields the two ball decomposition. (We shall explain this in section 8.)

(Milnor's topological arguments, which he presents with a meticulous care, became a common knowledge and can be now found in any textbook; his lemmas look apparent to a to-day topology student. The hardest for the modern reader is the final Milnor's lemma claiming that his function $\Sigma^7 \to \mathbb{R}$ is Morse with two critical points. Milnor is laconic at this point: "It is easy to verify" is all what he says.)

The 8-manifolds Θ^8_+ associated with Milnor's exotic Σ^7 can be triangulated with a single non-smooth point in such a triangulation. Yet, they admit no smooth structures compatible with these triangulations since their combinatorial Pontryagin numbers (defined by Rochlin-Schwartz and Thom) fail the divisibility condition issuing from the Thom formula $sig(X^8) = L_2[X^8]$; in fact, they are not *combinatorially bordant* to smooth manifolds.

Moreover, these Θ^8_+ are not even *topologically* bordant, and therefore, they are non-homeomorphic to smooth manifolds by (slightly refined) Novikov's topological Pontryagin classes theorem.

The number of *homotopy spheres*, i.e. of mutually non-diffeomorphic manifolds Σ^n which are homotopy equivalent to S^n is not that large. In fact, it is *finite* for all $n \neq 4$ by the work of Kervaire and Milnor [**39**], who, eventually, derive this from the Serre finiteness theorem. (One knows now-a-days that every smooth homotopy sphere Σ^n is *homeomorphic* to S^n according to the solution of the Poincaré conjecture by Smale for $n \geq 5$, by Freedman for $n = 4$ and by Perelman for $n = 3$, where "homeomorphic"$\Rightarrow$ "diffeomorphic" for $n = 3$ by *Moise's theorem.*)

Kervaire and Milnor start by showing that for every *homotopy sphere* Σ^n, there exists a smooth map $f : S^{n+N} \to S^N$, $N >> n$, such that the pullback $f^{-1}(s) \subset S^{n+N}$ of a generic point $s \in S^N$ is diffeomorphic to Σ^n. (The existence of such an f with

$f^{-1}(s) = \Sigma^n$ is equivalent to the existence of an *immersion* $\Sigma^n \to \mathbb{R}^{n+1}$ by the *Hirsch theorem.*)

Then, by applying *surgery* (see section 9) to the f_0-pullback of a point for a *given* generic map $f_0 : S^{n+N} \to S^N$, they prove that almost all homotopy classes of maps $S^{n+N} \to S^N$ come from homotopy n-spheres. Namely:

- If $n \neq 4k+2$, then every homotopy class of maps $S^{n+N} \to S^N$, $N >> n$, can be represented by a "Σ^n-map" f, i.e. where the pullback of a generic point is a homotopy sphere.

If $n = 4k+2$, then the homotopy classes of "Σ^n-maps" constitute a subgroup in the corresponding stable homotopy group, say $K_n^\perp \subset \pi_n^{\mathrm{st}} = \pi_{n+N}(S^N)$, $N >> n$, that has index 1 or 2 and which is expressible in terms of the *Kervaire-(Arf) invariant* classifying (similarly to the signature for $n = 4k$) properly defined "self-intersections" of $(k+1)$-cycles mod 2 in $(4k+2)$-manifolds.

One knows today by the work of Pontryagin, Kervaire-Milnor and Barratt-Jones-Mahowald see [**9**] that

- If $n = 2, 6, 14, 30, 62$, then the Kervaire invariant can be non-zero, i.e. $\pi_n^{\mathrm{st}}/K_n^\perp = \mathbb{Z}_2$.

Furthermore,

- The Kervaire invariant vanishes, i.e. $K_n^\perp = \pi_n^{\mathrm{st}}$, for $n \neq 2, 6, 14, 30, 62, 126$ (where it remains unknown if $\pi_{126}^{\mathrm{st}}/K_{126}^\perp$ equals $\{0\}$ or $\mathbb{Z}_2$).

In other words,

> every continuous map $S^{n+N} \to S^N$, $N >> n \neq 2, 6, ..., 126$, is homotopic to a smooth map $f : S^{n+N} \to S^N$, such that the f-pullback of a generic point is a homotopy n-sphere.

The case $n \neq 2^l - 2$ goes back to Browder (1969) and the case $n = 2^l - 2$, $l \geq 8$ is a recent achievement by Hill, Hopkins and Ravenel [**37**]. (Their proof relies on a generalized homology theory H_n^{gen} where $H_{n+256}^{\mathrm{gen}} = H_n^{\mathrm{gen}}$.)

If the pullback of a generic point of a smooth map $f : S^{n+N} \to S^N$, is *diffeomorphic* to S^n, the map f may be *non-contractible.* In fact, the set of the homotopy classes of such f makes a cyclic subgroup in the stable homotopy group of spheres, denoted $J_n \subset \pi_n^{\mathrm{st}} = \pi_{n+N}(S^N)$, $N >> n$ (and called the *image of the J-homomorphism* $\pi_n(SO(\infty)) \to \pi_n^{\mathrm{st}}$). The order of J_n is 1 or 2 for $n \neq 4k-1$; if $n = 4k-1$, then the order of J_n equals the denominator of $|B_{2k}/4k|$, where B_{2k} is the Bernoulli number. The first non-trivial J are

$$J_1 = \mathbb{Z}_2,\ J_3 = \mathbb{Z}_{24},\ J_7 = \mathbb{Z}_{240},\ J_8 = \mathbb{Z}_2,\ J_9 = \mathbb{Z}_2 \text{ and } J_{11} = \mathbb{Z}_{504}.$$

In general, the homotopy classes of maps f such that the f-pullback of a generic point is diffeomorphic to a given homotopy sphere Σ^n, make a J_n-coset in the stable homotopy group π_n^{st}. Thus the correspondence $\Sigma^n \rightsquigarrow f$ defines a map from the set $\{\Sigma^n\}$ of the diffeomorphism classes of homotopy spheres to the factor group π_n^{st}/J_n, say $\mu : \{\Sigma^n\} \to \pi_n^{\mathrm{st}}/J_n$.

The map μ (which, by the above, is surjective for $n \neq 2, 6, 14, 30, 62, 126$) is *finite-to-one* for $n \neq 4$, where the proof of this finiteness for $n \geq 5$ depends on *Smale's h-cobordism theorem*, (see section 8). In fact, the homotopy n-spheres make an Abelian *group* ($n \neq 4$) under the *connected sum operation* $\Sigma_1 \# \Sigma_2$ (see next section) and, by applying surgery to manifolds Θ^{n+1} with boundaries Σ^n, where these Θ^{n+1}

(unlike the above Milnor's Θ^8) come as pullbacks of generic points under smooth maps from $(n+N+1)$-balls B^{n+N+1} to S^N, Kervaire and Milnor show that

(★) $\mu : \{\Sigma^n\} \to \pi_n^{st}/J_n$ is a *homomorphism* with a finite ($n \neq 4$) kernel denoted $\mathcal{B}^{n+1} \subset \{\Sigma^n\}$ which is a cyclic group.

(The homotopy spheres $\Sigma^n \in \mathcal{B}^{n+1}$ bound $(n+1)$-manifolds with trivial tangent bundles.)

Moreover,

($\star$) The kernel $\mathcal{B}^{n+1}$ of μ is zero for $n = 2m \neq 4$.

If $n+1 = 4k+2$, then $\mathcal{B}^{n+1}$ is either zero or $\mathbb{Z}_2$, depending on the Kervaire invariant:

($\star$) If n equals 1, 5, 13, 29, 61 and, possibly, 125, then $\mathcal{B}^{n+1}$ is zero, and $\mathcal{B}^{n+1} = \mathbb{Z}_2$ for the rest of $n = 4k+1$.

($\star$) If $n = 4k-1$, then the cardinality (order) of $\mathcal{B}^{n+1}$ equals $2^{2k-2}(2^{2k-1}-1)$ times the numerator of $|4B_{2k}/k|$, where B_{2k} is the Bernoulli number.

The above and the known results on the stable homotopy groups π_n^{st} imply, for example, that there are no exotic spheres for $n = 5, 6$, there are 28 mutually non-diffeomorphic homotopy 7-spheres, there are 16 homotopy 18-spheres and 523264 mutually non-diffeomorphic homotopy 19-spheres.

By Perelman, there is a single smooth structure on the homotopy 3 sphere and the case $n = 4$ remains open. (Yet, every homotopy 4-sphere is *homeomorphic* to S^4 by Freedman's solution of the $4D$-Poincaré conjecture.)

7. Isotopies and Intersections

Besides constructing, listing and classifying manifolds X one wants to understand the topology of spaces of maps $X \to Y$.

The space $[X\to Y]_{\mathrm{smth}}$ of *all* C^∞ maps carries little geometric load by itself since this space is homotopy equivalent to $[X\to Y]_{\mathrm{cont(inuous)}}$.

An analyst may be concerned with completions of $[X\to Y]_{\mathrm{smth}}$, e.g. with Sobolev's topologies while a geometer is keen to study geometric structures, e.g. Riemannian metrics on this space.

But from a differential topologist's point of view the most interesting is the space of *smooth embeddings* $F : X \to Y$ which diffeomorphically send X onto a smooth submanifold $X' = f(X) \subset Y$.

If $\dim(Y) > 2\dim(X)$ then generic f are embeddings, but, in general, you can not produce them at will so easily. However, given such an embedding $f_0 : X \to Y$, there are plenty of smooth homotopies, called *(smooth) isotopies* f_t, $t \in [0,1]$, of it which remain embeddings for every t and which can be obtained with the following

THEOREM (Thom, 1954). *Let $Z \subset X$ be a compact smooth submanifold (boundary is allowed) and $f_0 : X \to Y$ is an embedding, where the essential case is where $X \subset Y$ and f_0 is the identity map.*

Then every isotopy of $Z \overset{f_0}{\to} Y$ can be extended to an isotopy of all of X. More generally, the restriction map $R_{|Z} : [X\to Y]_{emb} \to [Z\to Y]_{emb}$ is a fibration; in particular, the isotopy extension property holds for an arbitrary family of embeddings $X \to Y$ parametrized by a compact space.

This is similar to the *homotopy extension property* (mentioned in section 1) for spaces of continuous maps $X \to Y$—the "geometric" cornerstone of the algebraic topology.)

The proof easily reduces with the implicit function theorem to the case, where $X = Y$ and $\dim(Z) = \dim(W)$.

Since diffeomorphisms are *open* in the space of all smooth maps, one can extend "small" isotopies, those which only slightly move Z, and since diffeomorphisms of Y make a *group*, the required isotopy is obtained as a composition of small diffeomorphisms of Y. (The details are easy.)

Both "open" and "group" are crucial: for example, homotopies by *locally diffeomorphic* maps, say of a disk $B^2 \subset S^2$ to S^2 do not extend to S^2 whenever a map $B^2 \to S^2$ starts overlapping itself. Also it is much harder (yet possible, [**12**], [**40**]) to extend *topological* isotopies, since homeomorphisms are, by no means, open in the space of all continuous maps.

For example if $\dim(Y) \geq 2\dim(Z) + 2$. then a generic smooth homotopy of Z is an isotopy: Z does not, generically, cross itself as it moves in Y (unlike, for example, a circle moving in the 3-space where self-crossings are stable under small perturbations of homotopies). Hence, every generic homotopy of Z extends to a smooth isotopy of Y.

Mazur Swindle and Hauptvermutung. Let U_1, U_2 be compact n-manifolds with boundaries and $f_{12} : U_1 \to U_2$ and $f_{21} : U_2 \to U_1$ be embeddings which land in the interiors of their respective target manifolds.

Let W_1 and W_2 be the unions (inductive limits) of the infinite increasing sequences of spaces

$$W_1 = U_1 \subset_{f_{12}} U_2 \subset_{f_{21}} U_1 \subset_{f_{12}} U_2 \subset_{f_{12}} \ldots$$

and

$$W_2 = U_2 \subset_{f_{21}} U_1 \subset_{f_{12}} U_2 \subset_{f_{12}} U_1 \subset_{f_{12}} \ldots$$

Observe that W_1 and W_2 are open manifolds without boundaries and that they are diffeomorphic since dropping the first term in a sequence $U_1 \subset U_2 \subset U_3 \subset \ldots$ does not change the union.

Similarly, both manifolds are diffeomorphic to the unions of the sequences

$$W_{11} = U_1 \subset_{f_{11}} U_1 \subset_{f_{11}} \ldots \text{ and } W_{22} = U_2 \subset_{f_{22}} U_2 \subset_{f_{22}\ldots}$$

for

$$f_{11} = f_{12} \circ f_{21} : U_1 \to U_1 \text{ and } f_{22} = f_{21} \circ f_{12} : U_2 \to U_2.$$

If the self-embedding f_{11} is isotopic to the identity map, then W_{11} is diffeomorphic to the interior of U_1 by the isotopy theorem and the same applies to f_{22} (or any self-embedding for this matter).

Thus we conclude with the above, that, for example, the following holds.

> Open normal neighbourhoods U_1^{op} and U_2^{op} of two homotopy equivalent n-manifolds (and triangulated spaces in general) Z_1 and Z_2 in $\mathbb{R}^{n+N}$, $N \geq n + 2$, are diffeomorphic (Mazur 1961).

Anybody might have guessed that the "open" condition is a pure technicality and everybody believed so until Milnor's 1961 counterexample to the *Hauptvermutung*— the main conjecture of the combinatorial topology.

Milnor has shown that there are two free isometric actions A_1 and A_2 of the cyclic group $\mathbb{Z}_p$ on the sphere S^3, for every prime $p \geq 7$, such that *the quotient (lens) spaces $Z_1 = S^3/A_1$ and $Z_2 = S^3/A_2$ are homotopy equivalent, but their closed normal neighbourhoods U_1 and U_2 in any $\mathbb{R}^{3+N}$ are not diffeomorphic.* (This could not have happened to *simply connected* manifolds Z_i by the h-cobordism theorem.)

Moreover, *the polyhedra* P_1 *and* P_2 *obtained by attaching the cones to the boundaries of these manifolds admit no isomorphic simplicial subdivisions.* Yet, the interiors U_i^{op} of these U_i , $i = 1, 2$, are diffeomorphic for $N \geq 5$. In this case, P_1 *and* P_2 *are homeomorphic as the one point compactifications of two homeomorphic spaces* U_1^{op} *and* U_2^{op}.

It was previously known that these Z_1 and Z_2 are *homotopy equivalent* (J. H. C. Whitehead, 1941); yet, they are *combinatorially non-equivalent* (Reidemeister, 1936) and, hence, by Moise's 1951 positive solution of the Hauptvermutung for 3-manifolds, *non-homeomorphic.*

There are few direct truly geometric constructions of diffeomorphisms, but those available, are extensively used, e.g. *fiberwise linear diffeomorphisms of vector bundles.* Even the sheer existence of the humble homothety of $\mathbb{R}^n$, $x \mapsto tx$, combined with the isotopy theorem, effortlessly yields, for example, the following

Lemma ($[B{\to}Y]$-Lemma). *The space of embeddings* f *of the* n*-ball (or* $\mathbb{R}^n$*) into an arbitrary* $Y = Y^{n+k}$ *is homotopy equivalent to the space of tangent* n*-frames in* Y*; in fact the differential* $f \mapsto Df|0$ *establishes a homotopy equivalence between the respective spaces.*

For example, there is the following.

> The assignment $f \mapsto J(f)|0$ of the Jacobi matrix at $0 \in B^n$ is a homotopy equivalence of the space of embeddings $f : B \to \mathbb{R}^n$ to the linear group $GL(n)$.

Corollary (Ball Gluing Lemma). *Let* X_1 *and* X_2 *be* $(n+1)$*-dimensional manifolds with boundaries* Y_1 *and* Y_2*, let* $B_1 \subset Y_1$ *be a smooth submanifold diffeomorphic to the* n*-ball and let* $f : B_1 \to B_2 \subset Y_2 = \partial(A_2)$ *be a diffeomorphism. If the boundaries* Y_i *of* X_i *are connected, the diffeomorphism class of the* $(n+1)$*-manifold* $X_3 = X_1 +_f X_2$ *obtained by attaching* X_1 *to* X_2 *by* f *and (obviously canonically) smoothed at the "corner" (or rather the "crease") along the boundary of* B_1*, does not depend on* B_1 *and* f.

This X_3 is denoted $X_1 \#_\partial X_2$. For example, this "sum" of balls, $B^{n+1} \#_\partial B^{n+1}$, is again a smooth $(n+1)$-ball.

Connected Sum. The boundary $Y_3 = \partial(X_3)$ can be defined without any reference to $X_i \supset Y_i$, as follows. Glue the manifolds Y_1 an Y_2 by $f : B_1 \to B_2 \subset Y_2$ and then remove the interiors of the balls B_1 and of its f-image B_2.

If the manifolds Y_i (not necessarily anybody's boundaries or even being closed) are connected, then the resulting *connected sum* manifold is denoted $Y_1 \# Y_2$.

Isn't it a waste of glue? You may be wondering why bother glueing the interiors of the balls if you are going to remove them anyway. Wouldn't it be easier *first* to remove these interiors from both manifolds and *then* glue what remains along the spheres $S_i^{n-1} = \partial(B_i)$?

This *is* easier but also it is also a *wrong* thing to do: the result may depend on the diffeomorphism $S_1^{n-1} \leftrightarrow S_2^{n-1}$, as it happens for $Y_1 = Y_2 = S^7$ in Milnor's example; but the connected sum defined with balls is unique by the $[B{\to}Y]$-lemma.

The ball gluing operation may be used many times in succession; thus, for example, one builds "big $(n+1)$-balls" from smaller ones, where this lemma in lower dimension may be used for ensuring the ball property of the gluing sites.

Gluing and Bordisms. Take two closed oriented n-manifold X_1 and X_2 and let

$$X_1 \supset U_1 \underset{f}{\leftrightarrow} U_2 \subset X_2$$

be an *orientation reversing* diffeomorphisms between compact n-dimensional submanifolds $U_i \subset X_i$, $i = 1, 2$ with boundaries. If we glue X_1 and X_2 by f and remove the (glued together) interiors of U_i the resulting manifold, say $X_3 = X_1 +_{-U} X_2$ is naturally oriented and, clearly, it is orientably bordant to the disjoint union of X_1 and X_2. (This is similar to the geometric/algebraic cancellation of cycles mentioned in section 4.)

Conversely, one can give an alternative definition of the oriented bordism group $\mathcal{B}_n^o$ as of the Abelian group generated by oriented n-manifolds with the relations $X_3 = X_1 + X_2$ for all $X_3 = X_1 +_{-U} X_2$. This gives the same $\mathcal{B}_n^o$ even if the only U allowed are those diffeomorphic to $S^i \times B^{n-i}$ as it follows from the handle decompositions induced by Morse functions.

The isotopy theorem is not dimension specific, but the following construction due to Haefliger (1961) generalizing the *Whitney Lemma* of 1944 demonstrates something special about isotopies in high dimensions.

Let Y be a smooth n-manifold and $X', X'' \subset Y$ be smooth closed submanifolds in general position. Denote $\Sigma_0 = X' \cap X'' \subset Y$ and let X be the (abstract) *disjoint union* of X' and X''. (If X' and X'' are connected equividimensional manifolds, one could say that X is a smooth manifold with its two "connected components" X' and X'' being embedded into Y.)

Clearly,

$$\dim(\Sigma_0) = n - k' - k''$$

$$\text{for } n = \dim(Y),\ n - k' = \dim(X') \text{ and } n - k'' = \dim(X'').$$

Let $f_t : X \to Y$, $t \in [0, 1]$, be a smooth generic homotopy which disengages X' from X'', i.e. $f_1(X')$ does not intersect $f_1(X'')$, and let

$$\tilde{\Sigma} = \{(x', x'', t)\}_{f_t(x') = f_t(x'')} \subset X' \times X'' \times [0, 1],$$

i.e. $\tilde{\Sigma}$ consists of the triples (x', x'', t) for which $f_t(x') = f_t(x'')$.

Let $\Sigma \subset X' \cup X''$ be the union $S' \cup S''$, where $S' \subset X'$ equals the projection of $\tilde{\Sigma}$ to the X'-factor of $X' \times X'' \times [0, 1]$ and $S'' \subset X''$ is the projection of $\tilde{\Sigma}$ to X''.

Thus, there is a correspondence $x' \leftrightarrow x''$ between the points in $\Sigma = S' \cup S''$, where the two points correspond one to another if $x' \in S'$ meets $x'' \in S''$ at some moment t_* in the course of the homotopy, i.e.

$$f_{t_*}(x') = f_{t_*}(x'') \text{ for some } t_* \in [0, 1].$$

Finally, let $W \subset Y$ be the union of the f_t-paths, denoted $[x' *_t x''] \subset Y$, travelled by the points $x' \in S' \subset \Sigma$ and $x'' \in S'' \subset \Sigma$ until they meet at some moment t_*. In other words, $[x' *_t x''] \subset Y$ consists of the union of the points $f_t(x')$ and $f_t(x'')$ over $t \in [0, t_* = t_*(x') = t_*(x'')]$ and

$$W = \bigcup_{x' \in S'} [x' *_t x''] = \bigcup_{x'' \in S''} [x' *_t x''].$$

Clearly,

$$\dim(\Sigma) = \dim(\Sigma_0) + 1 = n - k' - k'' + 1 \text{ and}$$

$$\dim(W) = \dim(\Sigma) + 1 = n - k' - k'' + 2\,.$$

To grasp the picture look at X consisting of a round 2-sphere X' (where $k' = 1$) and a round circle X'' (where $k'' = 2$) in the Euclidean 3-space Y, where X and X' intersect at two points x_1, x_2 – our $\Sigma_0 = \{x_1, x_2\}$ in this case.

When X' an X'' move away one from the other by parallel translations in the opposite directions, their intersection points sweep W which equals the intersection of the 3-ball bounded by X' and the flat 2-disc spanned by X''. The boundary Σ of this W consists of two arcs $S' \subset X'$ and $S'' \subset X''$, where S' joins x_1 with x_2 in X' and S'' join x_1 with x_2 in X''.

