8 Vector space

8.1 Introduction

Our study of vectors in $\mathbf{R}^n$ has been based on the two basic vector operations, namely, vector addition and scalar multiplication. For instance, the notion of a linear combination of vectors,

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x} + \cdots + \alpha_s\mathbf{x}_s,$$

uses these two operations. And so do all of the definitions involving linear combinations, such as span, linear independence, basis, and coordinate vector. Even the idea of a linear function $L : \mathbf{R}^n \to \mathbf{R}^m$ is based on these two operations:

$$L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}), \quad L(\alpha\mathbf{x}) = \alpha L(\mathbf{x}).$$

There are sets besides $\mathbf{R}^n$ that also have naturally defined addition and scalar multiplication. For example, the set $\mathbf{M}_{2\times 3}$ of 2×3 matrices:

$$\begin{bmatrix} 1 & -3 & 4 \\ -2 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 1 & -2 \\ 7 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ 5 & -1 & 3 \end{bmatrix},$$

$$2\begin{bmatrix} -3 & 1 & 2 \\ -1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 2 & 4 \\ -2 & 8 & 0 \end{bmatrix}.$$

Both $\mathbf{R}^n$ and $\mathbf{M}_{2\times 3}$ are “vector spaces.” A vector space is a set having an addition and a scalar multiplication that satisfy some properties. In this section, we study vector spaces in general. This allows for efficiency in that we can apply anything we learn about a general vector space to any particular vector space that we encounter.

8.2 Definition

VECTOR SPACE.

A **vector space** is a set V (the elements of which are called **vectors**) with an addition and a scalar multiplication satisfying the following properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha, \beta \in \mathbf{R}$:

(V1) $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$,

(V2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$,

(V3) there exists a vector $\mathbf{0}$ in V such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$,

(V4) for each vector $\mathbf{v}$ in V, there exists a vector $-\mathbf{v}$ in V such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$,

(V5) $\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}$,

(V6) $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$,

(V7) $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$,

(V8) $1\mathbf{v} = \mathbf{v}$.

Although property (V3) allows for the possibility of more than one vector in V satisfying the stated property for $\mathbf{0}$, it can be shown using the other properties that there can be only one such vector (see Section 8.8); it is called the **zero vector**. Similarly, for any vector $\mathbf{v}$ in V, there is only one vector $-\mathbf{v}$ satisfying the stated property in (V4); it is called the **inverse of v**.

8.3 Example: Euclidean space

The set $V = \mathbf{R}^n$ is a vector space with usual vector addition and scalar multiplication. To verify this, one needs to check that all of the properties (V1)–(V8) are satisfied. Here, we check only a few of the properties (and in the special case $n = 2$) to give the reader an idea of how the verifications are done.

8.3.1 Example Show that $\mathbf{R}^2$ satisfies properties (V1), (V3), (V4), and (V5).

Solution

(V1) If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$, then

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} = \begin{bmatrix} y_1+x_1 \\ y_2+x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{y}+\mathbf{x}.$$

(V3) The vector $\mathbf{0} = [0,0]^T$ satisfies the property since, for each $\mathbf{x} \in \mathbf{R}^2$,

$$\mathbf{x}+\mathbf{0} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1+0 \\ x_2+0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}.$$

(V4) If $\mathbf{x} \in \mathbf{R}^2$, then the vector $-\mathbf{x} = [-x_1, -x_2]^T$ satisfies the property since

$$\mathbf{x}+(-\mathbf{x}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} x_1+(-x_1) \\ x_2+(-x_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}.$$

(V5) If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ and $\alpha \in \mathbf{R}$, then

$$\begin{aligned} \alpha(\mathbf{x}+\mathbf{y}) &= \alpha\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \alpha\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} = \begin{bmatrix} \alpha(x_1+y_1) \\ \alpha(x_2+y_2) \end{bmatrix} \\ &= \begin{bmatrix} \alpha x_1+\alpha y_1 \\ \alpha x_2+\alpha y_2 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} + \begin{bmatrix} \alpha y_1 \\ \alpha y_2 \end{bmatrix} = \alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \alpha\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \alpha\mathbf{x}+\alpha\mathbf{y}. \end{aligned}$$

□

The vectors in the properties (V1)–(V8) are written using letters like $\mathbf{v}$ and $\mathbf{w}$. Particular vector spaces usually already have a common notation for their vectors. For instance, $\mathbf{R}^n$ uses letters like $\mathbf{x}$ and $\mathbf{y}$ for its vectors. We use the common notation when we work with the particular vector space.

8.4 Example: Matrix space

The set $V = \mathbf{M}_{m\times n}$ of $m \times n$ matrices is a vector space with usual matrix addition and scalar multiplication. The $m \times n$ matrix with 0 in every entry satisfies (V3); it is called the **zero matrix** and it is denoted $\mathbf{0}$.

If $\mathbf{A}$ is a general matrix, we write a_{ij} for the entry in its ith row and jth column. For instance, if $\mathbf{A}$ is a 2×3 matrix, then

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

8.4.1 Example Show that $\mathbf{M}_{2\times 3}$ satisfies property (V7).

Solution For $\mathbf{A} \in \mathbf{M}_{2\times 3}$ and $\alpha, \beta \in \mathbf{R}$, we have

$$\begin{aligned}
(\alpha\beta)\mathbf{A} = (\alpha\beta)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} &= \begin{bmatrix} (\alpha\beta)a_{11} & (\alpha\beta)a_{12} & (\alpha\beta)a_{13} \\ (\alpha\beta)a_{21} & (\alpha\beta)a_{22} & (\alpha\beta)a_{23} \end{bmatrix} \\
&= \begin{bmatrix} \alpha(\beta a_{11}) & \alpha(\beta a_{12}) & \alpha(\beta a_{13}) \\ \alpha(\beta a_{21}) & \alpha(\beta a_{22}) & \alpha(\beta a_{23}) \end{bmatrix} = \alpha \begin{bmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \end{bmatrix} \\
&= \alpha\left(\beta \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = \alpha(\beta\mathbf{A}).
\end{aligned}$$

□

As vector spaces, $\mathbf{M}_{2\times 3}$ and $\mathbf{R}^6$ are essentially the same. The only difference is in the way the six entries are displayed (three columns with two entries each versus one column with six entries). More generally, $\mathbf{M}_{m\times n}$ is essentially the same as $\mathbf{R}^{mn}$.

8.5 Example: Polynomial space

We denote by $\mathbf{P}_n$ the set of all polynomials of degree less than n (the "degree" of a polynomial is the highest power of x that appears). For instance, $\mathbf{P}_3$ contains the polynomials

$$4x^2 + 5x - 3, \quad x - 7, \quad 5,$$

and generally any expression of the form $p = ax^2 + bx + c$ with a, b, and c real numbers (possibly zero). The set $\mathbf{P}_n$ has a naturally defined addition and scalar multiplication. For instance, in $\mathbf{P}_3$ we have

$$(4x^2+5x-3)+(3x^2-6x+9) = 7x^2-x+6 \quad \text{and} \quad 2(4x^2+5x-3) = 8x^2+10x-6.$$

With this addition and scalar multiplication the set $V = \mathbf{P}_n$ is a vector space. The role of the zero vector $\mathbf{0}$ is played by the **zero polynomial** 0. The inverse of a polynomial is obtained by distributing the negative sign. For instance,

$$-(4x^2 + 5x - 3) = -4x^2 - 5x + 3.$$

In order to check that $\mathbf{P}_n$ satisfies the vector space properties, we need to know what it means to say that two polynomials are equal. We will say that two polynomials are equal if and only if their corresponding coefficients are equal. For instance,

$$ax^2 + bx + c = a'x^2 + b'x + c'$$

if and only if $a = a'$, $b = b'$, and $c = c'$.

