Two Binomial Coefficient Analogues

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Outline

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1. Find a recursion for the a_n and use induction.
2. Find a combinatorial interpretation for the a_n. In other words, find sets $S_0, S_1, S_2, \ldots$ such that, for all n,

   $$a_n = \#S_n$$

   where $\#$ denotes cardinality.

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Since the sum of two integers is an integer, induction on n gives:

Corollary

For all $0 \leq k \leq n$ *we have* $\binom{n}{k} \in \mathbb{N}$.

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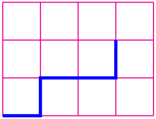

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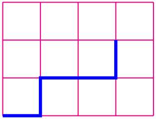
$$P = ENEEN =$$ 

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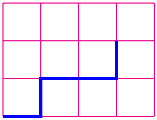

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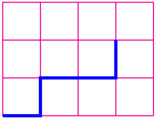

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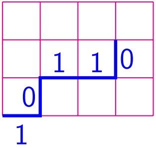

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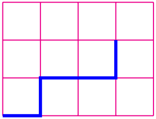

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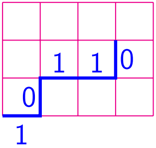

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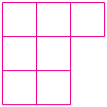
$\lambda = (3, 2, 2) =$ 

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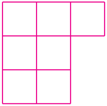
$\lambda = (3, 2, 2) =$ 

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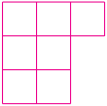
$\lambda = (3, 2, 2) =$  $\subseteq 3 \times 4$:

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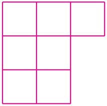
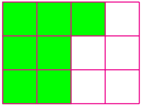
$\lambda = (3, 2, 2) =$  $\subseteq 3 \times 4$: 

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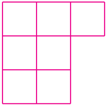
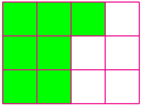
$\lambda = (3, 2, 2) =$  $\subseteq 3 \times 4$: 

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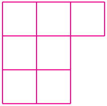
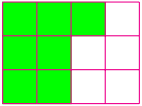
$\lambda = (3, 2, 2) =$  $\subseteq 3 \times 4$: 

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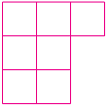
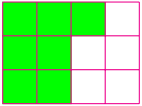
$\lambda = (3, 2, 2) =$  $\subseteq 3 \times 4$: 

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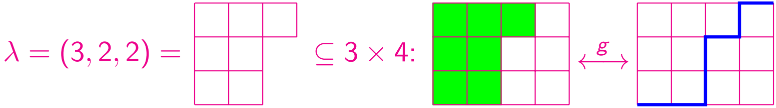

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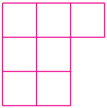
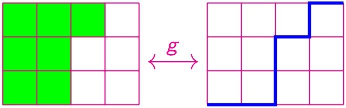
$\lambda = (3, 2, 2) =$  $\subseteq 3 \times 4$: 

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There is a bijection $g : \mathcal{L}_{n,k} \to \mathcal{P}_{n,k}$: given $\lambda \subseteq k \times (n-k)$, $P = g(\lambda)$ is formed by going from the SW corner of the rectangle to the NE corner along the rectangle and the SE boundary of λ.

Proposition

For $0 \leq k \leq n$ we have $\binom{n}{k} = \#\mathcal{L}_{n,k}$.

Outline

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Note that it is not clear from the definition that $\begin{bmatrix} n \\ k \end{bmatrix}$ is always in $\mathbb{N}[q]$, the set of polynomials in q with coefficients in $\mathbb{N}$.

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The q-binomial coefficients satisfy $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1$ *and, for* $0 < k < n$,

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Since sums and products of elements of $\mathbb{N}[q]$ are again in $\mathbb{N}[q]$, we immediately get the following result.

Corollary

For all $0 \leq k \leq n$ we have $\begin{bmatrix} n \\ k \end{bmatrix} \in \mathbb{N}[q]$.

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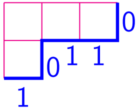

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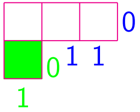
$w = 10110 \overset{h}{\longleftrightarrow} \lambda =$ 

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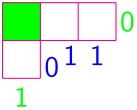

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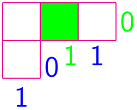
Composing $g : \mathcal{L}_{n,k} \to \mathcal{P}_{n,k}$ and $f : \mathcal{P}_{n,k} \to \mathcal{W}_{n,k}$ gives a bijection $h = f \circ g : \mathcal{L}_{n,k} \to \mathcal{W}_{n,k}$ such that, if $h(\lambda) = w$ then $|\lambda| = \operatorname{inv} w$. In fact, squares of λ correspond bijectively to elements of $\operatorname{Inv} w$.

Ex. When $n = 5$ and $k = 2$

$$w = 10110 \overset{h}{\longleftrightarrow} \lambda =$$



(c) Partitions.

If $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition then its *size* is

$$|\lambda| = \lambda_1 + \cdots + \lambda_m.$$

Ex. If $\lambda = (4, 3, 3, 2)$ then $|\lambda| = 4 + 3 + 3 + 2 = 12$.

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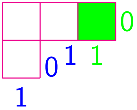

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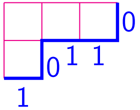

Theorem

For all $0 \le k \le n$ we have $\left[{n \atop k}\right] = S_{n,k}(q)$.

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Theorem (Knuth, 1971)

For all $0 \le k \le n$ and q a prime power we have $\left[{n \atop k} \right] = \# V_{n,k}(q)$.

Outline

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Corollary

For all $0 \leq k \leq n$ *we have* $\binom{n}{k}_F \in \mathbb{N}$.