Back to the general case, we want W to be, generically, a smooth submanifold *without double points* as well as without any other singularities, except for the unavoidable corner in its boundary Σ, where S' meet S'' along Σ_0. We need for this

$$2\dim(W) = 2(n - k' - k'' + 2) < n = \dim(Y) \text{ i.e. } 2k' + 2k'' > n + 4.$$

Also, we want to avoid an intersection of W with X' and with X'' away from $\Sigma = \partial(W)$. If we agree that $k'' \geq k'$, this, generically, needs

$$\dim(W) + \dim(X) = (n - k' - k'' + 2) + (n - k') < n \text{ i.e. } 2k' + k'' > n + 2.$$

These inequalities imply that $k' \geq k \geq 3$, and the lowest dimension where they are meaningful is the first Whitney case: $\dim(Y) = n = 6$ and $k' = k'' = 3$.

Accordingly, W is called *Whitney's disk*, although it may be non-homeomorphic to B^2 with the present definition of W (due to Haefliger).

Lemma (Haefliger Lemma: Whitney for $k + k' = n$). *Let the dimensions $n - k' = \dim(X')$ and $n - k'' = \dim(X'')$, where $k'' \geq k'$, of two submanifolds X' and X'' in the ambient n-manifold Y satisfy $2k' + k'' > n + 2$. Then every homotopy f_t of (the disjoint union of) X' and X'' in Y which disengages X' from X'', can be replaced by a disengaging homotopy f_t^{new} which is an isotopy, on both manifolds, i.e. $f_t^{\text{new}}(X')$ and $f^{\text{new}}(X'')$ reman smooth without self intersection points in Y for all $t \in [0,1]$ and $f_1^{\text{new}}(X')$ does not intersect $f_1^{\text{new}}(X'')$.*

Proof. Assume f_t is smooth generic and take a small neighbourhood $U_{3\varepsilon} \subset Y$ of W. By genericity, this f_t is an isotopy of X' as well as of X'' *within* $U_{3\varepsilon} \subset Y$: the intersections of $f_t(X')$ and $f_t(X'')$ with $U_{3\varepsilon}$, call them $X'_{3\varepsilon}(t)$ and $X''_{3\varepsilon}(t)$ are smooth submanifolds in $U_{3\varepsilon}$ for all t, which, moreover, do not intersect away from $W \subset U_{3\varepsilon}$.

Hence, by the Thom isotopy theorem, there exists an isotopy F_t of $Y \setminus U_\varepsilon$ which equals f_t on $U_{2\varepsilon} \setminus U_\varepsilon$ and which is constant in t on $Y \setminus U_{3\varepsilon}$.

Since f_t and F_t within $U_{3\varepsilon}$ are equal on the overlap $U_{2\varepsilon} \setminus U_\varepsilon$ of their definition domains, they make together a homotopy of X' and X'' which, obviously, satisfies our requirements. □

There are several immediate generalizations/applications of this theorem.

(1) One may allow self-intersections Σ_0 *within* connected components of X, where the necessary homotopy condition for removing Σ_0 (which was expressed with the disengaging f_t in the present case) is formulated in terms of maps $\tilde{f} : X \times X \to Y \times Y$ commuting with the involutions $(x_1, x_2) \leftrightarrow (x_2, x_1)$ in $X \times X$ and $(y_1, y_2) \leftrightarrow (y_2, y_1)$ in $Y \times Y$ and having the pullbacks $\tilde{f}^{-1}(Y_{\text{diag}})$ of the diagonal $Y_{\text{diag}} \subset Y \times Y$ equal $X_{\text{diag}} \subset X \times X$, [**33**].

(2) One can apply all of the above to p parametric families of maps $X \to Y$, by paying the price of the extra p in the excess of $\dim(Y)$ over $\dim(X)$, [**33**].

If $p = 1$, this yield an isotopy classification of embeddings $X \to Y$ for $3k > n + 3$ by homotopies of the above symmetric maps $X \times X \to Y \times Y$, which shows, for example, that there are *no knots* for these dimensions (Haefliger, 1961). *If* $3k > n + 3$, *then every smooth embedding* $S^{n-k} \to \mathbb{R}^n$ *is smoothly isotopic to the standard* $S^{n-k} \subset \mathbb{R}^n$.

But if $3k = n + 3$ and $k = 2l + 1$ is odd then there are *infinitely many isotopy of classes of embeddings* $S^{4l-1} \to \mathbb{R}^{6l}$ (Haefliger 1962).

Non-triviality of such a knot $S^{4l-1} \to \mathbb{R}^{6l}$ is detected by showing that a map $f_0 : B^{4l} \to \mathbb{R}^{6l} \times \mathbb{R}_+$ extending $S^{4l-1} = \partial(B^{4l})$ cannot be turned into an embedding, keeping it transversal to $\mathbb{R}^{6l} = \mathbb{R}^{6l} \times 0$ and with its boundary equal our knot $S^{4l-1} \subset \mathbb{R}^{6l}$.

The Whitney-Haefliger W for f_0 has dimension $6l + 1 - 2(2l + 1) + 2 = 2l + 1$ and, generically, it transversally intersects B^{4l} at several points.

The resulting (properly defined) intersection index of W with B is non-zero (otherwise one could eliminate these points by Whitney) and it does not depend on f_0. In fact, it equals the linking invariant of Haefliger. (This is reminiscent of the "higher linking products" described by Sullivan's minimal models, see Section 9.)

(3) In view of he above, one must be careful if one wants to relax the dimension constraint by an inductive application of the Whitney-Haefliger disengaging procedure, since obstructions/invariants for removal "higher" intersections which come on the way may be not so apparent. (The structure of "higher self-intersections" of this kind for Euclidean hypersurfaces carries a significant information on the stable homotopy groups of spheres.)

But this is possible, at least on the $\mathbb{Q}$-level, where one has a comprehensive algebraic control of self-intersections of all multiplicities for maps of codimension $k \geq 3$. Also, even without tensoring with $\mathbb{Q}$, the higher intersection obstructions tend to vanish in the combinatorial category.

For example, *there are no combinatorial knots of codimension* $k \geq 3$ (Zeeman, 1963).

The essential mechanism of knotting $X = X^n \subset Y = Y^{n+2}$ depends on the fundamental group Γ of the complement $U = Y \subset X$. The group Γ may look a nuisance when you want to untangle a knot, especially a surface X^2 in a 4-manifold, but these $\Gamma = \Gamma(X)$ for various $X \subset Y$ form beautifully intricate patterns which are poorly understood.

For example, the groups $\Gamma = \pi_1(U)$ capture *the étale cohomology of algebraic manifolds* and the *Novikov-Pontryagin classes of topological manifolds* (see section 10). Possibly, the groups $\Gamma(X^2)$ for surfaces $X^2 \subset Y^4$ have much to tell us about the smooth topology of 4-manifolds.

There are few systematic ways of constructing "simple" $X \subset Y$, e.g. immersed submanifolds, with "interesting" (e.g. far from being free) fundamental groups of their complements.

Offhand suggestions are pullbacks of (special singular) divisors X_0 in complex algebraic manifolds Y_0 under generic maps $Y \to Y_0$ and immersed subvarieties X^n in cubically subdivided Y^{n+2}, where X^n are made of n-sub-cubes $\square^n$ *inside* the cubes $\square^{n+2} \subset Y^{n+2}$ and where these interior $\square^n \subset \square^{n+2}$ are parallel to the n-faces of $\square^{n+2}$.

It remains equally unclear what is the possible topology of self-intersections of immersions $X^n \to Y^{n+2}$, say for $S^3 \to S^5$, where the self-intersection makes a link in S^3, and for $S^4 \to S^6$ where this is an immersed surface in S^4.

(4) One can control the position of the image of $f^{\text{new}}(X) \subset Y$, e.g. by making it to land in a given open subset $W_0 \subset W$, if there is no homotopy obstruction to this.

The above generalizes and simplifies in the combinatorial or "piecewise smooth" category, e.g. for "unknotting spheres", where the basic construction is as follows

Theorem (Engulfing). *Let X be a piecewise smooth polyhedron in a smooth manifold Y. If $n-k = dim(X) \le dim(Y) - 3$ and if $\pi_i(Y) = 0$ for $i = 1, ...dim(Y)$, then there exists a smooth isotopy F_t of Y which eventually (for $t = 1$) moves X to a given (small) neighbourhood $B_\circ$ of a point in Y*

Sketch of the Proof. Start with a generic f_t. This f_t does the job away from a certain W which has $\dim(W) \le n-2k+2$. This is $< \dim(X)$ under the above assumption and the proof proceeds by induction on $\dim(X)$. □

This is called "engulfing" since $B_\circ$, when moved by the time reversed isotopy, engulfs X; engulfing was invented by Stallings in his approach to the *Poincaré Conjecture* in the combinatorial category, which goes, roughly, as follows.

Let Y be a smooth n-manifold. Then, with a simple use of two mutually dual smooth triangulations of Y, one can decompose Y, for each i, into the union of *regular neighbourhoods* U_1 and U_2 of smooth subpolyhedra X_1 and X_2 in Y of dimensions i and $n-i-1$ (similarly to the *handle body decomposition* of a 3-manifold into the union of two thickened graphs in it), where, recall, a neighbourhood U of an $X \subset Y$ is regular if there exists an isotopy $f_t : U \to U$ which brings all of U arbitrarily close to X.

Now let Y be a homotopy sphere of dimension $n \ge 7$, say $n = 7$, and let $i = 3$ Then X_1 and X_2, and hence U_1 and U_2, can be engulfed by (diffeomorphic images of) balls, say by $B_1 \supset U_1$ and $B_2 \supset U_2$ with their centers denoted $0_1 \in B_1$ and $0_2 \in B_2$.

By moving the 6-sphere $\partial(B_1) \subset B_2$ by the radial isotopy in B_2 toward 0_2, one represents $Y \setminus 0_2$ by the union of an increasing sequence of isotopic copies of the ball B_1. This implies (with the isotopy theorem) that $Y \setminus 0_2$ is diffeomorphic to $\mathbb{R}^7$, hence, Y is homeomorphic to S^7.

(A refined generalization of this argument delivers the Poincaré conjecture in the combinatorial and topological categories for $n \ge 5$. See [**66**] for an account of techniques for proving various "Poincaré conjectures" and for references to the source papers.)

8. Handles and h-Cobordisms

The original approach of Smale to the Poincaré conjecture depends on *handle decompositions* of manifolds—counterparts to cell decompositions in the homotopy theory.

Such decompositions are more flexible, and by far more abundant than triangulations and they are better suited for a match with algebraic objects such as homology. For example, one can sometimes realize a basis in homology by suitably chosen cells or handles which is not even possible to formulate properly for triangulations.

Recall that an *i-handle of dimension n* is the ball B^n decomposed into the product $B^n = B^i \times B^{n-i}(\varepsilon)$ where one think of such a handle as an ε-thickening of the unit i-ball and where

$$A(\varepsilon) = S^i \times B^{n-1}(\varepsilon) \subset S^{n-1} = \partial B^n$$

is seen as an ε-neighbourhood of its *axial $(i-1)$-sphere* $S^{i-1} \times 0$ – *an equatorial i-sphere in S^{n-1}*.

If X is an n-manifold with boundary Y and $f : A(\varepsilon) \to Y$ a smooth embedding, one can attach B^n to X by f and the resulting manifold (with the "corner" along $\partial A(\varepsilon)$ made smooth) is denoted $X +_f B^n$ or $X +_{S^{i-1}} B^n$, where the latter subscript refers to the f-image of the axial sphere in Y.

The effect of this on the boundary, i.e. modification

$$\partial(X) = Y \rightsquigarrow_f Y' = \partial(X +_{S^{i-1}} B^n)$$

does not depend on X but only on Y and f. It is called an *i-surgery of Y at the sphere $f(S^{i-1} \times 0) \subset Y$.*

The manifold $X = Y \times [0,1] +_{S^{i-1}} B^n$, where B^n is attached to $Y \times 1$, makes a bordism between $Y = Y \times 0$ and Y' which equals the surgically modified $Y \times 1$-component of the boundary of X. If the manifold Y is oriented, so is X, unless $i = 1$ and the two ends of the 1-handle $B^1 \times B^{n-1}(\varepsilon)$ are attached to the same connected component of Y with opposite orientations.

When we attach an i-handle to an X along a *zero-homologous* sphere $S^{i-1} \subset Y$, we create a new i-cycle in $X +_{S^{i-1}} B^n$; when we attach an $(i+1)$-handle along an i-sphere in X which is *non-homologous to zero*, we "kill" an i-cycle.

These creations/annihilations of homology may cancel each other and a handle decomposition of an X may have by far more handles (balls) than the number of independent homology classes in $H_*(X)$.

Smale's argument proceeds in two steps.

(1) The overall algebraic cancellation is decomposed into "elementary steps" by "reshuffling" handles (in the spirit of J.H.C. Whitehead's theory of the *simple homotopy type*);

(2) each elementary step is implemented geometrically as in the example below (which does not elucidate the case $n = 6$).

Cancelling a 3-handle by a 4-handle. Let $X = S^3 \times B^4(\varepsilon_0)$ and let us attach the 4-handle $B^7 = B^4 \times B^3(\varepsilon)$, $\varepsilon << \varepsilon_0$, to the (normal) ε-neighbourhood $A_\sim$ of some sphere

$$S^3_\sim \subset Y = \partial(X) = S^3 \times S^3(\varepsilon_0) \text{ for } S^3(\varepsilon_0) = \partial B^4(\varepsilon_0).$$

by some diffeomorphism of $A(\varepsilon) \subset \partial(B^7)$ onto $A_\sim$.

If $S^3_\sim = S^3 \times b_0$, $b_0 \in S^3(\varepsilon_0)$, is the standard sphere, then the resulting $X_\sim = X +_{S^3_\sim} B^7$ is obviously diffeomorphic to B^7: adding $S^3 \times B^4(\varepsilon_0)$ to B^7 amounts to "bulging" the ball B^7 over the ε-neighbourhood $A(\varepsilon)$ of the axial 3-sphere on its boundary.

Another way to see it is by observing that this addition of $S^3 \times B^4(\varepsilon_0)$ to B^7 can be decomposed into gluing two balls in succession to B^7 as follows.

Take a ball $B^3(\delta) \subset S^3$ around some point $s_0 \in S^3$ and decompose $X = S^3 \times B^4(\varepsilon_0)$ into the union of two balls that are

$$B^7_\delta = B^3(\delta) \times B^4(\varepsilon_0)$$

and

$$B^7_{1-\delta} = B^3(1-\delta) \times B^4(\varepsilon_0) \text{ for } B^3(1-\delta) =_{def} S^3 \smallsetminus B^3(\delta).$$

Clearly, the attachment loci of $B^7_{1-\delta}$ to X and of B^7_δ to $X + B^7_{1-\delta}$ are diffeomorphic (after smoothing the corners) to the 6-ball.

Let us modify the sphere $S^3 \times b_0 \subset S^3 \times B^4(\varepsilon_0) = \partial(X)$ by replacing the original standard embedding of the 3-ball

$$B^3(1-\delta) \to B^7_{1-\delta} = B^3(1-\delta) \times S^3(\varepsilon_0) \subset \partial(X)$$

by another one, say,

$$f_\sim : B^3(1-\delta) \to B^7_{1-\delta} = B^3(1-\delta) \times S^3(\varepsilon_0) = \partial(X),$$

such that $f_\sim$ equals the original embedding near the boundary of $\partial(B^3(1-\delta)) = \partial(B^3(\delta)) = S^2(\delta)$.

Then the same "ball after ball" argument applies, since the first gluing site where $B^7_{1-\delta}$ is being attached to X, albeit "wiggled", remains diffeomorphic to B^6 by the isotopy theorem, while the second one does not change at all. So we conclude the following.

> Whenever $S^3_\sim \subset S^3 \times S^3(\varepsilon_0)$ transversally intersect $s_0 \times S^3(\varepsilon_0)$, $s_0 \in S^3$, at a single point, the manifold $X_\sim = X +_{S^3_\sim} B^7$ is diffeomorphic to B^7.

Finally, by Whitney's lemma, every embedding $S^3 \to S^3 \times S^3(\varepsilon_0) \subset S^3 \times B^4(\varepsilon_0)$ which is *homologous in* $S^3 \times B^4(\varepsilon_0)$ to the standard $S^3 \times b_0 \subset S^3 \times B^4(\varepsilon_0)$, can be isotoped to another one which meets $s_0 \times S^3(\varepsilon_0)$ transversally at a single point. Hence the following.

> The handles do cancel one another: if a sphere
>
> $$S^3_\sim \subset S^3 \times S^3(\varepsilon_0) = \partial(X) \subset X = S^3 \times B^4(\varepsilon_0),$$
>
> is homologous in X to
>
> $$S^3 \times b_0 \subset X = S^3 \times B^4(\varepsilon_0),\ b_0 \in B^4(\varepsilon),$$
>
> then the manifold $X +_{S^3_\sim} B^7$ is diffeomorphic to the 7-ball.

Let us show in this picture that Milnor's sphere Σ^7 minus a small ball is diffeomorphic to B^7. Recall that Σ^7 is fibered over S^4, say by $p : \Sigma^7 \to S^4$, with S^3-fibers and with the Euler number $e = \pm 1$.

Decompose S^4 into two round balls with the common S^3-boundary, $S^4 = B^4_+ \cup B^4_-$. Then Σ^7 decomposes into $X_+ = p^{-1}(B^4_+) = B^4_+ \times S^3$ and $X_- = p^{-1}(B^4_-) = B^4_- \times S^3$, where the gluing diffeomorphism between the boundaries $\partial(X_+) = S^3_+ \times S^3$ and $\partial(X_-) = S^3_- \times S^3$ for $S^3_\pm = \partial B^4_\pm$, is homologically the same as for the Hopf fibration $S^7 \to S^4$ for $e = \pm 1$.

Therefore, if we decompose the S^3-factor of $B^4_- \times S^3$ into two round balls, say $S^3 = B^3_1 \cup B^3_2$, then either $B^4_- \times B^3_1$ or $B^4_- \times B^3_2$ makes a 4-handle attached to X_+ to which the handle cancellation applies and shows that $X_+ \cup (B^4_- \times B^3_1)$ is a smooth 7-ball. (All what is needed of the Whitney's lemma is obvious here: the zero section

$X \subset V$ in an oriented $\mathbb{R}^{2k}$-bundle $V \to X = X^{2k}$ with $e(V) = \pm 1$ can be perturbed to $X' \subset V$ which *transversally* intersect X at a *single* point.)

The handles shaffling/cancellation techniques do not solve the existence problem for diffeomorphisms $Y \leftrightarrow Y'$ but rather reduce it to the existence of *h-cobordisms* between manifolds, where a compact manifold X with two boundary components Y and Y' is called an h-cobordism (between Y and Y') if the inclusion $Y \subset X$ is a homotopy equivalence.

Theorem (Smale h-Cobordism Theorem). *If an h-cobordism has* $dim(X) \geq 6$ *and* $\pi_1(X) = 1$ *then* X *is diffeomorphic to* $Y \times [0,1]$, *by a diffeomorphism keeping* $Y = Y \times 0 \subset X$ *fixed. In particular, h-cobordant simply connected manifolds of dimensions* ≥ 5 *are diffeomorphic.*

Notice that the Poincaré conjecture for the homotopy spheres Σ^n, $n \geq 6$, follows by applying this to Σ^n minus two small open balls, while the case $m = 1$ is solved by Smale with a construction of an h-cobordism between Σ^5 and S^5.

Also Smale's handle techniques deliver the following geometric version of the Poincaré connectedness/contractibilty correspondence (see Section 4).

> Let X be a closed n-manifold, $n \geq 5$, with $\pi_i(X) = 0$, $i = 1, ..., k$. Then X contains a $(n-k-1)$-dimensional smooth sub-polyhedron $P \subset X$, such that the complement of the open (regular) neighbourhood $U_\varepsilon(P) \subset X$ of P is diffeomorphic to the n-ball, (where the boundary $\partial(U_\varepsilon)$ is the $(n-1)$-sphere "ε-collapsed" onto $P = P^{n-k-1}$).

If $n = 5$ and if the *normal bundle of* X *embedded into some* $\mathbb{R}^{5+N}$ *is trivial*, i.e. if the normal Gauss map of X to the Grassmannian $Gr(\mathbb{R}^{5+N})$ is contractible, then Smale proves, assuming $\pi_1(X) = 1$, that one can choose $P = P^3 \subset X = X^5$ that equals the union of a smooth topological segment $s = [0,1] \subset X$ and several spheres S_i^2 and S_i^3, where each S_i^3 meets s at one point, and also transversally intersects S_i^2 at a single point and where there are no other intersections between s, S_i^2 and S_i^3. In other words,

> (Smale 1965) X is diffeomorphic to the connected sum of several copies of $S^2 \times S^3$.

The triviality of the bundle in this theorem is needed to ensure that all embedded 2-spheres in X have trivial normal bundles, i.e. their normal neighbourhoods split into $S^2 \times \mathbb{R}^3$ which comes handy when you play with handles.

If one drops this triviality condition, one has

Theorem (Classification of Simply Connected 5-Manifolds. Barden 1966). *There is a finite list of explicitly constructed* 5*-manifolds* X_i, *such that every closed simply connected manifold* X *is diffeomorphic to the connected sum of* X_i.

This is possible, in view of the above Smale theorem, since all simply connected 5-manifolds X have "almost trivial" normal bundles e.g. their only possible Pontryagin class $p_1 \in H^4(X)$ is zero. Indeed $\pi_1(X) = 1$ implies that $H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)] = 0$ and then $H^4(X) = H_1(X) = 0$ by the Poincaré duality.

When you encounter bordisms, the generecity sling launches you to the stratosphere of algebraic topology so fast that you barely discern the geometric string attached to it.

Smale's cells and handles, on the contrary, feel like slippery amebas which merge and disengage as they reptate in the swamp of unruly geometry, where n-dimensional cells continuously collapse to lower dimensional ones and keep squeezing through paper-thin crevices. Yet, their motion is governed, for all we know, by the rules dictated by some *algebraic K-theory* (theories?)

This motion hardly can be controlled by any traditional geometric flow. First of all, the "simply connected" condition cannot be encoded in geometry ([**52**], [**28**] [**53**] and also breaking the symmetry by dividing a manifold into handles along with "genericity" poorly fare in geometry.

Yet, some generalized "Ricci flow with partial collapse and surgeries" in the "space of (generic, random?) amebas" might split away whatever it fails to untangle and bring fresh geometry into the picture.

For example, take a compact locally symmetric space $X_0 = S/\Gamma$, where S is a non-compat irreducible symmetric space of rank ≥ and make a 2-surgery along some non-contractible circle $S^1 \subset X_0$. The resulting manifold X has finite fundamental group by Margulis' theorem and so a finite covering $\tilde{X} \to X$ is simply connected. What can a geometric flow do to these X_0 and $\tilde{X}$? Would it bring X back to X_0?