8.5.1 Example Show that $\mathbf{P}_3$ satisfies property (V5).

Solution Let $p = ax^2 + bx + c$ and $q = dx^2 + ex + f$ be polynomials in $\mathbf{P}_3$ and let $\alpha \in \mathbf{R}$. We have

$$\begin{aligned}
\alpha(p+q) &= \alpha((ax^2+bx+c)+(dx^2+ex+f)) = \alpha\big((a+d)x^2+(b+e)x+(c+f)\big) \\
&= \alpha(a+d)x^2+\alpha(b+e)x+\alpha(c+f) = (\alpha a+\alpha d)x^2+(\alpha b+\alpha e)x+(\alpha c+\alpha f) \\
&= \big((\alpha a)x^2+(\alpha b)x+(\alpha c)\big)+\big((\alpha d)x^2+(\alpha e)x+(\alpha f)\big) \\
&= \alpha(ax^2+bx+c)+\alpha(dx^2+ex+f) = \alpha p+\alpha q.
\end{aligned}$$

□

8.6 Example: Function space

We denote by $\mathbf{F}_I$ the set of all real-valued functions on the interval I. For instance, $\mathbf{F}_{[0,1]}$ contains the functions

$$\sin x, \quad \sqrt{1-x}, \quad \frac{1}{x+1},$$

and generally any function that is defined for all x in the interval $[0,1]$. (The function $1/x$ is not contained in $\mathbf{F}_{[0,1]}$ since it is not defined when x is 0.) We allow the possibility $I = \mathbf{R}$, so $\mathbf{F}_\mathbf{R}$ is the set of all functions that are defined for every input x in $\mathbf{R}$.

As usual, letters like f, g, and h, are used to refer to functions in $\mathbf{F}_I$. For instance, we might say "Let f be the function in $\mathbf{F}_{[0,1]}$ given by $f(x) = \sin x$."

If f and g are two functions in $\mathbf{F}_I$, their sum $f + g$ is defined by saying what it does to an input x:

$$(f+g)(x) = f(x) + g(x).$$

For instance, if $f(x) = \sin x$ and $g(x) = \sqrt{1-x}$, then

$$(f+g)(x) = f(x) + g(x) = \sin x + \sqrt{1-x}.$$

If f and g are defined on I, then their sum is as well, so this defines an addition on the set $\mathbf{F}_I$.

If f is a function in $\mathbf{F}_I$ and α is a real number, we define the product αf by

$$(\alpha f)(x) = \alpha f(x).$$

For instance, if $f(x) = \sin x$, then

$$(2f)(x) = 2f(x) = 2\sin x.$$

This defines a scalar multiplication on the set $\mathbf{F}_I$.

With this addition and scalar multiplication, the set $V = \mathbf{F}_I$ is a vector space. The role of the zero vector $\mathbf{0}$ is played by the **zero function**, denoted 0, given by $0(x) = 0$. The inverse of a function f in $\mathbf{F}_I$ is given by $(-f)(x) = -f(x)$. For instance, if $f(x) = \sin x$, then $(-f)(x) = -\sin x$.

Two functions f and g in $\mathbf{F}_I$ are equal if $f(x) = g(x)$ for all x in I. For instance, if f and g are the functions in $\mathbf{F}_\mathbf{R}$ given by $f(x) = \sin^2 x + \cos^2 x$ and $g(x) = 1$, then $f = g$ since

$$f(x) = \sin^2 x + \cos^2 x = 1 = g(x)$$

for all x in $\mathbf{R}$ (due to a trigonometric identity).

8.6.1 Example Show that $\mathbf{F}_I$ satisfies property (V6).

Solution Let $f \in \mathbf{F}_I$ and $\alpha, \beta \in \mathbf{R}$. We need to show that $(\alpha+\beta)f = \alpha f + \beta f$. Each side represents a function, so we need to show that $\big[(\alpha+\beta)f\big](x) = \big[\alpha f + \beta f\big](x)$ for all $x \in I$. For all $x \in I$, we have

$$\big[(\alpha+\beta)f\big](x) = (\alpha+\beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x) = \big[\alpha f + \beta f\big](x).$$

Therefore, $(\alpha+\beta)f = \alpha f + \beta f$. □

8.7 Nonexample

To check that a set with an addition and scalar multiplication is a vector space one has to verify that all of the properties (V1)–(V8) are satisfied.

On the other hand, to show that a set with an addition and scalar multiplication is *not* a vector space it is only necessary to exhibit a single explicit counterexample to one of the properties.

8.7.1 Example Let V be the set of ordered pairs (x, y) of real numbers. Define addition to be usual addition, but define scalar multiplication by the rule $\alpha(x, y) = (x^\alpha, y^\alpha)$. Show that V is not a vector space.

Solution Let $\mathbf{v} = (1, 1)$, $\mathbf{w} = (1, 1)$, and $\alpha = 3$. Then

$$\alpha(\mathbf{v}+\mathbf{w}) = 3\big((1,1)+(1,1)\big) = 3(2,2) = (2^3, 2^3) = (8,8),$$

while

$$\alpha\mathbf{v} + \alpha\mathbf{w} = 3(1,1) + 3(1,1) = (1^3, 1^3) + (1^3, 1^3) = (1,1) + (1,1) = (2,2).$$

Therefore, $\alpha(\mathbf{v}+\mathbf{w}) \neq \alpha\mathbf{v} + \alpha\mathbf{w}$ and property (V5) fails. □

8.8 Elementary theorems

The eight properties in the definition of a vector space are called the vector space axioms. These axioms can be used to prove other properties about vector spaces called theorems. Any theorem that is obtained can be used to prove other theorems. This is the way that the study of vector spaces proceeds.

In this section, we provide a couple of elementary theorems. They show that familiar properties of addition and scalar multiplication in $\mathbf{R}^n$ hold in an arbitrary vector space.

> THEOREM. *Let V be a vector space.*
>
> (i) *There is only one vector $\mathbf{0}$ in V such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$ in V.*
>
> (ii) *For each $\mathbf{v}$ in V, there is only one vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.*

Proof. (i) Suppose that $\mathbf{0}'$ also satisfies $\mathbf{v} + \mathbf{0}' = \mathbf{v}$ for all $\mathbf{v}$ in V. Then

$$\begin{aligned} \mathbf{0}' &= \mathbf{0}' + \mathbf{0} && \text{(V3)} \\ &= \mathbf{0} + \mathbf{0}' && \text{(V1)} \\ &= \mathbf{0} && \text{assumption about } \mathbf{0}' \end{aligned}$$

(ii) Let $\mathbf{v} \in V$ and suppose that $\mathbf{v}' \in V$ also satisfies $\mathbf{v} + (\mathbf{v}') = \mathbf{0}$. Then

$$\begin{aligned} \mathbf{v}' &= \mathbf{v}' + \mathbf{0} && \text{(V3)} \\ &= \mathbf{v}' + (\mathbf{v} + (-\mathbf{v})) && \text{(V4)} \\ &= (\mathbf{v}' + \mathbf{v}) + (-\mathbf{v}) && \text{(V2)} \\ &= \mathbf{0} + (-\mathbf{v}) && \text{(V1) and assumption about } \mathbf{v}' \\ &= -\mathbf{v} && \text{(V1) and (V3)} \end{aligned}$$

□

> THEOREM. *Let V be a vector space and let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.*
>
> (i) *If $\mathbf{v} + \mathbf{u} = \mathbf{w} + \mathbf{u}$, then $\mathbf{v} = \mathbf{w}$.*
>
> (ii) $0\mathbf{v} = \mathbf{0}$.
>
> (iii) *If $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.*
>
> (iv) $(-1)\mathbf{v} = -\mathbf{v}$.

Proof. (i) Assume that $\mathbf{v}+\mathbf{u}=\mathbf{w}+\mathbf{u}$. Then

$$\begin{aligned}
\mathbf{v}+\mathbf{u} &= \mathbf{w}+\mathbf{u} \\
(\mathbf{v}+\mathbf{u})+(-\mathbf{u}) &= (\mathbf{w}+\mathbf{u})+(-\mathbf{u}) && \text{(Add } -\mathbf{u} \text{ to both sides)} \\
\mathbf{v}+\big(\mathbf{u}+(-\mathbf{u})\big) &= \mathbf{w}+\big(\mathbf{u}+(-\mathbf{u})\big) && \text{(V2)} \\
\mathbf{v}+\mathbf{0} &= \mathbf{w}+\mathbf{0} && \text{(V4)} \\
\mathbf{v} &= \mathbf{w} && \text{(V3)}
\end{aligned}$$

(ii) We have

$$\begin{aligned}
(0+0)\mathbf{v} &= 0\mathbf{v} && 0+0=0 \\
0\mathbf{v}+0\mathbf{v} &= \mathbf{0}+0\mathbf{v} && \text{(V6),(V3),(V1)} \\
0\mathbf{v} &= \mathbf{0} && \text{part (i)}
\end{aligned}$$

(iii) Assume that $\mathbf{v}+\mathbf{u}=\mathbf{0}$. By part (ii) of the previous theorem, $-\mathbf{v}$ is the only vector in V such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$. Therefore, $\mathbf{u}=-\mathbf{v}$.

