

# Laurent Series and z-Transform

## - Geometric Series

## Permutations B

20241111 Mon

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$a^n$  $\times$  $R(n)$ 

$a^n$	$a^{-n}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

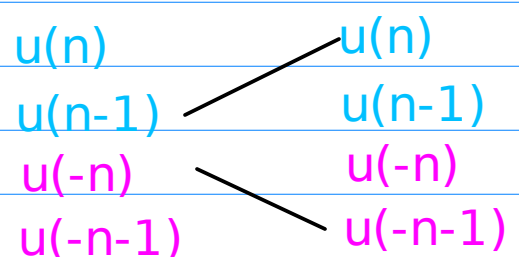
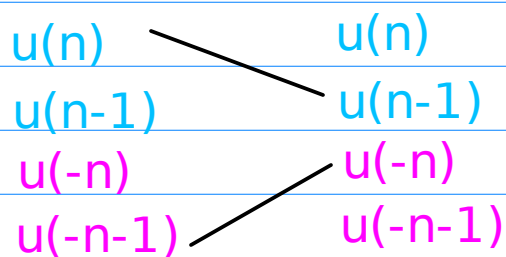
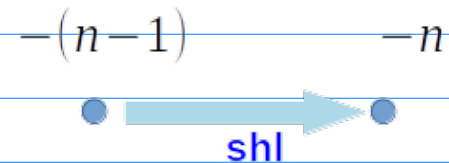
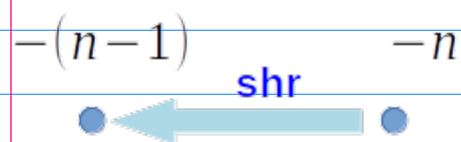
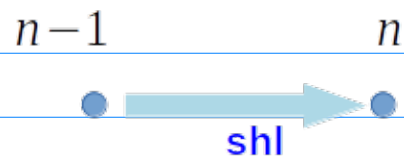
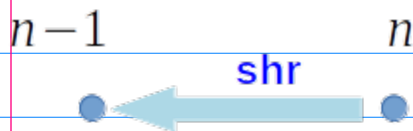
(1)	$a^n u(n)$	$a^{-n} u(n)$	(2)
(3)	$a^n u(-n)$	$a^{-n} u(-n)$	(4)
(5)	$a^n u(-n-1)$	$a^{-n} u(-n-1)$	(6)
(7)	$a^n u(n-1)$	$a^{-n} u(n-1)$	(8)

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

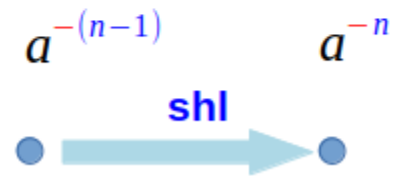
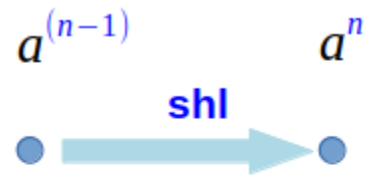
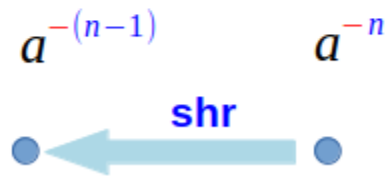
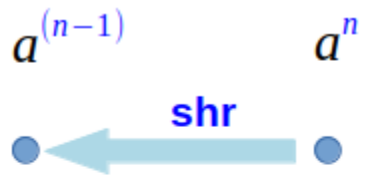
$$\text{shl}(R(n)) = R(n+1)$$

$$\text{shr}(R(n)) = R(n-1)$$

$u(n)$	$u(n-1)$	shr	$n \rightarrow n-1$
$u(n-1)$	$u(n)$	shl	$n \rightarrow n+1$
$u(-n)$	$u(-n-1)$	shl	$n \rightarrow n+1$
$u(-n-1)$	$u(-n)$	shr	$n \rightarrow n-1$



## Exponent Shifting

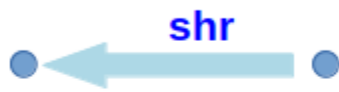


	$shl(b^n)$
$b^n$	$b^{n+1}$
$a^n$	$a^{(n+1)}$
$a^{-n}$	$a^{-(n+1)}$

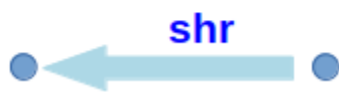
	$shr(b^n)$
$b^n$	$b^{n-1}$
$a^{(n+1)}$	$a^n$
$a^{-(n+1)}$	$a^{-n}$

## Range Shifting

$u((n-1))$        $u(n)$



$u(-(n-1))$        $u(-n)$



$u((n-1))$        $u(n)$



$u(-(n-1))$        $u(-n)$



$shl(R(n))$

$R(n)$

$R(n+1)$

$u(n-1)$

$u(n)$

$u(-n)$

$u(-(n+1))$

$shr(R(n))$

$R(n)$

$R(n-1)$

$u(n)$

$u(n-1)$

$u(-(n+1))$

$u(-n)$

$$\begin{array}{ccc} b^n & \longleftrightarrow & b^{n+1} \\ R(n) & \longleftrightarrow & R(n+1) \end{array}$$

$$\begin{array}{ccc} b^n & \longleftrightarrow & b^{-(n+1)} \\ R(n) & \longleftrightarrow & R(n+1) \end{array}$$

$$\begin{array}{ccc} b^n & \longleftrightarrow & b^{n-1} \\ R(n) & \longleftrightarrow & R(n-1) \end{array}$$

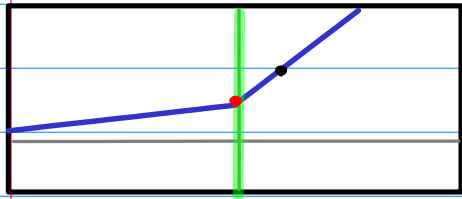
$$\begin{array}{ccc} b^n & \longleftrightarrow & b^{-(n-1)} \\ R(n) & \longleftrightarrow & R(n-1) \end{array}$$

# Exponent Shifting

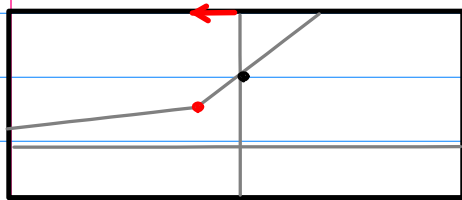
$$\text{shl}(b^n) = b^{n+1}$$

$$\text{shr}(b^n) = b^{n-1}$$

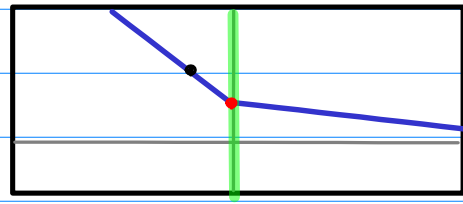
$$b^n \longleftrightarrow b^{n+1}$$

 $2^n$ 

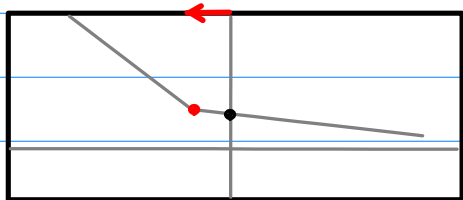
shift left  
 $n \leftarrow n+1$

 $2^{n+1}$ 

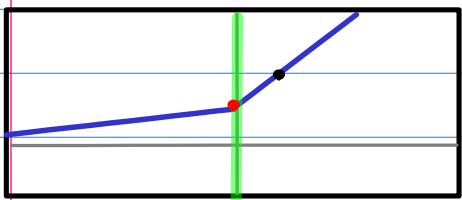
$$b^n \longleftrightarrow b^{-(n+1)}$$

 $2^{-n}$ 

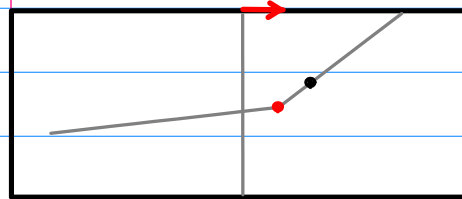
shift left  
 $n \leftarrow n+1$

 $2^{-(n+1)}$ 

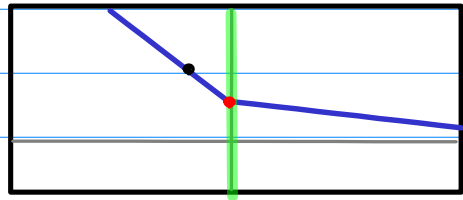
$$b^n \longleftrightarrow b^{n-1}$$

 $2^n$ 

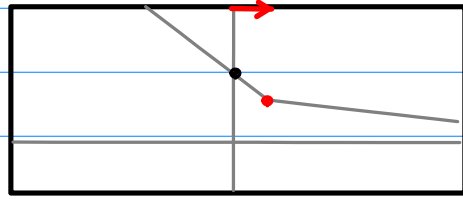
shift right  
 $n \leftarrow n-1$

 $2^{n-1}$ 

$$b^n \longleftrightarrow b^{-(n-1)}$$

 $2^{-n}$ 

shift right  
 $n \leftarrow n-1$

 $2^{-(n-1)}$ 



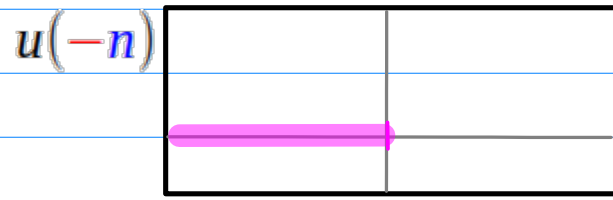
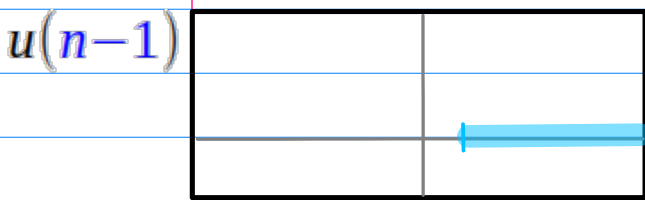
# Range Shifting

$$\text{shl}(R(n)) = R(n+1)$$

$$\text{shr}(R(n)) = R(n-1)$$

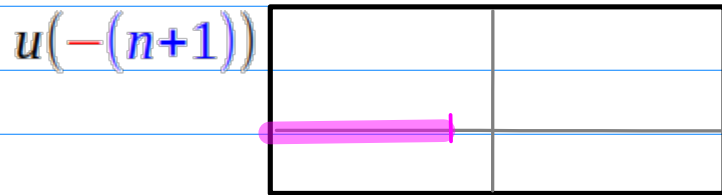
$R(n) \longleftrightarrow R(n+1)$

$R(n) \longleftrightarrow R(n+1)$



↓ shift left  
 $n \leftarrow n+1$

↓ shift left  
 $n \leftarrow n+1$



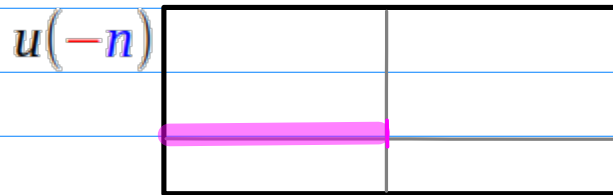
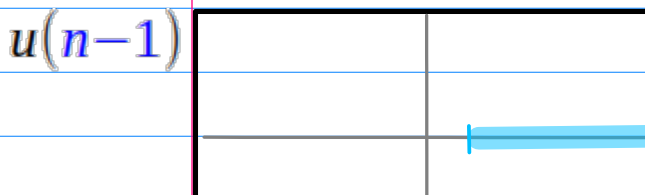
$R(n) \longleftrightarrow R(n-1)$

$R(n) \longleftrightarrow R(n-1)$



↓ shift right  
 $n \leftarrow n-1$

↓ shift right  
 $n \leftarrow n-1$



$a^n$  $\times$  $R(n)$ 

$a^{n+1}$	$a^{-n-1}$
$a^{n-1}$	$a^{-n+1}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1')	$a^{n+1}u(n)$	$a^{-n-1}u(n)$	(2')
(3')	$a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	(4')
(5')	$a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	(6')
(7')	$a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	(8')