9. Manifolds under Surgery

The Atiyah–Thom construction and Serre's theory allows one to produce "arbitrarily large" manifolds X for the *m-domination* $X_1 \succ_m X_2$, $m > 0$, meaning that there is a map $f : X_1 \to X_2$ of degree m.

Every such f between closed connected oriented manifolds induces a *surjective* homomorphisms $f_{*i} : H_i(X_1;\mathbb{Q}) \to H_i(X_1;\mathbb{Q})$ for all $i = 0,1,...,n$, (as we know from section 4), or equivalently, an injective cohomology homomorphism $f^{*i} : H^i(X_2;\mathbb{Q}) \to H^i(X_2;\mathbb{Q})$.

Indeed, by the Poincaré $\mathbb{Q}$-duality, the cup-product (this the common name for the product on cohomology) pairing $H^i(X_2;\mathbb{Q}) \otimes H^{n-i}(X_2;\mathbb{Q}) \to \mathbb{Q} = H^n(X_2;\mathbb{Q})$ is faithful; therefore, if f^{*i} vanishes, then so does f^{*n}. But the latter amounts to multiplication by $m = deg(f)$,

$$H^n(X_2;\mathbb{Q}) = \mathbb{Q} \to_{\cdot d} \mathbb{Q} = H^n(X_1;\mathbb{Q}).$$

(The main advantage of the cohomology product over the intersection product on homology is that the former is preserved by all continuous maps,

$$f^{*i+j}(c_1 \cdot c_2) = f^{*i}(c_1) \cdot f^{*j}(c_2)$$

for all $f : X \to Y$ and all $c_1 \in H^i(Y)$, $c_2 \in H^j(Y)$.)

If $m = 1$, then (by the full cohomological Poincaré duality) the above remains true for all coefficient fields $\mathbb{F}$; moreover, the induced homomorphism $\pi_i(X_1) \to \pi_i(X_2)$ is surjective as it is seen by looking at the lift of $f : X_1 \to X_2$ to the induced map from the covering $\tilde{X}_1 \to X_1$ induced by the universal covering $\tilde{X}_2 \to X_2$ to $\tilde{X}_2$. (A map of degree $m > 1$ sends $\pi_1(X_1)$ to a subgroup in $\pi_1(X_2)$ of a finite index dividing m.)

Let us construct manifolds starting from *pseudo-manifolds*, where a compact oriented n-dimensional pseudo-manifold is a triangulated n-space X_0, such that the following holds.

- Every simplex of dimension $< n$ in X_0 lies in the boundary of an n-simplex,

- The complement to the union of the $(n-2)$-simplices in X_0 is an oriented manifold.

Pseudo-manifolds are infinitely easier to construct and to recognize than manifolds: essentially, these are simplicial complexes with exactly two n-simplices adjacent to every $(n-1)$-simplex.

There is no comparably simple characterization of triangulated n-manifolds X where the links $L^{n-i-1} = L_{\Delta^i} \subset X$ of the i-simplices must be topological $(n-i-1)$-spheres. But even deciding if $\pi_1(L^{n-i-1}) = 1$ is *an unsolvable problem* except for a couple of low dimensions.

Accordingly, it is very hard to produce manifolds by combinatorial constructions; yet, one can "dominate" any pseudo-manifold by a manifold, where, observe, the notion of degree perfectly applies to oriented pseudo-manifolds.

Theorem. *Let X_0 be a connected oriented n-pseudomanifold. Then there exists a smooth closed connected oriented manifold X and a continuous map $f : X \to X_0$ of degree $m > 0$.*

Moreover, given an oriented $\mathbb{R}^N$-bundle $V_0 \to X_0$, $N \geq 1$, one can find an m-dominating X, which also admits a smooth embedding $X \subset \mathbb{R}^{n+N}$, such that our $f : X \to X_0$ of degree $m > 0$ induces the normal bundle of X from V_0.

Proof. Since that the first $N-1$ homotopy groups of the Thom space of $V_\bullet$ of V_0 vanish (see section 5), Serre's m-sphericity theorem delivers a map $f_\bullet : S^{n+N} \to V_\bullet$ a non-zero degree m, provided $N > n$. Then the "generic pullback" X of $X_0 \subset V_0$ (see section 3) does the job as it was done in Section 5 for Thom's bordisms. □

In general, if $1 \leq N \leq n$, the m-sphericity of the fundamental class $[V_\bullet] \in H_{n+N}(V_\bullet)$ is proven with the *Sullivan's minimal models*, see Theorem 24.5 in [**19**]

The minimal model, of a space X is a *free (skew)commutative differential algebra* which, in a way, extends the cohomology algebra of X and which faithfully encodes *all* homotopy $\mathbb{Q}$-invariants of X. If X is a smooth N-manifold it can be seen in terms of "higher linking" in X.

For example, if two cycles $C_1, C_2 \subset X$ of codimensions i_1, i_2, satisfy $C_1 \sim 0$ and $C_1 \cap C_2 = 0$, then the (first order) *linking class* between them is an element in the quotient group $H_{N-i_1-i_2-1}(X)/(H_{N-i_1-1}(X) \cap [C_2])$ which is defined with a plaque $D_1 \in \partial^{-1}(C_1)$, i.e. such that $\partial(D_1) = C_1$, as the image of $[D_1 \cap C_2]$ under the quotient map

$$H_{N-i_1-i_2-1}(X) \ni [D_1 \cap C_2] \mapsto H_{N-i_1-i_2-1}(X)/(H_{N-i_1-1}(X) \cap [C_2]).$$

Surgery and the Browder-Novikov Theorem. (1962 [**8**],[**54**]). Let X_0 be a smooth closed simply connected oriented n-manifold, $n \geq 5$, and $V_0 \to X_0$ be a *stable* vector bundle where "stable" means that $N = \mathrm{rank}(V) >> n$. We want to modify the smooth structure of X_0 keeping its homotopy type unchanged but with its original normal bundle in $\mathbb{R}^{n+N}$ replaced by V_0.

There is an obvious algebraic-topological obstruction to this highlighted by Atiyah in [**2**] which we call *$[V_\bullet]$-sphericity* and which means that there exists a *degree one*, map $f_\bullet$ of S^{n+N} to the Thom space $V_\bullet$ of V_0, i.e. $f_\bullet$ sends the generator $[S^{n+N}] \in H_{n+N}(S^{n+N}) = \mathbb{Z}$ (for some orientation of the sphere S^{n+N}) to the fundamental class of the Thom space, $[V_\bullet \in H_{n+N}(V_\bullet) = \mathbb{Z}$, which is distinguished by the orientation in X. (One has to be pedantic with orientations to keep track of possible/impossible algebraic cancellations.)

However, this obstruction is "$\mathbb{Q}$-nonessential", [**2**] : the set of the vector bundles admitting such an $f_\bullet$ constitutes a coset of a subgroup of *finite index* in Atiyah's (reduced) *K-group* by Serre's finiteness theorem.

Recall that $K(X)$ is the Abelian group formally generated by the isomorphism classes of vector bundles V over X, where $[V_1]+[V_2] =_{\rm def} 0$ whenever the *Whitney sum* $V_1 \oplus V_2$ is isomorphic to a trivial bundle.

The *Whitney sum* of an $\mathbb{R}^{n_1}$-bundle $V_1 \to X$ with an $\mathbb{R}^{n_2}$-bundle $V_2 \to X$, is the $\mathbb{R}^{n_1+n_2}$-bundle over X. which equals the fiber-wise Cartesian product of the two bundles.

For example the Whitney sum of the tangent bundle of a smooth submanifold $X^n \subset W^{n+N}$ and of its normal bundle in W equals the tangent bundle of W restricted to X. Thus, it is *trivial* for $W = \mathbb{R}^{n+N}$, i.e. it isomorphic to $\mathbb{R}^{n+N} \times X \to X$, since the tangent bundle of $\mathbb{R}^{n+N}$ is, obviously, trivial.

Granted an $f_\bullet : S^{n+N} \to V_\bullet$ of degree 1, we take the "generic pullback" X of X_0,

$$X \subset \mathbb{R}^{n+N} \subset \mathbb{R}^{n+N}_\bullet = S^{n+N},$$

and denote by $f : X \to X_0$ the restriction of $f_\bullet$ to X, where, recall, f induces the normal bundle of X from V_0. .

The map $f : X_1 \to X_0$, which is clearly *onto*, is far from being injective – it may have uncontrollably complicated folds. In fact, it is not even a homotopy equivalence – the homology homomorphism induced by f

$$f_{*i} : H_i(X_1) \to H_i(X_0),$$

is, as we know, surjective and it may (and usually does) have non-trivial kernels $\ker_i \subset H_i(X_1)$. However, these kernels can be "killed" by a "surgical implementation" of the obstruction theory (generalizing the case where $X_0 = S^n$ due to Kervaire-Milnor) as follows.

Assume $\ker_i = 0$ for $i = 0, 1, ..., k-1$, invoke Hurewicz' theorem and realize the cycles in $\ker_k$ by k-spheres mapped to X_1, where, observe, the f-images of these spheres are contractible in X_0 by a relative version of the (elementary) Hurewicz theorem.

Furthermore, if $k < n/2$, then these spheres $S^k \subset X_1$ are generically *embedded* (no self-intersections) and have *trivial* normal bundles in X_1, since, essentially, they come from $V \to X_1$ via contractible maps. Thus, small neighbourhoods (ε-annuli) $A = A_\varepsilon$ of these spheres in X_1 split: $A = S^k \times B^{n-k}_\varepsilon \subset X_1$.

It follows, that the corresponding spherical cycles can be killed by $(k+1)$-surgery (where X_1 now plays the role of Y in the definition of the surgery); moreover, it is not hard to arrange a map of the resulting manifold to X_0 with the same properties as f.

If $n = \dim(X_0)$ is odd, this works up to $k = (n-1)/2$ and makes *all* $\ker_i$, including $i > k$, equal zero by the Poincaré duality.

Since

> a continuous map between *simply connected* spaces which induces an *isomorphism on homology* is a *homotopy equivalence* by the (elementary) *Whitehead theorem*,

the resulting manifold X is a *homotopy equivalent* to X_0 via our surgically modified map f, call it $f_{srg} : X \to X_0$.

Besides, by the construction of f_{srg}, this map induces the normal bundle of X from $V \to X_0$. Thus we conclude,

> the Atiyah $[V_\bullet]$-sphericity is the only condition for realizing a stable vector bundle $V_0 \to X_0$ by the normal bundle of a smooth manifold X in the homotopy class of a given odd dimensional simply connected manifold X_0.

If n is even, we need to kill k-spheres for $k = n/2$, where an extra obstruction arises. For example, if k is even, the surgery does not change the signature; therefore, the Pontryagin classes of the bundle V *must satisfy* the Rokhlin-Thom-Hirzebruch formula to start with.

(There is an additional constraint for the tangent bundle $T(X)$ – the equality between the Euler characteristic $\chi(X) = \sum_{i=0,\dots,n}(-1)^i \mathrm{rank}_{\mathbb{Q}}(H_i(X))$ and the Euler number $e(T(X))$ that is the self-intersection index of $X \subset T(X)$.)

On the other hand the equality $L(V)[X_0] = \mathrm{sig}(X_0)$ (obviously) implies that $\mathrm{sig}(X) = \mathrm{sig}(X_0)$. It follows that

> the intersection form on $\ker_k \subset H_k(X)$ has zero signature,

since all $h \in \ker_k$ have zero intersection indices with the pullbacks of k-cycles from X_0.

Then, assuming $\ker_i = 0$ for $i < k$ and $n \neq 4$, one can use Whitney's lemma and realize a basis in $\ker_k \subset H_k(X_1)$ by $2m$ *embedded* spheres $S^k_{2j-1}, S^k_{2j} \subset X_1$, $i = 1, ...m$, which have zero self-intersection indices, one point crossings between S^k_{2j-1} and S^k_{2j} and no other intersections between these spheres.

Since the spheres $S^k \subset X$ with $[S^k] \in \ker_k$ have trivial *stable* normal bundles $U^\perp$ (i.e. their Whitney sums with trivial 1-bundles, $U^\perp \oplus \mathbb{R}$, are trivial), the normal bundle $U^\perp = U^\perp(S^k)$ of such a sphere S^k is trivial if and only if the Euler number $e(U^\perp)$ vanishes.

Indeed any oriented k-bundle $V \to B$, such that $V \times \mathbb{R} = B \times \mathbb{R}^{k+1}$, is induced from the tautological bundle V_0 over the oriented Grassmannian $\mathrm{Gr}^{\mathrm{or}}_k(\mathbb{R}^{k+1})$, where $\mathrm{Gr}^{\mathrm{or}}_k(\mathbb{R}^{k+1}) = S^k$ and V_0 is the tangent bundle $T(S^k)$. Thus, the Euler class of V is induced from that of $T(S^k)$ by the classifying map, $G : B \to S^k$. If $B = S^k$ then the Euler number of $e(V)$ equals $2\deg(G)$ and if $e(V) = 0$ the map G is contractible which makes $V = S^k \times \mathbb{R}^k$.

Now, observe, $e(U^\perp(S^k))$ is conveniently equal to the self-intersection index of S^k in X. ($e(U^\perp(S^k))$ equals, by definition, the self-intersection of $S^k \subset U^\perp(S^k)$ which is the same as the self-intersection of this sphere in X.)

Then it easy to see that the $(k+1)$-surgeries applied to the spheres S^k_{2j}, $j = 1, ..., m$, kill all of $\ker_k$ and make $X \to X_0$ a homotopy equivalence.

There are several points to check (and to correct) in the above argument, but everything fits amazingly well in the lap of the linear algebra (The case of odd k is more subtle due to the Kervaire-Arf invariant.)

Notice, that our starting X_0 does not need to be a manifold, but rather a *Poincaré (Browder) n-space*, i.e. a finite cell complex satisfying the Poincaré duality: $H_i(X_0, \mathbb{F}) = H^{n-i}(X_0, \mathbb{F})$ for all coefficient fields (and rings) $\mathbb{F}$, where these "equalities" must be coherent in an obvious sense for different $\mathbb{F}$.

Also, besides the *existence* of smooth n-manifolds X, the above *surgery argument* applied to a bordism Y between homotopy equivalent manifolds X_1 and X_2. Under suitable conditions on the normal bundle of Y, such a bordism can be surgically modified to an h-cobordism. Together with the h-cobordism theorem, this leads to an algebraic *classification* of smooth structures on simply connected manifolds of dimension $n \geq 5$. (see [**54**]).

Then the Serre finiteness theorem implies that

> there are at most finitely many smooth closed simply connected n-manifolds X in a given a homotopy class and with given Pontryagin classes $p_k \in H^{4k}(X)$.

Summing up, the question "What are manifolds?" has the following

1962 Answer. *Smooth closed simply connected n-manifolds for $n \geq 5$, up to a "finite correction term", are "just" simply connected Poincaré n-spaces X with distinguished cohomology classes $p_i \in H^{4i}(X)$, such that $L_k(p_i)[X] = sig(X)$ if $n = 4k$.*

This is a fantastic answer to the "manifold problem" undreamed of 10 years earlier. Yet,

- Poincaré spaces are not classifiable. Even the candidates for the cohomology rings are not classifiable over $\mathbb{Q}$.

Are there special "interesting" classes of manifolds and/or coarser than $diff$ classifications? (Something mediating between bordisms and h-cobordisms maybe?)

- The $\pi_1 = 1$ is very restrictive. The surgery theory extends to manifolds with an arbitrary fundamental group Γ and, modulo *the Novikov conjecture* – a non-simply connected counterpart to the relation $L_k(p_i)[X] = \text{sig}(X)$ (see next section) – delivers a comparably exhaustive answer in terms of the "Poincaré complexes over (the group ring of) Γ" (see [**78**]).

But this does not tells you much about "topologically interesting" Γ, e.g. fundamental groups of n-manifold X with the universal covering $\mathbb{R}^n$ (see [**13**] [**14**] about it).

10. Elliptic Wings and Parabolic Flows

The geometric texture in the topology we have seen so far was all on the side of the "entropy"; topologists were finding gentle routes in the rugged landscape of all possibilities, you do not have to sweat climbing up steep energy gradients on these routs. And there was no essential new analysis in this texture for about 50 years since Poincaré.

Analysis came back with a bang in 1963 when Atiyah and Singer discovered the index theorem.

The underlying idea is simple: the "difference" between dimensions of two spaces, say Φ and Ψ, can be defined and be *finite* even if the spaces themselves are *infinite dimensional*, provided the spaces come with a linear (sometimes non-linear) *Fredholm* operator $D : \Phi \to \Psi$. This means, there exists an operator $E : \Psi \to \Phi$ such that $(1 - D \circ E) : \Psi \to \Psi$ and $(1 - E \circ D) : \Phi \to \Phi$ are compact operators. (In the non-linear case, the definition(s) is local and more elaborate.)

If D is Fredholm, then the spaces $\ker(D)$ and $\text{coker}(D) = \Psi/D(\Phi)$ are finite dimensional and the *index* $\text{ind}(D) = \dim(\ker(D)) - \dim(\text{coker}(D))$ is (by a simple argument) is a *homotopy invariant* of D in the space of *Fredholm* operators.

If, and this is a "big IF", you can associate such a D to a geometric or topological object X, this index will serve as an invariant of X.

It was known since long that *elliptic* differential operators, e.g. the ordinary Laplace operator, are Fredholm under suitable (boundary) conditions but most of these "natural" operators are self-adjoint and always have zero indices: they are of no use in topology.

"Interesting" elliptic differential operators D are scarce: the ellipticity condition is a tricky inequality (or, rather, non-equality) between the coefficients of D. In fact, all such (linear) operators currently in use descend from a single one: *the Atiyah-Singer-Dirac operator on spinors.*

Atiyah and Singer have computed the indices of their geometric operators in terms of traditional topological invariants, and thus discovered new properties of the latter.

For example, they expressed the signature of a closed smooth Riemannian manifold X as an index of such an operator D_{sig} acting on differential forms on X. Since the *parametrix* operator E for an elliptic operator D can be obtained by piecing together local parametrices, the very existence of D_{sig} implies the multiplicativity of the signature.

The elliptic theory of Atiyah and Singer and their many followers, unlike the classical theory of PDE, is functorial in nature as it deals with many interconnected operators at the same time in coherent manner.

Thus smooth structures on potential manifolds (Poincaré complexes) define a functor from the homotopy category to the category of "Fredholm diagrams" (e.g. operators—one arrow diagrams); one is tempted to forget manifolds and study such functors per se. For example, a closed smooth manifold represents a homology class in Atiyah's K-theory – the index of D_{sig}, twisted with vector bundles over X with connections in them.

Interestingly enough, one of the first topological applications of the index theory, which equally applies to all dimensions be they big or small, was the solution (Massey, 1969) of the *Whitney 4D-conjecture* of 1941 which, in a simplified form, says the following.

> The number $N(Y)$ of possible normal bundles of a closed connected non-orientable surface Y embedded into the Euclidean space $\mathbb{R}^4$ equals $|\chi(Y) - 1| + 1$, where χ denotes the Euler characteristic.Equivalently, there are $|\chi(Y) - 1| = 1$ possible homeomorphisms types of small normal neighbourhoods of Y in $\mathbb{R}^4$.

If Y is an orientable surface then $N(Y) = 1$, since a small neighbourhood of such a $Y \subset \mathbb{R}^4$ is homeomorphic to $Y \times \mathbb{R}^2$ by an elementary argument.

If Y is non-orientable, Whitney has shown that $N(Y) \geq |\chi(Y) - 1| + 1$ by constructing $N = |\chi(Y) - 1| + 1$ embeddings of each Y to $\mathbb{R}^4$ with different normal bundles and then conjectured that one could not do better.

Outline of Massey's Proof. Take the (unique in this case) ramified double covering X of $S^4 \supset \mathbb{R}^4 \supset Y$ branched at Y with the natural involution $I : X \to X$. Express the *signature of* I, that is the quadratic form on $H_2(X)$ defined by the intersection of cycles C and $I(C)$ in X, in terms of the *Euler number* $e^{\perp}$ of the normal bundle of $Y \subset \mathbb{R}^4$ as $sig = e^{\perp}/2$ (with suitable orientation and sign conventions) by applying the *Atiyah-Singer equivariant signature theorem.* Show that $rank(H_2(X)) = 2 - \chi(Y)$ and thus establish the bound $|e^{\perp}/2| \leq 2 - \chi(Y)$ in agreement with Whitney's conjecture.

(The experience of the high dimensional topology would suggest that $N(Y) = \infty$. Now-a-days, multiple constrains on topology of embeddings of surfaces into 4-manifolds are derived with Donaldson's theory.)

Non-simply Connected Analytic Geometry. The Browder-Novikov theory implies that, besides the Euler-Poincaré formula, there is a *single* "$\mathbb{Q}$-essential (i.e. non-torsion) homotopy constraint" on tangent bundles of closed *simply connected* $4k$-manifolds– the Rokhlin-Thom-Hirzebruch signature relation.

But in 1966, Sergey Novikov, in the course of his proof of the *topological* invariance of the of the *rational Pontryagin classes*, i.e. of the homology homomorphism $H_*(X^n;\mathbb{Q}) \to H_*(\mathrm{Gr}_N(\mathbb{R}^{n+N});\mathbb{Q})$ induced by the normal Gauss map, found the following new relation for *non-simply connected* manifolds X.

Let $f : X^n \to Y^{n-4k}$ be a smooth map. Then the signature of the $4k$-dimensional pullback manifold $Z = f^{-1}(y)$ of a generic point, $\mathrm{sig}[f] = \mathrm{sig}(Z)$, does not depend on the point and/or on f within a given homotopy class $[f]$ by the generic pull-back theorem and the cobordism invariance of the signature, but it may change under a homotopy equivalence $h : X_1 \to X_2$.

By an elaborate (and, at first sight, circular) *surgery + algebraic K-theory* argument, Novikov proves that

$$\text{if } Y \text{ is a } k\text{-torus, then } \mathrm{sig}[f \circ h] = \mathrm{sig}[f],$$

where the simplest case of the projection $X \times \mathbb{T}^{n-4k} \to \mathbb{T}^{n-4k}$ is (almost all) what is needed for the topological invariance of the Pontryagin classes. (See [**27**] for a simplified version of Novikov's proof and [**61**] for a different approach to the topological Pontryagin classes.)