(iv) We have

$$\begin{aligned}
\mathbf{v}+(-1)\mathbf{v} &= 1\mathbf{v}+(-1)\mathbf{v} && \text{(V8)} \\
&= (1+(-1))\mathbf{v} && \text{(V6)} \\
&= 0\mathbf{v} && 1+(-1)=0 \\
&= \mathbf{0} && \text{part (ii)}
\end{aligned}$$

Therefore, by part (iii), we have $(-1)\mathbf{v}=-\mathbf{v}$. □

8.9 Linear function

The main example of a vector space is $\mathbf{R}^n$. Working with $\mathbf{R}^n$ we have introduced the notions of linear combination, linear function, subspace, span, linear independence, basis, coordinate vector, and dimension. These notions involve only addition and scalar multiplication, so they make sense for a general vector space as well (in the definitions we need only replace $\mathbf{R}^n$ with V). For instance, a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s$ in a vector space V is an expression of the form

$$\alpha_1\mathbf{v}_1+\alpha_2\mathbf{v}_2+\cdots+\alpha_s\mathbf{v}_s.$$

This is the same as the definition for $\mathbf{R}^n$.

In this section and the next few, we provide the definitions and some theorems in the general setting and give some examples using the vector spaces $\mathbf{R}^n$, $\mathbf{M}_{m\times n}$, $\mathbf{P}_n$, and $\mathbf{F}_I$.

LINEAR FUNCTION.

Let V and V' be vector spaces. A function $L : V \to V'$ is **linear** if

(i) $L(\mathbf{v} + \mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w})$,

(ii) $L(\alpha\mathbf{v}) = \alpha L(\mathbf{v})$,

for all $\mathbf{v}, \mathbf{w} \in V$, $\alpha \in \mathbf{R}$.

8.9.1 Example Show that the function $L : \mathbf{R}^2 \to \mathbf{P}_2$ given by

$$L(\mathbf{x}) = x_1 x + x_2$$

is linear.

Solution We check the two properties:

(i) For all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$, we have

$$\begin{aligned} L(\mathbf{x} + \mathbf{y}) &= L\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = (x_1 + y_1)x + (x_2 + y_2) = (x_1 x + x_2) + (y_1 x + y_2) \\ &= L(\mathbf{x}) + L(\mathbf{y}). \end{aligned}$$

(ii) For all $\mathbf{x} \in \mathbf{R}^2$ and $\alpha \in \mathbf{R}$, we have

$$L(\alpha\mathbf{x}) = L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = (\alpha x_1)x + (\alpha x_2) = \alpha(x_1 x + x_2) = \alpha L(\mathbf{x}).$$

$\square$

8.9.2 Example Show that the function $L : \mathbf{M}_{2\times 2} \to \mathbf{R}$ given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

is linear. (Here, $\mathbf{R}$ is viewed as the vector space $\mathbf{R}^1$ with $[x]$ written as just x.)

Solution We check the two properties:

(i) For all $\mathbf{A}, \mathbf{B} \in \mathbf{M}_{2\times 2}$, we have

$$\begin{aligned}
L(\mathbf{A}+\mathbf{B}) &= L\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = L\left(\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{22} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}\right) \\
&= (a_{11}+b_{11}) + (a_{22}+b_{22}) = (a_{11}+a_{22}) + (b_{11}+b_{22}) \\
&= L\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) + L\left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) \\
&= L(\mathbf{A}) + L(\mathbf{B}).
\end{aligned}$$

(ii) For all $\mathbf{A} \in \mathbf{M}_{2\times 2}$ and $\alpha \in \mathbf{R}$, we have

$$\begin{aligned}
L(\alpha\mathbf{A}) &= L\left(\alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = L\left(\begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}\right) \\
&= \alpha a_{11} + \alpha a_{22} = \alpha(a_{11}+a_{22}) = \alpha L\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) \\
&= \alpha L(\mathbf{A}).
\end{aligned}$$

□

The next example shows that differentiation can be construed as a linear function.

8.9.3 Example Let $\mathbf{D}_{\mathbf{R}}$ denote the set of differentiable functions on $\mathbf{R}$ (this is a vector space using function addition and scalar multiplication). Define $L : \mathbf{D}_{\mathbf{R}} \to \mathbf{F}_{\mathbf{R}}$ by $L(f)(x) = \frac{d}{dx}[f(x)]$. Show that L is linear.

Solution We check the two properties:

(i) For $f, g \in \mathbf{D}_{\mathbf{R}}$, we have

$$\begin{aligned}
L(f+g)(x) &= \frac{d}{dx}[(f+g)(x)] = \frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \\
&= L(f)(x) + L(g)(x) = [L(f)+L(g)](x)
\end{aligned}$$

for every $x \in \mathbf{R}$, where we have used the derivative sum rule for the third equality. Therefore, $L(f+g) = L(f) + L(g)$.

(ii) For $f \in \mathbf{D}_{\mathbf{R}}$ and $\alpha \in \mathbf{R}$, we have

$$L(\alpha f)(x) = \frac{d}{dx}[(\alpha f)(x)] = \frac{d}{dx}[\alpha f(x)] = \alpha \frac{d}{dx}[f(x)] = \alpha L(f)(x) = [\alpha L(f)](x)$$

for every $x \in \mathbf{R}$, where we have used the derivative constant multiple rule for the third equality. Therefore, $L(\alpha f) = \alpha L(f)$.

Therefore, L is linear. □

Integration can also be construed as a linear function (see Exercise 8–10).

8.10 Subspace

> SUBSPACE.
>
> Let V be a vector space. A subset S of V is called a **subspace** if the following hold:
>
> (i) $\mathbf{0} \in S$,
>
> (ii) $\mathbf{u}, \mathbf{v} \in S$ implies $\mathbf{u} + \mathbf{v} \in S$,
>
> (iii) $\mathbf{v} \in S, \alpha \in \mathbf{R}$ implies $\alpha\mathbf{v} \in S$.

8.10.1 Example Let S be the subset of $\mathbf{M}_{2\times 2}$ consisting of all matrices of the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

Show that S is a subspace of $\mathbf{M}_{2\times 2}$.

Solution We check the three properties of subspace:

(i) We have

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -0 \end{bmatrix} \in S$$

(the $(2,2)$-entry is the negative of the $(1,1)$-entry).

(ii) Let $\mathbf{A}, \mathbf{B} \in S$. (Must show that $\mathbf{A} + \mathbf{B} \in S$.) Since $\mathbf{A}$ and $\mathbf{B}$ are in S, they can be written

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} d & e \\ f & -d \end{bmatrix}.$$

Therefore,

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} + \begin{bmatrix} d & e \\ f & -d \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ c+f & -a-d \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ c+f & -(a+d) \end{bmatrix} \in S$$

(the $(2,2)$-entry is the negative of the $(1,1)$-entry).

(iii) Let $\mathbf{A} \in S$ and $\alpha \in \mathbf{R}$. (Must show that $\alpha\mathbf{A} \in S$.) Since $\mathbf{A}$ is in S, it can be written as in part (ii), so that

$$\alpha\mathbf{A} = \alpha \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & -\alpha a \end{bmatrix} \in S$$

Therefore, S is a subspace of $\mathbf{M}_{2\times 2}$. □

8.10.2 Example Let S be the subset of $\mathbf{F}_{\mathbf{R}}$ consisting of all functions f for which $f(1) = 0$. Show that S is a subspace of $\mathbf{F}_{\mathbf{R}}$.

Solution We check the three properties of subspace:

(i) The role of the zero vector $\mathbf{0}$ is played by the zero function 0 given by $0(x) = 0$. We have $0(1) = 0$, so $0 \in S$.

(ii) Let $f, g \in S$. (Must show that $f + g \in S$.) We have

$$\begin{aligned} (f+g)(1) &= f(1) + g(1) && \text{definition of } f+g \\ &= 0 + 0 && f, g \in S \\ &= 0, \end{aligned}$$

so $f + g \in S$.

(iii) Let $f \in S$ and $\alpha \in \mathbf{R}$. (Must show that $\alpha f \in S$.) We have

$$\begin{aligned} (\alpha f)(1) &= \alpha f(1) && \text{definition of } \alpha f \\ &= \alpha 0 && f \in S \\ &= 0, \end{aligned}$$

so $\alpha f \in S$.