(1'')	$a^{n-1}u(n)$	$a^{-n+1}u(n)$	(2'')
(3'')	$a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	(4'')
(5'')	$a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	(6'')
(7'')	$a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	(8'')

$a^n$  $\times$  $R(n)$ 

$a^{n+1}$	$a^{-n-1}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

$$(1') \quad a^{n+1} u(n) \quad a^{-n-1} u(n) \quad (2')$$

$$(3') \quad a^{n-1} u(-n) \quad a^{-n+1} u(-n) \quad (4')$$

$$(5') \quad a^{n+1} u(-n-1) \quad a^{-n-1} u(-n-1) \quad (6')$$

$$(7') \quad a^{n-1} u(n-1) \quad a^{-n+1} u(n-1) \quad (8')$$

$a^n$  $\times$  $R(n)$ 

$a^{n+1}$	$a^{-n-1}$
$a^{n-1}$	$a^{-n+1}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1'')  $a^{n-1} u(n)$      $a^{-n+1} u(n)$  (2'')

(3'')  $a^{n+1} u(-n)$      $a^{-n-1} u(-n)$  (4'')

(5'')  $a^{n-1} u(-n-1)$      $a^{-n+1} u(-n-1)$  (6'')

(7'')  $a^{n+1} u(n-1)$      $a^{-n-1} u(n-1)$  (8'')

**Unshifted Sequence  $x$**

**Shifted Sequence 1  $x'$**

**Shifted Sequence 2  $x''$**

(1)  $a^n u(n)$

$shl(b^n) a^{n+i} u(n) (1')$

$shr(b^n) a^{n-i} u(n) (1'')$

(2)  $a^{-n} u(n)$

$shl(b^n) a^{-n-i} u(n) (2')$

$shr(b^n) a^{-n+i} u(n) (2'')$

(3)  $a^n u(-n)$

$shr(b^n) a^{n-i} u(-n) (3')$

$shl(b^n) a^{n+i} u(-n) (3'')$

(4)  $a^{-n} u(-n)$

$shr(b^n) a^{-n+i} u(-n) (4')$

$shl(b^n) a^{-n-i} u(-n) (4'')$

(5)  $a^n u(-n-1)$

$shl(b^n) a^{n+i} u(-n-1) (5')$

$shr(b^n) a^{n-i} u(-n-1) (5'')$

(6)  $a^{-n} u(-n-1)$

$shl(b^n) a^{-n-i} u(-n-1) (6')$

$shr(b^n) a^{-n+i} u(-n-1) (6'')$

(7)  $a^n u(n-1)$

$shr(b^n) a^{n-i} u(n-1) (7')$

$shl(b^n) a^{n+i} u(n-1) (7'')$

(8)  $a^{-n} u(n-1)$

$shr(b^n) a^{-n+i} u(n-1) (8')$

$shl(b^n) a^{-n-i} u(n-1) (8'')$

$shl(b^n) \longleftrightarrow shr(b^n)$

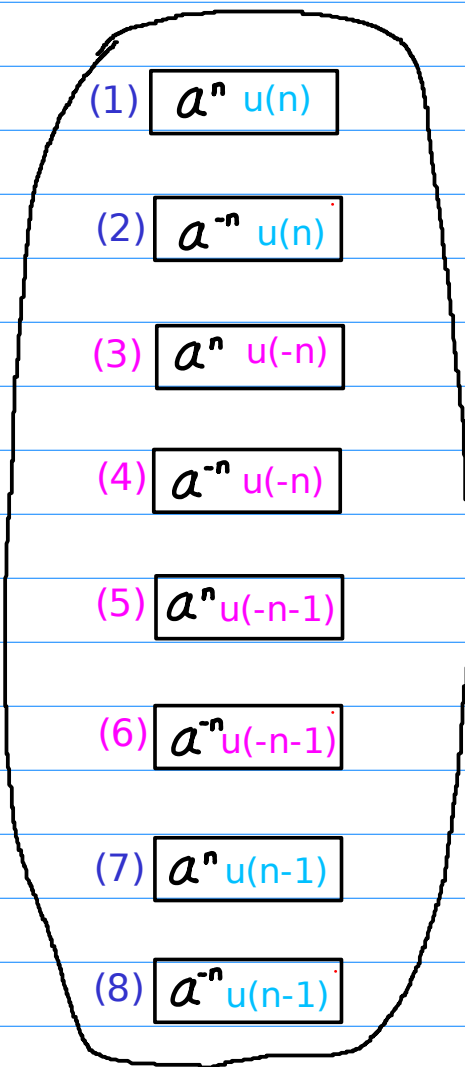
$shr(b^n) \longleftrightarrow shl(b^n)$

many possible permutations are possible  
but consider these two

**Unshifted Sequence  $x$**



**Shifted Sequence 1  $x'$**



$shl(b^n)$

$a^{n+1} u(n)$  (1')

$shl(b^n)$

$a^{-n-1} u(n)$  (2')

$shr(b^n)$

$a^{n-1} u(-n)$  (3')

$shr(b^n)$

$a^{-n+1} u(-n)$  (4')

$shl(b^n)$

$a^{n+1} u(-n-1)$  (5')

$shl(b^n)$

$a^{-n-1} u(-n-1)$  (6')

$shr(b^n)$

$a^{n-1} u(n-1)$  (7')

$shr(b^n)$

$a^{-n+1} u(n-1)$  (8')

**Inter-permutations over unshifted sequence and shifted sequence**

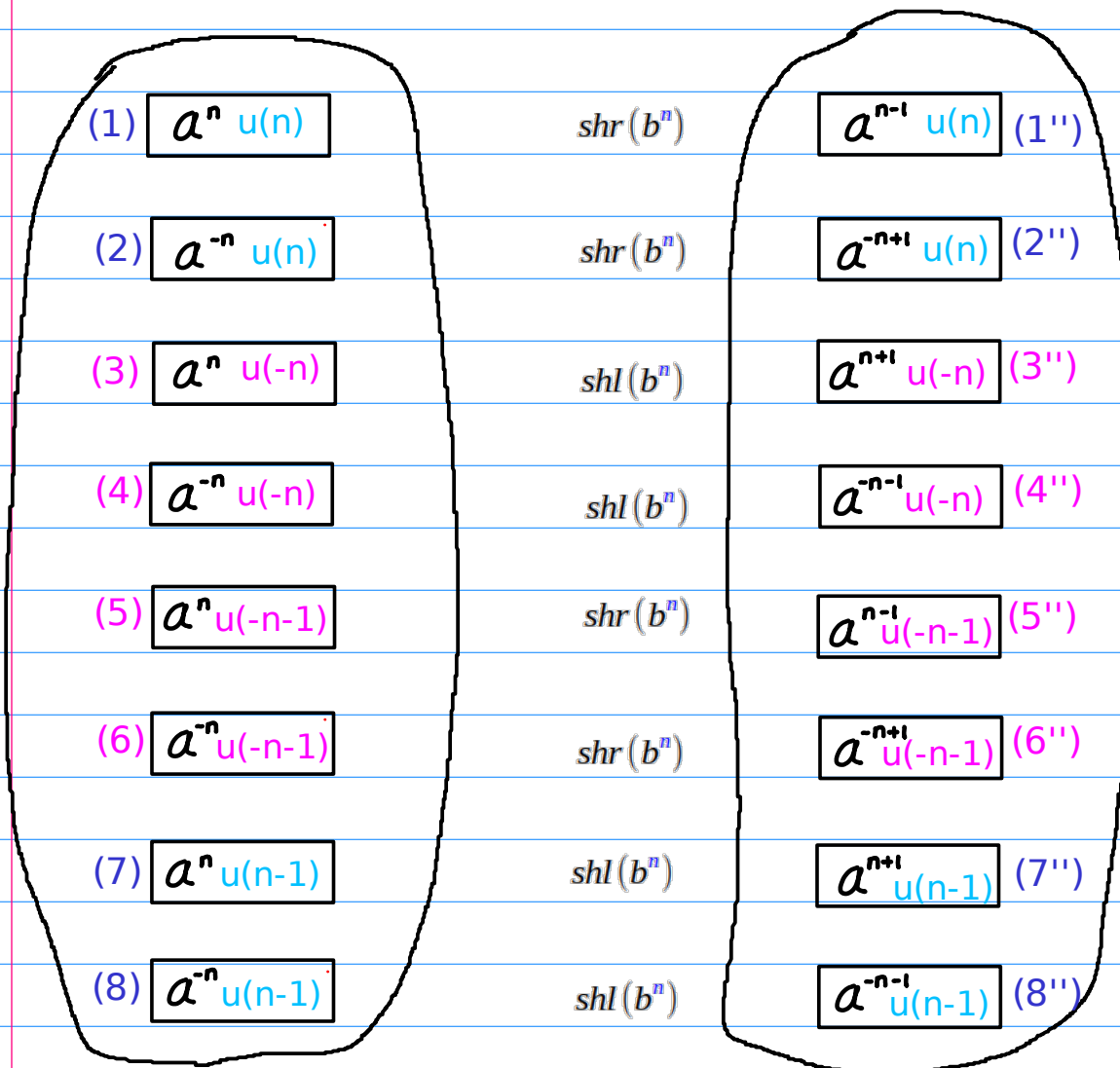
**Intra-permutations over unshifted sequence**

**Intra-permutations over shifted sequence**

**Unshifted Sequence  $x$**



**Shifted Sequence 2  $x''$**



**Inter-permutations over unshifted sequence and shifted sequence**

**Intra-permutations over unshifted sequence**

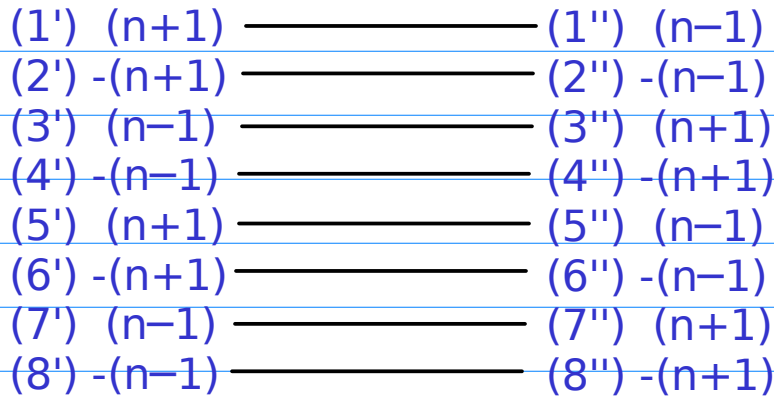
**Intra-permutations over shifted sequence**

**(x')**  $\longleftrightarrow$  **(x'')**  
 (1')~(8') (1'')~(8'')

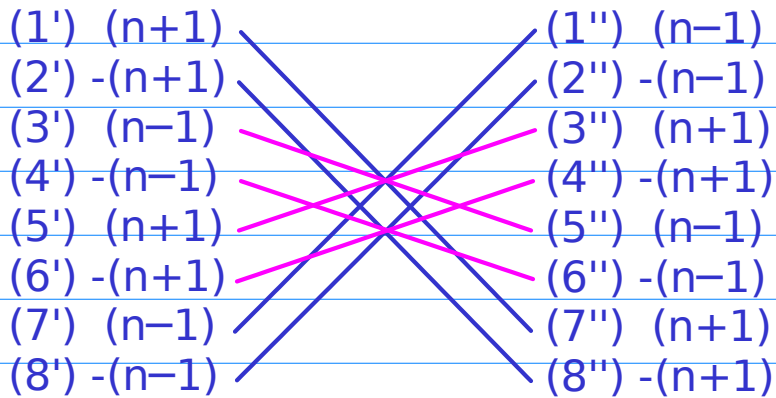
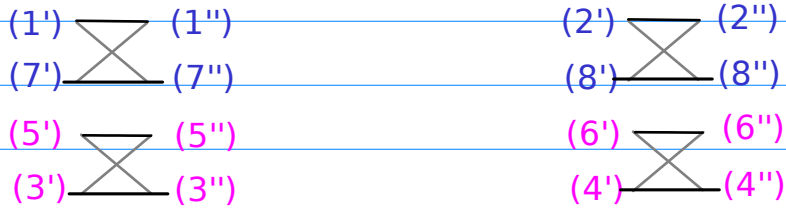
### Exponent Conversion