Novikov conjectured (among other things) that a similar result holds for an arbitrary closed manifold Y with *contractible* universal covering. (This would imply, in particular, that if an oriented manifold Y' is orientably homotopy equivalent to such a Y, then it is bordant to Y.) Mishchenko (1974) proved this for manifolds Y admitting metrics of *non-positive curvature* with a use of an index theorem for operators on infinite dimensional bundles, thus linking the Novikov conjecture to geometry.

(Hyperbolic groups also enter Sullivan's existence/uniqueness theorem of *Lipschitz structures* on topological manifolds of dimensions ≥ 5.

A bi-Lipschitz homeomorphism may look very nasty. Take, for instance, infinitely many disjoint round balls $B_1, B_2, ...$ in $\mathbb{R}^n$ of radii $\to 0$, take a diffeomorphism f of B_1 fixing the boundary $\partial(B_1)$ an take the scaled copy of f in each B_i. The resulting homeomorphism, fixed away from these balls, becomes quite complicated whenever the balls accumulate at some closed subset, e.g. a hypersurface in $\mathbb{R}^n$. Yet, one can extend the signature index theorem and some of the Donaldson theory to this unfriendly bi-Lipschitz, and even to quasi-conformal, environment.)

The Novikov conjecture remains unsolved. It can be reformulated in purely group theoretic terms, but the most significant progress which has been achieved so far depends on geometry and on the index theory.

In a somewhat similar vein, Atiyah (1974) introduced square integrable (also called L_2) cohomology on non-compact manifolds $\tilde{X}$ with cocompact discrete group actions and proved the L_2-*index theorem.* For example, he has shown that

> if a compact Riemannian $4k$-manifolds has non-zero signature, then the universal covering $\tilde{X}$ admits a non-zero square summable harmonic $2k$-form.

This L_2-index theorem was extended to measurable *foliated spaces* (where "measurable" means the presence of *transversal measures*) by Alain Connes, where the two basic manifolds' attributes– the smooth structure and the measure—are separated: the *smooth structures in the leaves* allow differential operators while the *transversal measures* underly integration and where the two cooperate in the "non-commutative world" of Alain Connes.

If X is a compact measurably and smoothly n-foliated (i.e. almost all leaves are smooth n-manifolds) leaf-wise oriented space then one naturally defines Pontryagin's numbers which are real numbers in this case.

(Every closed manifold X can be regarded as a measurable foliation with the "transversal Dirac δ-measure" supported on X. Also complete Riemannian manifolds of *finite volume* can be regarded as such foliations, provided the universal coverings of these have locally bounded geometries [**11**].)

There is a natural notion of bordisms between measurable foliated spaces, where the Pontryagin numbers are obviously, bordism invariant.

Also, the L_2-signature, (which is also defined for leaves being $\mathbb{Q}$-manifolds) is bordism invariant by Poincaré duality.

The corresponding L_k-number, $k = n/4$, satisfies here the Hirzebruch formula with the L_2-signature (sorry for the mix-up in notation: $L_2 \neq L_{k=2}$): $L_k(X) = \mathrm{sig}(X)$ by the Atiyah-Connes L_2-index theorem [**11**].

It seems not hard to generalize this to measurable foliated spaces where leaves are topological (or even topological $\mathbb{Q}$) manifolds.

Questions. Let X be a measurable leaf-wise oriented n-foliated space with zero Pontryagin numbers, e.g. $n \neq 4k$. Is X orientably bordant to zero, provided every leaf in X has measure zero.

What is the counterpart to the Browder-Novikov theory for measurable foliations?

Measurable foliations can be seen as transversal measures on some *universal topological foliation*, such as the *Hausdorff moduli space* X of the isometry classes of pointed complete Riemannian manifolds L with uniformly *locally bounded geometries* (or locally bounded covering geometries [**11**]), which is tautologically foliated by these L. Alternatively, one may take the space of pointed triangulated manifolds with a uniform bound on the numbers of simplices adjacent to the points in L.

The simplest transversal measures on such an X are weak limits of convex combinations of Dirac's δ-measures supported on closed leaves, but most (all?) known interesting examples descent from group actions, e.g. as follows.

Let L be a Riemannian symmetric space (e.g. the complex hyperbolic space $\mathbb{C}H^n$ as in section 5), let the isometry group G of L be embedded into a locally compact group H and let $O \subset H$ be a compact subgroup such that the intersection $O \cap G$ equals the (isotropy) subgroup $O_0 \subset G$ which fixes a point $l_0 \in L$. For example, H may be the special linear group $SL_N(\mathbb{R})$ with $O = SO(N)$ or H may be an adelic group.

Then the quotient space $\tilde{X} = H/O$ is naturally foliated by the H-translate copies of $L = G/O_0$.

This foliation becomes truly interesting if we pass from $\tilde{X}$ to $X = \tilde{X}/\Gamma$ for a discrete subgroup $\Gamma \subset H$, where H/Γ has finite volume. (If we want to make sure that all leaves of the resulting foliation in X are manifolds, we take Γ without torsion, but singular *orbifold* foliations are equally interesting and amenable to the general index theory.)

The full vector of the Pontryagin numbers of such an X depends, up to rescaling, only on L but it is unclear if there are "natural (or any) bordisms" between different X with the same L.

Linear operators are difficult to delinearize keeping them topologically interesting. The two exceptions are the *Cauchy–Riemann operator* and the signature operator in dimension 4. The former is used by Thurston (starting from late 70s) in his $3D$-geometrization theory and the latter, in the form of the *Yang–Mills equations*, begot Donaldson's $4D$-theory (1983) and the Seiberg–Witten theory (1994).

The logic of Donaldson's approach resembles that of the index theorem. Yet, his operator $D : \Phi \to \Psi$ is *non-linear* Fredholm and instead of the index he studies the bordism-like invariants of (finite dimensional!) pullbacks $D^{-1}(\psi) \subset \Phi$ of suitably generic ψ.

These invariants for the Yang-Mills and Seiberg-Witten equations unravel an incredible richness of the smooth $4D$-topological structures which remain invisible from the perspectives of pure topology" and/or of linear analysis.

The non-linear Ricci flow equation of Richard Hamilton, the parabolic relative of Einstein, does not have any built-in topological intricacy; it is similar to the plain heat equation associated to the ordinary Laplace operator. Its potential role is not in exhibiting new structures but, on the contrary, in showing that these do not exist by ironing out bumps and ripples of Riemannian metrics. This potential was realized in dimension 3 by Perelman in 2003:

> The Ricci flow on Riemannian 3-manifolds, when manually redirected at its singularities, eventually brings every closed Riemannian 3-manifold to a canonical geometric form predicted by Thurston.

(Possibly, there is a non-linear analysis on foliated spaces, where solutions of, e.g. parabolic Hamilton-Ricci for $3D$ and of elliptic Yang-Mills/Seiberg-Witten for $4D$, equations fast, e.g. L_2, decay on each leaf and where "decay" for non-linear objects may refer to a decay of distances between pairs of objects.)

There is hardly anything in common between the proofs of Smale and Perelman of the Poincaré conjecture. Why the statements look so similar? Is it the same "Poincaré conjecture" they have proved? Probably, the answer is "no" which raises another question: what is the high dimensional counterpart of the Hamilton-Perelman $3D$-structure?

To get a perspective let us look at another, seemingly remote, fragment of mathematics – the theory of algebraic equations, where the numbers $2, 3$ and 4 also play an exceptional role.

If topology followed a contorted path 2→5...→4→3, algebra was going straight 1→2→3→4→5... and it certainly did not stop at this point.

Thus, by comparison, the Smale–Browder–Novikov theorems correspond to non-solvability of equations of degree ≥ 5 while the present day $3D$- and $4D$-theories are brethren of the magnificent formulas solving the equations of degree 3 and 4.

What does, in topology, correspond to the Galois theory, class field theory, the modularity theorem... ?

Is there, in truth, anything in common between this algebra/arthmetic and geometry?

It seems so, at least on the surface of things, since the reason for the particularity of the numbers 2, 3, 4 in both cases arises from the same formula:

$$4 =_3 2 + 2 :$$

a 4 element set has exactly 3 partitions into two 2-element subsets and where, observe $3 < 4$. No number $n \geq 5$ admits a similar class of decompositions.

In algebra, the formula $4 =_3 2+2$ implies that the alternating group $A(4)$ admits an epimorphism onto $A(3)$, while the higher groups $A(n)$ are simple non-Abelian.

In geometry, this transforms into the splitting of the Lie algebra so(4) into so(3) ⊕ so(3). This leads to the splitting of the space of the 2-forms into self-dual and anti-self-dual ones which underlies the Yang–Mills and Seiberg–Witten equations in dimension 4.

In dimension 2, the group SO(2) "unfolds" into the geometry of Riemann surfaces and then, when extended to $homeo(S^1)$, brings to light the conformal field theory.

In dimension 3, Perelman's proof is grounded in the infinitesimal O(3)-symmetry of Riemannian metrics on 3-manifolds (which is broken in Thurston's theory and even more so in the high dimensional topology based on surgery) and depends on the irreducibility of the space of traceless curvature tensors.

It seems, the geometric topology has a long way to go in conquering high dimensions with all their symmetries.

11. Crystals, Liposomes and Drosophila

Many geometric ideas were nurtured in the cradle of manifolds; we want to follow these ideas in a larger and yet unexplored world of more general "spaces".

Several exciting new routes were recently opened to us by the high energy and statistical physics, e.g. coming from around the string theory and non-commutative geometry—somebody else may comment on these, not myself. But there are a few other directions where geometric spaces may be going.

Infinite Cartesian Products and Related Spaces. A crystal is a collection of identical molecules $mol_\gamma = mol_0$ positioned at certain sites γ which are the elements of a discrete (crystallographic) group Γ.

If the space of states of each molecule is depicted by some "manifold" M, and the molecules do not interact, then the *space X of states* of our "crystal" equals the Cartesian power $M^\Gamma = \times_{\gamma\in\Gamma} M_\gamma$.

If there are inter-molecular constrains, X will be a subspace of M^Γ; furthermore, X may be a quotient space of such a subspace under some equivalence relation, where, e.g. two states are regarded equivalent if they are indistinguishable by a certain class of "measurements".

We look for mathematical counterparts to the following physical problem. Which properties of an individual molecule can be determined by a given class of measurement of the whole crystal?

Abstractly speaking, we start with some category $\mathcal{M}$ of "spaces" M with Cartesian (direct) products, e.g. a category of finite sets, of smooth manifolds or of algebraic manifolds over some field. Given a countable group Γ, we enlarge this category as follows.

Γ-Power Category $\Gamma^{\mathcal{M}}$. The objects $X \in \Gamma^{\mathcal{M}}$ are projective limits of finite Cartesian powers M^{Δ} for $M \in \mathcal{M}$ and finite subsets $\Delta \subset \Gamma$. Every such X is naturally acted upon by Γ and the admissible morphisms in our Γ-category are Γ-*equivariant* projective limits of morphisms in $\mathcal{M}$.

Thus each morphism, $F : X = M^{\Gamma} \to Y = N^{\Gamma}$ is defined by a single morphism in $\mathcal{M}$, say by $f : M^{\Delta} \to N = N$ where $\Delta \subset \Gamma$ is a *finite* (sub)set. Namely, if we think of $x \in X$ and $y \in Y$ as M- and N-valued functions $x(\gamma)$ and $y(\gamma)$ on Γ then the value $y(\gamma) = F(x)(\gamma) \in N$ is evaluated as follows:

translate $\Delta \subset \Gamma$ to $\gamma\Delta \subset \Gamma$ by γ, restrict $x(\gamma)$ to $\gamma\Delta$ and apply
f to this restriction $x|\gamma\Delta \in M^{\gamma\Delta} = M^{\Delta}$.

In particular, every morphism $f : M \to N$ in $\mathcal{M}$ tautologically defines a morphism in $\mathcal{M}^{\Gamma}$, denoted $f^{\Gamma} : M^{\Gamma} \to N^{\Gamma}$, but $\mathcal{M}^{\Gamma}$ has many other morphisms in it.

Which concepts, constructions, properties of morphisms and objects, etc. from $\mathcal{M}$ "survive" in $\Gamma^{\mathcal{M}}$ for a given group Γ? In particular, what happens to topological invariants which are multiplicative under Cartesian products, such as the Euler characteristic and the signature?

For instance, let M and N be manifolds. Suppose M admits *no topological embedding* into N (e.g. $M = S^1$, $N = [0,1]$ or $M = \mathbb{R}P^2$, $N = S^3$). When does M^{Γ} admit an injective morphism to N^{Γ} in the category $\mathcal{M}^{\Gamma}$?

(One may meaningfully reiterate these questions for *continuous* Γ-equivariant maps between Γ-Cartesian products, since *not all* continuous Γ-equivariant maps lie in $\mathcal{M}^{\Gamma}$.)

Conversely, let $M \to N$ be a map of non-zero degree. When is the corresponding map $f^{\Gamma} : M^{\Gamma} \to N^{\Gamma}$ equivariantly homotopic to a *non-surjective* map?

Γ-Subvarieties. Add new objects to $\mathcal{M}^{\Gamma}$ defined by equivariant systems of equations in $X = M^{\Gamma}$, e.g. as follows.

Let M be an algebraic variety over some field $\mathbb{F}$ and $\Sigma \subset M \times M$ a subvariety, say, a generic algebraic hypersurface of bi-degree (p,q) in $\mathbb{C}P^n \times \mathbb{C}P^n$.

Then every directed graph $G = (V, E)$ on the vertex set V defines a subvariety, in M^V, say $\Sigma(G) \subset M^V$ which consists of those M-valued functions $x(v)$, $v \in V$, where $(x(v_1), x(v_2)) \in \Sigma$ whenever the vertices v_1 and v_2 are joined by a directed edge $e \in E$ in G. (If $\Sigma \subset M \times M$ is symmetric for $(m_1, m_2) \leftrightarrow (m_2, m_1)$, one does not need directions in the edges.)

Notice that even if Σ is non-singular, $\Sigma(G)$ may be singular. (I doubt, this ever happens for generic hypersurfaces in $\mathbb{C}P^n \times \mathbb{C}P^n$.) On the other hand, if we have a "sufficiently ample" *family* of subvarieties Σ in $M \times M$ (e.g. of (p,q)-hypersurfaces in $\mathbb{C}P^n \times \mathbb{C}P^n$) and, for each $e \in E$, we take a generic representative $\Sigma_{\text{gen}} = \Sigma_{\text{gen}}(e) \subset M \times M$ from this family, then the resulting generic subvariety in $M \times M$, call it $\Sigma_{gen}(G)$ is non-singular and, if $\mathbb{F} = \mathbb{C}$, its topology does not depend on the choices of $\Sigma_{\text{gen}}(e)$.

We are manly interested in $\Sigma(G)$ and $\Sigma_{\text{gen}}(G)$ for infinite graphs G with a cofinite action of a group Γ, i.e. where the quotient graph G/Γ is finite. In particular, we want to understand "infinite dimensional (co)homology" of these spaces, say for $\mathbb{F} = \mathbb{C}$ and the "cardinalities" of their points for finite fields $\mathbb{F}$ (see [**5**] for some results and references). Here are test questions.

Let Σ be a hypersurface of bi-degree (p,q) in $\mathbb{C}P^n \times \mathbb{C}P^n$ and $\Gamma = \mathbb{Z}$. Let $P_k(s)$ denote the Poincaré polynomial of $\Sigma_{\text{gen}}(G/k\mathbb{Z})$, $k = 1, 2, ...$ and let

$$P(s,t) = \sum_{k=1}^{\infty} t^k P(s) = \sum_{k,i} t^k s^i rank(H_i(\Sigma_{gen}(G/k\mathbb{Z})).$$

Observe that the function $P(s,t)$ depends only on n, and (p,q).

Is $P(s,t)$ meromorphic in the two complex variables s and t? Does it satisfy some "nice" functional equation?

Similarly, if $\mathbb{F} = \mathbb{F}_p$, we ask the same question for the generating function in two variables counting the $\mathbb{F}_{p^l}$-points of $\Sigma(G/k\mathbb{Z})$.

Γ-Quotients. These are defined with equivalence relations $R \subset X \times X$ where R are subobjects in our category.

The transitivity of (an *equivalence* relation) R, and it is being a *finitary* defined sub-object are hard to satisfy simultaneously. Yet, *hyperbolic dynamical systems* provide encouraging examples at least for the category $\mathcal{M}$ of *finite sets.*

If $\mathcal{M}$ is the category of finite sets then subobjects in $\mathcal{M}^\Gamma$, defined with subsets $\Sigma \subset M \times M$ are called *Markov* Γ*-shifts.* These are studied, mainly for $\Gamma = \mathbb{Z}$, in the context of *symbolic dynamics* [**43**], [**7**].

Γ-*Markov quotients* Z of Markov shifts are defined with equivalence relations $R = R(\Sigma') \subset Y \times Y$ which are Markov subshifts. (These are called *hyperbolic* and/or *finitely presented* dynamical systems [**20**], [**26**].)

If $\Gamma = \mathbb{Z}$, then the counterpart of the above $P(s,t)$, now a function only in t, is, essentially, what is called the ζ*-function* of the dynamical system which counts the number of periodic orbits. It is shown in [**20**] with a use of *(Sinai-Bowen) Markov partitions* that this function is rational in t for all $\mathbb{Z}$-Markov quotient systems.

The local topology of Markov quotient (unlike that of shift spaces which are Cantor sets) may be quite intricate, but some are *topological manifolds.*

For instance, classical *Anosov systems* on *infra-nilmanifolds* V and/or *expanding endomorphisms* of V are representable as a $\mathbb{Z}$- Markov quotient via Markov partitions [**35**].

Another example is where Γ is the fundamental group of a closed n-manifold V of negative curvature. The *ideal boundary* $Z = \partial_\infty(\Gamma)$ is a topological $(n-1)$-sphere with a Γ-action which admits a Γ-Markov quotient presentation [**26**].

Since the topological S^{n-1}-bundle $S \to V$ associated to the universal covering, regarded as the principle Γ bundle, is, obviously, isomorphic to the unit tangent bundle $UT(V) \to V$, the Markov presentation of $Z = S^{n-1}$ defines the topological Pontryagin classes p_i of V in terms of Γ.

Using this, one can reduce the homotopy invariance of the Pontryagin classes p_i of V to the ε*-topological* invariance.

Recall that an ε*-homeomorphism* is given by a pair of maps $f_{12} : V_1 \to V_2$ and $f_{21} : V_2 \to V_1$, such that the composed maps $f_{11} : V_1 \to V_1$ and $f_{22} : V_2 \to V_2$ are ε-close to the respective identity maps for some metrics in V_1, V_2 and a small $\varepsilon > 0$ depending on these metrics.

Most known proofs, starting from Novikov's, of invariance of p_i under homeomorphisms equally apply to ε-homeomorphisms.

This, in turn, implies the *homotopy* invariance of p_i if the homotopy can be "rescaled" to an ε-homotopy.

For example, if V is a *nil-manifold* $\tilde{V}/\Gamma$, (where $\tilde{V}$ is a nilpotent Lie group homeomorphic to $\mathbb{R}^n$) with an *expanding endomorphism* $E : V \to V$ (such a V is a $\mathbb{Z}$-Markov quotient of a shift), then a large negative power $\tilde{E}^{-N} : \tilde{V} \to \tilde{V}$ of the lift $\tilde{E} : \tilde{V} \to \tilde{V}$ brings any homotopy close to identity. Then the ε-topological invariance of p_i implies the homotopy invariance for these V. (The case of $V = \mathbb{R}^n/\mathbb{Z}^n$ and $\tilde{E} : \tilde{v} \to 2\tilde{v}$ is used by Kirby in his topological *torus trick*.)

A similar reasoning yields the homotopy invariance of p_i for many (manifolds with fundamental) groups Γ, e.g. for *hyperbolic groups*.

Questions. Can one effectively describe the local and global topology of Γ-Markov quotients Z in combinatorial terms? Can one, for a given (e.g. hyperbolic) group Γ, "classify" those Γ-Markov quotients Z which are *topological manifolds* or, more generally, *locally contractible* spaces?

For example, can one describe the classical Anosov systems Z in terms of the combinatorics of their $\mathbb{Z}$-Markov quotient representations? How restrictive is the assumption that Z is a topological manifold? How much the topology of the local dynamics at the periodic points in Z restrict the topology of Z (e.g. we want to incorporate pseudo-Anosov automorphisms of surfaces into the general picture.)

It seems, as in the case of the hyperbolic groups, (irreducible) $\mathbb{Z}$-Markov quotients becomes more scarce/rigid/symmetric as the topological dimension and/or the local topological connectivity increases.

Are there interesting Γ-Markov quotients over categories $\mathcal{M}$ besides finite sets? For example, can one have such an object over the category of algebraic varieties over $\mathbb{Z}$ with non-trivial (e.g. positive dimensional) topology in the spaces of its $\mathbb{F}_{p^i}$-points?

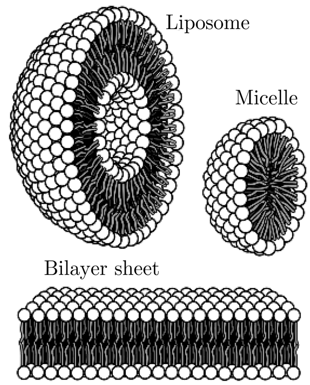
Liposomes and *Micelles* are surfaces of *membranes* surrounded by water which are assembled of rod-like (phospholipid) molecules oriented normally to the surface of the membrane with *hydrophilic* "heads" facing the exterior and the interior of a cell while the *hydrophobic* "tails" are buried inside the membrane.

These surfaces satisfy certain partial differential equations of rather general nature (see [**30**]). If we heat the water, membranes dissolve: their constituent molecules become (almost) randomly distributed in the water; yet, if we cool the solution, the surfaces and the equations they satisfy re-emerge.

QUESTION. *Is there a (quasi)-canonical way of associating statistical ensembles $\mathcal{S}$ to geometric system S of PDE, such that the equations emerge at low temperatures T and also can be read from the properties of high temperature states of $\mathcal{S}$ by some "analytic continuation" in T?*

The architectures of liposomes and micelles in an ambient space, say W, which are composed of "somethings" normal to their surfaces $X \subset W$, are reminiscent of Thom-Atiyah representation of submanifolds with their normal bundles by generic maps $f_\bullet : W \to V_\bullet$, where $V_\bullet$ is the Thom space of a vector bundle V_0 over some space X_0 and where manifolds $X = f_\bullet^{-1}(X_0) \subset W$ come with their normal bundles induced from the bundle V_0.