Therefore, S is a subspace of $\mathbf{F}_{\mathbf{R}}$. □

Let S be a subspace of a vector space V. By the closure properties of subspace, S has an addition and a scalar multiplication. In fact, with these operations, S is a vector space in its own right.

THEOREM. *If S is a subspace of a vector space V, then S is a vector space using the same addition and scalar multiplication as that in V.*

Proof. The vector space axioms (V1)–(V8) need to be checked. By the definition of subspace, S contains the zero vector $\mathbf{0}$ of V, which acts as a zero vector for S, so (V3) is satisfied. If $\mathbf{v}$ is a vector in S, then $-\mathbf{v} = (-1)\mathbf{v}$ is in S as well (using the second theorem of Section 8.8 and closure of S under scalar multiplication), and $-\mathbf{v}$ acts as an inverse for $\mathbf{v}$ in S, so (V4) is satisfied. The other axioms hold for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V, so they automatically hold for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in S. Therefore, S is a vector space. □

This theorem applies in particular to the case $V = \mathbf{R}^3$ to show that every line through the origin is a vector space and every plane through the origin is a vector space.

Let $L : V \to V'$ be a linear function. The kernel of L and the image of L are defined just as before:

$$\ker L = \{\mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0}\},$$

$$\operatorname{im} L = \{L(\mathbf{v}) \mid \mathbf{v} \in V\}.$$

Also as before, $\ker L$ is a subspace of V and $\operatorname{im} L$ is a subspace of V'.

8.10.3 Example The function $L : \mathbf{M}_{2\times 2} \to \mathbf{R}$ given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

is linear (see Example 8.9.2).

(a) Find $\ker L$.

(b) Find $\operatorname{im} L$.

Solution

(a) A matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in $\ker L$ if and only if $L(\mathbf{A}) = 0$, that is $a+d = 0$, which is the same as $d = -a$. Therefore,

$$\ker L = \{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbf{R}\}.$$

(In Example 8.10.1, we saw that this last set is a subspace of $\mathbf{M}_{2\times 2}$, so this example illustrates the fact that $\ker L$ is always a subspace.)

(b) From the definition of L, we see that $\operatorname{im} L \subseteq \mathbf{R}$. We claim that in fact $\operatorname{im} L = \mathbf{R}$. We need to show that $\mathbf{R} \subseteq \operatorname{im} L$, that is, we need to show that every element of $\mathbf{R}$ is an output. If $x \in \mathbf{R}$, then

$$x = x - 0 = L\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) \in \operatorname{im} L,$$

so $\mathbf{R} \subseteq \operatorname{im} L$. Therefore, $\operatorname{im} L = \mathbf{R}$.

□

8.11 Basis

> SPAN.
>
> Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ be a set of vectors in a vector space V. The **span of** $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ (denoted $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$) is the set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$. In symbols,
>
> $$\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\} = \{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_s\mathbf{v}_s \mid \alpha_i \in \mathbf{R}\}.$$

The span of a set of vectors in V is a subspace of V (cf. Section 4.4).

8.11.1 Example In the vector space $\mathbf{P}_3$, determine whether $-5x + 3$ is in

$$\mathrm{Span}\{x^2 - 2x + 3,\ x + 4,\ -2x^2 + 1\}.$$

Solution We are wondering whether there are scalars α_1, α_2, and α_3 such that

$$\begin{aligned} -5x + 3 &= \alpha_1(x^2 - 2x + 3) + \alpha_2(x + 4) + \alpha_3(-2x^2 + 1) \\ &= (\alpha_1 - 2\alpha_3)x^2 + (-2\alpha_1 + \alpha_2)x + (3\alpha_1 + 4\alpha_2 + \alpha_3) \end{aligned}$$

Since two polynomials are equal if and only if their corresponding coefficients are equal, we get a system of equations with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ -2 & 1 & 0 & -5 \\ 3 & 4 & 1 & 3 \end{array}\right] \begin{matrix} 2 \; -3 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -4 & -5 \\ 0 & 4 & 7 & 3 \end{array}\right] \begin{matrix} \\ -4 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 23 & 23 \end{array}\right]$$

A solution α_1, α_2, and α_3 exists, so $-5x + 3$ is in the span of the given polynomials. □

LINEAR DEPENDENCE/INDEPENDENCE.

We say that vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ in a vector space V are **linearly dependent** if there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_s$ *not all zero* such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_s\mathbf{v}_s = \mathbf{0}.$$

We say that the vectors are **linearly independent** if they are not linearly dependent, that is, if

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_s\mathbf{v}_s = \mathbf{0} \quad \text{implies} \quad \alpha_i = 0 \text{ for all } i.$$

8.11.2 Example Let $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = \sin 2x$. Determine whether the functions f, g and h are linearly independent in the vector space $\mathbf{F}_{\mathbf{R}}$.

Solution Suppose we have

$$\alpha_1 f + \alpha_2 g + \alpha_3 h = 0. \qquad (*)$$

(Must show that $\alpha_1, \alpha_2, \alpha_3 = 0$.) Each side of this equation is a function, so according to the definition of equality of functions, we have for all $x \in \mathbf{R}$

$$\begin{aligned} [\alpha_1 f + \alpha_2 g + \alpha_3 h](x) &= 0(x) \\ (\alpha_1 f)(x) + (\alpha_2 g)(x) + (\alpha_3 h)(x) &= 0 \\ \alpha_1 f(x) + \alpha_2 g(x) + \alpha_3 h(x) &= 0. \end{aligned}$$

where we have used the definitions of function addition and scalar multiplication. Therefore,

$$\alpha_1 \sin x + \alpha_2 \cos x + \alpha_3 \sin 2x = 0 \qquad (**)$$

for all $x \in \mathbf{R}$. (For future reference, note that the equation involving function names (*) gives rise to an equation involving the output expressions (**), which holds for all x.) Since the above equation holds for all $x \in \mathbf{R}$, it holds in particular for $x = 0$:

$$\begin{aligned} \alpha_1 \sin 0 + \alpha_2 \cos 0 + \alpha_3 \sin 2(0) &= 0 \\ \alpha_1(0) + \alpha_2(1) + \alpha_3(0) &= 0. \end{aligned}$$

Therefore, $\alpha_2 = 0$ and the middle term goes away leaving $\alpha_1 \sin x + \alpha_3 \sin 2x = 0$. Next, setting $x = \pi/2$ gives

$$\begin{aligned} \alpha_1 \sin(\frac{\pi}{2}) + \alpha_3 \sin 2(\frac{\pi}{2}) &= 0 \\ \alpha_1(1) + \alpha_3(0) &= 0, \end{aligned}$$

so that $\alpha_1 = 0$, leaving $\alpha_3 \sin 2x = 0$. Finally, setting $x = \pi/4$ gives

$$\alpha_3 \sin 2(\frac{\pi}{4}) = 0$$
$$\alpha_3(1) = 0,$$

so that $\alpha_3 = 0$.

We have shown that $\alpha_1, \alpha_2, \alpha_3 = 0$. Therefore, the functions are linearly independent. □

> Basis.
>
> Let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s$ be vectors in a vector space V. The set $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s\}$ is a **basis** for V if
>
> (i) $\text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_s\} = V$.
>
> (ii) $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_2$ are linearly independent.

8.11.3 Example Show that $\{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22}\}$ is a basis for $\mathbf{M}_{2\times 2}$, where

$$\mathbf{e}_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{e}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution We check the two properties of basis:

(i) ($\text{Span}\{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22}\} = \mathbf{M}_{2\times 2}$?) Since every linear combination of 2×2 matrices is a 2×2 matrix, the first set is contained in the second set. For the other inclusion, we note that if $\mathbf{A} \in \mathbf{M}_{2\times 2}$, then

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= a_{11}\mathbf{e}_{11} + a_{12}\mathbf{e}_{12} + a_{21}\mathbf{e}_{21} + a_{22}\mathbf{e}_{22} \in \text{Span}\{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22}\}.\end{aligned}$$

Therefore, $\text{Span}\{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22}\} = \mathbf{M}_{2\times 2}$.

(ii) ($\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22}$ linearly independent?) Suppose we have

$$\alpha_1\mathbf{e}_{11} + \alpha_2\mathbf{e}_{12} + \alpha_3\mathbf{e}_{21} + \alpha_4\mathbf{e}_{22} = \mathbf{0}.$$

Then

$$\begin{aligned}\alpha_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Since matrices are equal only if their corresponding entries are equal, we conclude that $\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 0$, so that the matrices are linearly independent.