$shl(b^n) \longleftrightarrow shr(b^n)$

$shr(b^n) \longleftrightarrow shl(b^n)$



### Identical Exponent Elements





(1')	(n+1)	shl(n)	(1'')	(n-1)	shr(n)
(7')	(n-1)	shr(n)	(7'')	(n+1)	shl(n)

(2')	-(n+1)	shl(n)	(2'')	-(n-1)	shr(n)
(8')	-(n-1)	shr(n)	(8'')	-(n+1)	shl(n)



(3')	(n-1)	shr(n)	(3'')	(n+1)	shl(n)
(5')	(n+1)	shl(n)	(5'')	(n-1)	shr(n)

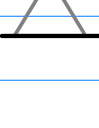

(4')	-(n-1)	shr(n)	(4'')	-(n+1)	shl(n)
(6')	-(n+1)	shl(n)	(6'')	-(n-1)	shr(n)

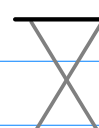
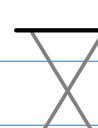
**(x')**  $\longleftrightarrow$  **(x'')**

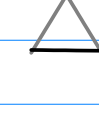
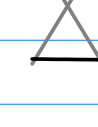
(1')~(8')

(1'')~(8'')

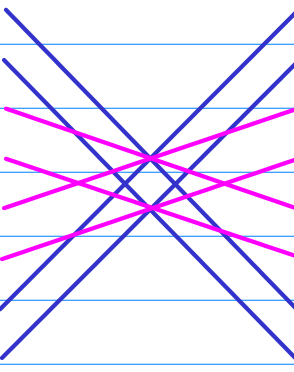
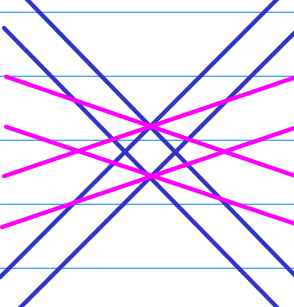
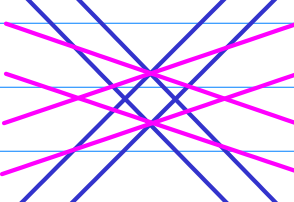
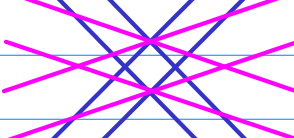




(1')  $a^{n+1}u(n)$   (1'')  $a^{n-1}u(n)$   $a^{-n-1}u(n)$  (2')   $a^{-n+1}u(n)$  (2'')

(7')  $a^{-n-1}u(n-1)$   (7'')  $a^{n+1}u(n-1)$   $a^{-n+1}u(n-1)$  (8')   $a^{-n-1}u(n-1)$  (8'')

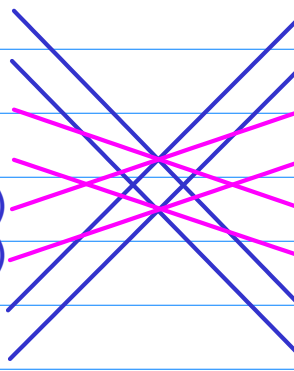
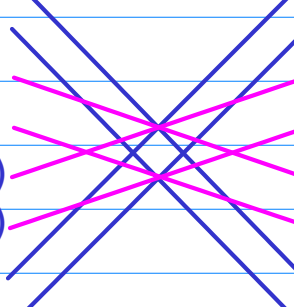






(3')  $a^{-n-1}u(-n)$   (3'')  $a^{n+1}u(-n)$   $a^{-n+1}u(-n)$  (4')   $a^{-n-1}u(-n)$  (4'')

(5')  $a^{n+1}u(-n-1)$   (5'')  $a^{-n-1}u(-n-1)$   $a^{-n+1}u(-n-1)$  (6')   $a^{-n-1}u(-n-1)$  (6'')

### Identical Exponent Elements

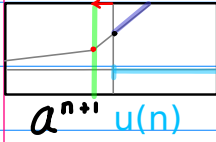
(1') (n+1)		(1'') (n-1)
(2') -(n+1)		(2'') -(n-1)
(3') (n-1)		(3'') (n+1)
(4') -(n-1)		(4'') -(n+1)
(5') (n+1)		(5'') (n-1)
(6') -(n+1)		(6'') -(n-1)
(7') (n-1)		(7'') (n+1)
(8') -(n-1)		(8'') -(n+1)

### Shifted Range Elements

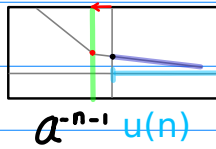
(1') u(n)		(1'') u(n)
(2') u(n)		(2'') u(n)
(3') u(-n)		(3'') u(-n)
(4') u(-n)		(4'') u(-n)
(5') u(-n-1)		(5'') u(-n-1)
(6') u(-n-1)		(6'') u(-n-1)
(7') u(n-1)		(7'') u(n-1)
(8') u(n-1)		(8'') u(n-1)

**(x')**  $\longleftrightarrow$  **(x'')**  
 (1')~(8') (1'')~(8'')

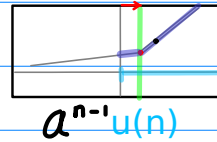
(1') 1000



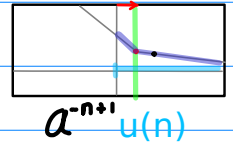
(2') 1001



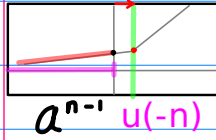
(1'') 1000



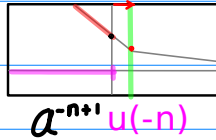
(2'') 1001



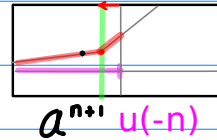
(3') 1010



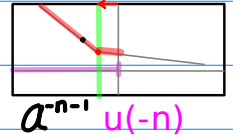
(4') 1011



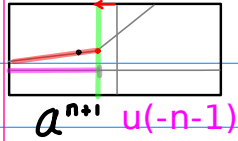
(3'') 1010



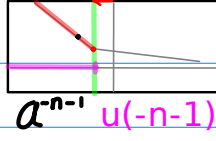
(4'') 1011



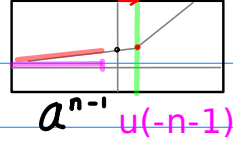
(5') 1100



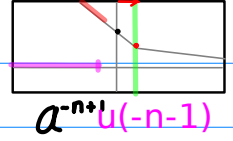
(6') 1101



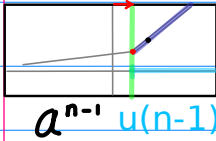
(5'') 1100



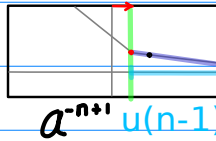
(6'') 1101



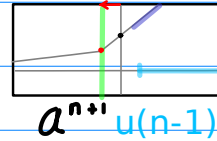
(7') 1110



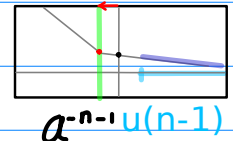
(8') 1111



(7'') 1110



(8'') 1111



### Conversion between x' and x''

$$shl(b^n) \longleftrightarrow shr(b^n)$$

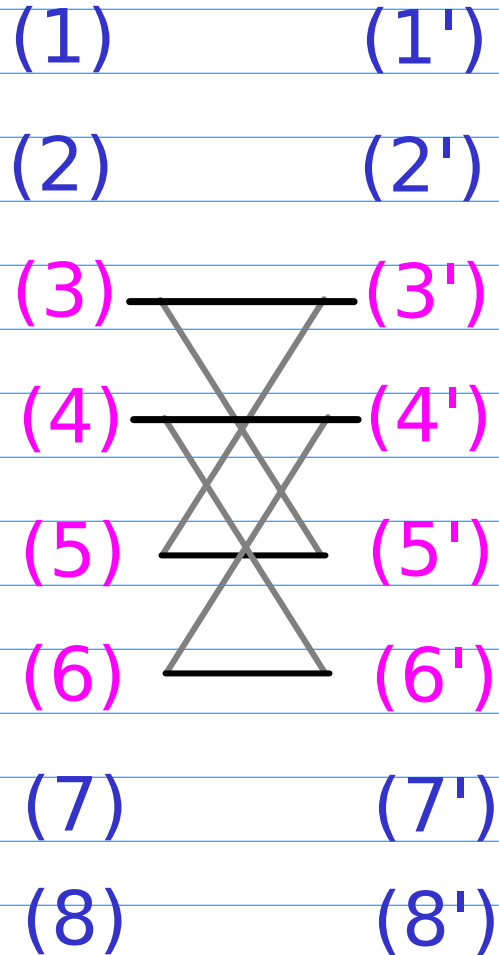
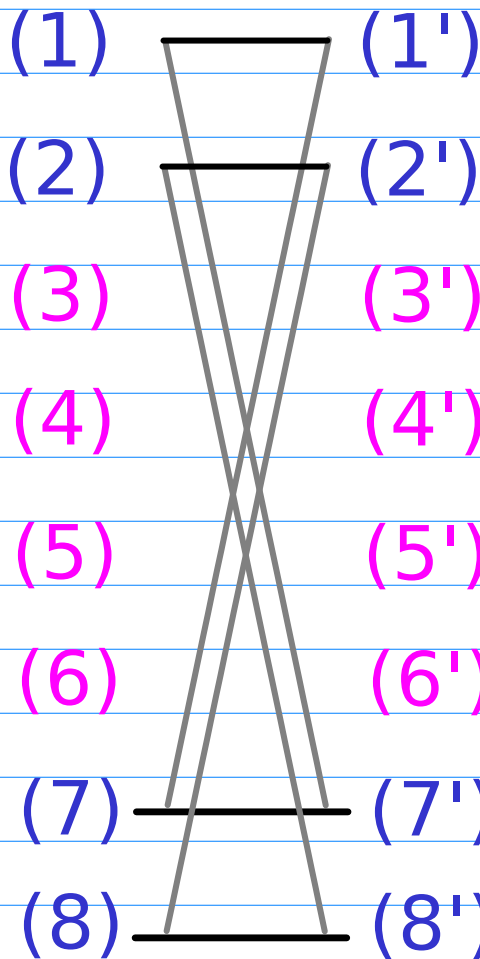
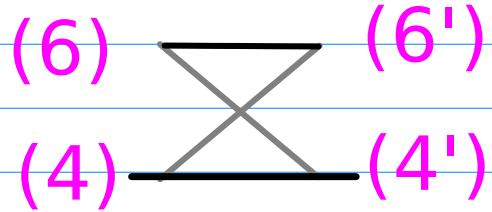
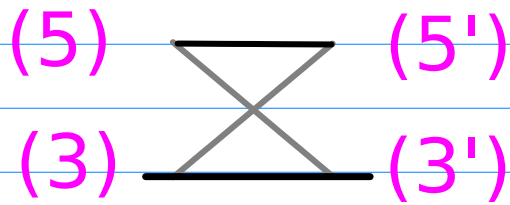
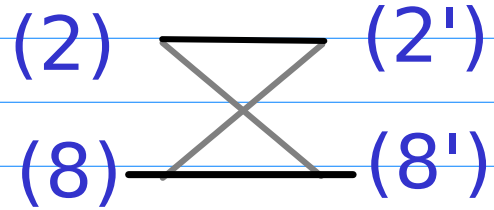
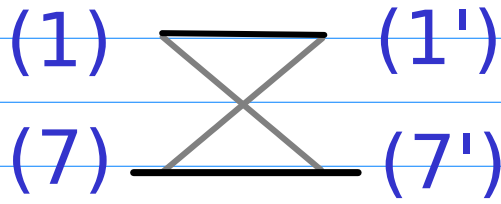
$$shr(b^n) \longleftrightarrow shl(b^n)$$



(1) - (1')  
(7) - (7')  
(5) - (5')  
(3) - (3')

(2) - (2')  
(8) - (8')  
(6) - (6')  
(4) - (4')