The space of these "generic maps" $f_\bullet$ looks as an intermediate between an individual "deterministic" liposome X and its high temperature randomization. Can one make this precise?



Poincaré-Sturtevant Functors. All that the brain knows about the geometry of the space is a flow S_{in} of electric impulses delivered to it by our sensory organs. All what an alien browsing through our mathematical manuscripts would directly perceive, is a flow of symbols on the paper, say G_{out}.

Is there a natural functorial-like transformation $\mathcal{P}$ from sensory inputs to mathematical outputs, a map between "spaces of flows" $\mathcal{P} : \mathcal{S} \to \mathcal{G}$ such that $\mathcal{P}(S_{\text{in}})$"="$G_{\text{out}}$?

It is not even easy to properly state this problem as we neither know what our "spaces of flows" are, nor what the meaning of the equality "=" is.

Yet, it is an essentially mathematical problem a solution of which (in a weaker form) is indicated by Poincaré in [**58**]. Besides, we all witness the solution of this problem by our brains.

An easier problem of this kind presents itself in the classical genetics.

> What can be concluded about the geometry of a genome of an organism by observing the phenotypes of various representatives of the same species (with no molecular biology available)?

This problem was solved in 1913, long before the advent of the molecular biology and discovery of DNA, by 19-year old Alfred Sturtevant (then a student in T. H. Morgan's lab) who reconstructed *the linear structure* on the set of genes on a chromosome of Drosophila melanogaster from *samples of a probability measure* on the space of gene linkages.

Here mathematics is more apparent: the geometry of a space X is represented by something like a measure on the set of subsets in X; yet, I do not know how to formulate clear-cut mathematical questions in either case (compare [**29**], [**31**]).

Who knows where manifolds are going?

References

1. P. Akhmet'ev, Geometric approach to stable homotopy groups of spheres. The Adams-Hopf invariants, Fundam. Prikl. Mat., 13:8, pp. 3-15, (2007). MR2475578 (2009m:55010)
2. M. Atiyah, Thom complexes. Proc. London Math. Soc. (3) 11. 291-310. (1961). MR0131880 (24:A1727)
3. M. Atiyah, I. Singer, The Index of Elliptic Operators on Compact Manifolds, Bull. Amer. Math. Soc. 69: 322-433, (1963). MR0157392 (28:626)
4. D. Barden, Simply connected five manifolds. Ann. Math., 82, 365-385 (1965). MR0184241 (32:1714)

5. M. Bertelson, Topological invariant for discrete group actions. Lett. Math. Phys. 62, 147-156 (2004). MR2104441 (2005m:37057)
6. S. Buoncristiano, C. P. Rourke, B. J. Sanderson, A geometric approach to homology theory. London Mathematical Society Lecture Note Series, no. 18, Cambridge Univ. Press, (1976) MR0413113 (54:1234)
7. M. Boyle, Open problems in symbolic dynamics. url http://www-users.math.umd.edu/mmb/open/, (2008). MR2478466 (2010h:37023)
8. W. Browder, Homotopy type of differentiable manifolds, Colloq. Algebraic Topology, pp. 42-46, Aarhus University, (1962). reprinted in "Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)", 97-100, LMS Lecture Notes 226 (1995) MR1388298 (97d:57044)
9. E. Brown Jr., The Kervaire invariant and surgery theory. www.math.rochester.edu/u/faculty/doug/otherpapers/brown2.pdf
10. J. Cheeger, Spectral geometry of singular Riemannian spaces, J. Differential Geom. V. 18. 4. pp. 575-657, (1983). MR730920 (85d:58083)
11. J. Cheeger and M. Gromov, On the characteristic numbers of complete manifolds of bounded curvature and finite volume, Differential geometry and complex analysis, H. E. Rauch Memorial Volume, Springer-Verlag, Berlin, (1985). MR780040 (86h:58131)
12. A. Chernavskii, Local contractibility of the group of homeomorphisms of a manifold, Math. USSR Sb. , 8 : 3 pp. 287-333, (1969),
13. M. Davis, Poincaré duality groups, from: "Surveys on surgery theory, Vol. 1", Ann. of Math. Stud. 145, Princeton Univ. Press, Princeton, NJ, 167-193, (2000). MR1747535 (2001b:57001)
14. M. Davis, The Geometry and Topology of Coxeter Groups,. London Mathematical Society Monographs, Princeton University Press, (2008). MR2360474 (2008k:20091)
15. S. Donaldson, An Application of Gauge Theory to Four Dimensional Topology, Journal of Differential Geometry 18 (2): 279-315, (1983). MR710056 (85c:57015)
16. P. Eccles, Multiple points of codimension one immersions of oriented manifolds. Math. Proc. Cambridge Philos. Soc. 87, 213-220 (1980). MR553578 (81j:55014)
17. P. Eccles, Codimension one immersions and the Kervaire invariant one problem, Math. Proc. Camb. Phil. Soc. 90, 483-493 (1981). MR628831 (83c:57015)
18. S. Eilenberg, Cohomology and continuous mappings, Ann. of Math., 41, 231-251, (1940). MR0001351 (1:222b)
19. Y. Felix, S. Halperin, J.-C. Thomas, Rational homotopy theory, Springer, (2001). MR1802847 (2002d:55014)
20. D. Fried, Finitely presented dynamical systems. Ergodic Theory and Dynamical Systems, 7, pp 489-507, (1987). MR922362 (89h:58157)
21. D. Fuks, Classical Manifolds. In Topology-1. Sovr. Probl. Math. Fund. Napr. (Itogi. Nauki-Tech.), M., VINITI, (1986). MR895593 (88j:57002)
22. M. Furuta, Homology cobordism group of homology 3-spheres, Invent. Math. 100, 339-355, (1990). MR1047138 (91c:57039)
23. A. Gaifullin, The construction of combinatorial manifolds with prescribed sets of links of vertices, Izvestiya RAN: Ser. Mat. 72:5, 3-62, (2008). MR2473771 (2010d:51036)
24. A. Gaifullin, Configuration spaces, bistellar moves, and combinatorial formulae for the first Pontryagin class, Tr. Mat. Inst. Steklova, Volume 268, pp. 76-93, (2010). MR2724336 (2011k:57032)
25. M. Gromov, On the number of simplices of subdivisions of finite complexes, Mat. Zametki, 3:5, pp. 511-522, (1968). MR0227971 (37:3555)
26. M. Gromov, Hyperbolic manifolds, groups and actions, Ann. Math. Studies 97, 183-215, Princeton University Press, Princeton, (1981). MR624814 (82m:53035)
27. M. Gromov, Positive curvature, macroscopic dimension, spectral gaps and higher signatures, Functional analysis on the eve of the 21st century, Vol. II , 1-213, Progr. Math., 132, Birkh́auser, (1996). MR1389019 (98d:53052)
28. M. Gromov. Spaces and questions, Geom. Funct. Anal., Special Volume, Part I:118-161, (2000). MR1826251 (2002e:53056)
29. M. Gromov, Mendelian Dynamics and Sturtevant's Paradigm, in "Geometric and Probabilistic Structures", Contemporary Mathematics Series v. 469, (ed, Keith Burns,, Dmitry Dolgopyat and Yakov Pesin), pp 227-242, American Mathematical Society, Providence RI (2007). MR2478473 (2009k:92066)

30. M. Gromov, Crystals, Proteins, Stability and Isoperimetry. Submitted to Bulletin of AMS, (2010).
http://www.ihes.fr/˜gromov/PDF/pansu-crystals-isoper.pdf MR2774091 (2012a:92196)
31. M. Gromov, Structures, Learning and Ergosystems.
www.ihes.fr/˜gromov/PDF/ergobrain.pdf
32. R. Hamilton Three-manifolds with positive Ricci curvature. J. Differential Geom. Volume 17, Number 2, 255-306, (1982). MR664497 (84a:53050)
33. A. Haefliger, Plongements differentiables de variétés dans variétés, Comment. Math. Helv. 36, 47-82, (1961). MR0145538 (26:3069)
34. A. Haefliger, Knotted (4k-1)-spheres in 6k-space. Ann. of Math. (2) 75, 452-466, (1962). MR0145539 (26:3070)
35. B. Hasselblatt, Hyperbolic dynamical systems. Handbook of Dynamical Systems 1A, 239-319, Elsevier North Holland, (2002). MR1928520 (2004b:37047)
36. H. Hopf, Abbildungsklassen n-dimensionaler Mannigfaltigkeiten. Math. Annalen 96, pp. 225-250, (1926).
37. M.Hill, M. Hopkins, D.Ravenel. On the non-existence of elements of Kervaire invariant one.
www.math.rochester.edu/u/faculty/doug/kervaire_082609.pdf
38. W. Hurewicz, Beitr$^{\acute{}}$ age zur Topologie der Deformationen I-II. Proc. Ned. Akad. Weten. Ser. A , 38, pp. 112-119; 521-528, (1935).
39. M. Kervaire, J. Milnor, Groups of homotopy spheres. I Ann. of Math., 77:3, pp. 504-537 (1963). MR0148075 (26:5584)
40. R. Kirby, L. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings and Triangulations, Ann. of Math. Studies 88 (1977). MR0645390 (58:31082)
41. M. Kreck, Differential Algebraic Topology, Graduate Studies in Mathematics, Volume 110, (2010). MR2641092 (2011i:55001)
42. N. Levitt, C. Rourke, The existence of combinatorial formulae for characteristic classes, Trans. Amer. Math. Soc. 239, 391-397, (1978). MR0494134 (58:13063)
43. D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, (1995). MR1369092 (97a:58050)
44. N. Martin, On the difference between homology and piecewise-linear bundles, J. London Math. Soc. (2), 6:2, 197-204, (1973). MR0314061 (47:2613)
45. W. Massey. Proof of a conjecture of Whitney. Pacific J. Math. Volume 31, Number 1 (1969), 143-156. MR0250331 (40:3570)
46. B. Mazur, Stable equivalence of differentiable manifolds, Bull. Amer. Math. Soc. 67, 377-384, (1961). MR0130697 (24:A557)
47. B. Mazur, Bernoulli numbers and the unity of mathematics.
www.math.wisc.edu/˜boston/Bernoulli.pdf
48. J. Milnor, On manifolds homeomorphic to the 7-sphere, Ann. of Math. (2) 64 (1956) 399-405, (1956). MR0082103 (18:498d)
49. J. Milnor, Two complexes which are homeomorphic but combinatorially distinct, Annals of Mathematics 74 (2): 575-590 (1961). MR0133127 (24:A2961)
50. J. Milnor and J. Stasheff, Characteristic classes,. Princeton. Univ. Press (1974). MR0440554 (55:13428)
51. A. Mishchenko, Infinite-dimensional representations of discrete groups, and higher signatures, Math. USSR-Izv., 8:1, 85-111, (1974). MR0362407 (50:14848)
52. A. Nabutovsky, Combinatorics of the space of Riemannian structures and logic phenomena of Euclidean Quantum Gravity, in "Perspectives in Riemannian Geometry", ed. by V. Apostolov et al., CRM Proceedings and Lecture Notes, vol. 40, 223-248, AMS, Providence, RI, (2006). MR2237112 (2007h:53056)
53. A. Nabutovsky, S. Weinberger, The fractal geometry of Riem/Diff, Geometriae Dedicata 101, 1-54, (2003). MR2017894 (2005d:53063)
54. S. Novikov, Homotopy equivalent smooth manifolds I. (In Russian). Izv. Akad. Nauk S.S.S.R. ser. mat. 28 (2) (1964),365-474.Also, Translations Amer. Math. Soc. 48, 271-396, (1965). MR0162246 (28:5445)
55. S. Novikov, On manifolds with free Abelian fundamental group and their application, Izv. Akad. Nauk SSSR, v. 30, N 1, 207-246, (1966). MR0196765 (33:4951)
56. G. Perelman, Ricci flow with surgery on three-manifolds.
arXiv:math.DG/0303109, (2003).

57. S. Podkorytov, An alternative proof of a weak form of Serre's theorem, J. of Math. Sci. vol 110, No.4 2875-2881, (2002). MR1758429 (2001d:55009)
58. H. Poincaré, Science and hypothesis, London and Newcastle-on-Tyne: The Walter Scott Publishing Co., (1905).
59. L. Pontryagin, Classification of continuous transformations of a complex into a sphere, Dokl. Akad. Nauk SSSR , 19, pp. 361-363, (In Russian) (1938).
60. L. Pontryagin, Homotopy classification of the mappings of an $(n+2)$-dimensional sphere on an n-dimensional one, Dokl. Akad. Nauk SSSR (N.S.) 70, 957-959, (Russian) (1950). MR0042121 (13:57b)
61. A. Ranicki, M. Weiss, On the construction and topological invariance of the Pontryagin classes arXiv:0901.0819, (2009). MR2721630 (2011j:57040)
62. V. Rokhlin, Summary of results in homotopy theory of continuous transformations of a sphere into a sphere, Uspekhi Mat. Nauk, 5:6(40), 88-101, (1950). MR0039250 (12:519h)
63. V. Rokhlin, New results in the theory of 4-dimensional manifolds, Dokl. Akad. Nauk. SSSR 84, 221-224, (Russian) (1952). MR0052101 (14:573b)
64. V. Rokhlin, On Pontrjagin characteristic classes, Dokl. Akad. Nauk SSSR 113, 276-279, (1957). MR0094806 (20:1318)
65. V. Rokhlin, A. Schwarz, The combinatorial invariance of Pontryagin classes, Dokl. Akad Nauk SSSR, 114, 490-493, (1957). MR0102070 (21:865)
66. C. Rourke, Essay on the Poincaré conjecture, http://msp.warwick.ac.uk/˜cpr/.
67. A. Russon, Orangutans: Wizards of the Rain Forest, Key Porter books, (2004).
68. J-P. Serre, Homologie singulière des espaces fibrés. Applications, Annals of Mathematics, 54 (3): 425-505, (1951). MR0045386 (13:574g)
69. J. Simons, Minimal varieties in riemannian manifolds, Ann. Math. 88, 62-105, (1968). MR0233295 (38:1617)
70. S. Smale, Generalized Poincaré's conjecture in dimensions greater than four. Ann. of Math. (2) 74,1961 391-406, (1961). MR0137124 (25:580)
71. S. Smale: On the structure of 5-manifolds. Ann. Math., 75, 38-46, (1962). MR0141133 (25:4544)
72. J. Stallings, Polyhedral homotopy-spheres, Bull. Amer. Math. Soc. Volume 66, Number 6, 485-488, (1960). MR0124905 (23:A2214)
73. D. Sullivan, Hyperbolic geometry and homeomorphisms, Geometric Topology., Proc. Georgia Topology Conf., Athens, Ga., Academic Press, New York, pp. 543-555, (1977). MR537749 (81m:57012)
74. A. Szùcs, Cobordism of singular maps, Geometry and Topology 12, 2379-2452 (2008). MR2443969 (2009g:57044)
75. R. Thom, Quelques propriétés globales des variétés différentiables, Commentarii Mathematici Helvetici 28: 17-86, (1954). MR0061823 (15:890a)
76. R. Thom, Les classes caractéristiques de Pontrjagin des variétés triangulées,Topologia Algebraica, Mexico, 54-67, (1958). MR0102071 (21:866)
77. W. Thurston, The geometry and topology of 3-manifolds, Princeton lecture notes (1978-1981).
78. C. T. C. Wall, Surgery on compact manifolds, Mathematical Surveys and Monographs, 69 (2nd ed.), Providence, R.I.: American Mathematical Society, (1999). MR1687388 (2000a:57089)
79. H. Whitney, On the topology of differentiable manifolds, from: "Lectures in Topology", University of Michigan Press,101-141, (1941). MR0005300 (3:133a)
80. H. Whitney, The self-intersection of a smooth n-manifold in $2n$-space, Ann. of Math. 45, 220-246, (1944). MR0010274 (5:273g)
81. E. Witten, Monopoles and four-manifolds, Math. Res. Lett. 1 (6): 769-796,(1994). MR1306021 (96d:57035)
82. E. Zeeman, Unknotting combinatorial balls. Ann. of Math. (2) 78, 501–526, (1963). MR0160218 (28:3432)

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Geometric Analysis on 4-Manifolds

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Abstract. In this expository paper, we will discuss some geometric analytic approaches to studying the topology and geometry of 4-manifolds. We will start with a brief summary on 2-manifolds and then recall some aspects of Perelman's resolution of the Geometrization conjecture for 3-manifolds by using Hamilton's Ricci flow. Then we discuss geometric approaches and progress on studying 4-manifolds. For simplicity, we assume that all manifolds in this paper are closed and oriented.

1. Geometrization of 2-manifolds

Let M be a 2-dimensional manifold. Any Riemannian metric g on M gives rise to a conformal structure and makes M into a Riemann surface. It then follows from complex analysis that the universal covering of M is conformal to either S^2 or $\mathbb{R}^2$ or the hyperbolic disc D. In particular, the topology of M is determined by its fundamental group. Moreover, since each of the standard spaces above has a canonical metric with constant curvature, we can conclude that there is a metric $\tilde{g}$ with constant curvature and conformal to g. In fact, such a $\tilde{g}$ is unique if the volume is normalized.

Another approach to studying 2-manifolds is to construct metrics with constant curvature by solving partial differential equations. This is more analytic and opens the possibility of generalization to higher dimensions. Given a Riemannian metric g on M, consider a new metric $\tilde{g} = e^{\varphi} g$ for some smooth function. A simple computation shows that $\tilde{g}$ has constant curvature μ if and only if

$$-\Delta\varphi + K(g) = \mu e^{\varphi}, \tag{1.1}$$

where $K(g)$ denotes the curvature of g and Δ is the Laplacian operator of g. This equation has been studied a lot: see Chapter 6 of [**Au**] for a detailed discussion. Here we give a summary for the readers' convenience. If $\mu = 0$, it is a linear equation and has a solution by the standard theory. If $\mu < 0$, by the Maximum Principle, there is a uniform L^∞-bound on φ. Standard elliptic theory can then be used to derive a prior bounds on all the derivatives on any solutions of the above equation; consequently, one can establish existence. When $\mu > 0$, the problem is more tricky and is often referred as the Nirenberg problem. Many prominent mathematicians, including Nirenberg, Kazdan–Warner, Aubin et al. studied this problem. It has been shown that (1.1) always has a solution (see Section 4 of

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Chapter 6 in [**Au**]). Therefore, given any g, there is a metric $\tilde{g}$ conformal to g and with constant curvature. A classical uniformization theorem in differential geometry (cf. Chapter 8, [**DoC**]) then implies that modulo scaling, the universal covering of M with the induced metric from $\tilde{g}$ is isometric to S^2 or $\mathbb{R}^2$ or the hyperbolic disc D with the standard metric. The advantage of this approach is that one gets a full understanding of geometry and topology of 2-manifolds by solving a partial differential equation.

A more recent method of finding metrics with constant curvature is to use the Ricci flow introduced by R. Hamilton [**Ha82**]:

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g), \; g(0) = g_0. \tag{1.2}$$

In [**Ha88**], [**Ch90**], it was proven that given any initial g_0, (1.2) has a global solution $g(t)$ after normalization and $g(t)$ converges to a metric g_∞ on M. One can show that g_∞ is of constant curvature. The proof is trivial if the Euler number of M is non-positive and is contained in [**CLT06**] if M has positive Euler number. Thus the Ricci flow gives rise to another approach to geometrizing 2-manifolds.

2. Geometrization of 3-manifolds

Can one extend what we said about surfaces to higher dimensions? First we need to introduce the notion of Einstein metrics.

Definition 2.1. g is Einstein if $\mathrm{Ric}(g) = \lambda g$, where $\lambda = -(n-1), 0, n-1$.

Note that $\mathrm{Ric}(g) = (R_{ij})$ denotes the Ricci curvature of g. It measures the deviation of volume form from the Euclidean one. In dimension 2, an Einstein metric is simply a metric with constant Gauss curvature.

Now assume that M is a compact 3-manifold. In this case, an Einstein metric has constant sectional curvature, and the classical uniformization theorem in differential geometry (cf. Chapter 8, [**DoC**]) then states that if M admits an Einstein metric, then its universal covering is of the form S^3/Γ, $\mathbb{R}^3/\Gamma$ or $\mathbb{H}^3/\Gamma$, where $\Gamma \simeq \pi_1(M)$ and $\mathbb{H}^3$ denotes the hyperbolic space of dimension 3. Thus, if we can always construct an Einstein metric, then we have a similar picture for 3-manifolds as we have for surfaces. However, not every 3-manifold admits an Einstein metric. One can easily construct such examples, such as $\Sigma \times S^1$ for any surface Σ of genus greater than 1. This is because its fundamental group is the product of a surface group with $\mathbb{Z}$ which is neither an abelian group nor the fundamental group of any hyperbolic compact 3-manifold (cf. [**Th97**]).

It was known [**Kn29**] that any closed 3-manifold can be decomposed along embedded 2-spheres into irreducible 3-manifolds; moreover, such a decomposition is essentially unique. Thurston's Geometrization Conjecture claims (cf. [**Th97**], [**CHK00**]) that *any irreducible* 3*-manifold can be decomposed along incompressible tori into finitely many complete Einstein* 3*-manifolds plus some Graph manifolds.* The famous Poincare conjecture is a special case of this Geometrization Conjecture.

This conjecture has been solved by Perelman (cf. [**Per02**], [**Per03**]) using the Ricci flow introduced by R. Hamilton in early 80's:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \qquad g(0) = \text{a given metric}. \tag{2.1}$$

R. Hamilton and later DeTurck proved that for any initial metric, there is a unique solution $g(t)$ on $M \times [0, T)$ for some $T > 0$. R. Hamilton also established an analytic theory for Ricci flow.

If the Ricci flow has a solution $g(t)$, then we can choose a scaling $\lambda(t) > 0$ and a reparametrization $t = t(s)$ with $\lambda(0) - 1$ and $t(0) = 0$ such that $\tilde{g}(x) = \lambda(s)g(t(s))$ has fixed volume and satisfies the normalized Ricci flow:

$$\frac{\partial g_{ij}}{\partial t} = -2\big(R_{ij} - R(t)g_{ij}\big) . \tag{2.2}$$

If the normalized Ricci flow (2.2) has a global solution $\tilde{g}(s)$ for all $s \geq 0$ and $\tilde{g}(s)$ converges to a smooth metric g_∞ as s goes to ∞, then its limiting metric g_∞ is an Einstein metric, so the universal covering of M is standard. The Geometrization Conjecture follows.