Therefore $\{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22}\}$ is a basis for $\mathbf{M}_{2\times 2}$. □

More generally, if $\mathbf{e}_{ij}$ denotes the matrix with 1 in the ith row and jth column and zeros elsewhere, then

$$\{\mathbf{e}_{ij} \,|\, 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

is a basis for $\mathbf{M}_{m\times n}$. It is called the **standard basis for $\mathbf{M}_{m\times n}$**. (This coincides with the standard basis for $\mathbf{R}^{mn}$ if we form a single column matrix from an $m\times n$ matrix by writing the columns one after the other.)

8.11.4 Example Show that $\{x^2, x, 1\}$ is a basis for $\mathbf{P}_3$.

Solution We check the two properties of basis:

(i) (Span$\{x^2, x, 1\} = \mathbf{P}_3$?) The span of x^2, x, and 1 consists of all linear combinations of these three polynomials, so all polynomials of the form $\alpha_1 x^2 + \alpha_2 x + \alpha_3 1$. Therefore,

$$\text{Span}\{x^2, x, 1\} = \{\alpha_1 x^2 + \alpha_2 x + \alpha_3 1 \,|\, \alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}\} = \mathbf{P}_3.$$

(ii) ($x^2, x, 1$ linearly independent?) Suppose we have $\alpha_1 x^2 + \alpha_2 x + \alpha_1 1 = 0$. Then

$$\alpha_1 x^2 + \alpha_2 x + \alpha_1 1 = 0x^2 + 0x + 0 \cdot 1,$$

and, since two polynomials are equal only if their corresponding coefficients are equal, we conclude that $\alpha_1, \alpha_2, \alpha_3 = 0$, so that the polynomials are linearly independent.

Therefore, $\{x^2, x, 1\}$ is a basis for $\mathbf{P}_3$. □

More generally, $\{x^{n-1}, x^{n-2}, \ldots, x, 1\}$ is a basis for $\mathbf{P}_n$. It is called the **standard basis for $\mathbf{P}_n$**.

8.11.5 Example Show that $\{2, \cos^2 x\}$ is a basis for the subspace $S =$ Span$\{1, \cos 2x\}$ of $\mathbf{F}_{\mathbf{R}}$. (Hint: $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.)

Solution We have introduced a new notation. For instance, the expression $\cos^2 x$ is to be interpreted as the unnamed function $x \mapsto \cos^2 x$, that is, the function that sends the input x to the output $\cos^2 x$. This keeps us from having to name the function by writing something like "Let f be the function given by $f(x) = \cos^2 x$."

We check the two properties of basis:

(i) (Span$\{2, \cos^2 x\} = S$?) First,

$$2 = 2 \cdot 1 + 0 \cos 2x \quad \text{and} \quad \cos^2 x = \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cos 2x$$

for all $x \in \mathbf{R}$ (where we have used the hint). Therefore 2 and $\cos^2 x$ are in Span$\{1, \cos 2x\} = S$. Now S is a subspace of $\mathbf{F}_{\mathbf{R}}$ (since it is a span of vectors), so every linear combination of 2 and $\cos^2 x$ is also in S, that is, Span$\{2, \cos^2 x\} \subseteq S$.

For the other inclusion, we first note that

$$1 = \tfrac{1}{2} \cdot 2 + 0 \cos^2 x \quad \text{and} \quad \cos 2x = -\tfrac{1}{2} \cdot 2 + 2 \cos^2 x$$

for all $x \in \mathbf{R}$ (again using the hint), so $S = \text{Span}\{1, \cos 2x\} \subseteq \text{Span}\{2, \cos^2 x\}$. Therefore, Span$\{2, \cos^2 x\} = S$.

(ii) ($2, \cos^2 x$ linearly independent?) Suppose that we have $\alpha_1 \cdot 2 + \alpha_2 \cos^2 x = 0$ for all $x \in \mathbf{R}$ (cf. solution to Example 8.11.2). Since the equation holds for all $x \in \mathbf{R}$, it holds in particular for $x = \pi/2$, so

$$\begin{aligned} \alpha_1 2 + \alpha_2 \cos^2 \frac{\pi}{2} &= 0 \\ \alpha_1 2 + \alpha_2(0) &= 0, \end{aligned}$$

Therefore, $\alpha_1 = 0$, leaving $\alpha_2 \cos^2 x = 0$. Letting $x = 0$ then shows that $\alpha_2 = 0$. We have shown that $\alpha_1, \alpha_2 = 0$, so the functions are linearly independent.

Therefore, $\{2, \cos^2 x\}$ is a basis for S. □

The next theorem says that a linear function can be prescribed by just saying where it sends basis vectors.

> THEOREM. *Let V and V' be vector spaces, let $\{\mathbf{b}_1, \ldots, \mathbf{b}_s\}$ be a basis for V, and let $\{\mathbf{c}_1, \ldots, \mathbf{c}_s\}$ be vectors in V'. There is a unique linear function $L : V \to V'$ such that*
>
> $$L(\mathbf{b}_i) = \mathbf{c}_i$$
>
> *for each i.*

Proof. We prove only the special case with $s = 2$ since this shows all of the main ideas. Since $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for V, every vector in V can be written uniquely as a linear combination of $\mathbf{b}_1, \mathbf{b}_2$, so we can define a function $L : V \to V'$ by

$$L(\alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2) = \alpha_1 \mathbf{c}_1 + \alpha_2 \mathbf{c}_s.$$

By letting $\alpha_1 = 1$ and $\alpha_1 = 0$, we get $L(\mathbf{b}_1) = \mathbf{c}_1$. Similarly, $L(\mathbf{b}_2) = \mathbf{c}_2$, so L satisfies the given equation.

We claim that L is linear. Let $\mathbf{v}, \mathbf{w} \in V$. We can write $\mathbf{v} = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2$ and $\mathbf{w} = \beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2$. Then

$$\begin{aligned} L(\mathbf{v} + \mathbf{w}) &= L\big((\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2) + (\beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2)\big) \\ &= L\big((\alpha_1 + \beta_1)\mathbf{b}_1 + (\alpha_2 + \beta_2)\mathbf{b}_2\big) \\ &= (\alpha_1 + \beta_1)\mathbf{c}_1 + (\alpha_2 + \beta_2)\mathbf{c}_2 \\ &= (\alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2) + (\beta_1\mathbf{c}_1 + \beta_2\mathbf{c}_2) \\ &= L(\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2) + L(\beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2) \\ &= L(\mathbf{v}) + L(\mathbf{w}). \end{aligned}$$

Similarly, $L(\alpha\mathbf{v}) = \alpha L(\mathbf{v})$. Therefore, L is linear.

Finally, if $L' : V \to V'$ is a linear function satisfying $L'(\mathbf{b}_1) = \mathbf{c}_1$ and $L'(\mathbf{b}_2) = \mathbf{c}_2$, then

$$\begin{aligned} L'(\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2) &= \alpha_1 L'(\mathbf{b}_1) + \alpha_2 L'(\mathbf{b}_2) \\ &= \alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2 \\ &= L(\alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2) \end{aligned}$$

for all $\alpha_1, \alpha_2 \in \mathbf{R}$, so $L' = L$. This shows that L is unique. □

8.11.6 Example The vector space $\mathbf{P}_3$ has standard basis $\{x^2, x, 1\}$. Let $L : \mathbf{P}_3 \to \mathbf{R}_2$ be the unique linear function satisfying

$$L(x^2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad L(x) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad L(1) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Find $L(6x^2 - 2x + 4)$.

Solution Using the two properties of linear function, we get

$$\begin{aligned} L(6x^2 - 2x + 4) &= L(6x^2 + (-2)x + (4)1) \\ &= 6L(x^2) + (-2)L(x) + 4L(1) \\ &= 6\begin{bmatrix} -1 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} 3 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 16 \end{bmatrix}. \end{aligned}$$

□

8.12 Matrix of a linear function

Coordinate vector.

Let V be a vector space with ordered basis $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$, and let v be a vector in V. The **coordinate vector of** v **relative to** $\mathcal{B}$ is

$$[v]_{\mathcal{B}} = [\alpha_1, \alpha_2, \dots, \alpha_n]^T,$$

where $v = \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \cdots + \alpha_s\mathbf{b}_n$.