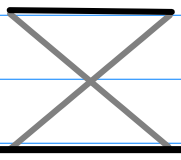

**(x)**  **(x')**  
(1)~(8) (1')~(8')

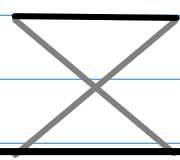



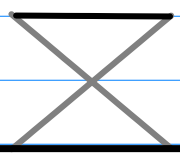

(1) - (1')  
(7) - (7')  
(5) - (5')  
(3) - (3')

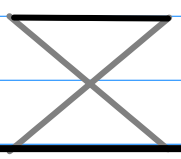

(2) - (2')  
(8) - (8')  
(6) - (6')  
(4) - (4')

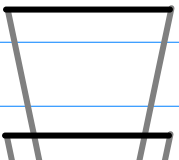
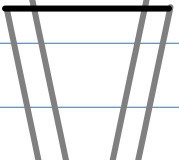
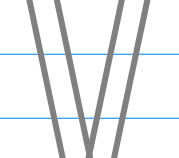
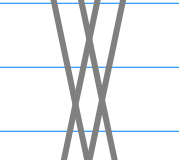
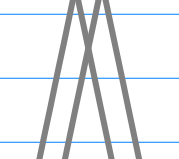



**(x)**  **(x'')**  
(1)~(8) (1')~(8')





(1)  (1'')  
(7)  (7'')

(2)  (2'')  
(8)  (8'')

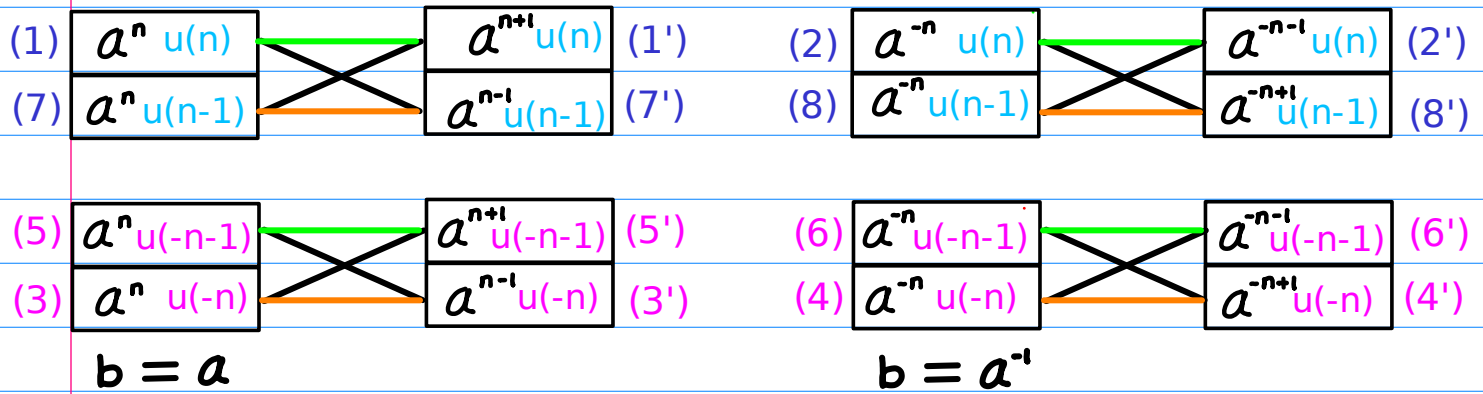
(5)  (5'')  
(3)  (3'')

(6)  (6'')  
(4)  (4'')

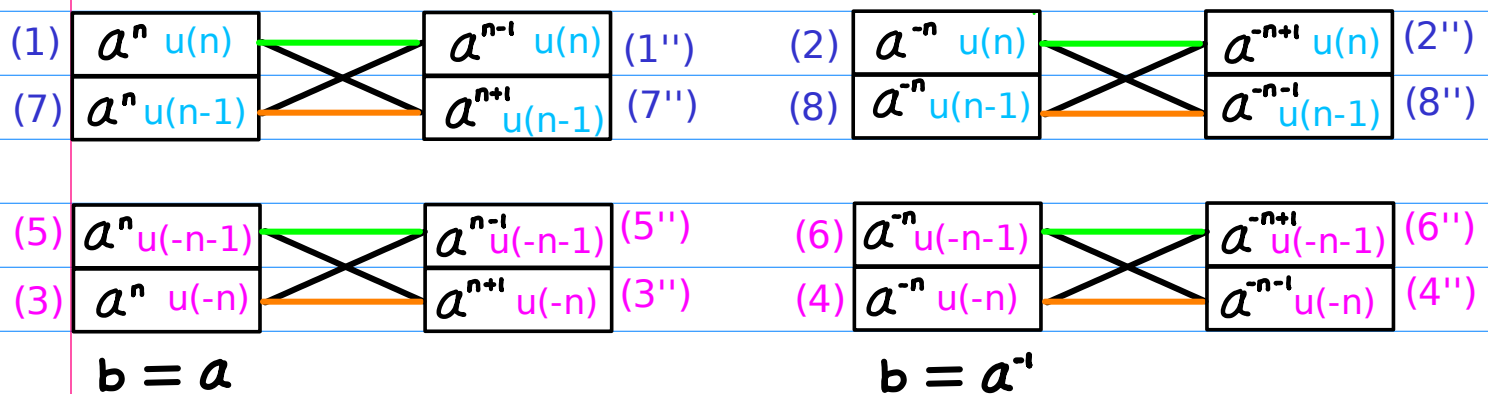
(1)  (1'')  
(2)  (2'')  
(3)  (3'')  
(4)  (4'')  
(5)  (5'')  
(6)  (6'')  
(7)  (7'')  
(8)  (8'')

(1) (1'')  
(2) (2'')  
(3)  (3'')  
(4)  (4'')  
(5)  (5'')  
(6)  (6'')  
(7) (7'')  
(8) (8'')

### A Shifting Shifted Sequence 1



### B Shifting Shifted Sequence 2

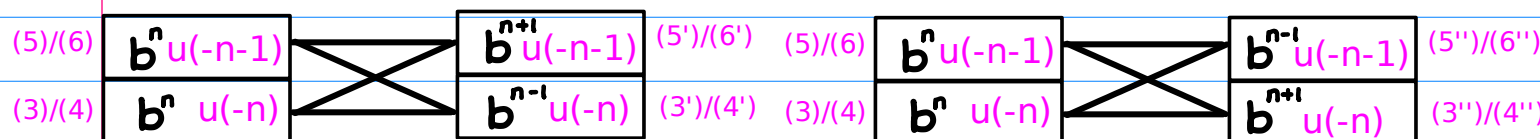
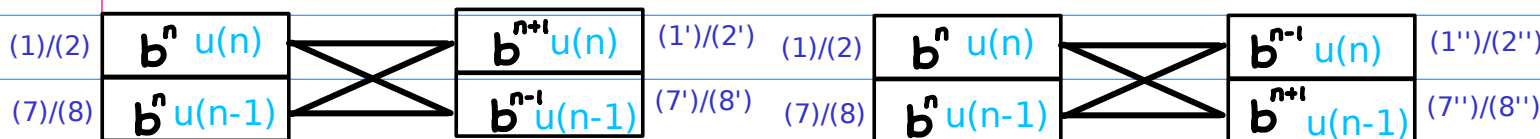


## A Shifting Shifted Sequence 1

(1) Exponent Shifting  
(2) Range Shifting

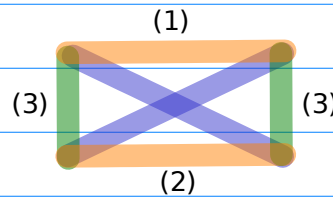
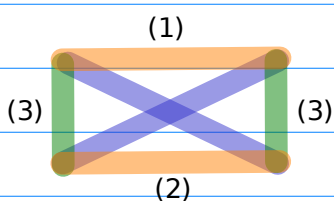
## B Shifting Shifted Sequence 2

(1) Exponent Shifting  
(2) Range Shifting



$$b = a/a^i$$

$$b = a/a^i$$



$$b^n \xleftrightarrow{(1)} b^{n+1}$$

$$b^n \xleftrightarrow{(1)} b^{n-1}$$

$$b^n \xleftrightarrow{(2)} b^{n-1}$$

$$b^n \xleftrightarrow{(2)} b^{n+1}$$

$$R(n) \xleftrightarrow{(3)} R(n-1)$$

$$R(n) \xleftrightarrow{(3)} R(n-1)$$

### Exponent Shifting

$b^n$	$b^{sh(n)}$	
$a^n$	$a^{(n+1)}$	$shl(b^n)$
$a^n$	$a^{(n-1)}$	$shr(b^n)$
$a^{-n}$	$a^{-(n+1)}$	$shl(b^n)$
$a^{-n}$	$a^{-(n-1)}$	$shr(b^n)$

### Exponent Shifting

$b^n$	$b^{sh(n)}$	
$a^n$	$a^{(n-1)}$	$shr(b^n)$
$a^n$	$a^{(n+1)}$	$shl(b^n)$
$a^{-n}$	$a^{-(n-1)}$	$shr(b^n)$
$a^{-n}$	$a^{-(n+1)}$	$shl(b^n)$