The first successful case was done by R. Hamilton in 1982: *If M has a metric of positive Ricci curvature, then the normalized Ricci flow has a global solution which converges smoothly to a metric of constant positive curvature, consequently, M is a quotient of S^3 by a finite group.*

However, in general, the Ricci flow develops a singularity at finite time. The singularity can be either forced by the topology or caused by complexity in metric behavior even if the manifold has simple topology. The latter singularity can occur along proper subsets of the manifolds, but not the entire manifold. Therefore, one is led to studying a more general evolution process called *Ricci flow with surgery*, which was first introduced by Hamilton for 4-manifolds with positive isotropic curvature. This evolution process is still parametrized by an interval in time, so that for each t in the interval of definition there is a compact Riemannian 3-manifold M_t. But there is a discrete set of times at which the manifolds and metrics undergo topological and metric discontinuities (surgeries). In each of the complementary intervals to the singular times, the evolution is the usual Ricci flow, though the topological type of M_t changes as t moves from one complementary interval to the next.

It is crucial for the topological applications that we do surgery only along two spheres rather than surfaces of higher genus. Surgery along two spheres produces the connected sum decomposition, which is well-understood topologically, while surgeries along tori can completely destroy the topology, changing any 3-manifold into any other. Perelman's first technical advance is that one needs to do surgeries only along two spheres. He understood completely the change in topology. More precisely, he established the following:

THEOREM 2.2. *Let (M, g_0) be a closed orientable Riemannian 3-manifold. Then there is a Ricci flow with surgery $(M_t, g(t))$ defined for all $t \in [0, \infty)$ with initial conditions (M, g_0). The set of discontinuity times for this Ricci flow with surgery is a discrete subset of $[0, \infty)$. The topological change in the 3-manifold as one crosses a surgery time is a connected sum decomposition together with removal of connected components, each of which is diffeomorphic to one of $S^2 \times S^1$, $RP^3 \sharp RP^3$, or a manifold admitting a metric of constant positive curvature.*

If M is simply-connected or its fundamental group is not too big, Perelman showed us in [**Perel**] why M_t has to become empty for t sufficiently large, i.e., the Ricci flow with surgery becomes extinct in finite time. In particular, the Poincaré conjecture follows. One can find detailed proof of this finite extinction in [**MT06**] and [**CM07**].

To solve Thurston's Geometrization Conjecture, one needs to study the asymptotic behavior of the above Ricci flow with surgery[1] and to prove a result on collapsing 3-manifolds with curvature locally bounded from below and with geodesically convex boundary. A detailed proof of this result on collapsing can be found in [**MT08**], [**CG09**] and [**KL10**]. Also we should point out that this result on collapsing of closed 3-manifolds was a weaker version of the main conclusion of Shioya-Yamaguchi in a series of papers ranging from 2000 to 2005 (see [**SY06**], [**SY06**]). But its proof relies crucially on a hard stability theorem first shown by Perelman in an unpublished preprint in 1992. In 2007, V. Kapovitch gave an alternative proof of this stability result. His proof was published in [**Ka07**].

Let me mention a few crucial ingredients in establishing Theorem 2.2 before Perelman's work. First the analytic theory established by R. Hamilton plays a fundamental role in the proof, particularly, the compactness theorem for Ricci flow. The second is the Hamilton-Ivey curvature pinching estimate [**Ha99**]: *There is a function* ϕ *with* $\phi(s) = 0$ *for* $s \leq 0$ *and* $\lim_{s\to\infty} \phi(s) = 0$ *such that*

$$Rm \geq -\phi(R)\, R - C_0,$$

where R *denotes the scalar curvature and* C_0 *is a constant depending only on the initial metric.* This is only true in dimension 3! The third is the work of Cheeger-Gromoll et al. in late 60's on manifolds with non-negative curvature. Also, in the proof of Theorem 2.2, one needs to use the Harnack-type inequality for Ricci flow proved by Hamilton which is a non-linear extension of the Harnack-Li-Yau inequality for the heat equation (see [**LY86**], [**Ha93**]).

Perelman's first technical advance is a new length function $\mathcal{L}$ for Ricci flow. He called it reduced length. He developed a theory for $\mathcal{L}$ analogous to the theory for the usual length function on Riemannian manifolds. Using the reduced length, Perelman defined the reduced volume and proved that the reduced volume is non-decreasing under the associated backward Ricci flow (backwards in time). This is the fundamental tool used by Perelman for proving a crucial non-collapsing result. More precisely, he proved: If $g(t)$ is a solution of Ricci flow on $[0, T)$ for some $T < \infty$, there is a $\kappa > 0$ depending only on T and $g(0)$ such that whenever $|Rm(g(t))| \leq r^{-2}$ on $B(x, t, r) \times (t - r^2, t]$, $\mathrm{Vol}(B(x, t, r)) \geq \kappa\, r^3$, where $B(x, t, r)$ is the geodesic ball of $g(t)$ centered at $x \in M$ and with radius r.

Together with the curvature pinching estimate and the work on manifolds of non-negative curvature, this non-collapsing result is used to classify topologically all the κ-solutions in dimension 3 which characterize finite-time singularity, in particular, one needs to do surgery only along 2-spheres.

With slight modifications to the domain of integration, one can extend the definitions and the analysis of the reduced length and the reduced volume as well as its monotonicity in the context of the Ricci flow with surgery.

Perelman's second major technical breakthrough is to prove that the region of big curvature is approximated by a κ-solution. This is fundamental in his approach towards geometrization or the Poincaré conjecture. It implies: there is a $r_0 > 0$, which depends on initial metric and is bounded away from zero in any given time interval, such that all points of scalar curvature $\geq r_0^{-2}$ have canonical neighborhoods, that is, neighborhoods which can be described in a topologically and analytically controlled way.

[1] In 2012, R. Bamler found and fixed a gap in Section 6.4 of [**Perel**] which had been missed.

Theorem 2.2 can be proved by applying these techniques in a wise way.

3. Canonical structure in dimension 4

The situation in dimension 4 and upwards is very different from dimension 2 or 3. The Ricci curvature no longer determines the full curvature tensor, so even if M admits an Einstein metric, it may not be a quotient of S^4, $\mathbb{R}^4$ or H^4. The simplest example is $\mathbb{C}P^2$ which admits a homogeneous Einstein metric, the Fubini-Study metric, but it is not a quotient space of this form.

What is special in dimension 4? The special feature in dimension 4 is the occurence of self-duality. It arises because the Lie algebra $so(4)$ can be written as a direct sum of two $so(3)$, where $so(k)$ denotes the Lie algebra of skew-symmetric $k \times k$ matrices. More explicitly, this splitting can be described in terms of the Hodge operator $\star : \Lambda^2\mathbb{R}^4 \mapsto \Lambda^2\mathbb{R}^4$: If $e_1, \cdots, e_4$ form an orthonormal basis of $\mathbb{R}^4$ with respect to the Euclidean inner product, then $\star^2 = \mathrm{Id}$ and

$$\star(e_1 \wedge e_2) = e_3 \wedge e_4, \star(e_1 \wedge e_3) = e_4 \wedge e_2, \star(e_1 \wedge e_4) = e_2 \wedge e_3$$

Then we have $\wedge^2\mathbb{R}^4 = \Lambda_+\mathbb{R}^4 \oplus \Lambda_-\mathbb{R}^4$, where $\Lambda_+\mathbb{R}^4$ is the self-dual part and $\Lambda_-\mathbb{R}^4$ is the anti-self-dual part according to eigenspaces of $\star$ with eigenvalues ± 1. Note that $\Lambda^2\mathbb{R}^k$ can be naturally identified with $so(k)$.

The self-dual structure plays a fundamental role in Donaldson's theory of smooth 4-manifolds (see [**Do86**]). The basic building blocks in Donaldson's theory are the anti-self-dual solutions, also called instantons, to the Yang-Mills equation. Many beautiful results were proved by using the moduli of anti-self-dual solutions, for example, the theorem that a definite intersection form of a smooth simply-connected 4-manifold is diagnosable over $\mathbb{Z}$ and the construction of Donaldson invariants for smooth 4-manifolds. Later, in the middle of 90's, the Seiberg-Witten invariants were introduced [**Wi94**]. The associated Seiberg-Witten equation also uses the self-dual structure in dimension 4.

In line with my interest, I will concentrate on the metric geometry of 4-manifolds and its connections to the topology. Given a 4-dimensional Riemannian manifold (M, g), we have a family of spaces T_pM with the inner product $g_p(\cdot,\cdot)$ $(p \in M)$. We then have a Hodge operator $\star : \Lambda^2 M \mapsto \Lambda^2 M$, which can be characterized by

$$\varphi \wedge \star\psi \;=\; g(\varphi, \psi)\, dV_g,$$

where $\varphi, \psi \in \Lambda^2 M$ and dV_g denotes the volume form of g. Accordingly, we have a decomposition of $\wedge^2 M$ into the self-dual part $\Lambda_+ M$ and the anti-self-dual part $\Lambda_- M$, and consequently, we have the following decomposition of the curvature operator:

$$Rm \;=\; \begin{pmatrix} W_+ + \dfrac{S}{12} & Z \\ Z & W_- + \dfrac{S}{12} \end{pmatrix}$$

where $W = W_+ + W_-$ is the Weyl tensor whose vanishing implies that g is locally conformally flat, Z is the traceless Ricci curvature, that is,

$$Z \;=\; \mathrm{Ric}(g) - \frac{S}{4} g,$$

and S is the scalar curvature.

If g is an Einstein metric, then $Z = 0$, i.e., $Rm(g)$ is self-dual. Another class of canonical metrics are anti-self-dual.

DEFINITION 3.1. A metric g is *anti-self-dual* and of *constant scalar curvature* if $W_+ = 0$ and $S = \text{constant}$.

More generally, one can perturb the anti-self-dual equation: we call a metric g a *generalized anti-self-dual metric* if there is a section f of $S^2\Lambda_+M$ over M, where Λ_+M denotes the self-dual part of $\Lambda^2 M$ with respect to g, such that f is harmonic and g satisfies

$$W_+ = S \cdot f, \ S = \text{const.} \tag{3.1}$$

For simplicity, we often call g an *f-asd metric*.

If M is a complex surface with complex structure J, there is a natural decomposition $\Lambda^2 M = \Lambda_{-J}M \oplus \Lambda_J M$ according to anti-J-invariance and J-invariance, i.e., $\Lambda_{-J}M$ consists of all φ with $\varphi \cdot J = -\varphi$ and $\Lambda_J M$ consists of all φ with $\varphi \cdot J = \varphi$. Then we have a section f of the form

$$f|_{\Lambda_J M} = \frac{1}{6}\text{Id} \quad \text{and} \quad f|_{\Lambda_{-J}M} = -\frac{1}{12}\text{Id}.$$

Kähler metrics of constant scalar curvature are f-asd for this f since f is parallel and any Kähler metric has $W_+ = S \cdot f$.

The above canonical metrics also impose topological constraints on underlying 4-manifolds. Recall that the cup product induces a non-degenerate intersection form I on $H^2(M, \mathbb{Z})$. There is a natural decomposition $H^2(M, \mathbb{R}) = H^2_+(M) \oplus H^2_-(M)$ induced by the Hodge operator $*$. This decomposition is orthogonal with respect to the real form $I_{\mathbb{R}}$ of I on $H^2(M, \mathbb{R})$. A famous theorem of M. Freedman [**Fr82**] says that if two simply-connected smooth 4-manifolds have the same intersection form, then they are homeomorphic to each other.

By the Index Theorem, the signature $\tau(M) = \dim_{\mathbb{R}} H^2_+(M) - \dim_{\mathbb{R}} H^2_-(M)$ is given by

$$\tau(M) = \frac{1}{8\pi^2}\int_M (|W_+|^2 - |W_-|^2)\, \mathrm{d}V \tag{3.2}$$

On the other hand, the Gauss-Bonnet-Chern formula gives

$$\chi(M) = \frac{1}{12\pi^2}\int_M (|W|^2 - |Z|^2 + \frac{S^2}{24})\, \mathrm{d}V, \tag{3.3}$$

where $\chi(M)$ denotes the Euler number.

It follows from these: if M admits an Einstein metric, then we have the Hitchin–Thorpe inequality

$$|\tau(M)| \le \frac{2}{3}\chi(M).$$

Furthermore, in [**Hi74**], Hitchin proved that the equality holds if and only if M is diffeomorphic to $K3$ surfaces, which include quartic surfaces in $\mathbb{C}P^3$.

As a corollary of the Hitchin–Thorpe inequality, we can easily show that $\mathbb{C}P^2 \sharp k \overline{\mathbb{C}P^2}$ does not have Einstein metrics for $k \ge 9$, while it does admit an Einstein metric for $k \le 8$, see [**TY87**], [**CLW08**].

If M is also a complex surface, then there is a stronger Miyaoka–Yau inequality [**Ya77**]: $\tau(M) \le \frac{1}{3}\chi(M)$. In [**Le95**], Lebrun extended this inequality and proved the same inequality for any smooth 4-manifold M which admits an Einstein metric and has non-vanishing Seiberg–Witten invariant. This provides an obstruction to the existence of Einstein metrics. Recently, a new obstruction was given by Ishida [**Is12**].

We can state another corollary of the Hitchin–Thorpe inequality: *If a simply-connected spin 4-manifold M is a connected sum of smooth 4-manifolds which are homeomorphic to Einstein 4-manifolds, then* $8\, b_2(M) \geq -11\, \tau(M)$.

Anti-self-dual metrics also impose constraints on M. First the signature $\tau(M)$ has to be non-positive if M admits an anti-self-dual metric. The following theorem can be easily proved.

THEOREM 3.2. *If $\tau(M) = 0$ and M admits an anti-self-dual metric, then M is locally conformal flat. If M is further simply-connected, then M has to be a standard 4-sphere.*

Basic question on canonical metrics include existence, uniqueness, compactness and regularity theory for related equations.

4. Ricci flow on 4-manifolds

The Ricci flow (2.1) provides a way of constructing Einstein metrics on 4-manifolds. However, unlike the 2- or 3-dimensional cases, it is much more difficult to study the flow. For instance, we do not have an analogue of the Hamilton-Ivey pinching estimate for curvature. Even the static solutions of the flow, i.e., the Einstein metrics, may develop singularities, which never happens in lower dimensions.

If the initial metric g_0 has positive curvature operator, Hamilton proved in [**Ha86**] that after normalizing the volume, the flow has a global solution which converges to a metric of constant sectional curvature. If g_0 has positive isotropic curvature, so does $g(t)$ along the Ricci flow. The structure of such 4-manifolds can be analyzed by using the Ricci flow (cf. [**Ha97**],[**CZ06**]). In general, not much is known about the flow in dimension 4 except for Kähler surfaces.

The Ricci flow has the following property: if the initial metric g_0 is Kähler, so is $g(t)$ along the Ricci flow. In this case, we understand completely the singularity formation. Let us give a brief tour of what we know about Ricci flow on Kähler surfaces.

Assume that (M, g_0) is a compact Kähler surface with Kähler form ω_0. It is proved in [**TZ06**] that (2.1) has a maximal solution $g(t)$ for $t \in [0, T)$, where

$$T = \sup\{t \,|\, [\omega_0] - tc_1(M) > 0\}.$$

Moreover, each $g(t)$ is Kähler with the Kähler class $[\omega_0] - tc_1(M)$.

If M contains a holomorphic sphere C with $c_1(M)([C]) > 0$, then this implies that $[\omega_0] - tc_1(M)$ ceases to be a Kähler class for some

$$T \,\leq\, [\omega_0]([C])/c_1(M)([C]) < \infty,$$

so (2.1) develops singularity at T. If C is irreducible, $c_1(M)([C]) > 0$ is the same as saying that the self-intersection of C is greater than -2. It turns out that those holomorphic curves are the only reason for the finite-time singularity of (2.1) on Kähler surfaces; this is because $[\omega_0] - tc_1(M) > 0$ for any $t \geq 0$ if no such curves exist. As t tends to T, $(M, g(t))$ collapses to a point or a Riemann surface or contracts certain holomorphic spheres of self-intersection number -1. In the first case, $[\omega_0] = T\, c_1(M)$ and consequently M is a del Pezzo surface.

In the second case, M has to be a ruled surface over a Riemann surface Σ and $g(t)$ converges to a (1,1)-current g_T on Σ in the weak sense, possibly in the Gromov–Hausdorff topology. Then it can be proved that there is a Ricci flow $g(t)$ $(t > T)$ on Σ with $\lim_{t\to T} g(t) = g_T$ in the sense of currents and $g(t)$ converges to a metric of constant curvature on Σ after suitable normalization.

In the third case, there will be a curve C (may have several connected components) such that

$$([\omega_0] - Tc_1(M))([C]) = 0,$$

so $c_1(M)([C]) > 0$ and C must be made of disjoint holomorphic 2-spheres of self-intersection -1. Hence, we can blow down C to get a new complex manifold M_1, furthermore, there is a holomorphic map $\pi : M \mapsto M_1$ such that $\pi|_{M\backslash C}$ is a biholomorphism onto its image and $\pi(C)$ consists of finitely many isolated points. It was known that $g(t)$ converges to g_T in the sense of currents, moreover, it is proved by Tian–Zhang [**TZ06**] that g_T is a smooth Kähler metric and $g(t)$ converges to g_T in the smooth topology on $M\backslash C$. It is easy to see that $g_T = \pi^*\bar{g}_T$ for some $\bar{g}_T$ on M_1. In fact, Song–Weinkove [**SW**] proved that $g(t)$ converges to $\bar{g}_T$ in the Gromov–Hausdorff topology. This can be also proved by using La Nave–Tian's V-soliton equation [**LT**].

It is proved in [**ST09**] that the Ricci flow (2.1) can continue on M_1 with initial metric $\bar{g}_T$. Then we can repeat the above process, and it is clear that after finitely many blow-downs, we will either run into a collapsed space or arrive at a complex surface M_ℓ without holomorphic spheres of self-intersection number > -2, so the Ricci flow has a global solution on M_ℓ. In particular, if M is not birational to $\mathbb{C}P^2$ or a ruled surface, we have a global solution $g(t)$ with surgery for (2.1) such that each M_t is a Kähler surface obtained by blowing down rational curves of self-intersection -1 successively from previous $M_{t'}$, topologically, M is a connected sum of M_t and finitely many copies of $\mathbb{C}P^2$ with reversed orientation.

There are three possibilities for asymptotic behaviors of $g(t)$ as t tends to ∞ according to the Kodaira dimension $\kappa(M)$ of M:

1. If $\kappa(M) = 0$, then $c_1(M)_{\mathbb{R}} = 0$ or a finite cover of M_ℓ is either a K3 surface or an Abelian surface. In this case, the solution $\tilde{\omega}_t$ on M_ℓ converges to a Ricci flat Kähler metric.

In the other two cases, it is better to use the normalized Kähler-Ricci flow on M:

$$\frac{\partial\tilde{\omega}(s)}{\partial s} = -\mathrm{Ric}(\tilde{\omega}(s)) - \tilde{\omega}(s),\ \tilde{\omega}(0) = \omega_0,$$

where $t = e^s - 1$ and $\tilde{\omega}(s) = e^{-s}\tilde{\omega}_t$.

2. If $\kappa(M) = 1$, then M_ℓ is a minimal elliptic surface: $\pi : M_\ell \mapsto \Sigma$. It was proved in [**ST07**] that as $s \to \infty$, $\tilde{\omega}(s)$ converges to a positive current of the form $\pi^*(\tilde{\omega}_\infty)$ and the convergence is in the $C^{1,1}$-topology on any compact subset outside singular fibers $F_{p_1}, \cdots, F_{p_k}$, where $p_1, \cdots, p_k \in \Sigma$. Furthermore, $\tilde{\omega}_\infty$ satisfies the generalized Kähler-Einstein equation:

$$\mathrm{Ric}(\tilde{\omega}_\infty) = -\tilde{\omega}_\infty + f^*\omega_{WP},\ \text{on}\ \Sigma\backslash\{p_1, \cdots .p_k\},$$

where f is the induced holomorphic map from $\Sigma\backslash\{p_1, \cdots .p_k\}$ into the moduli of elliptic curves.

3. If $\kappa(M) = 2$, then M_ℓ is a surface of general type and its canonical model $M_{\rm can}$ is a Kähler orbifold with possibly finitely many rational double points and ample canonical bundle. By the version of the Aubin–Yau Theorem for orbifolds, there is an unique Kähler–Einstein metric $\tilde\omega_\infty$ on $X_{\rm can}$ with scalar curvature -2. It was proved in [**TZ06**] that as $s \to \infty$, $\tilde\omega(s)$ converges to $\tilde\omega_\infty$ and converges in the C^∞-topology outside those rational curves over the rational double points.

Let us end this section with an interesting example: Suppose that $\pi : M \mapsto S^2$ is a simply-connected minimal elliptic surface. Write $F_x = \pi^{-1}(x)$ and let $p_1, \cdots, p_\ell$ be all the points in S^2 over which the fiber is singular. Let $\pi' : M' \mapsto \Sigma$ be the elliptic surface obtained by performing logarithmic transformations along two non-singular fibers F_p and F_q of coprime multiplicities k and l. Then M' is a minimal elliptic surface homeomorphic but not diffeomorphic to M. If we run the Kähler-Ricci flow on both M and M', we get two generalized Kähler-Einstein metrics g on $S^2\backslash\{p_1, \cdots, p_\ell\}$ and g' on $S^2\backslash\{p_1, \cdots, p_\ell, p, q\}$. The asymptotic behaviors of g and g' are the same at each p_i, while g is smooth at p or q and g' has a conic angle of $2\pi(k-1)/k$ at p or $2\pi(l-1)/l$ q. Even though M and M' are far from each other, e.g., they are not diffeomorphic to each other, g and g' resemble each other and may be deformed to each other by smoothing out the angles at p and q. *Do we have a similar picture for general homeomorphic 4-manifolds which are not diffeomorphic to each other*?

5. Symplectic curvature flow

Symplectic 4-manifolds form a very important class of 4-manifolds and include Kähler surfaces as special cases. It will be desirable to extend what we have about Ricci flow on Kähler surfaces to symplectic 4-manifolds. But the Ricci flow does not preserve the symplectic structure, so we need a new flow for studying symplecic manifolds. Fortunately, J. Streets and I found a new curvature flow for symplectic manifolds, see [**StT10**].