8.12.1 Example Find the coordinate vector of $p = 5x^2 + 3x - 2$ relative to the standard ordered basis $\mathcal{B} = (x^2, x, 1)$ of $\mathbf{P}_3$.

Solution Since $p = 5x^2 + 3x + (-2)1$, we have $[p]_{\mathcal{B}} = [5, 3, -2]^T$. □

8.12.2 Example Find the coordinate vector of

$$\mathbf{A} = \begin{bmatrix} 7 & -2 \\ 0 & 3 \end{bmatrix}$$

relative to the ordered basis $\mathcal{B} = (\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22})$ of $\mathbf{M}_{2\times 2}$. (See Example 8.11.3 and following remark for notation.)

Solution Since

$$\mathbf{A} = \begin{bmatrix} 7 & -2 \\ 0 & 3 \end{bmatrix} = 7\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 7\mathbf{e}_{11} + (-2)\mathbf{e}_{12} + 0\mathbf{e}_{21} + 3\mathbf{e}_{22},$$

we have $[\mathbf{A}]_{\mathcal{B}} = [7, -2, 0, 3]^T$. □

8.12.3 Example Let $\mathcal{B} = (2, \cos^2 x)$, an ordered basis for the subspace $S = \operatorname{Span}\{1, \cos 2x\}$ of $\mathbf{F}_{\mathbf{R}}$ (see Example 8.11.5).

(a) Find $[5 + 4\cos 2x]_{\mathcal{B}}$.

(b) Find $f(\pi)$ given that $[f]_{\mathcal{B}} = [-3, 4]^T$.

Solution

(a) We need to write $5 + 4\cos 2x$ as a linear combination of 2 and $\cos^2 x$. Using the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ we get

$$5 + 4\cos 2x = 5 + 4\big(2\cos^2 x - 1\big) = 1 + 8\cos^2 x = (\tfrac{1}{2})2 + 8\cos^2 x.$$

Therefore $[5 + 4\cos 2x]_{\mathcal{B}} = [\frac{1}{2}, 8]^T$.

(b) By the definition of coordinate vector,

$$f = -3 \cdot 2 + 4\cos^2 x = 4\cos^2 x - 6,$$

so $f(\pi) = 4\cos^2 \pi - 6 = 4(-1)^2 - 6 = -2$.

□

MATRIX OF A LINEAR FUNCTION.

Let $L : V \to V'$ be a linear function and let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ and $\mathcal{B}' = (\mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_m)$ be ordered bases of V and V', respectively. There is a unique $m \times n$ matrix $\mathbf{A}$ such that

$$[L(\mathbf{v})]_{\mathcal{B}'} = \mathbf{A}[\mathbf{v}]_{\mathcal{B}},$$

for every vector $\mathbf{v}$ in V. Moreover,

$$\mathbf{A} = \begin{bmatrix} [L(\mathbf{b}_1)]_{\mathcal{B}'} & [L(\mathbf{b}_2)]_{\mathcal{B}'} & \cdots & [L(\mathbf{b}_n)]_{\mathcal{B}'} \end{bmatrix}.$$

The matrix $\mathbf{A}$ is the **matrix of L relative to the bases $\mathcal{B}$ and $\mathcal{B}'$**.

Proof. Both $[L(\mathbf{v})]_{\mathcal{B}'}$ and $\mathbf{A}[\mathbf{v}]_{\mathcal{B}}$ are linear functions of $\mathbf{v}$ as the reader can check. By the last theorem of the previous section, these two functions are equal if they produce the same output for each basis vector $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$. Now $[\mathbf{b}_1]_{\mathcal{B}} = [1, 0, \ldots, 0]^T$, so $\mathbf{A}[\mathbf{b}_1]_{\mathcal{B}}$ is the first column of $\mathbf{A}$, which is $[L(\mathbf{b}_1)]_{\mathcal{B}'}$. Similarly, $\mathbf{A}[\mathbf{b}_i]_{\mathcal{B}} = [L(\mathbf{b}_i)]_{\mathcal{B}'}$ for all i. This proves the theorem. □

If $\mathcal{E}$ is the standard ordered basis for $\mathbf{R}^n$, then $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$ for each $\mathbf{x} \in \mathbf{R}^n$. Therefore, when $V = \mathbf{R}^n$ and $V' = \mathbf{R}^m$, and $\mathcal{B}$ and $\mathcal{B}'$ are the standard ordered bases, there is no need to compute coordinate vectors and we get the earlier theorem (see Section 3.5).

8.12.4 Example Let $L : \mathbf{P}_3 \to \mathbf{P}_2$ be the derivative function:

$$L(p) = p'.$$

(a) Find the matrix $\mathbf{A}$ of L relative to the standard ordered bases $\mathcal{B} = (x^2, x, 1)$ and $\mathcal{B}' = (x, 1)$ of $\mathbf{P}_3$ and $\mathbf{P}_2$, respectively.

(b) Use part (a) to find $L(3x^2 - 2x - 5)$ and check the answer by computing this quantity directly.

Solution

(a) According to the theorem,

$$\mathbf{A} = \begin{bmatrix}[L(x^2)]_{\mathcal{B}'} & [L(x)]_{\mathcal{B}'} & [L(1)]_{\mathcal{B}'}\end{bmatrix}$$

so we can construct $\mathbf{A}$ one column at a time. We have

$$[L(x^2)]_{\mathcal{B}'} = [2x]_{\mathcal{B}'} = [2x + 0 \cdot 1]_{\mathcal{B}'} = \begin{bmatrix}2\\0\end{bmatrix},$$
$$[L(x)]_{\mathcal{B}'} = [1]_{\mathcal{B}'} = [0x + 1 \cdot 1]_{\mathcal{B}'} = \begin{bmatrix}0\\1\end{bmatrix},$$
$$[L(1)]_{\mathcal{B}'} = [0]_{\mathcal{B}'} = [0x + 0 \cdot 1]_{\mathcal{B}'} = \begin{bmatrix}0\\0\end{bmatrix},$$

so

$$\mathbf{A} = \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\end{bmatrix}.$$

(b) The theorem tells us how to get the coordinate vector of $L(3x^2 - 2x - 5)$:

$$[L(3x^2 - 2x - 5)]_{\mathcal{B}'} = \mathbf{A}[3x^2 - 2x - 5]_{\mathcal{B}} = \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}3\\-2\\-5\end{bmatrix} = \begin{bmatrix}6\\-2\end{bmatrix}.$$

Therefore, $L(3x^2 - 2x - 5) = 6x + (-2)1 = 6x - 2$.

Computing directly, we get $L(3x^2 - 2x - 5) = (3x^2 - 2x - 5)' = 6x - 2$, in agreement with what we just found.

□

8.12.5 Example Let S be the subspace of $\mathbf{F}_{\mathbf{R}}$ spanned by the set $\{\sin x, \cos x\}$. Let $L : S \to \mathbf{M}_{2\times 2}$ be the linear function given by

$$L(f) = \begin{bmatrix}f(0) & f(\pi/2)\\f(\pi) & f(3\pi/2)\end{bmatrix}$$

(see Exercise 8–9).

(a) Find the matrix $\mathbf{A}$ of L relative to the ordered bases $\mathcal{B} = (\sin x, \cos x)$ and $\mathcal{B}' = (\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22})$ of S and $\mathbf{M}_{2\times 2}$, respectively.

(b) Use part (a) to find $L(2\cos x - \sin x)$ and check the answer by computing this quantity directly.