### Range Shifting

$R(n)$	$R(sh(n))$	
$u(n)$	$u(n-1)$	$shr(R(n))$
$u(-(n+1))$	$u(-n)$	$shr(R(n))$

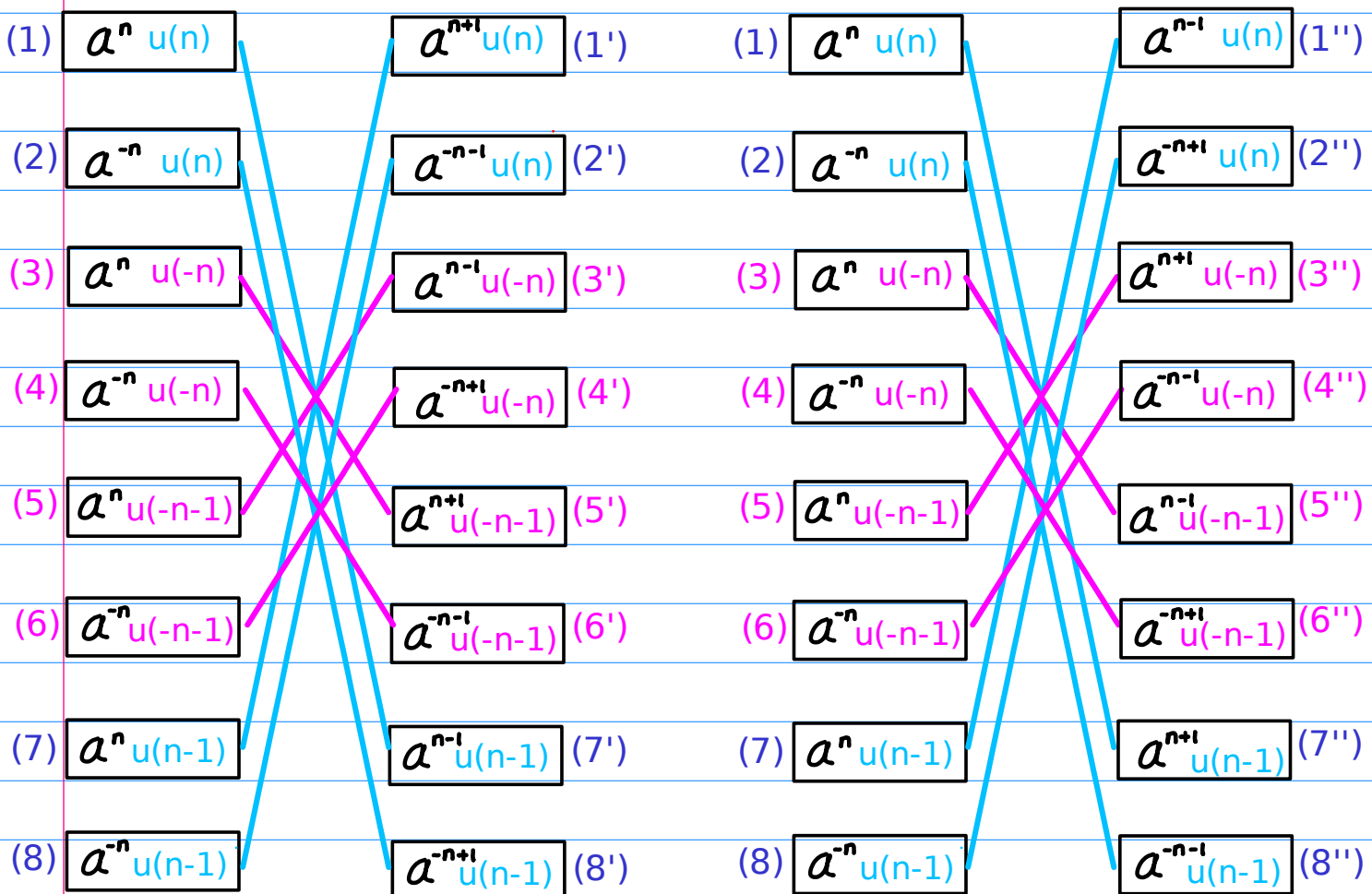
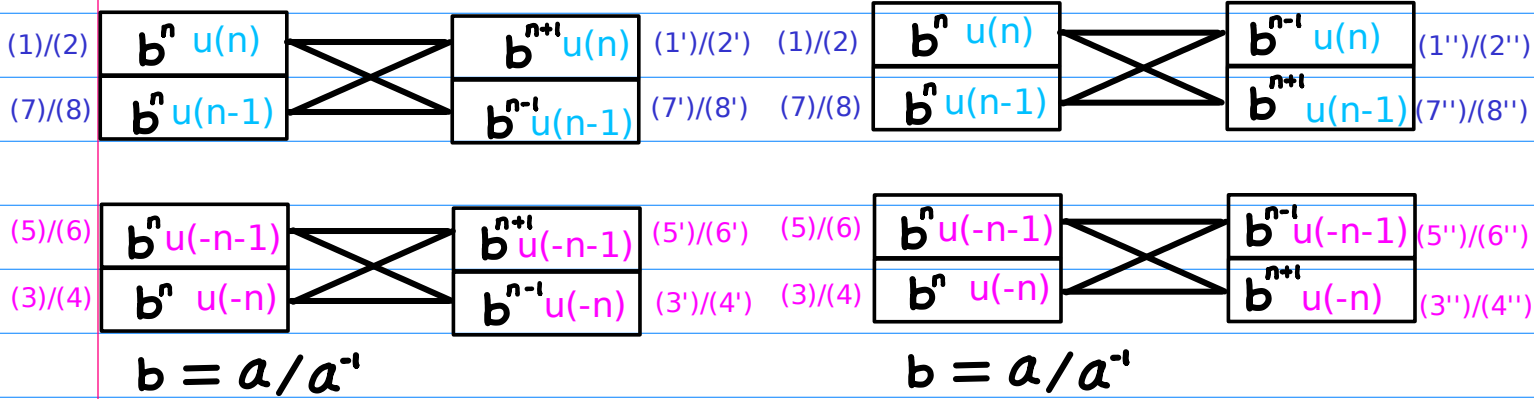
### Range Shifting

$R(n)$	$R(sh(n))$	
$u(n)$	$u(n-1)$	$shr(R(n))$
$u(-(n+1))$	$u(-n)$	$shr(R(n))$

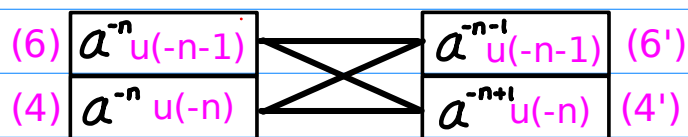
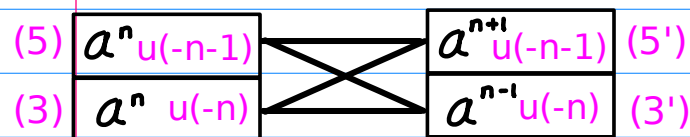
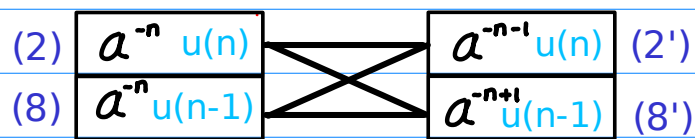
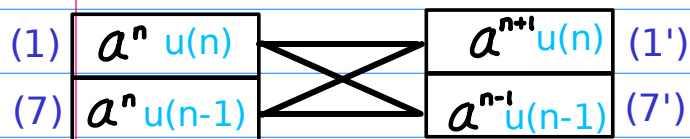


## A Shifting Shifted Sequence 1

## B Shifting Shifted Sequence 2

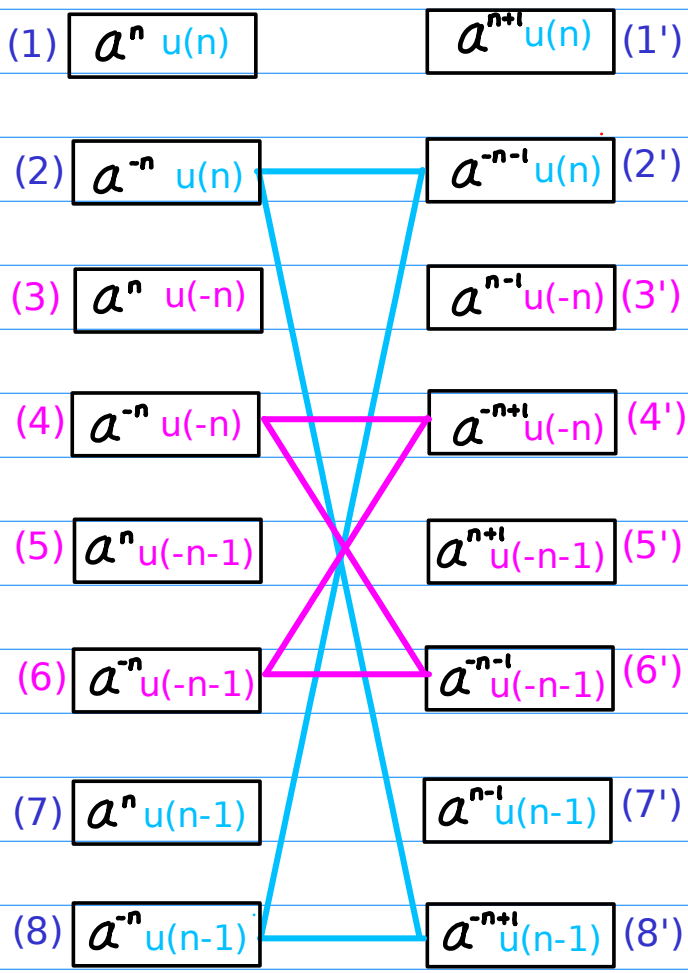
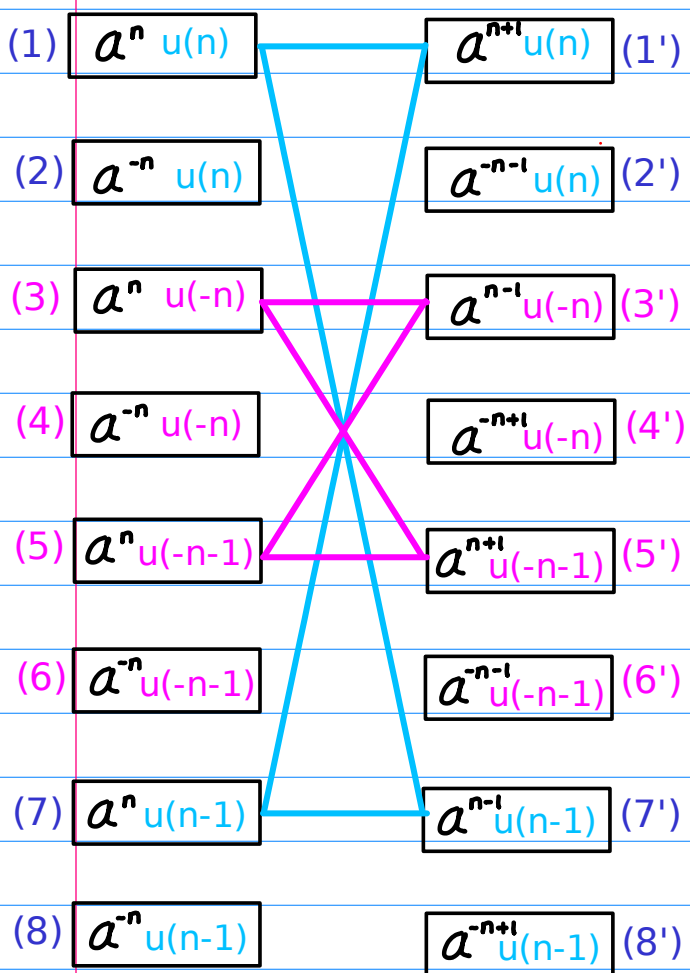


# A Shifting Shifted Sequence 1

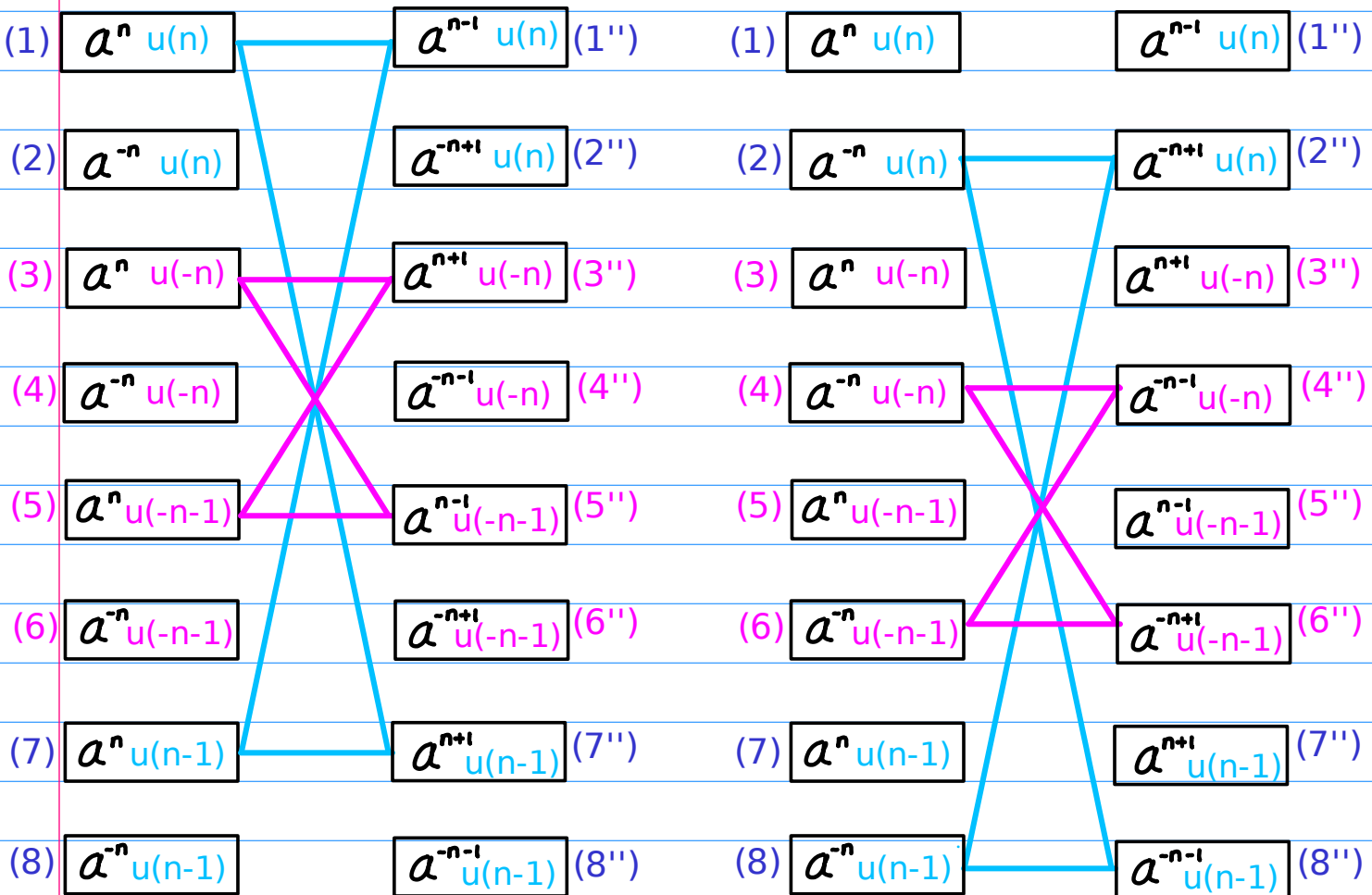
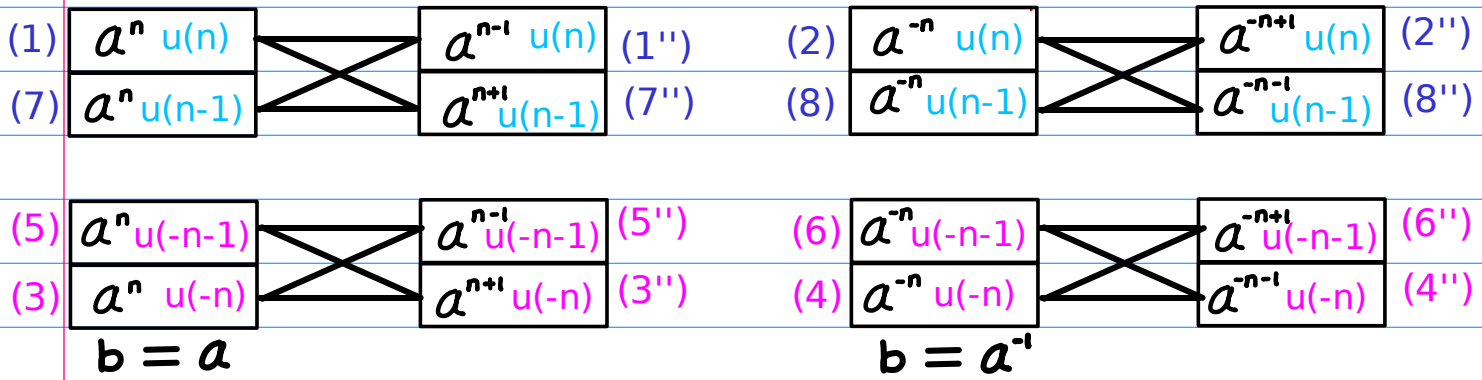


