9 Inner product

9.1 Dot product

In calculus, the "dot product" of two vectors $\vec{x} = \langle 2, -3\rangle$ and $\vec{y} = \langle 5, 1\rangle$ is

$$\vec{x} \cdot \vec{y} = \langle 2, -3\rangle \cdot \langle 5, 1\rangle = (2)(5) + (-3)(1) = 7$$

(multiply corresponding entries and add). In linear algebra we write these same vectors as

$$\mathbf{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 5 \\ 1 \end{bmatrix},$$

and express the dot product as

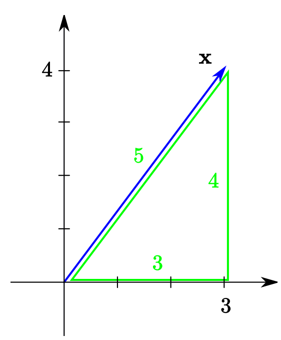
$$\mathbf{x}^T\mathbf{y} = \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix} \quad \text{(or just 7)}$$

(so $\vec{x} \cdot \vec{y}$ becomes $\mathbf{x}^T\mathbf{y}$).

Length

The length of a vector $\mathbf{x}$ is denoted $\|\mathbf{x}\|$. A formula for this length comes from the Pythagorean theorem. For instance, if $\mathbf{x} = [3, 4]^T$, then

$$\begin{aligned} \|\mathbf{x}\| &= \sqrt{3^2 + 4^2} \\ &= \sqrt{25} \\ &= 5. \end{aligned}$$



Note that

$$\mathbf{x}^T\mathbf{x} = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3^2 + 4^2,$$

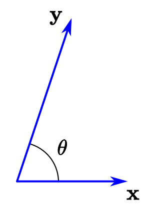
which is what appears under the square root. In general we have

$$\boxed{\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}}$$

Angle

The angle θ between two vectors $\mathbf{x}$ and $\mathbf{y}$ is related to the dot product by the formula

$$\boxed{\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta}$$



9.1.1 Example Find the angle between $\mathbf{x} = [2, -3]^T$ and $\mathbf{y} = [3, 2]^T$.

Solution We solve the equation above to get

$$\begin{aligned}\cos\theta &= \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \\ &= \frac{\begin{bmatrix}2 & -3\end{bmatrix}\begin{bmatrix}3 \\ 2\end{bmatrix}}{\sqrt{2^2+(-3)^2}\sqrt{3^2+2^2}} \\ &= \frac{0}{13} = 0,\end{aligned}$$

so $\theta = \cos^{-1} 0 = 90°$. □

This example shows that

$$\boxed{\mathbf{x} \perp \mathbf{y} \quad \iff \quad \mathbf{x}^T\mathbf{y} = 0}$$

($\mathbf{x} \perp \mathbf{y}$ is read "$\mathbf{x}$ is orthogonal (or perpendicular) to $\mathbf{y}$").

9.2 Definition

We have seen that in $\mathbf{R}^2$ the length of a vector and the angle between two vectors can be expressed using the dot product. So in a sense the dot product is what gives rise to the geometry of vectors. It is certain properties of the dot product that make this work.

The generalization of the dot product to an arbitrary vector space is called an "inner product." Just like the dot product, this is a certain way of putting two vectors together to get a number. The properties it satisfies are enough to get a geometry that behaves much like the geometry of $\mathbf{R}^2$ (for instance, the Pythagorean theorem holds).

INNER PRODUCT. Let V be a vector space. An **inner product** on V is a rule that assigns to each pair $\mathbf{v}, \mathbf{w} \in V$ a real number $\langle \mathbf{v}, \mathbf{w} \rangle$ such that, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbf{R}$,

(i) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$,

(ii) $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$,

(iii) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$,

(iv) $\langle \alpha \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{v}, \mathbf{w} \rangle$.

Note that, combining (iii) and (iv) with (ii), we get the properties

$$\begin{aligned} \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle, \\ \langle \mathbf{v}, \alpha \mathbf{w} \rangle &= \alpha \langle \mathbf{v}, \mathbf{w} \rangle. \end{aligned}$$

An **inner product space** is a vector space with an inner product. Each of the vector spaces $\mathbf{R}^n$, $\mathbf{M}_{m \times n}$, $\mathbf{P}_n$, and $\mathbf{F}_I$ is an inner product space:

9.3 Example: Euclidean space

We get an inner product on $\mathbf{R}^n$ by defining, for $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}.$$

To verify that this is an inner product, one needs to show that all four properties hold. We check only two of them here.

(i) We have
$$\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^T \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0,$$
with equality if and only if $x_i = 0$ for all i, that is, $\mathbf{x} = \mathbf{0}$.