Solution (a) According to the theorem,

$$\mathbf{A} = \begin{bmatrix}[L(\sin x)]_{\mathcal{B}'} & [L(\cos x)]_{\mathcal{B}'}\end{bmatrix}$$

so we can construct $\mathbf{A}$ one column at a time. We have

$$\begin{aligned}[L(\sin x)]_{\mathcal{B}'} &= \left[\begin{bmatrix}\sin 0 & \sin\frac{\pi}{2}\\ \sin\pi & \sin\frac{3\pi}{2}\end{bmatrix}\right]_{\mathcal{B}'} = \left[\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}\right]_{\mathcal{B}'}\\ &= [0\mathbf{e}_{11} + 1\mathbf{e}_{12} + 0\mathbf{e}_{21} + (-1)\mathbf{e}_{22}]_{\mathcal{B}'} = \begin{bmatrix}0\\1\\0\\-1\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}[L(\cos x)]_{\mathcal{B}'} &= \left[\begin{bmatrix}\cos 0 & \cos\frac{\pi}{2}\\ \cos\pi & \cos\frac{3\pi}{2}\end{bmatrix}\right]_{\mathcal{B}'} = \left[\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}\right]_{\mathcal{B}'}\\ &= [1\mathbf{e}_{11} + 0\mathbf{e}_{12} + (-1)\mathbf{e}_{21} + 0\mathbf{e}_{22}]_{\mathcal{B}'} = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}\end{aligned}$$

so

$$\mathbf{A} = \begin{bmatrix}0 & 1\\ 1 & 0\\ 0 & -1\\ -1 & 0\end{bmatrix}.$$

(b) The theorem tells us how to get the coordinate vector of $L(2\cos x - \sin x)$:

$$[L(2\cos x - \sin x)]_{\mathcal{B}'} = \mathbf{A}[2\cos x - \sin x]_{\mathcal{B}} = \begin{bmatrix}0 & 1\\ 1 & 0\\ 0 & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}2\\-1\\-2\\1\end{bmatrix}.$$

Therefore, $L(2\cos x - \sin x) = 2\mathbf{e}_{11} + (-1)\mathbf{e}_{12} + (-2)\mathbf{e}_{21} + 1\mathbf{e}_{22} = \begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}$.

Computing directly, we get

$$\begin{aligned}L(2\cos x - \sin x) &= \begin{bmatrix}2\cos 0 - \sin 0 & 2\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\\ 2\cos\pi - \sin\pi & 2\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2}\end{bmatrix}\\ &= \begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}\end{aligned}$$

in agreement with what we just found.

□

8.13 Dimension

We defined the dimension of a subspace S of $\mathbf{R}^n$ to be the number of vectors in any basis for S. This definition made sense only after we proved that any two bases for S must have the same number of vectors. That same proof shows that any two bases of *any* vector space V must have the same number of vectors so we can define the dimension of V to be the number of vectors any any basis of V just as before.

There is a difficulty that arises in the general case that did not arise for subspaces of $\mathbf{R}^n$: a vector space might not have a basis consisting of finitely many vectors (see Example 8.13.2). When this happens, we say that the vector space is "infinite dimensional."

> DIMENSION.
>
> Let V be a vector space. If V has a basis consisting of n vectors, we say that V is **finite dimensional** and has **dimension** n (written $\dim V = n$). If V does not have a basis consisting of finitely many vectors, we say that V is **infinite dimensional**.

8.13.1 Example

(a) Find $\dim \mathbf{M}_{2\times 3}$.

(b) Find $\dim \mathbf{P}_5$.

Solution

(a) The set $\{\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{21}, \mathbf{e}_{22}, \mathbf{e}_{23}\}$ is a basis for $\mathbf{M}_{2\times 3}$ (see Example 8.11.3 and following remark). Therefore, $\dim \mathbf{M}_{2\times 3} = 6$.

(b) The set $\{x^4, x^3, x^2, x, 1\}$ is a basis for $\mathbf{P}_5$ (see Example 8.11.4 and following remark). Therefore, $\dim \mathbf{P}_5 = 5$.

□

8.13.2 Example The set of all polynomials $\mathbf{P}_\infty$ (of all possible degrees) is a vector space using usual addition and scalar multiplication of polynomials. Show that $\mathbf{P}_\infty$ is infinite dimensional.

Solution We need to show that $\mathbf{P}_\infty$ does not have a basis consisting of finitely many polynomials. We can do this by supposing it does have such a basis and deriving a contradiction.

Suppose that $\mathcal{B} = \{p_1, p_2, \dots, p_n\}$ is a basis for $\mathbf{P}_\infty$. Let m be the largest degree of the polynomials $p_1, p_2, \dots, p_n$. Then x^{m+1} is a polynomial in $\mathbf{P}_\infty$ that is not in the span of the polynomials $p_1, p_2, \dots, p_n$ (since any linear combination of these vectors has degree at most m).

This shows that $\text{Span}\{p_1, p_2, \ldots, p_n\}$ is not equal to $\mathbf{P}_\infty$ contradicting our assumption that $\mathcal{B}$ is a basis for $\mathbf{P}_\infty$. We conclude that $\mathbf{P}_\infty$ is infinite dimensional. □

The theorems about dimension stated for subspaces of $\mathbf{R}^n$ (see Section 7.3) have analogs for a general vector space V. We state two of the analogs here and give examples.

THEOREM. *If V is a vector space of dimension n and S is a subspace of V, then $\dim S \leq n$.*

8.13.3 Example Show that $\mathbf{F}_{\mathbf{R}}$ is infinite dimensional.

Solution The vector space $\mathbf{P}_\infty$ of all polynomials can be viewed as a subspace of $\mathbf{F}_{\mathbf{R}}$ if we regard a polynomial as a function on $\mathbf{R}$ (addition and scalar multiplication of polynomials and the definition of equality of polynomials do not change when polynomials are viewed as functions). If $\mathbf{F}_{\mathbf{R}}$ were finite dimensional, then $\mathbf{P}_\infty$ would be finite dimensional as well according to the preceding theorem. But this would contradict Example 8.13.2. Therefore, $\mathbf{F}_{\mathbf{R}}$ is infinite dimensional. □

THEOREM. *If $L : V \to V'$ is a linear function and $\dim V = n$, then*
$$\dim \operatorname{im} L + \dim \ker L = n.$$

8.13.4 Example Verify that the linear function $L : \mathbf{M}_{2\times 2} \to \mathbf{R}$ given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

satisfies the preceding theorem.

Solution In Example 8.10.3 we found that

$$\operatorname{im} L = \mathbf{R}, \qquad \ker L = \{\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in \mathbf{R}\}.$$

We claim that the set $\{\mathbf{e}_{11} - \mathbf{e}_{22}, \mathbf{e}_{12}, \mathbf{e}_{21}\}$ is a basis for $\ker L$.

(i) ($\text{Span}\{\mathbf{e}_{11} - \mathbf{e}_{22}, \mathbf{e}_{12}, \mathbf{e}_{21}\} = \ker L$?) The matrix $\mathbf{e}_{11} - \mathbf{e}_{22}$ is in $\ker L$ ($a = 1$, $b = 0$, $c = 0$) and similarly $\mathbf{e}_{12}$ and $\mathbf{e}_{21}$ are in $\ker L$. Since $\ker L$ is a subspace of $\mathbf{M}_{2\times 2}$ every linear combination of these matrices is in $\ker L$, so $\operatorname{Span}\{\mathbf{e}_{11}-\mathbf{e}_{22}, \mathbf{e}_{12}, \mathbf{e}_{21}\} \subseteq \ker L$. On the other hand,

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = a(\mathbf{e}_{11}-\mathbf{e}_{22}) + b\mathbf{e}_{12} + c\mathbf{e}_{21},$$

so $\ker L \subseteq \operatorname{Span}\{\mathbf{e}_{11}-\mathbf{e}_{22}, \mathbf{e}_{12}, \mathbf{e}_{21}\}$. Therefore, these sets are equal.

(ii) ($\mathbf{e}_{11}-\mathbf{e}_{22}, \mathbf{e}_{12}, \mathbf{e}_{21}$ linearly independent?) Suppose that

$$\alpha_1(\mathbf{e}_{11}-\mathbf{e}_{22}) + \alpha_2\mathbf{e}_{12} + \alpha_3\mathbf{e}_{21} = \mathbf{0}.$$

Then, combining the terms on the left, we get

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & -\alpha_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so that $\alpha_1, \alpha_2, \alpha_3 = 0$. Therefore, $\mathbf{e}_{11}-\mathbf{e}_{22}, \mathbf{e}_{12}, \mathbf{e}_{21}$ are linearly independent.

This establishes the claim that $\{\mathbf{e}_{11}-\mathbf{e}_{22}, \mathbf{e}_{12}, \mathbf{e}_{21}\}$ is a basis for $\ker L$. Therefore, $\dim \ker L = 3$. Also, $\dim \operatorname{im} L = \dim \mathbf{R} = \dim \mathbf{R}^1 = 1$ and $\dim \mathbf{M}_{2\times 2} = 4$ (see Example 8.11.3), so

$$\dim \operatorname{im} L + \dim \ker L = 1 + 3 = 4 = \dim \mathbf{M}_{2\times 2}$$

in agreement with the preceding theorem. □

8 – Exercises

8–1 Show that $\mathbf{M}_{2\times 3}$ satisfies property (V6).

