Reduced Row Echelon Form

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Based on
A First Course in Linear Algebra, R. A. Beezer
http://linear.ups.edu/fcla/front-matter.html

Outline

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Matrix (1)

Matrix

An $m \times n$ matrix is

a rectangular layout of numbers from C
having m rows and n columns.

- Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top)
- columns will be referenced starting from the left (i.e. column 1 is at the left).

Matrix (2)

Matrix

- use upper-case Latin letters from the start of the alphabet (A,B,C,. . .) to denote matrices and
- squared-off brackets to delimit the layout.
- For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A.

Column Vector

Column Vector

A column vector of size m is an ordered list of m numbers,

which is written in order vertically, starting at the top
and proceeding to the bottom.

- Column vectors will be written in bold,

  usually with lower case Latin letter
  from the end of the alphabet such as u, v, w, x, y, z.
- Some books like to write vectors with arrows, such as $\overrightarrow{u}$.
- Writing by hand, some like to put arrows on top of the symbol,
  or a tilde underneath the symbol, as in $\tilde{v}$.
- To refer to the entry or component of vector v in location i of the list,
  we write $[v]_i$.

Zero Column Vector

Zero Column Vector

The zero vector of size m is the column vector of size m where each entry is the number zero,

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly, $[0]i=0$ for $1 \leq i \leq m$.

Coefficient Matrix

Coefficient Matrix

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the coefficient matrix is the m×n matrix

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
 & & & & \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}$$

Vector of Constants

Vector of Constants

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad &\qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the vector of constants is the column vector of size m

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \\ b_m \end{bmatrix}$$

Solution Vector

Solution Vector

For a system of linear equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

the solution vector is the column vector of size n

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ x_n \end{bmatrix}$$

Matrix Representation of a Linear System

Matrix Representation

If $\boldsymbol{A}$ is the coefficient matrix of a system of linear equations and $\boldsymbol{b}$ is the vector of constants, then we will write $LS(\boldsymbol{A}, \boldsymbol{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the matrix representation of the linear system.

Augmented Matrix

Augmented Matrix

Suppose we have a system of m equations in n variables,
with coefficient matrix A and vector of constants b.

Then the augmented matrix of the system of equations is

the $m \times (n+1)$ matrix whose first n columns are the columns of A
and whose last column $(n+1)$ is the column vector b.
When described symbolically, this matrix will be written as $[\boldsymbol{A}|\boldsymbol{b}]$.

Augmented Matrix Example

Augmented Matrix Example

Archetype A is the following system of 3 equations in 3 variables.

$$\begin{array}{cccc} x_1 & -x_2 & +2x_3 & = 1 \\ 2x_1 & +x_2 & +x_3 & = 8 \\ x_1 & +x_2 & & = 5 \end{array}$$

Here is its augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ 1 & 1 & 0 & 5 \end{bmatrix}$$

Row Operations

Augmented Matrix Example

The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a row operation.

1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j
2. αR_i: Multiply row i by the nonzero scalar α
3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j

Row Equivalent Matrices

Row Equivalent Matrices

Two matrices, A and B, are row-equivalent
if one can be obtained from the other by a sequence of row operations.

Two Row Equivalent Matrices

Row Equivalent Matrix Example

The matrices

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & 2 & -2 & 3 \\ 1 & 1 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 6 \\ 3 & 0 & -2 & -9 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

are row equivalent

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & 2 & -2 & 3 \\ 1 & 1 & 0 & 6 \end{bmatrix} : R_1 \leftrightarrow R_3 \quad \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 6 \\ 5 & 2 & -2 & 3 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & 6 \\ 5 & 2 & -2 & 3 \\ 2 & -1 & 3 & 4 \end{bmatrix} : -2R_1 + R_2 \quad \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 6 \\ 3 & 0 & -2 & -9 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

Row Equivalent Matrices

Equivalent Systems

Suppose that A and B are row-equivalent augmented matrices.

Then the two systems of linear equations
represented by A and B are equivalent systems

Reduced Row-Echelon Form (1)

Reduced Row-Echelon Form

A matrix is in reduced row-echelon form
if it meets all of the following conditions:

1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
2. The leftmost nonzero entry of a row is equal to 1.
3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s>i, then t>j

Reduced Row-Echelon Form (2)

a **zero row**, a **leading 1**, a **pivot column**

A row of only zero entries is called a **zero row** and
the leftmost nonzero entry of a nonzero row is a **leading 1**.
A column containing a leading 1 will be called a **pivot column**.

The number of nonzero rows will be denoted by r,
which is also equal to the number of **leading 1**'s
and the number of **pivot columns**.

The set of column indices for the **pivot columns**
will be denoted by $D = d_1, d_2, d_3, \ldots, d_r$ where $d_1 < d_2 < d_3 < \cdots < d_r$,

while the columns that are not **pivot columns**
will be denoted as $F = f_1, f_2, f_3, \ldots, f_{n-r}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

A Reduced Row Echelon Matrix Example

Reduced Row Echelon Matrix Example

The matrix C is in reduced row-echelon form.

$$C = \begin{bmatrix} 1 & -3 & 0 & 6 & 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix has two zero rows and three pivot columns.
So $r = 3$. Columns 1, 5, and 6 are the three pivot columns,
so $D = 1,5,6$ and then $F = 2,3,4,7,8$.