$b = a$

$b = a^{-1}$

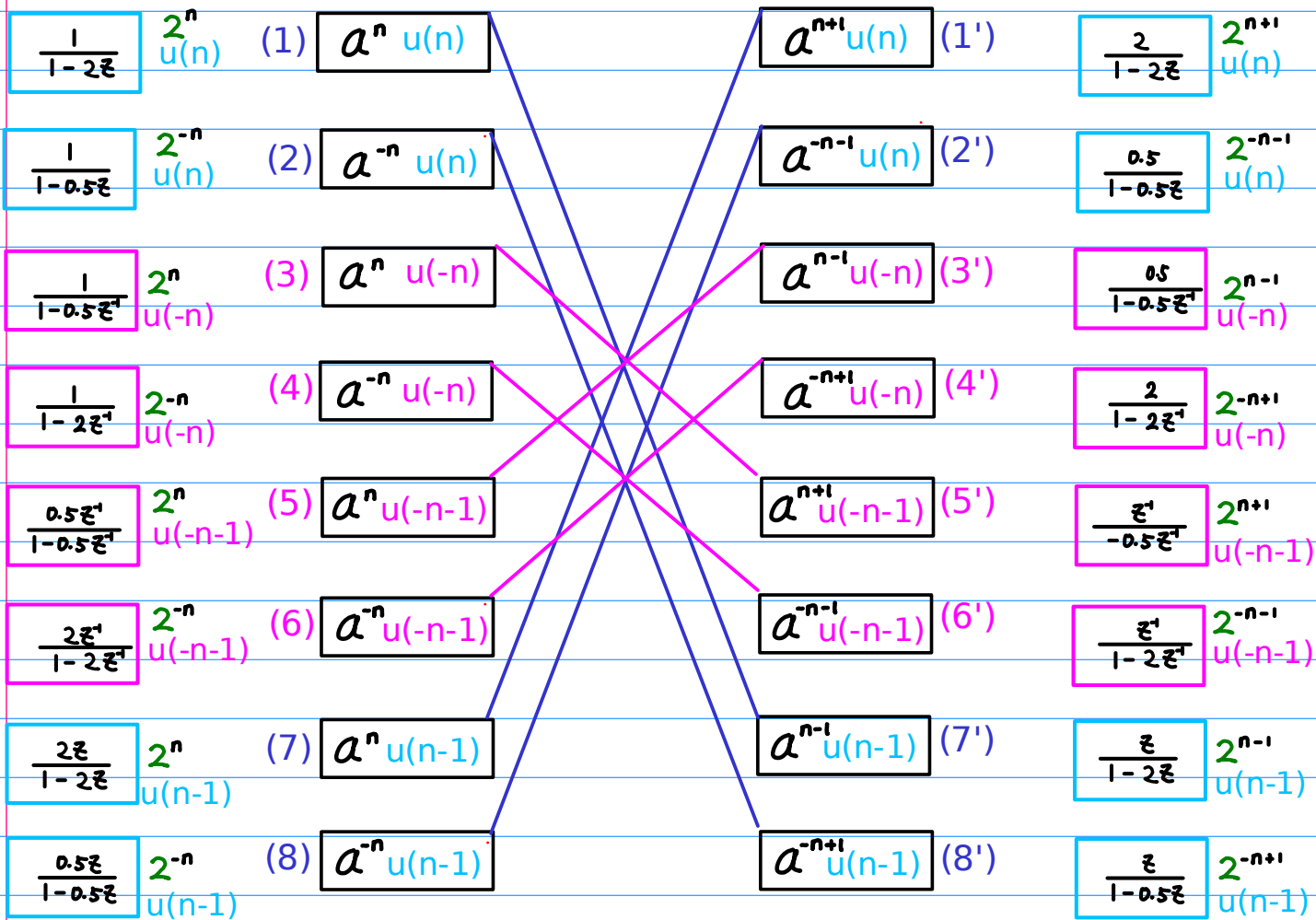
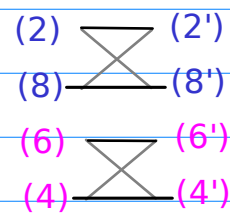
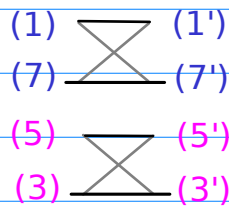


## B Shifting Shifted Sequence 2

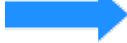


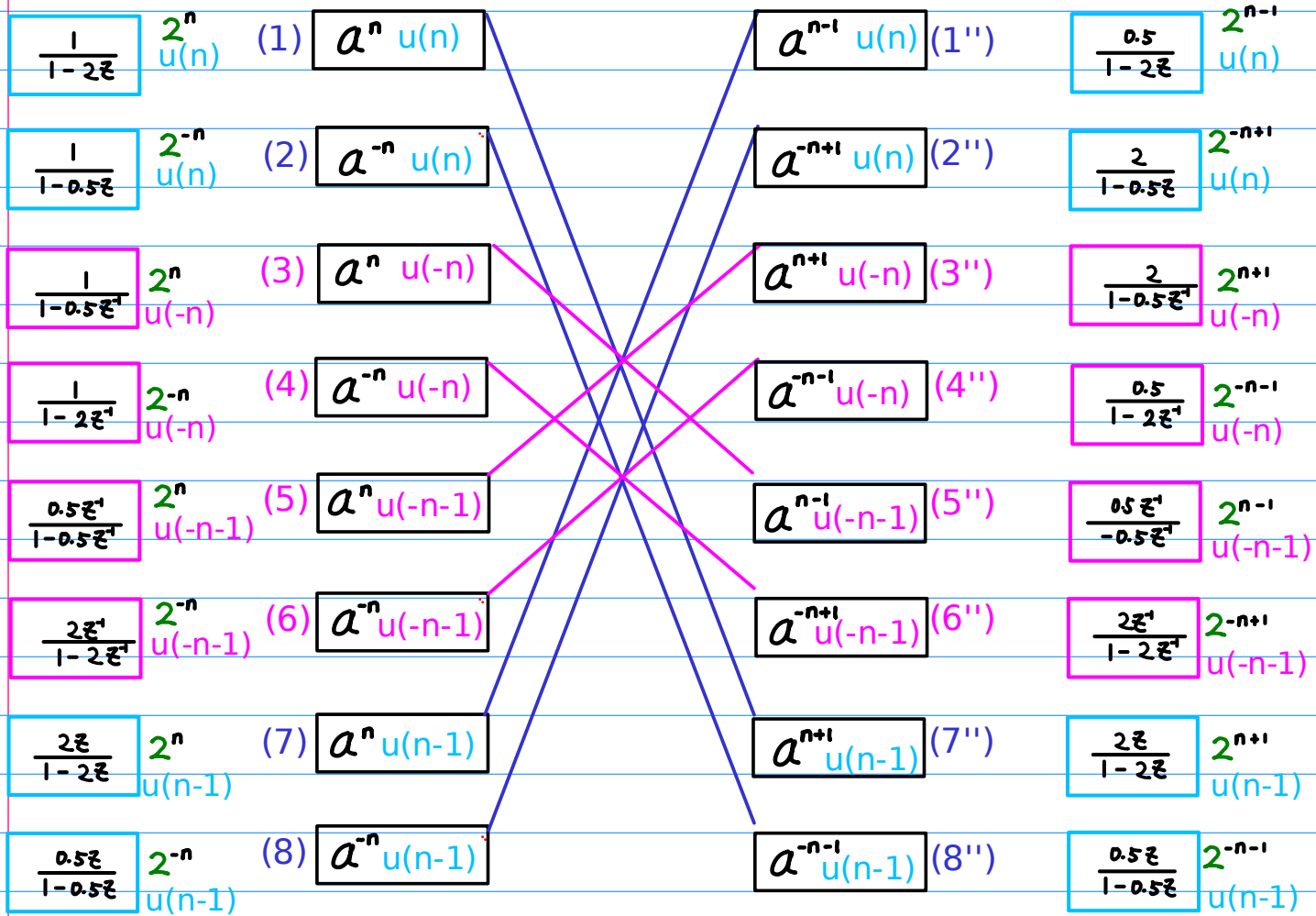
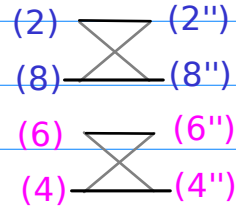
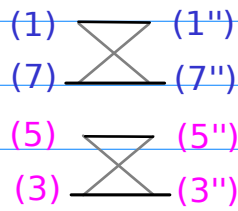
# A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