(iii) We have
$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = (\mathbf{x} + \mathbf{y})^T \mathbf{z} = (\mathbf{x}^T + \mathbf{y}^T) \mathbf{z} = \mathbf{x}^T \mathbf{z} + \mathbf{y}^T \mathbf{z} = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle,$$
where we have used that matrix multiplication distributes over addition.

9.4 Example: Matrix space

We get an inner product on $\mathbf{M}_{m\times n}$ by defining, for $\mathbf{A}, \mathbf{B} \in \mathbf{M}_{m\times n}$,

$$\langle \mathbf{A}, \mathbf{B} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$$

(multiply corresponding entries and add). For instance,

$$\begin{aligned}\left\langle \begin{bmatrix} 2 & -1 & 3 \\ 5 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -2 \end{bmatrix} \right\rangle &= (2)(1) + (-1)(3) + (3)(8) + (5)(0) + (0)(1) + (4)(-2) \\ &= 15.\end{aligned}$$

This inner product is identical to the dot product on $\mathbf{R}^{mn}$ if an $m \times n$ matrix is viewed as an $mn \times 1$ matrix by stacking its columns.

9.5 Example: Polynomial space

Let $x_1, x_2, \ldots, x_n$ be fixed numbers. We get an inner product on $\mathbf{P}_n$ by defining, for $p, q \in \mathbf{P}_n$,

$$\begin{aligned}\langle p, q \rangle &= \sum_{i=1}^{n} p(x_i) q(x_i) \\ &= p(x_1)q(x_1) + p(x_2)q(x_2) + \cdots p(x_n)q(x_n).\end{aligned}$$

For instance, if $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$, then for $p = x^2$ and $q = x + 1$, we get

$$\begin{aligned}\langle p, q \rangle &= p(-1)q(-1) + p(0)q(0) + p(1)q(1) \\ &= (1)(0) + (0)(1) + (1)(2) \\ &= 2.\end{aligned}$$

Different choices of the numbers $x_1, x_2, \ldots, x_n$ produce different inner products.

9.6 Example: Function space

We get an inner product on $\mathbf{C}_{[a,b]}$ (= vector space of continuous functions on the interval $[a, b]$) by defining, for $f, g \in \mathbf{C}_{[a,b]}$,

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx.$$

We check the first property of inner product:

(i) We have

$$\langle f, f\rangle = \int_a^b f(x)f(x)\,dx = \int_a^b [f(x)]^2\,dx.$$

This last integral gives the (signed) area between the graph of $y = [f(x)]^2$ and the x-axis from $x = a$ to $x = b$. Since $[f(x)]^2$ does not drop below the x-axis, the integral is ≥ 0 with equality if and only if $f(x) = 0$ for all x, that is, f is the zero function.

9.6.1 Example Find $\langle \sin x, \cos x\rangle$ using $a = -\pi$ and $b = \pi$.

Solution We have

$$\begin{aligned}\langle \sin x, \cos x\rangle &= \int_{-\pi}^{\pi} \sin x \cos x\,dx \\ &= \int_0^0 u\,du \qquad (u = \sin x,\ du = \cos x\,dx) \\ &= 0\end{aligned}$$

□

Since the inner product generalizes the dot product, it is reasonable to say that two vectors are "orthogonal" (or "perpendicular") if their inner product is zero. With this definition, we see from the preceding example that $\sin x$ and $\cos x$ are orthogonal (on the interval $[-\pi, \pi]$).

9.7 Geometry

Let V be an inner product space and let $\mathbf{v} \in V$. The **norm** (or **length**) of $\mathbf{v}$ is denoted $\|\mathbf{v}\|$ and is defined by

$$\boxed{\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}}$$

(the square root is defined by property (i) of inner product).

9.7.1 Example With the inner product on $\mathbf{M}_{2\times 2}$ defined as in Section 9.4 find $\|\mathbf{A}\|$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}.$$

Solution We have

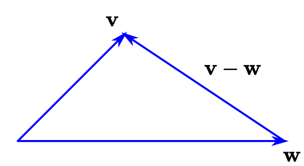
$$\|\mathbf{A}\| = \sqrt{\langle \mathbf{A}, \mathbf{A} \rangle} = \sqrt{\left\langle \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \right\rangle}$$
$$= \sqrt{(1)(1) + (2)(2) + (-1)(-1) + (4)(4)} = \sqrt{22}.$$

□

In the preceding example, $\|\mathbf{A}\|$ is called the "Frobenius norm" of the matrix $\mathbf{A}$.

The **distance** between two vectors in V is the norm of their difference:

$$\text{dist}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$$



9.7.2 Example Of the functions x and x^3, which is closer to x^2 on the interval $[0, 1]$ (using the inner product of Section 9.6)?