8–2 Show that $\mathbf{P}_3$ satisfies property (V1).

8–3 Show that $\mathbf{F}_I$ satisfies property (V2).

8–4 Let V be the set of positive real numbers. Define a new addition $\oplus$ and scalar multiplication $\odot$ on V by

$$x \oplus y = xy, \qquad \alpha \odot x = x^\alpha.$$

(We have used $\oplus$ and $\odot$ to keep from confusing these new operations with usual addition and multiplication of numbers.) For instance, $2 \oplus 5 = (2)(5) = 10$ and $3 \odot 2 = 2^3 = 8$. Then V is a vector space.

(a) What element of V plays the role of the zero vector $\mathbf{0}$ in property (V3)? Explain.

(b) Given $x \in V$, what element of V plays the role of $-x$ in property (V4)? Explain.

(c) Show that V satisfies property (V5).

8–5 Let V be the set of ordered pairs (x, y) of real numbers. Define addition to be usual addition, but define scalar multiplication by the rule $\alpha(x, y) = (\alpha x, y)$. Show that V is not a vector space.

8–6 Let V be a vector space and let $\mathbf{v} \in V$ and $\alpha \in \mathbf{R}$. Prove each of the following and give a reason for each step.

(a) $\alpha\mathbf{0} = \mathbf{0}$. (Hint: $\mathbf{0} + \mathbf{0} = \mathbf{0}$. See proof of part (ii) in second theorem of 8.8)

(b) If $\alpha\mathbf{v} = \mathbf{0}$ and $\alpha \neq 0$, then $\mathbf{v} = \mathbf{0}$. (Hint: If $\alpha \neq 0$, then α^{-1} exists. At some point, use part (a).)

8–7 Let V be a vector space and let $\mathbf{v} \in V$ and $\alpha \in \mathbf{R}$. Prove that $(-\alpha)\mathbf{v} = -(\alpha\mathbf{v})$ and give a reason for each step.

Hint: See proof of part (iv) in second theorem of 8.8.

8–8 Show that the function $L : \mathbf{P}_2 \to \mathbf{M}_{2\times 2}$ given by

$$L(ax+b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

is linear.

8–9 Show that the function $L : \mathbf{F}_{\mathbf{R}} \to \mathbf{M}_{2\times 2}$ given by

$$L(f) = \begin{bmatrix} f(0) & f(\pi/2) \\ f(\pi) & f(3\pi/2) \end{bmatrix}$$

is linear.

8–10 Let $\mathbf{C}_{\mathbf{R}}$ denote the set of continuous functions on $\mathbf{R}$ (this is a vector space using function addition and scalar multiplication). Define $L : \mathbf{C}_{\mathbf{R}} \to \mathbf{F}_{\mathbf{R}}$ by

$$L(f)(x) = \int_0^x f(t)\,dt.$$

Show that L is linear.

8–11 Let S be the subset of $\mathbf{F}_{\mathbf{R}}$ consisting of all functions f for which $f(-x) = f(x)$ for all $x \in \mathbf{R}$. Show that S is a subspace of $\mathbf{F}_{\mathbf{R}}$.

8–12 Let $L : \mathbf{M}_{2\times 2} \to \mathbf{P}_2$ be the linear function given by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-d)x + (b+c).$$

(a) Find $\ker L$.

(b) Find $\operatorname{im} L$.

8–13 Let $L : V \to V'$ be a linear function, let $\mathbf{a} \in V'$ and let $\mathbf{v}_p$ be a particular vector in V such that $L(\mathbf{v}_p) = \mathbf{a}$.

(a) Show that if $\mathbf{v}_0$ is in $\ker L$, then $L(\mathbf{v}_p + \mathbf{v}_0) = \mathbf{a}$.

(b) Show that if $\mathbf{v}$ is a vector in V such that $L(\mathbf{v}) = \mathbf{a}$, then $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_0$ for some $\mathbf{v}_0 \in \ker L$. (Hint: If you show that $\mathbf{v} - \mathbf{v}_p \in \ker L$, then you can conclude that $\mathbf{v} - \mathbf{v}_p = \mathbf{v}_0$ for some $\mathbf{v}_0 \in \ker L$.)

(c) Recall that $L^{-1}(\mathbf{a})$ is the set of all $\mathbf{v}$ in V such that $L(\mathbf{v}) = \mathbf{a}$. Show that $L^{-1}(\mathbf{a}) = \{\mathbf{v}_p + \mathbf{v}_0 \mid \mathbf{v}_0 \in \ker L\}$. (Hint: Use parts (a) and (b) to show that each set is contained in the other.)

8–14 Let $L : \mathbf{D}_{\mathbf{R}} \to \mathbf{F}_R$ be the function given by

$$L(y) = y' + 3y$$

($\mathbf{D}_{\mathbf{R}}$ is the vector space of differentiable functions).

(a) Show that L is linear.

(b) Let y_p be a particular solution to the linear differential equation $y' + 3y = 1$. Show that every solution of $y' + 3y = 1$ is of the form $y = y_p + y_0$ where y_0 is a solution to the corresponding homogeneous equation $y' + 3y = 0$. (Hint: Use part (a) and Exercise 8–13(b).)

8–15 In the vector space $\mathbf{M}_{2\times 2}$, determine whether $\begin{bmatrix} 5 & 6 \\ 1 & -5 \end{bmatrix}$ is in

$$\operatorname{Span}\{\begin{bmatrix} 1 & -3 \\ -4 & -1 \end{bmatrix}, \begin{bmatrix} -2 & 6 \\ 8 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}\}.$$

8–16 Let

$$f(x) = 1 + \frac{3}{x}, \quad g(x) = 2 - \frac{4}{x}, \quad h(x) = 1 + \frac{8}{x}.$$

Determine whether the functions f, g, and h are linearly independent in the vector space $\mathbf{F}_{\mathbf{R}^+}$, where $\mathbf{R}^+$ is the set of positive real numbers.

Hint: At some point, make three choices for x to get a system of three equations.

8–17 Show that $\{x^2 - 1, 2x^2 + x\}$ is a basis for the subspace $S = \operatorname{Span}\{-x - 2, x^2 + x + 1\}$ of $\mathbf{P}_3$.

8–18 The set $\{2, \cos^2 x\}$ is a basis for the subspace $S = \text{Span}\{1, \cos 2x\}$ of $\mathbf{F_R}$ (see Example 8.11.5). Let $L : S \to \mathbf{R}$ be the unique linear function satisfying

$$L(2) = 3 \quad \text{and} \quad L(\cos^2 x) = 1.$$

Find $L(4\cos 2x - 6)$.

Hint: $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

8–19 Let $\mathcal{B} = (x^2 - 1, 2x^2 + x)$, an ordered basis for the subspace $S = \text{Span}\{-x - 2, x^2 + x + 1\}$ of $\mathbf{P}_3$ (see Exercise 8–17).

(a) Find $[4x^2 + 3x + 2]_{\mathcal{B}}$ (if defined).

(b) Find p given that $[p]_{\mathcal{B}} = [5, -1]^T$.

8–20 Let $L : \mathbf{P}_2 \to \mathbf{M}_{2\times 2}$ be the linear function given by

$$L(ax + b) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

(see Exercise 8–8).

(a) Find the matrix $\mathbf{A}$ of L relative to the standard ordered bases $\mathcal{B} = (x, 1)$ and $\mathcal{B}' = (\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}, \mathbf{e}_{22})$ of $\mathbf{P}_2$ and $\mathbf{M}_{2\times 2}$, respectively.

(b) Use part (a) to find $L(4x - 3)$ and check the answer by computing this quantity directly.

8–21 Let $L : \mathbf{R}_2 \to \mathbf{R}_2$ be the linear function given by

$$L(\mathbf{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

(projection onto the x_1-axis).

(a) Find the matrix $\mathbf{A}$ of L relative to the ordered basis $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ of $\mathbf{R}^2$, where $\mathbf{b}_1 = [1, 1]^T$ and $\mathbf{b}_2 = [-1, 1]^T$ (with $\mathcal{B}'$ this same ordered basis).

(b) Use part (a) to find $L([2, 2]^T)$ and check the answer by computing this quantity directly.

8–22 Verify that the linear function $L : \mathbf{P}_3 \to \mathbf{P}_2$ given by

$$L(p) = p'$$

satisfies the theorem before Example 8.13.4.