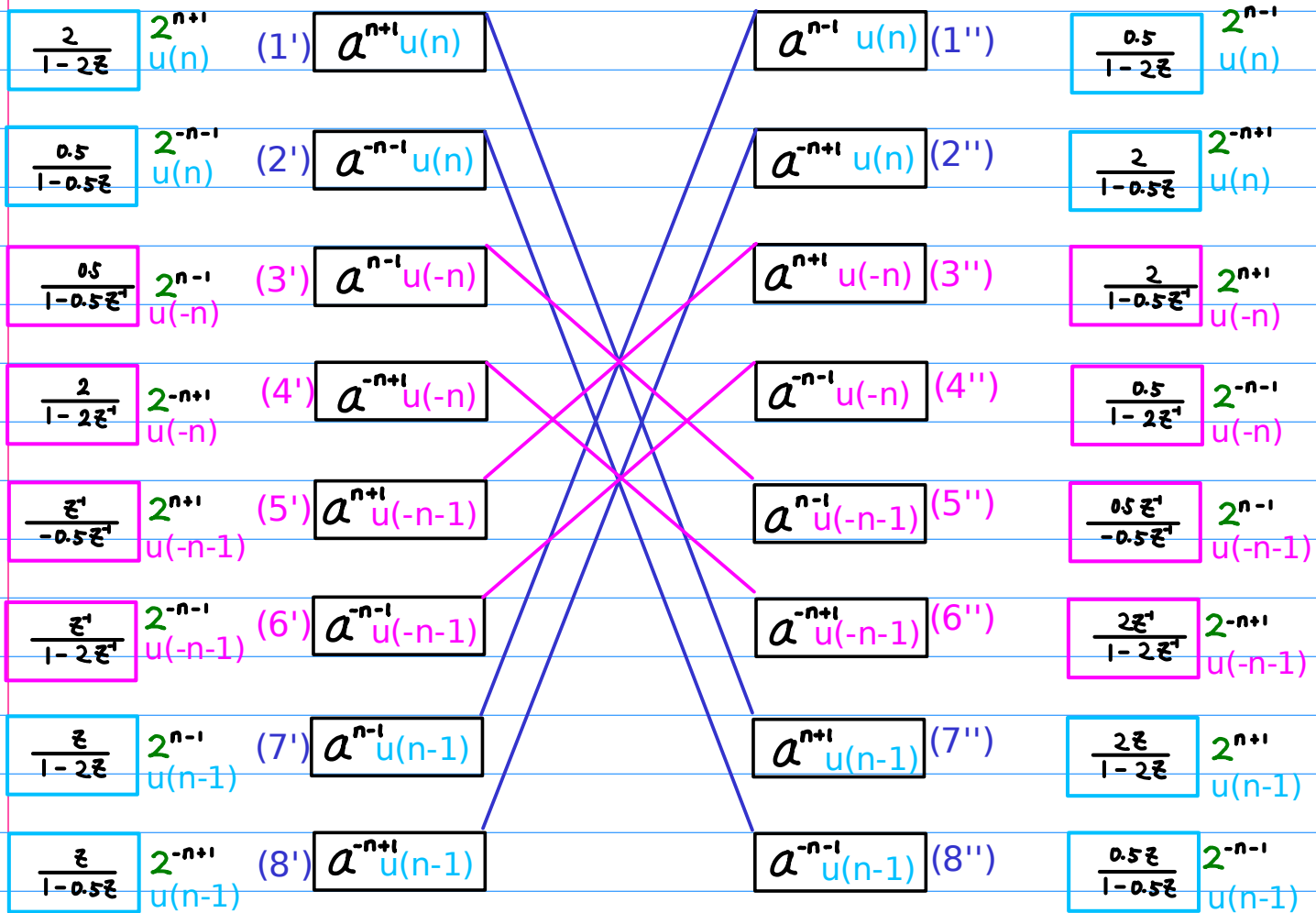
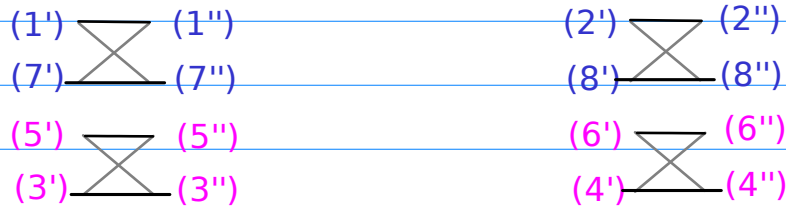


## B Shifting Shifted Sequence 2

Unshifted Sequence  $x$   Shifted Sequence 2  $x''$

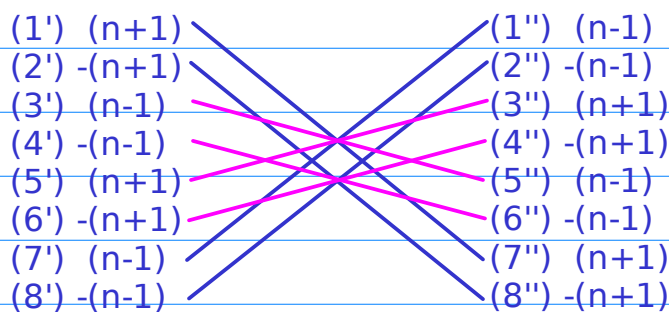


Unshifted Sequence  $x'$   Shifted Sequence  $2x''$



$shl(b^n) \iff shr(b^n)$

$shr(b^n) \iff shl(b^n)$



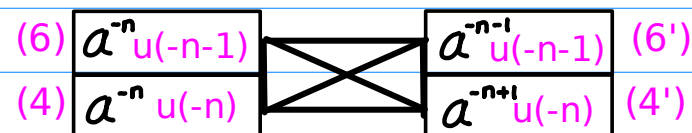
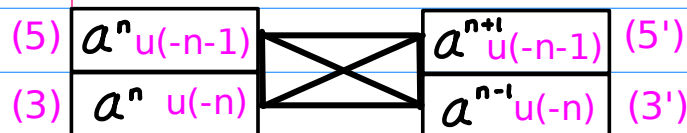
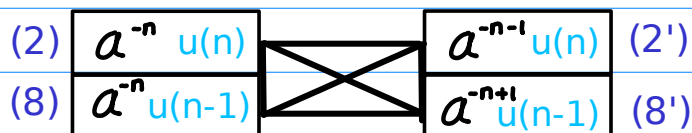
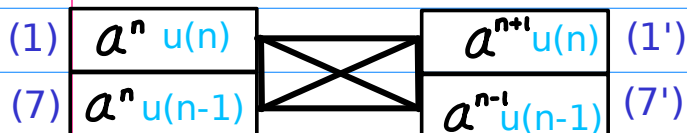
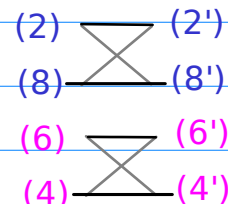
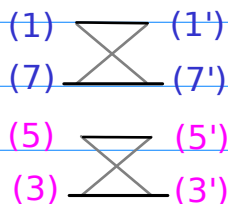
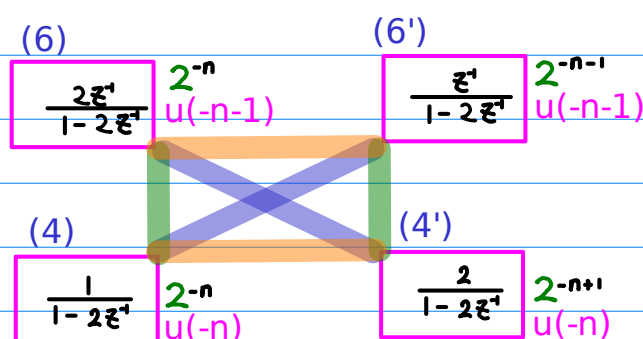
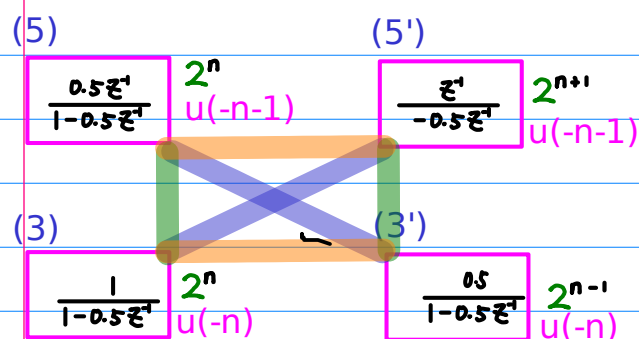
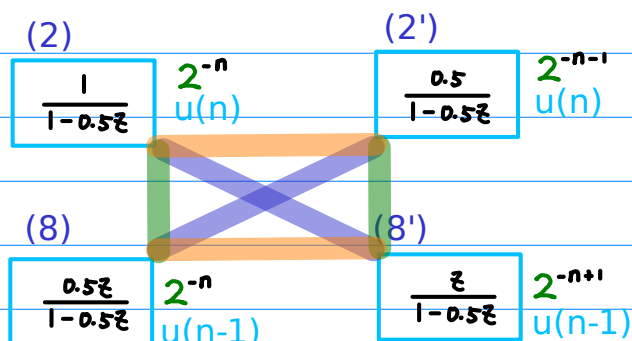
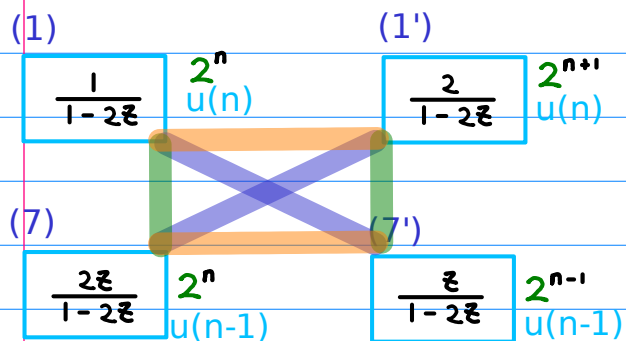
# Inter-permutation

## A Shifting Shifted Sequence 1

(x)  $\longrightarrow$  (x')

(1)~(8)

(1')~(8')

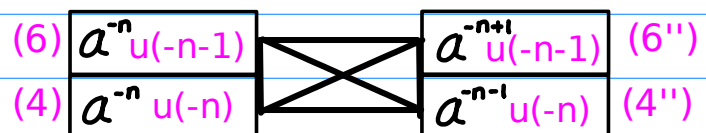
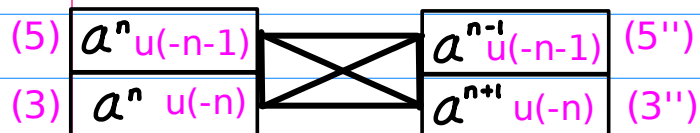
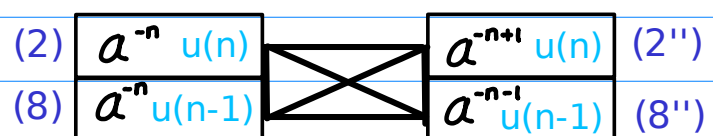
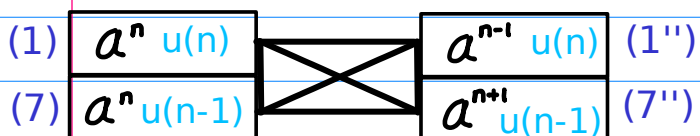
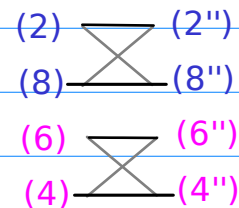
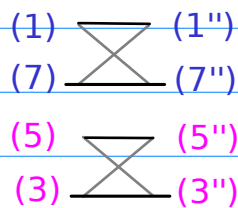
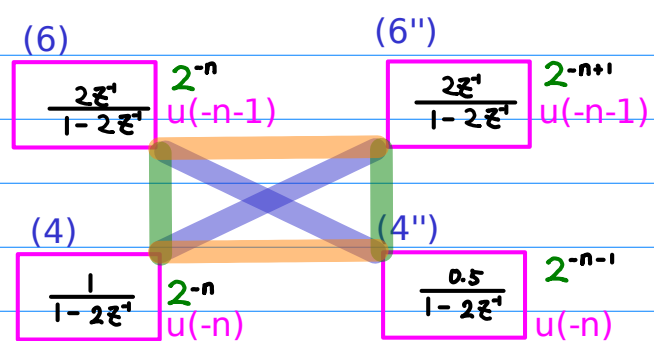
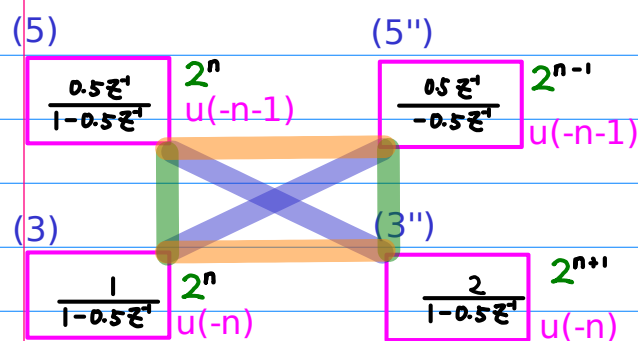
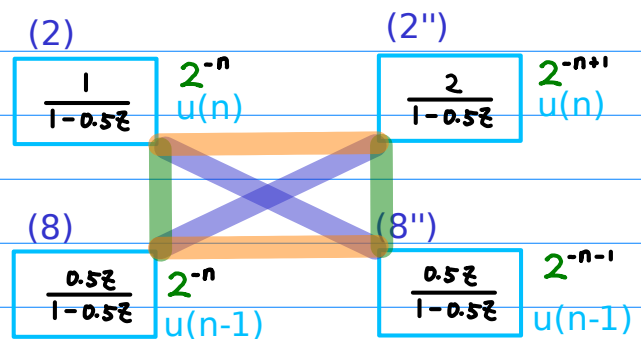
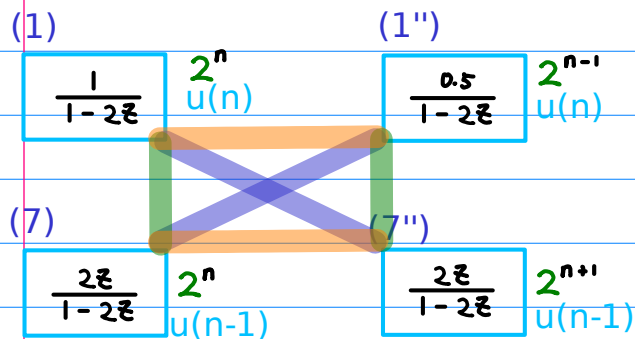