Solution We compute the two distances:

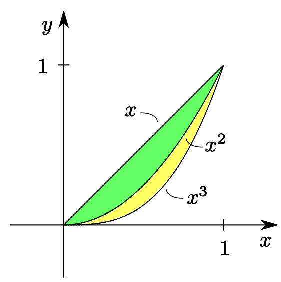
$$\text{dist}(x, x^2) = \|x - x^2\| = \sqrt{\langle x - x^2, x - x^2 \rangle}$$
$$= \sqrt{\int_0^1 (x - x^2)(x - x^2)\, dx} = \sqrt{\int_0^1 (x^2 - 2x^3 + x^4)\, dx}$$
$$= \sqrt{\left(\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right) \Big|_0^1}$$
$$= \frac{1}{\sqrt{30}}$$

and

$$\begin{aligned}\operatorname{dist}(x^3, x^2) &= \|x^3 - x^2\| = \sqrt{\langle x^3 - x^2, x^3 - x^2\rangle} \\ &= \sqrt{\int_0^1 (x^3 - x^2)(x^3 - x^2)\, dx} = \sqrt{\int_0^1 (x^6 - 2x^5 + x^4)\, dx} \\ &= \sqrt{\left(\frac{x^7}{7} - \frac{2x^6}{6} + \frac{x^5}{5}\right)\Bigg|_0^1} \\ &= \frac{1}{\sqrt{105}}.\end{aligned}$$

Therefore, x^3 is closer to x^2.

The distance between functions is a measure of the space between the graphs of the functions (although it is not the exact area). One can see that the space between x^3 and x^2 (yellow) is less than the space between x and x^2 (green).



□

In order to define the angle between two vectors, we need a theorem:

> Cauchy-Schwarz theorem.
>
> For all $\mathbf{v}, \mathbf{w} \in V$,
> $$|\langle \mathbf{v}, \mathbf{w}\rangle| \leq \|\mathbf{v}\|\|\mathbf{w}\|.$$

Proof. If $\mathbf{w} = \mathbf{0}$, then both sides are zero and the inequality holds. Assume that $\mathbf{w} \neq \mathbf{0}$ and put

$$\alpha = \frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\langle \mathbf{w}, \mathbf{w}\rangle}.$$

Using the properties of inner product we get

$$\begin{aligned} 0 \le \|\mathbf{v} - \alpha\mathbf{w}\|^2 &= \langle \mathbf{v} - \alpha\mathbf{w}, \mathbf{v} - \alpha\mathbf{w}\rangle \\ &= \langle \mathbf{v}, \mathbf{v}\rangle - \langle \mathbf{v}, \alpha\mathbf{w}\rangle - \langle \alpha\mathbf{w}, \mathbf{v}\rangle + \langle \alpha\mathbf{w}, \alpha\mathbf{w}\rangle && \text{(iii) and (iv)} \\ &= \langle \mathbf{v}, \mathbf{v}\rangle - 2\alpha\langle \mathbf{v}, \mathbf{w}\rangle + \alpha^2\langle \mathbf{w}, \mathbf{w}\rangle && \text{(ii) and (iv)} \\ &= \langle \mathbf{v}, \mathbf{v}\rangle - 2\frac{\langle \mathbf{v}, \mathbf{w}\rangle^2}{\langle \mathbf{w}, \mathbf{w}\rangle} + \frac{\langle \mathbf{v}, \mathbf{w}\rangle^2}{\langle \mathbf{w}, \mathbf{w}\rangle} \\ &= \langle \mathbf{v}, \mathbf{v}\rangle - \frac{\langle \mathbf{v}, \mathbf{w}\rangle^2}{\langle \mathbf{w}, \mathbf{w}\rangle}, \end{aligned}$$

so

$$\langle \mathbf{v}, \mathbf{w}\rangle^2 \le \langle \mathbf{v}, \mathbf{v}\rangle\langle \mathbf{w}, \mathbf{w}\rangle$$

and taking the square root of both sides gives

$$|\langle \mathbf{v}, \mathbf{w}\rangle| \le \|\mathbf{v}\|\|\mathbf{w}\|.$$

□

If $\mathbf{v}, \mathbf{w} \neq \mathbf{0}$, then the theorem implies that

$$-1 \le \frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\|\mathbf{w}\|} \le 1,$$

so we can define the **angle** θ between $\mathbf{v}$ and $\mathbf{w}$ by

$$\boxed{\theta = \cos^{-1}\frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}}$$

(This gives $\langle \mathbf{v}, \mathbf{w}\rangle = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$ which generalizes the dot product formula in Section 9.1.) Since $\theta = 90^\circ$ if and only if $\langle \mathbf{v}, \mathbf{w}\rangle = 0$, we say that $\mathbf{v}$ is **orthogonal** (or **perpendicular**) to $\mathbf{w}$ if and only if $\langle \mathbf{v}, \mathbf{w}\rangle = 0$:

$$\boxed{\mathbf{v} \perp \mathbf{w} \quad \Longleftrightarrow \quad \langle \mathbf{v}, \mathbf{w}\rangle = 0}$$

9.7.3 Example Let $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$ in the definition of the inner product on $\mathbf{P}_3$ given in Section 9.5. Find the angle θ between $p = x^2$ and $q = x + 1$.