Let (M, ω) be a symplectic manifold. An almost-Kähler metric g on M is a Riemannian metric such that

$$g(u, v) = \omega(u, Jv),\ \omega(Ju, Jv) = \omega(u, v),$$

where J is an almost complex structure. Such a triple (M, ω, J) is called an *almost Kähler.*

Besides the Levi-Civita connection ∇, there is another canonical connection, called Chern connection, given by

$$D_X Y = \nabla_X Y - \tfrac{1}{2} J(\nabla_X J)(Y).$$

The Chern connection D preserves the metric g and almost complex structure J, but it may have torsion.

The Chern connection D induces a Hermitian connection on the anticanonical bundle, and we denote the curvature form of this connection by P. Alternatively, if $\Omega = \{\Omega_{ijkl}\}$ denotes the curvature of D, one has

$$P_{ij} = \omega^{kl}\Omega_{ijkl}, \tag{5.1}$$

where $\{\omega^{ij}\}$ is the inverse of ω. By the general Chern-Weil theory, P is a closed form and represents $\pi c_1(M, J)$.

Define an endomorphism

$$\mathcal{R}_i^j = J_i^k \mathrm{Ric}_k^j - \mathrm{Ric}_i^k J_k^j \tag{5.2}$$

where the index on the Ricci tensor has been raised with respect to the associated metric. This $\mathcal{R}$ is actually the (2,0)+(0,2) part of the Ricci curvature of the associated metric.

The symplectic curvature flow in [**StT10**] is given by

$$\frac{d\omega}{dt} = -2P \text{ and } \frac{dJ}{dt} = -2g^{-1}\left[P^{(2,0)+(0,2)}\right] + \mathcal{R}. \tag{5.3}$$

A direct computation shows:

$$g^{-1}\left[P^{(2,0)+(0,2)}\right] = \frac{1}{2}\left[\nabla^*\nabla J - \mathcal{N}\right], \tag{5.4}$$

where

$$\mathcal{N}_b^a = g^{ij} J_c^a \nabla_i J_b^p \nabla_j J_p^c. \tag{5.5}$$

It follows that (5.3) can be written as

$$\frac{d\omega}{dt} = -2P \text{ and } \frac{dJ}{dt} = -\nabla^*\nabla J + \mathcal{N} + \mathcal{R}. \tag{5.6}$$

Clearly, (5.3) is invariant under diffeomorphisms of M, hence, like the Ricci flow, this symplectic curvature flow is parabolic only modulo the group of diffeomorphism. More precisely, we have

THEOREM 5.1. *Let (M, ω_0, J_0) be an almost Kähler manifold. There exists a unique solution to (*5.3*) with initial condition (ω_0, J_0) on $[0, T)$ for some $T > 0$.*

Our new flow resembles the Ricci flow. In terms of associated metrics $g(t)$, the flow becomes

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) + B^1 + B^2, \tag{5.7}$$

where

$$B_{ij}^1 = g^{kl} g_{pq} \nabla_i J_k^p \nabla_j J_l^q \text{ and } B_{ij}^2 = g^{kl} g_{pq} \nabla_k J_i^p \nabla_l J_j^q.$$

Like the Ricci flow, one can derive induced evolution equations on the curvature $Rm(g)$ of g and its derivatives as well as derivatives of J and apply the Maximum Principle to proving that if for some fixed α depending only on ω_0,

$$\sup_{M\times[0,\alpha/K]} \{|Rm(g), |\nabla J|^2, |\nabla^2 J|\} \le K,$$

then we have

$$\sup_{M\times[0,\alpha/K]} \{|\nabla^m Rm(g)|, |\nabla^{m+2} J|\} \le C_m K t^{-\frac{m}{2}}.$$

However, in dimension 4, we have a much better estimate:

THEOREM 5.2. *If* $\dim M = 4$ *and* $(\omega(t), J(t))$ *is a solution to* (5.3) *on* $[0, T]$ *satisfying*

$$\sup_{M\times[0,T]} |Rm| = K,$$

then there exists a constant $C(K, \omega(0), J(0), T)$ such that

$$\sup_{M\times[0,T]} (|\nabla J|^2 + |\nabla^2 J|) \le C.$$

It follows that whenever the curvature of $g(t)$ is bounded, so are all the derivatives of the curvature and $J(t)$, in other words, the deviation of J from being parallel or integrable is controlled by the curvature. We expect that all finite-time singularities of (5.3) are modeled on Kähler spaces, more precisely, we expect

CONJECTURE 5.3. The maximum existence time for (5.3) is given by

$$T = \sup\{t \mid [\omega_0] - t\pi c_1(M) \text{ is represented by a sympletic form }\}. \tag{5.8}$$

Moreover, if $T < \infty$, then there will be a holomorphic 2-sphere $C \subset M$ of self-intersection number greater than -2 along which $[\omega_0] - t\pi c_1(M)$ vanishes.

This conjecture implies that all finite-time singularities are caused by holomorphic 2-spheres of self-intersection number greater than -2. Clearly, (5.3) cannot have a solution at or beyond T, the problem is whether or not there is a solution up to T. As one expects, if J_0 is integrable, so is $J(t)$ and (5.3) becomes the Kähler-Ricci flow. In this case, the conjecture is indeed true as shown by Tian–Zhang in [**TZ06**]. We also know exactly how the flow behaves at time ∞ for Kähler metrics as shown at the end of last section.

We expect that the flow (5.3) exhibits a picture for symplectic 4-manifolds similar to the one for Kähler surfaces. First we point out that the static solutions of (5.3) are classified. By a static solution, we mean an almost Kähler metric (ω, J) satisfying:

$$P = \lambda\,\omega,\ 2\,P^{(2,0)+(0,2)} = g \cdot \mathcal{R},$$

where λ is a constant. These are equivalent to $P = \lambda\,\omega$ and $\mathcal{R} = 0$. It follows from a result of Apostolov et al (see [**StT10**], Section 9) that the latter system has only Kähler-Einstein metrics as solutions on a closed 4-manifold. Therefore, all static solutions are Kähler-Einstein metrics.

In view of this and what we know about Kähler-Ricci flow, we may hope that (5.3) *has only finitely many finite-time singularities which are caused by holomorphic 2-spheres of self-intersection number* > -2 *and thereafter becomes either extinct, or has a global solution; if* (5.3) *become extinct at finite time, then the underlying 4-manifold is essentially a rational surface; if* (5.3) *has a global solution, then after appropriate normalization, the solution converges to a sum of Kähler–Einstein surfaces and symplectic 4-manifolds which admit a fiberation over a lower dimensional space with k-dimensional tori as generic fibers, where* $1 \leq k \leq 4$, *glued along 3-dimensional tori with appropriate properties.*

This is analogous to Thurston's Geometrization Conjecture for 3-manifolds.

6. Pluri-closed flow

The Ricci flow has another generalization: the pluri-closed flow on Hermitian manifolds. This flow provides solutions of the B-field renormalization group flow in theoretical physics, but it is not exactly a case of the B-field renormalization group flow since the latter does not say anything about how complex structures evolve.

Let (M, J) be a complex manifold of complex dimension n and ω be a Hermitian metric on M. We denote a Hermitian metric by its Kähler form ω. We say a Hermitian metric ω *pluri-closed* if

$$\partial\bar{\partial}\omega = 0\,. \tag{6.1}$$

This condition is weaker than the Kähler condition. By a result of Gauduchon, every complex surface admits pluri-closed metrics.

The pluri-closed flow is given by

$$\frac{\partial \omega}{\partial t} = \partial\partial^*\omega + \bar\partial\bar\partial^*\omega + \frac{\sqrt{-1}}{2}\partial\bar\partial \log\det g\,. \tag{6.2}$$

This was introduced by Streets and myself in 2008, see [**StT09**]. It was proved there that *for any pluri-closed metric ω_0 on M, there is a unique solution $\omega(t)$ for (6.2) on $[0,T)$ for some $T>0$ such that $\omega(0)=\omega_0$; moreover, all $\omega(t)$ staisfy pluri-closed condition. If ω_0 is Kähler, so is $\omega(t)$, so this flow reduces to the Kähler-Ricci flow.*

A newer and better formulation of (6.2) is to use the Bismut connection. Recall the Bismut connection ∇ is defined as follows:

$$g(\nabla_X Y, Z) = g(D_X Y, Z) + \frac{1}{2}\, d\omega(JX, JY, JZ), \tag{6.3}$$

where D denotes the Levi-Civita connection. The Bismut connection can be characterized as the unique connection compatible with both g and J such that the torsion is a three-form. In terms of the Bismut connection, this torsion 3-form is closed if and only if ω is pluriclosed. Note that when ω is not closed, the Bismut connection is different from the Chern connection. Let Ω denote the curvature of the Bismut connection ∇, and let P denote the curvature of the induced connection on the canonical bundle, i.e., $P_{ab} = \omega^{ij}\Omega_{abij}$. Then (6.2) becomes

$$\frac{\partial \omega}{\partial t} = -P^{1,1}, \tag{6.4}$$

where $P^{1,1}$ denotes the (1,1)-component of P. Using this formulation, it is proved in [**StT11**] that *for any initial ω, (6.4) has a solution $\omega(t)$ on $[0,T)$ such that either $T=\infty$ or $|P^{1,1}(\omega(t))|$ blows up as t tends to T.* This is analogous to Sesum's result for Ricci flow [**Se05**].

Another consequence of this new formulation is to enable us to relate (6.2), or equivalently (6.4), to the B-field renormalization group flow of string theory. Recall the B-field flow studied in [**OSW06**]:

$$\frac{\partial g_{ij}}{\partial t} = -2\,\mathrm{Rc}_{ij} + \frac{1}{2}T_i^{pq}T_{jpq} \text{ and } \frac{\partial T}{\partial t} = \Delta_{LB} T\,, \tag{6.5}$$

where T is a 3-form and Δ_{LB} denotes the Laplace-Beltrami operator of the time-dependent metric. It is shown in [**OSW06**] that the system (6.5) is the gradient flow of the following Perelman-type functional

$$\lambda(g,T) = \inf\left\{\int_M \left[R - \frac{1}{12}T^2 + f^2\right] e^{-f}\mathrm{d}V \;\Big|\; \int_M e^{-f}\mathrm{d}V = 1\right\}. \tag{6.6}$$

Now let $\omega(t)$ be a solution to (6.2) by pluri-closed metrics on a complex manifold M with complex structure J. Denote by $T(t)$ the torsion of the Bismut connections associated to $\omega(t)$. It is proved in [**StT10**] that *if X is the time-dependent field dual to $-\frac{1}{2}Jd^*\omega$ and $\phi(t)$ is the integral curve of X, then $(\phi^*\omega, \phi^*T)$ is a solution of* (6.5).[2] This enables us to obtain two monotonic quantities, one is the functional λ, the other is an expanding entropy (see [**StT11**]).

It is a difficult yet significant problem to study the long-time existence of (6.2). In view of progress on Kähler-Ricci flow, especially, the sharp existence theorem

[2]Note that complex structures $\phi(t)^*J$ vary in t, so it is highly unclear how a solution of (6.5) goes back to that of the pluri-closed flow since (6.5) does not tell how complex structures vary.

of Zhang–Tian [**TZ06**], we expect the following: first we note that a pluri-closed metric defines a class in the finite dimensional Aeppli cohomology group

$$\mathcal{H}^{1,1}_{\partial+\bar{\partial}} = \frac{\{\mathrm{Ker}\partial\bar{\partial} : \Lambda^{1,1}_{\mathbb{R}} \to \Lambda^{2,2}_{\mathbb{R}}\}}{\{\partial\gamma + \bar{\partial}\gamma \mid \gamma \in \Lambda^{0,1}\}}. \tag{6.7}$$

Define the space $\mathcal{P}_{\partial+\bar{\partial}}$ to be the cone of the classes in $\mathcal{H}_{\partial+\bar{\partial}}$ which contain pluri-closed metrics.

Conjecture 6.1. Let (M, ω_0, J) be a compact complex manifold with a pluri-closed metric ω_0. Define

$$\tau^* := \sup_{t\geq 0}\{t \mid [\omega_0 - tc_1] \in \mathcal{P}_{\partial+\bar{\partial}}\}. \tag{6.8}$$

Then the solution to ((6.2)) with initial condition ω_0 exists on $[0, \tau^*)$, and τ^* is the maximal time of existence.

A positive resolution of Conjecture 6.1 would have geometric consequences for non-Kähler surfaces. This is because one can characterize the cone $\mathcal{P}_{\partial+\bar{\partial}}$ in a rather easy way (see [**StT11**]). Combining this with the fact that every complex surface is pluri-closed, we can use (6.2) to study geometry and topology of complex surfaces, particularly, the still largely mysterious Class VII$^+$ surfaces. Class VII$^+$ surfaces are those minimal compact complex surfaces with Betti number $b_1 = 1$ and $b_2 > 0$. In fact, we can show that the resolution of Conjecture 6.1 implies the classification of Class VII$^+$ surfaces with $b_2 = 1$, a result recently obtained by Teleman by using gauge theory [**Te10**]. We believe that through a finer analysis of limiting solutions of (6.2), we can expect to deduce a full classification of all Class VII$^+$ surfaces from the above conjecture.

Recently, through the correspondence between our pluri-closed and the B-field flow, J. Streets and I show that the B-field flow preserves the generalized Kähler structures which were introduced by N. Hitchin [**Hi11**]. Therefore, our pluri-closed flow can be also used to studying generalized Kähler manifolds.

7. Einstein 4-manifolds

In this section, we discuss some estimates for Einstein metrics on 4-manifolds. This is a step in my program with Cheeger on completely understanding how Einstein metrics develop a singularity [**CT06**]. This section as well as the next two follow corresponding sections in [**Ti06**].

Consider the normalized Einstein equation

$$\mathrm{Ric}(g) = \lambda g, \tag{7.1}$$

where the Einstein constant λ is $-3, 0$ or 3, and, if $\lambda = 0$, we add the additional normalization that the volume is equal to 1.

The first main result in our program is the following;

Theorem 7.1. [**CT06**] (ϵ-regularity.) *There exist uniform constants $\epsilon > 0$, $c > 0$, such that the following holds: If g is a solution to (7.1) and $B_r(p)$ is a geodesic ball of radius $r \leq 1$ satisfying:*

$$\int_{B_r(p)} |Rm(g)|^2 \leq \epsilon, \tag{7.2}$$

then

$$\sup_{B_{\frac{1}{2}r}(p)} |Rm(g)| \leq c \cdot r^{-2}, \tag{7.3}$$

where $Rm(g)$ denotes the sectional curvature of g.

REMARK 7.2. If $\lambda = 0$, we can drop the assumption $r \leq 1$. In particular, it implies that a complete Ricci-flat 4-manifold with finite L^2-norm of curvature has quadratic curvature decay.

The usual ϵ-regularity theorems for Yang-Mills and harmonic maps can be proved by the Moser iteration using the Sobolev inequality. Since the domain involved is effectively a standard ball, the Sobolev inequality holds. In [**An89**], [**Na88**], [**Ti89**], this Moser iteration argument was applied to Einstein 4-manifolds and a version of Theorem 7.1 was proved under the assumption that the L^2-norm of the curvature is sufficiently small against the Sobolov constant.

The proof of Theorem 7.1 is considerably more difficult than those of the earlier ϵ-regularity theorems and employs entirely different techniques. Also, as a consequence of our ϵ-regularity, we know essentially the topology of the geodesic ball $B_{r/2}(p)$, that is, it is essentially a quotient of an euclidean ball by Euclidean isometries. This also tells a difference between our theorem and previous ones for Yang–Mills etc.: *we determine the topology as well as analytic property of the geodesic ball considered at the same time.*

The second main result in our program gives an estimate on the injectivity radius.

THEOREM 7.3. [**CT06**](Lower Bound on Injectivity Radius.) *For any $\delta >$ and $v > 0$, there exists $w = w(v, \delta, \chi) > 0$, such that if (M, g) is a complete Einstein 4-manifold with L^2-norm of curvature equal to $12\pi^2\chi$ [3], $\mathrm{vol}(M, g) \geq v$ and $\lambda = \pm 3$, then the set $\mathbf{S}_w$ of $p \in M$ where the injectivity radius at p is less than w has measure less than δ.*

The proof for both theorems above is based on an effective version of Chern's transgression for the Gauss–Bonnet–Chern formula for the Euler number. Let P_χ be the Gauss–Bonnet–Chern form. On subsets of Riemannian manifolds which are sufficiently collapsed with locally bounded curvature, there is an essentially canonical transgression form $\mathcal{T}P_\chi$, satisfying

$$\mathrm{d}\,\mathcal{T}P_\chi = P_\chi, \text{ and } |\mathcal{T}P_\chi| \leq c(4) \cdot (r_{|Rm|}(p))^{-3},$$

where $r_{|Rm|}(p)$ is the supremum of those r such that the curvature Rm is bounded by $1/r^2$ on $B_r(p)$. In fact, this can be done for any dimensions. However, if (M, g) is an Einstein 4-manifold, we have

$$P_\chi = \frac{1}{8\pi^2} \cdot |R|^2 \cdot \mathrm{d}V. \tag{7.4}$$

It follows that near those points where (M, g) is sufficiently collapsed with locally bounded curvature, local L^2-norm of curvature for g can be controlled by the curvature on the boundary in a weaker norm. Then the above theorems can be deduced by applying this estimate. We refer the readers to [**CT06**] for details.

[3] If M is compact, χ is just the Euler number of M.

If $g(t)$ is a global solution of the normalized Ricci flow:

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric}(g) - \frac{r}{4}g), \tag{7.5}$$

where r denotes the integral of scalar curvature. Then the volume of $g(t)$ stays as a constant. We expect that the curvature and injectivity radius estimates in [**CT06**] hold for $g(t)$ as t goes to ∞.

The above theorems can be used to construct a compactifcation of moduli of Einstein 4-manifolds (see [**CT06**] for more details).

Denote by $\mathcal{M}(\lambda, \chi, v)$ the moduli of all solutions to (7.1) with volume equal to v and such that the underlying manifold is closed and has the Euler number χ. This moduli is usually non-compact. We would like to give a natural compactification analogous to the Deligne-Mumford compactification of Riemann surfaces of genus greater than 1.

Let (M_i, g_i) be a sequence of Einstein 4-manifolds with fixed Euler number, Einstein constant and volume. Let $y_i \in M_i$ be a sequence of base points. After passing to a subsequence if necessary, there is a limit (M_∞, y_∞) of the sequence (M_i, g_i, y_i) in a suitable weak geometric sense, the pointed Gromov-Hausdorff sense. This limit space can be thought of as a weakly Einstein spaces with singularities, although a priori they are length spaces and might not have any smooth points whatsoever. Our program provides understanding of the smooth structure of this limit space in general cases.

If $\lambda = 3$, then the diameter of (M, g) in $\mathcal{M}(\lambda, \chi, v)$ is uniformly bounded. It is a non-collapsing case. It has been known since the late 80's that the limit M_∞ is an Einstein orbifold with isolated singularities and that (M_i, g_i) converges to M_∞ in the Cheeger-Gromov topology[4], see [**An89**], [**Na88**], and [**Ti89**] in the Kähler case. As a consequence, the moduli can be compactified by adding Einstein 4-dimensional orbifolds with isolated singularities.

If $\lambda = -3$, the diameter for a sequence of Einstein metrics can diverge to ∞. It is crucial to bound the injectivity radius uniformly from below at almost every point in order to have a fine structure for the limit M_∞. Using Theorem 7.1 and 7.3, in [**CT06**], Cheeger and I proved the following

THEOREM 7.4. *Let (M_i, g_i) be a sequence of Einstein 4-manifolds in $\mathcal{M}(-3, \chi, v)$. Then by taking a subsequence if necessary, there is a sequence of N-tuples $(y_{i,1}, \cdots, y_{i,N})$ satisfying:*

1. *N is bounded by a constant depending only on χ;*
2. *$y_{i,\alpha} \in M_i$ and for any distinct α, β, $\lim_{i\to\infty} d_{g_i}(y_{i,\alpha}, y_{i,\beta}) = \infty$;*
3. *For each $1 \le \alpha \le N$, in the Cheeger-Gromov topology, $(M_i, g_i, y_{i,\alpha})$ converges to a complete Einstein orbifold $(M_{\infty,\alpha}, g_{\infty,\alpha}, y_{\infty,\alpha})$ with only finitely many isolated quotient singularities and $\lim_{i\to\infty} y_{i,\alpha} = y_{\infty,\alpha}$;*
4. *$\lim_{i\to\infty} \mathrm{vol}(M_i, g_i) = \mathrm{vol}(M_\infty, g_\infty)$, where*

$$M_\infty = \bigcup_\alpha M_{\infty,\alpha} \text{ and } g_\infty\big|_{M_{\infty,\alpha}} = g_{\infty,\alpha}.$$

If $\lambda = 0$, the sequence (M_i, g_i) can collapse, i.e., the injectivity radius can go uniformly to 0. We can scale the metrics g_i such that $(M_i, \mu_i g_i)$ has diameter 1.

[4]This means that for any $\epsilon > 0$, there is a compact subset K of the smooth part of M_∞ and diffeomorphisms ϕ_i from a neighborhood of K into M_i such that $M_i \backslash \phi_i(K)$ has measure less than ϵ and $\phi_i^* g_i$ converges to the Einstein orbifold metric of M_∞ in the smooth topology.

Then by taking a subsequence, this has a limit M_∞ in the Gromov–Hausdorff topology. Moreover, by Theorem 7.1, the curvature of g_i is uniformly bounded outside finitely many points, so one has some understanding of the topology of M_i compared to that of M_∞.

It remains to understand collapsing limits of Einstein 4-manifolds. More precisely, if (M_i, g_i, x_i) converges to a collapsed limit (Y, d) in the Gromov–Hausdorff topology, where d is a metric inducing the length structure on Y. This includes two cases:

1. When $\lambda = 0$, the diameter is uniformly bounded;
2. When $\lambda = -1$, x_i diverges to infinity from any points where the local volume is bounded from below.

We have known that outside finitely many points, Y is a limit of spaces with bounded curvature, so it has smooth points. In fact, it is shown in [**NT08**] that Y is a quotient by a smooth manifold by a group action outside finitely many points.

We can say more when M is a Kähler surface. Then $\mathcal{M}(\lambda, \chi, v)$ contains a component $\mathcal{M}(\Omega, c_2)$ of Kähler–Einstein metrics with Kähler class Ω, where c_1 and c_2 denotes the first and second Chern classes of M.