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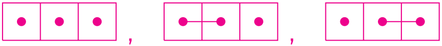
$\mathcal{T}_3 = \{$  $\}$

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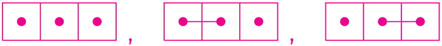

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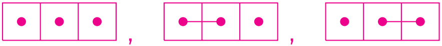

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Proposition

For all $n \geq 1$ we have $F_n = \#\mathcal{T}_{n-1}$.

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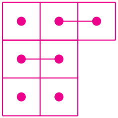
Ex. If $\lambda = (3, 2, 2)$ then  $\in \mathcal{T}_{(3,2,2)}$,

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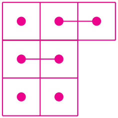

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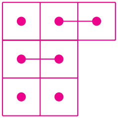
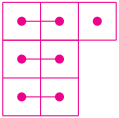
Ex. If $\lambda = (3, 2, 2)$ then  $\in \mathcal{T}_{(3,2,2)}$,  $\in \mathcal{D}_{(3,2,2)}$.

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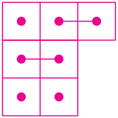
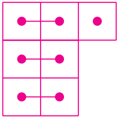
Ex. If $\lambda = (3, 2, 2)$ then  $\in \mathcal{T}_{(3,2,2)}$,  $\in \mathcal{D}_{(3,2,2)}$.

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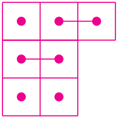
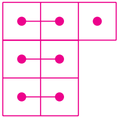
Ex. If $\lambda = (3, 2, 2)$ then  $\in \mathcal{T}_{(3,2,2)}$,  $\in \mathcal{D}_{(3,2,2)}$.

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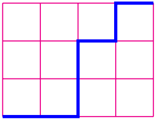

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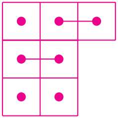
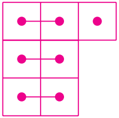
Ex. If $\lambda = (3, 2, 2)$ then  $\in \mathcal{T}_{(3,2,2)}$,  $\in \mathcal{D}_{(3,2,2)}$.

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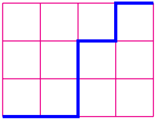

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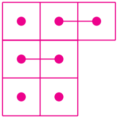
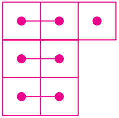
Ex. If $\lambda = (3, 2, 2)$ then  $\in \mathcal{T}_{(3,2,2)}$,  $\in \mathcal{D}_{(3,2,2)}$.

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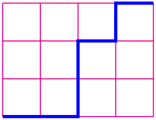
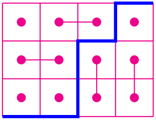
 and  $\in \mathcal{F}_{7,3}$

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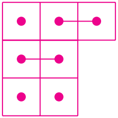
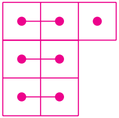
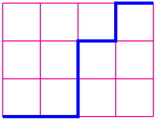
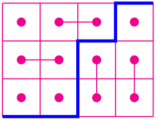
$$\begin{aligned} \mathcal{T}_\lambda &= \{T \ : T \text{ is a tiling of } \lambda\}, \\ \mathcal{D}_\lambda &= \{T \in \mathcal{T}_\lambda \ : \text{every } \lambda_i \text{ begins with a domino}\}. \end{aligned}$$

Ex. If $\lambda = (3, 2, 2)$ then  $\in \mathcal{T}_{(3,2,2)}$,  $\in \mathcal{D}_{(3,2,2)}$.

If $\lambda \subseteq k \times l$ then there is a *dual partition* $\lambda^* = (\lambda_1^*, \dots, \lambda_r^*)$ where the λ_j^* are the column lengths of $(k \times l) - \lambda$. Let

$$\mathcal{F}_{n,k} = \bigcup_{\lambda \subseteq k \times (n-k)} (\mathcal{T}_\lambda \times \mathcal{D}_{\lambda^*}).$$

Ex. If $\lambda = (3, 2, 2) \subseteq 3 \times 4$ then $\lambda^* = (3, 2)$:

 and  $\in \mathcal{F}_{7,3}$

Proposition (S. and Savage, 2010)

For $0 \le k \le n$ we have $\binom{n}{k}_F = \#\mathcal{F}_{n,k}$.

Outline

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It is not hard to show $C_{n.F} \in \mathbb{N}$ for all n. Lou Shapiro asked: Can one find a combinatorial interpretation?

References.

1. Andrews, George E. The theory of partitions. Reprint of the 1976 original. Cambridge Mathematical Library. *Cambridge University Press, Cambridge,* (1998).
2. Benjamin, A. T., and Plott, S. S. A combinatorial approach to Fibonomial coefficients, *Fibonacci Quart.* **46/47** (2008/09), 7–9.
3. Gessel, I., and Viennot, G. Binomial determinants, paths, and hook length formulae. *Adv. in Math.* **58** (1985), 300–321.
4. Knuth, Donald E. Subspaces, subsets, and partitions. *J. Combinatorial Theory Ser. A* **10** (1971) 178–180.
5. MacMahon, Percy A. Combinatory Analysis. Volumes 1 and 2. Reprint of the 1916 original. *Dover, New York,* (2004).
6. Sagan, Bruce E., and Savage, Carla D. Combinatorial interpretations of binomial coefficient analogues related to Lucas sequences. *Integers* **10** (2010), A52, 697–703.
7. Stanley, Richard P. Enumerative combinatorics. Volume 1. Second edition. Cambridge Studies in Advanced Mathematics, 49. *Cambridge University Press, Cambridge,* (2012).

THANKS FOR LISTENING!