Solution In Section 9.5 we found that $\langle p, q\rangle = 2$. We also have

$$\begin{aligned} \|p\| &= \sqrt{\langle p, p\rangle} = \sqrt{p(-1)^2 + p(0)^2 + p(1)^2} = \sqrt{1 + 0 + 1} = \sqrt{2}, \\ \|q\| &= \sqrt{\langle q, q\rangle} = \sqrt{q(-1)^2 + q(0)^2 + q(1)^2} = \sqrt{0 + 1 + 4} = \sqrt{5}, \end{aligned}$$

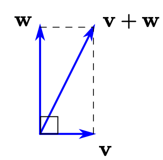
so

$$\theta = \cos^{-1}\frac{\langle p,q\rangle}{\|p\|\|q\|} = \cos^{-1}\frac{2}{\sqrt{2}\sqrt{5}} = \cos^{-1}\frac{2}{\sqrt{10}} \approx 51^\circ$$

□

Pythagorean theorem.

Let $\mathbf{v},\mathbf{w}\in V$. If $\mathbf{v}\perp\mathbf{w}$, then $\|\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.



Proof. Assume that $\mathbf{v}\perp\mathbf{w}$. Using a property of inner product, we get

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\|^2 &= \langle \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}\rangle \\ &= \langle \mathbf{v},\mathbf{v}\rangle + \langle \mathbf{v},\mathbf{w}\rangle + \langle \mathbf{w},\mathbf{v}\rangle + \langle \mathbf{w},\mathbf{w}\rangle && \text{(iii)} \\ &= \langle \mathbf{v},\mathbf{v}\rangle + 0 + 0 + \langle \mathbf{w},\mathbf{w}\rangle && \mathbf{v}\perp\mathbf{w} \\ &= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.\end{aligned}$$

□

9.7.4 Example Verify the Pythagorean theorem for $\sin x, \cos x \in \mathbf{C}_{[-\pi,\pi]}$.

Solution In Example 9.6.1 we found that $\langle \sin x, \cos x\rangle = 0$ so $\sin x \perp \cos x$ and the Pythagorean theorem applies. We have

$$\|\sin x\|^2 = \langle \sin x, \sin x\rangle = \int_{-\pi}^{\pi} \sin^2 x\,dx.$$

Now, using the identity $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ we get

$$\begin{aligned}\int_{-\pi}^{\pi}\sin^2 x\,dx &= \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2x)\,dx \\ &= \frac{1}{4}\int_{-2\pi}^{2\pi}(1-\cos u)\,du && (u=2x,\ du=2dx) \\ &= \frac{1}{4}(u-\sin u)\Big|_{-2\pi}^{2\pi} \\ &= \pi,\end{aligned}$$

so $\|\sin x\|^2 = \pi$. Similarly, using the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ we get $\|\cos x\|^2 = \pi$. On the other hand,

$$\begin{aligned}
\|\sin x + \cos x\|^2 = \langle \sin x + \cos x, \sin x + \cos x\rangle &= \int_{-\pi}^{\pi} (\sin x + \cos x)^2\, dx \\
&= \int_{-\pi}^{\pi} \sin^2 x\, dx + 2\int_{-\pi}^{\pi} \sin x \cos x\, dx + \int_{-\pi}^{\pi} \cos^2 x\, dx \\
&= \pi + 0 + \pi = 2\pi,
\end{aligned}$$

where we have used Example 9.6.1 to see that the second integral is 0. Therefore,

$$\|\sin x\|^2 + \|\cos x\|^2 = \pi + \pi = 2\pi = \|\sin x + \cos x\|^2$$

so the Pythagorean theorem is verified. □

9 – Exercises

9–1 Find the angle between $\mathbf{x} = [1, 2, -1, 2]^T$ and $\mathbf{y} = [2, -1, 1, 3]^T$ using the inner product of Section 9.3.

9–2 Show that $x \perp x^2$ in $\mathbf{P}_5$ using the inner product of Section 9.5 with $x_1 = -2$, $x_2 = -1$, $x_3 = 0$, $x_4 = 1$, and $x_5 = 2$.

9–3 Verify the Pythagorean theorem for the functions 1 and x in $\mathbf{C}_{[-1,1]}$ using the inner product of Section 9.6. (The theorem requires that $1 \perp x$ so this needs to be checked first.)