If $c_1 > 0$ and $\mathcal{M}(c_1/3, c_2)$ is of positive dimension, then M is a Del-Pezzo surface obtained by blowing up $\mathbb{C}P^2$ at m points in general position, where $5 \leq m \leq 8$. In [**Ti89**], it was proved that $\mathcal{M}(c_1/3, c_2)$ can be compactified by adding Kähler–Einstein orbifolds, and furthermore, there are strong constraints on quotient singularities. It was conjectured in [**Ti89**] that $\mathcal{M}(c_1/3, c_2)$ *can be compactified by adding Kähler–Einstein orbifolds whose singularities are only rational double points possibly with very few exceptional cases.* Indeed, it is true when M is a blow-up of $\mathbb{C}P^2$ at 5 point, see [**MM93**].

If $c_1 = 0$, then M is either a complex 2-torus or a K3 surface. This is a collapsing case and is related to problems on large complex limits in the Mirror symmetry.

If $c_1 < 0$, let (M, g_i) be a sequence in $\mathcal{M}(-c_1/3, c_2)$, let (M_∞, g_∞) be one of its limits as in Theorem 7.4, and let $M_{\infty,1}, \cdots, M_{\infty,N}$ be its irreducible components. We know that each $(M_{\infty,\alpha}, g_\infty)$ is a complete Kähler–Einstein orbifold. It should be possible to identify these irreducible components more explicitly. For simplicity, assume that $M_{\infty,\alpha}$ is smooth, then we expect:

> $M_{\infty,\alpha}$ *is of the form* $\bar{M} \backslash D$, *where* $\bar{M}$ *is a projective surface and* D *is a divisor with normal crossings, such that* $K_{\bar{M}} + D$ *is positive outside* D *and each component of* D *has either positive genus, or at least two intersection points with components of* D.

Note that the main theorem in [**TY86**] implies: given $\bar{M}$ and D as above, there is a Kähler–Einstein metric on $\bar{M} \backslash D$.

8. Complete Calabi-Yau 4-manifolds

To understand Y near those finitely many non-smooth points, we are led to classifying all complete Ricci-flat 4-manifolds (M, g) with finite L^2-norm of curvature. Almost all known examples of such complete Ricci-flat manifolds are Calabi–Yau ones, so we will focus on complete Calabi–Yau metrics. Non-flat Calabi–Yau metrics were first constructed on a minimal resolution of the quotient of $\mathbb{C}^2$ by a finite group in $SU(2)$ by physicists, by Hitchin and Calabi explicitly or by using the Twistor theory. Further complete Calabi–Yau 4-manifolds were constructed in [**Kr89**], [**TY90**], [**CK99**], [**CH05**] and [**He10**] et al.. A natural question is to see if they are all the complete Calabi–Yau 4-manifolds with L^2-bounded curvature. The work of Cheeger and myself may shed a light on answering this question.

THEOREM 8.1. [**CT06**] *If (M, g) is a complete Ricci-flat manifold with L^2-bounded curvature, then its curvature decays quadratically.*

A related question is the uniqueness of Calabi-Yau metrics on $\mathbb{C}^2$ raised by Calabi a long time ago. There are indeed complete non-flat Calabi-Yau metrics, like the Taub–Nut metric. However, many years ago, I proved the following result in an published note.

THEOREM 8.2. *Any complete Calabi–Yau metric on $\mathbb{C}^2$ with maximal volume growth must be flat.*

Its outlined proof appeared in [**Ti06**] and we refer the readers to there for more discussions.

REMARK 8.3. In fact, by the same arguments, one can actually show that any complete Calabi–Yau 4-manifolds with maximal volume growth must be a minimal resolution of the quotient of $\mathbb{C}^2$ by a finite subgroup in $SU(2)$.

I also conjectured many years ago that the same holds for higher dimensional cases, that is, any complete Calabi-Yau metrics on $\mathbb{C}^n$ with maximal volume growth is flat.

9. Metrics of anti-self-dual type

Anti-self-dual metrics impose strong constraints on underlying 4-manifolds. There have been many works on constructing anti-self-dual metrics by the gluing method or the twistor method[5] (see [**Fl91**], [**DoFr**], [**Kr89**], [**Le93**]). In particular, Taubes proved that given any 4-manifold M, after making connected sum of it with sufficiently many copies of $\overline{\mathbb{C}P^2}$, the resulting 4-manifold admits one anti-self-dual metric ([**Ta92**]). A fundamental question remains open: *how to deform a metric on any given 4-manifold towards an anti-self-dual metric* as we did for the geometrization of 3-manifolds when the Ricci flow is used. For this purpose, we need to develop some analytic estimates for the anti-self-dual equation.

Consider the anti-self-dual equation:

$$W_+ = 0, \; S = \text{const.} \tag{9.1}$$

(9.1) is an elliptic equation modulo diffeomorphisms. Let us show why it is: regarding the curvature Rm as a symmetric tensor on $\Lambda^2 M$, the symbol $\sigma(Rm)$: $T_xM \times S^2T_x^*M \mapsto S^2\Lambda_x^2 M$ of the linearized operator of Rm at (x, g) is given by

$$\sigma(Rm)(\xi, h)(e, e') = -2\, h(e(\xi), e'(\xi)),$$

where $\xi \in T_xM$, $h \in S^2T_x^*M$ and $e, e' \in \Lambda_x^2 M$. Here we have identified $e, e' \in \Lambda_x^2 M$ as endomorphisms of T_xM through the metric g. It follows that the symbol $\sigma(S)$

[5]There were also works on the moduli of anti-self-dual metrics, e.g., [**AHS**], [**KK92**]. In this section, our emphasis on the moduli is different and more towards compactifying the moduli.

of the scalar curvature is

$$\sigma(S)(\xi, h) \;=\; -4\sum_\alpha h(e_\alpha(\xi), e_\alpha(\xi)),$$

where $\{e_\alpha\}$ is an orthonormal basis of $\Lambda T_x M$. Notice that

$$W_+ \,+\, \frac{S}{12} \;=\; Rm|_{\Lambda_+ T_x M}.$$

So one can deduce the symbol

$$\sigma(W_+)(\xi,h)(e,e') \;=\; -2\,h(e(\xi),e'(\xi)) \;-\; \frac{2}{3}\sum_i h(e_i(\xi),e_i(\xi))\,g_x(e,e'),$$

where $\{e_i\}$ is an orthonormal basis of $\Lambda_+ T_x M$. The ellipticity of (9.1) modulo diffeomorphisms means that for any given unit ξ, $h = 0$ whenever if $i_\xi h = 0$ and $\sigma(W_+)(\xi,h) = 0$ and $\sigma(S)(\xi,h) = 0$. It follows directly from the above computations of symbols.

As for Einstein metrics, we need to study the following problem: *given a sequence of an anti-self-dual metrics g_i, or more general f-asd metrics, on M, what are possible limits of g_i as i tends to ∞?*

By using the Index Formula for the signature, we have

$$\int_M ||W(g_i)||^2 \mathrm{d}V_{g_i} \;=\; 12\pi^2\tau(M).$$

By the Gauss–Bonnet–Chern formula, we can deduce from this

$$\int_M ||Rm(g_i)||^2 \mathrm{d}V_{g_i} \;=\; 24\pi^2\tau(M) \;-\; 8\pi^2\chi(M) \;+\; \frac{1}{12}\int_M S(g_i)^2 \mathrm{d}V_{g_i}.$$

If the scalar curvature $S(g_i)$ has uniformly bounded L^2-norm, then we have a priori L^2-bound on curvature tensor $Rm(g_i)$.

Since the Weyl tensor is a conformal invariant, we can make conformal changes to g_i. Recall the Yamabe constant:

$$Q(M,g_i) = \inf_{u>0} \frac{\int_M \left(|\nabla_{g_i} u|^2 + S(g_i)u^2\right)\mathrm{d}V_{g_i}}{\left(\int_M u^4 \mathrm{d}V_{g_i}\right)^{\frac{1}{2}}}.$$

Using the Aubin–Schoen solution of the Yamabe conjecture, there is a u attaining $Q(M,g_i)$, so we simply take g_i with volume 1 and such that the scalar curvature $S(g_i)$ is $Q(M,g_i)$. Then we have

$$\int_M S(g_i)^2 \mathrm{d}V_{g_i} = Q(M,g_i)^2.$$

Therefore, if g_i form a sequence of anti-self-dual metrics with bounded Yamabe constant, then we may assume that their curvatures are uniformly L^2-bounded and that they have fixed volume. One can ask two questions:

1. *Given a compact 4-manifold, is there a uniform bound on the Yamabe constant for anti-self-dual metrics?*
2. *What are possible limits of anti-self-dual metrics g_i with uniformly bounded Yamabe constant?*

As a corollary of Theorem 1.3 in [**TV05**], one has the following partial answer to the second question:

THEOREM 9.1. *Let g_i be a sequence of anti-self-dual metrics on M with bounded Yamabe constant. We further assume that there is a uniform constant c such that for any function f,*

$$\left(\int_M f^4 \mathrm{d}V_{g_i}\right)^{\frac{1}{2}} \leq c \int_M (|df|^2_{g_i} + f^2)\mathrm{d}V_{g_i}.$$

Then by taking a subsequence if necessary, we have that g_i converges to a multi-fold (M_∞, g_∞) in the Cheeger–Gromov topology[6].

REMARK 9.2. A compactness result can be proved for Kähler metrics with constant scalar curvature by the same arguments.

It is possible to have an ϵ-regularity theorem similar to Theorem 7.1. The following can be proved by extending the effective transgression method in [**CT06**] and bounding Sobolev constants for collapsing 4-manifolds with bounded curvature.

THEOREM 9.3.[7] *There exist uniform constants $\epsilon > 0$, $c > 0$, such that the following holds: If g is an anti-self-dual metric or a Kähler metric with constant scalar curvature ± 12 or 0 and $B_r(p)$ is a geodesic ball of radius $r \leq 1$ satisfying:*

$$\int_{B_r(p)} |Rm(g)|^2 \leq \epsilon, \tag{9.2}$$

then

$$\sup_{B_{\frac{1}{2}r}(p)} |Rm(g)| \leq c \cdot r^{-2}. \tag{9.3}$$

It is hoped that the moduli space of anti-self-dual metrics can be used for constructing new differentiable invariants for 4-manifolds. If such an invariant exists, one can compute it and use it to establish existence of anti-self-dual metrics on a 4-manifold. However, there are two major difficulties to be overcome in order to define the invariant: *compactness and transversality.* We have discussed the compactness. For transversality, the readers can find some discussions in [**Ti06**].

References

[An89] M. Anderson, Ricci curvature bounds and Einstein metrics on compact manifolds. J. Amer. Math. Soc. 2 (1989), no. 3, 455–490. MR999661 (90g:53052)

[AHS] M. Atiyah, N. Hitchin and I. Singer, Self-duality in four-dimensional Riemannian geometry. Proc. Roy. Soc. London Ser. A 362 (1978), 425–461. MR506229 (80d:53023)

[Au] T. Aubin, Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics, Sringer, 1998. MR1636569 (99i:58001)

[CG09] J.G. Cao and J. Ge, Perelman's collapsing theorem for 3-manifolds. To appear in The Journal of Geometric Analysis, arXiv:0908.3229.

[CLT06] X.X.Chen, P. Lu and G. Tian, A note on uniformization of Riemann surfaces by Ricci flow. Proc. Amer. Math. Soc. 134 (2006), no. 11, 3391–3393 MR2231924 (2007d:53109)

[CLW08] X.X.Chen, C. Lebrun and B. Weber, On conformally Kähler, Einstein manifolds. J. Amer. Math. Soc. 21 (2008), no. 4, 1137–1168. MR2425183 (2010h:53054)

[Ch90] B. Chow, The Ricci flow on the 22-sphere. J. Differential Geom. 33 (1991), no. 2, 325–334. MR1094458 (92d:53036)

[CM07] T. Colding and W. Minicozzi, Width and finite extinction time of Ricci flow. Geom. Topol. 12 (2008), no. 5, 2537–2586. MR2460871 (2009k:53166)

[6] A multi-fold is a connected sum of finitely many orbifolds, see [**TV05**] for definition.

[7] This was announced in some of my lectures in early 2010, but a written proof has not appeared yet and is in preparation.

[CT06] J. Cheeger and G. Tian, Curvature and injectivity radius estimates for Einstein 4-manifolds. J. Amer. Math. Soc. 19 (2006), no. 2, 487–525. MR2188134 (2006i:53042)

[CH05] S. Cherkis and N. Hitchin, Gravitational instantons of type D_k. Comm. Math. Phys. 260 (2005), no. 2, 299–317. MR2177322 (2007a:53095)

[CHK00] D. Cooper, C. Hodgson, S. Kerchoff, 3-dimensional orbifolds and cone-manifolds. Math. Soc. Japan Memoirs, Vol. 5, Tokyo, 2000.

[CK99] S. Cherkis and A. Kapustin, Singular monopoles and gravitational instantons. Comm. Math. Phys. 203 (1999), no. 3, 713–728. MR1700937 (2000m:53062)

[CZ06] B-L Chen and X-P Zhu, Ricci flow with surgery on four-manifolds with positive isotropic curvature. J. Differential Geom. 74 (2006), no. 2, 177–264. MR2258799 (2007f:53077)

[DoC] M. doCarmo, Riemannian geometry. Mathematics: Theory and Applications, Birkhäuser, 1992, Boston-Basel-Berlin. MR1138207 (92i:53001)

[Do86] S. Donaldson, The geometry of 4-manifolds. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 43–54, Amer. Math. Soc., Providence, RI, 1987. MR934214 (89d:57002)

[DoFr] S. Donaldson and R. Friedman, Connected sums of self-dual manifolds and deformations of singular spaces. Nonlinearity 2 (1989), no. 2, 197–239. MR994091 (90e:32027)

[Fl91] A. Floer, Self-dual conformal structures on $\ell\mathbb{C}P^2$. J. Diff. Geom. 33 (1991), no. 2, 551–573. MR1094469 (92e:53049)

[Fr82] M. Freedman, The topology of four-dimensional manifolds. J. Diff. Geom. 17 (1982), no. 3, 357–453. MR679066 (84b:57006)

[Ha82] R. Hamilton, Three-manifolds with positive Ricci curvature. J. Differential Geom. 17 (1982), no. 2, 255–306. MR664497 (84a:53050)

[Ha86] R. Hamilton, Four-manifolds with positive curvature operator. J. Differential Geom. 24 (1986), no. 2, 153–179. MR862046 (87m:53055)

[Ha93] R. Hamilton, The Harnack estimate for the Ricci flow. J. Diff. Geom. 37 (1993), no. 1, 225–243. MR1198607 (93k:58052)

[Ha88] R. Hamilton, The Ricci flow on surfaces. Mathematics and general relativity. 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988. MR954419 (89i:53029)

[Ha97] R. Hamilton, Four-manifolds with positive isotropic curvature. Comm. Anal. Geom. 5 (1997), no. 1, 1–92. MR1456308 (99e:53049)

[Ha99] R. Hamilton, Nonsingular solutions of the Ricci flow on three-manifolds. Comm. Anal. Geom. 7 (1999), 695–729. MR1714939 (2000g:53034)

[He10] Hans-Joachim Hein, Complete Calabi-Yau metrics from $\mathbb{C}P^2\#9\overline{\mathbb{C}P}^2$. arXiv:1003.2646.

[Hi74] N. Hitchin, Compact four-dimensional Einstein manifolds. J. Differential Geom. 9 (1974), 435–441. MR0350657 (50:3149)

[Hi11] N. Hitchin, Poisson modules and generalized geometry. Geometry and analysis. No. 1, 403–417, Adv. Lect. Math. (ALM), 17, Int. Press, 2011. MR2882431

[Is12] M. Ishida, Einstein metrics and exotic smooth structures. Pacific J. Math. 258 (2012), no. 2, 327–348. MR2981957

[Ka07] V. Kapovitch, Perelman's stability theorem. Surveys in differential geometry. Vol. XI, 103-136. MR2408265 (2009g:53057)

[KK92] A. King and D. Kotschick, The deformation theory of anti-self-dual conformal structures. Math. Ann. 294 (1992), no. 4, 591–609. MR1190446 (93j:58021)

[KL10] B. Kleiner and J. Lott, Locally collapsed 3-manifolds. arXiv:1005.5106.

[Kn29] H. Kneser, Geschlossene Flächen in drei-dimesnionalen Mannigfaltigkeiten. Jahresber. Deutsch. Math. Verbein 38 (1929), 248–260.

[Kr89] P. Kronheimer, The construction of ALE spaces as hyper-Kähler quotients. J. Differential Geom. 29 (1989), no. 3, 665–683. MR992334 (90d:53055)

[Le93] C. LeBrun, Self-dual manifolds and hyperbolic geometry. Einstein metrics and Yang-Mills connections (Sanda, 1990), 99–131, Lecture Notes in Pure and Appl. Math., 145, Dekker, New York, 1993. MR1215284 (94h:53060)

[Le95] C. LeBrun, Einstein metrics and Mostow rigidity. Math. Res. Lett. 2 (1995), 1–8. MR1312972 (95m:53067)

[LT] G. La Nave and G. Tian, Soliton-type metrics and Kähler-Ricci flow on symplectic quotients. arXiv:0903.2413.

[LY86] P. Li and S.T. Yau, On the parabolic kernel of the Schrödinger operator. Acta Math. 156 (1986), no. 3-4, 153–201. MR834612 (87f:58156)

[MM93] T. Mabuchi and S. Mukai, Stability and Einstein–Kähler metric of a quartic del Pezzo surface. Einstein metrics and Yang-Mills connections, 133–160, Lecture Notes in Pure and Appl. Math., 145, Dekker, New York, 1993. MR1215285 (94m:32043)

[MT06] J. Morgan and G. Tian, Ricci flow and the Poincaré conjecture. Clay Mathematics Monographs, 3. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007, 521 pp. MR2334563 (2008d:57020)

[MT08] J. Morgan and G. Tian, Completion of the proof of the Geometrization Conjecture. arXiv:0809.4040.

[Na88] H. Nakajima, Hausdorff convergence of Einstein 4-manifolds. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 35 (1988), no. 2, 411–424. MR945886 (90e:53063)

[NT08] A. Naber and G. Tian, Geometric Structures of Collapsing Riemannian Manifolds I. arXiv:0804.2275. MR2906936

[OSW06] T. Oliynyk, V. Suneeta and E. Woolgar, A gradient flow for worldsheet nonlinear sigma models, Nuclear Phys. B 739 (2006), no. 3, 441–458. MR2214659 (2006m:81185)

[Per02] G. Perelman, The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159.

[Per03] G. Perelman, Ricci flow with surgery on three-manifolds. arXiv:math/0303109.

[Perel] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. arXiv:math/0307245

[Se05] N. Sesum, Curvature tensor under the Ricci flow. Amer. J. Math. 127 (2005), no. 6, 1315–1324. MR2183526 (2006f:53097)

[ST07] J. Song and G. Tian, The Kähler–Ricci flow on surfaces of positive Kodaira dimension. Invent. Math. 170 (2007), no. 3, 609–653. MR2357504 (2008m:32044)

[ST09] J. Song and G. Tian, The Kähler–Ricci flow through singularities. arXiv:0909.4898.

[StT09] J. Streets and G. Tian, A parabolic flow of pluriclosed metrics. Int. Math. Res. Not. IMRN 2010, no. 16, 3101–3133. MR2673720 (2011h:53091)

[StT10] J. Streets and G. Tian, Symplectic curvature flow. arXiv:1012.2104.

[StT11] J. Streets and G. Tian, Regularity theory for pluriclosed flow. C. R. Math. Acad. Sci. Paris 349 (2011), no. 1-2, 1–4. MR2755684 (2012a:32025)

[SW] J. Song and B. Weinkove, Contracting exceptional divisors by the Kähler–Ricci flow. Duke Math. J. 162 (2013), no. 2, 367–415. MR3018957

[SY06] T. Shioya and T. Yamaguchi, Collapsing three-manifolds under a lower curvature bound. J. Diff. Geom., 56 (2000), no. 1, 1–66. MR1863020 (2002k:53074)

[SY06] T. Shioya and T. Yamaguchi, Volume collapsed three-manifolds with a lower curvature bound. Math. Ann. 333 (2005), no. 1, 131–155. MR2169831 (2006j:53050)

[Ta92] C. Taubes, The existence of anti-self-dual conformal structures. J. Differential Geom. 36 (1992), no. 1, 163–253. MR1168984 (93j:53063)

[Te10] A. Teleman, Instantons and curves on class VII surfaces. Ann. of Math. (2) 172 (2010), no. 3, 1749–1804. MR2726099 (2011h:32020)

[Th97] W. Thurston, Three-dimensional geometry and topology, vol. 1, Princeton Math. Ser., vol. 35, Princeton Univ. Press, 1997. MR1435975 (97m:57016)

[Ti89] G. Tian, On Calabi's conjecture for complex surfaces with positive first Chern class. Invent. Math. 101 (1990), no. 1, 101–172. MR1055713 (91d:32042)

[Ti06] G. Tian, Aspects of metric geometry of four manifolds. Inspired by S. S. Chern, 381–397, Nankai Tracts Math., 11, World Sci. Publ., 2006. MR2313343 (2008i:53044)

[TV05] G. Tian and J. Viaclovsky, Moduli spaces of critical Riemannian metrics in dimension four. Adv. Math. 196 (2005), no. 2, 346–372. MR2166311 (2006i:53051)

[TY86] G. Tian and S.T. Yau, Existence of Kähler–Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. Mathematical aspects of string theory, 574–628, World Sci. Publishing, 1987. MR915840

[TY87] G. Tian and S.T. Yau, Kähler–Einstein metrics on complex surfaces with $C_1 > 0$. Comm. Math. Phys. 112 (1987), no. 1, 175–203. MR904143 (88k:32070)

[TY90] G. Tian and S.T. Yau, Complete Kähler manifolds with zero Ricci curvature. I. J. Amer. Math. Soc. 3 (1990), no. 3, 579–609. MR1040196 (91a:53096)

[TZ06] G. Tian and Z. Zhang, On the Kähler–Ricci flow on projective manifolds of general type. Chinese Ann. Math. Ser. B 27 (2006), no. 2, 179–192. MR2243679 (2007c:32029)

[Wi94] E. Witten, Monopoles and four-manifolds. Math. Res. Lett. 1 (1994), no. 6, 769–796. MR1306021 (96d:57035)

[Ya77] S.T. Yau, Calabi's conjecture and some new results in algebraic geometry. Proceedings of the National Academy of Sciences of the USA, 74 (1977), 1798–1799. MR0451180 (56:9467)

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