# Inter-permutation

## B Shifting Shifted Sequence 2

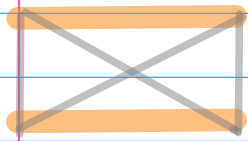
(x)  $\longrightarrow$  (x'')

(1)~(8) (1'')~(8'')

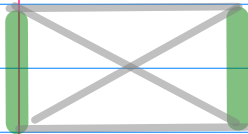




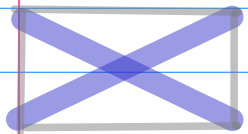
# Decomposing Shift Operations



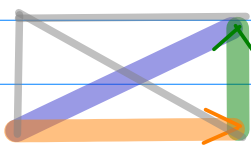
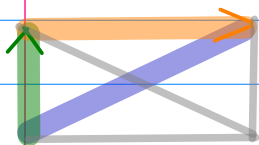
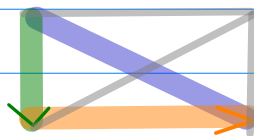
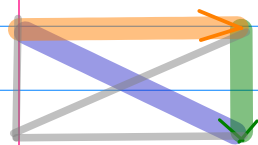
( exponent shift, identity )



( identity, range shift )



( exponent shift, range shift )  
= ( exponent shift, identity )  
+ ( identity, range shift )



**A Shifting Shifted Sequence 1**  
**(1) Exponent Shifting**  
**(2) Range Shifting**

$$\begin{aligned}(\text{SR}, \text{id}) + (\text{id}, \text{SR}) &= (\text{SR}, \text{SR}) \\ (\text{SL}, \text{id}) + (\text{id}, \text{SL}) &= (\text{SL}, \text{SL})\end{aligned}$$

**B Shifting Shifted Sequence 2**  
**(1) Exponent Shifting, Flipping**  
**(2) Range Shifting**

$$\begin{aligned}(\text{SR}, \text{id}) + (\text{id}, \text{SL}) &= (\text{SR}, \text{SL}) \\ (\text{SL}, \text{id}) + (\text{id}, \text{SR}) &= (\text{SL}, \text{SR})\end{aligned}$$

# A Shifting Shifted Sequence 1

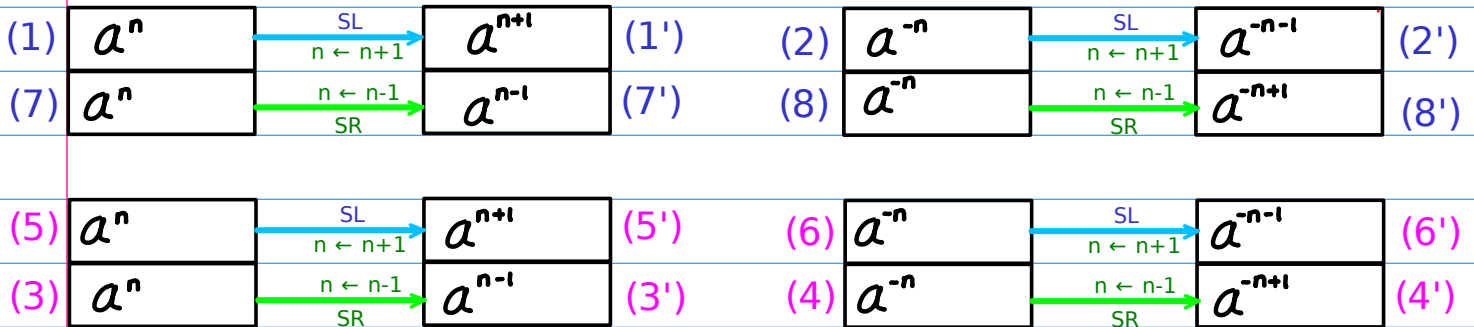
(1) Exponent Shifting

(2) Range Shifting

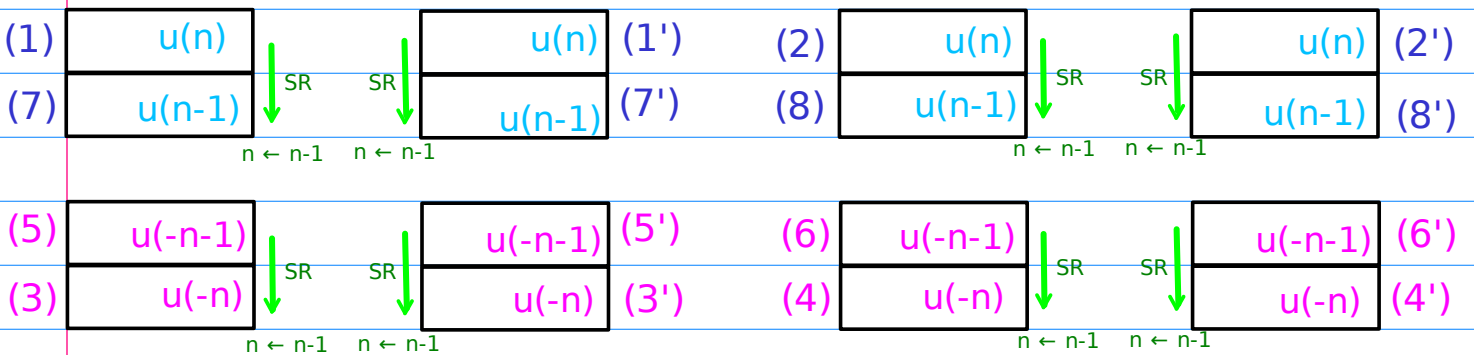
(SR, id) or (SL, id)

(id, SR) or (id, SL)

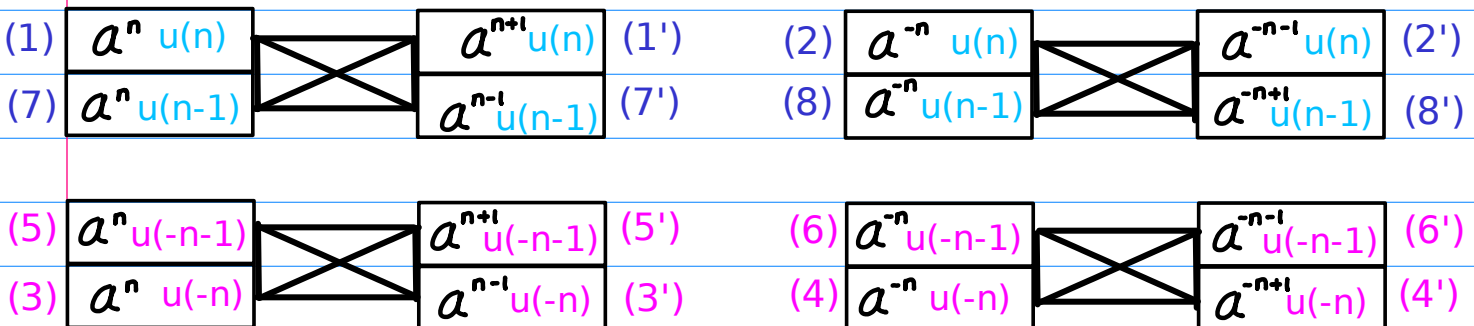
## Exponent Shifts : (SR, id) or (SL, id)



## Range Shifts : (id, SR) or (id, SL)



## Exponent & Range Permutations



## Decomposition

$$(EP, RP) = (EP, id) + (id, RP)$$

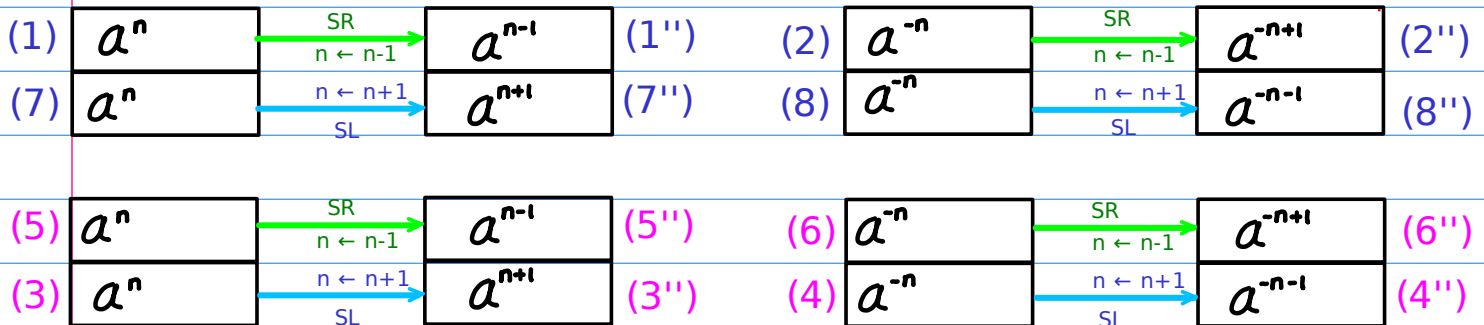
EP : Exponent Permutations

RP : Range Permutations

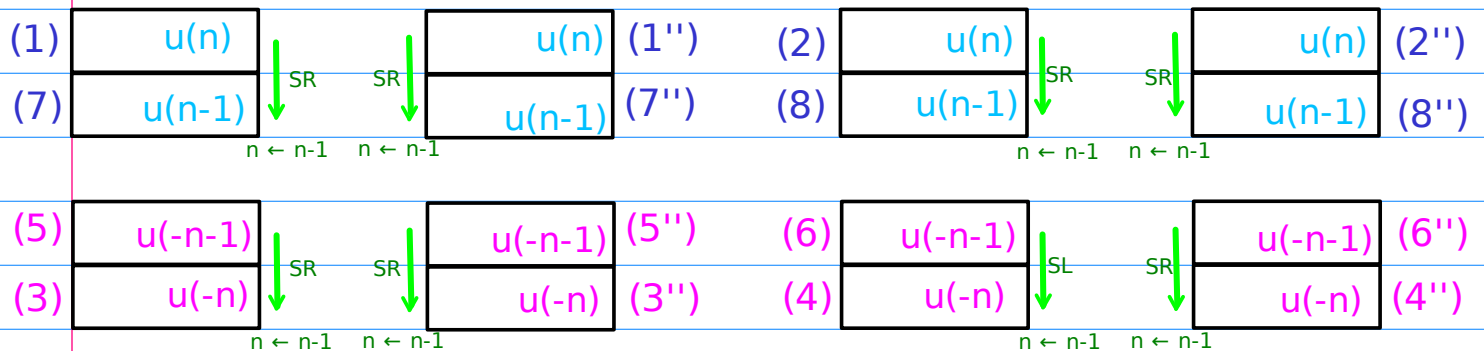
## B Shifting Shifted Sequence 2

- (1) Exponent Shifting, Flipping (SR, id) or (SL, id)  
 (2) Range Shifting (id, SR) or (id, SL)

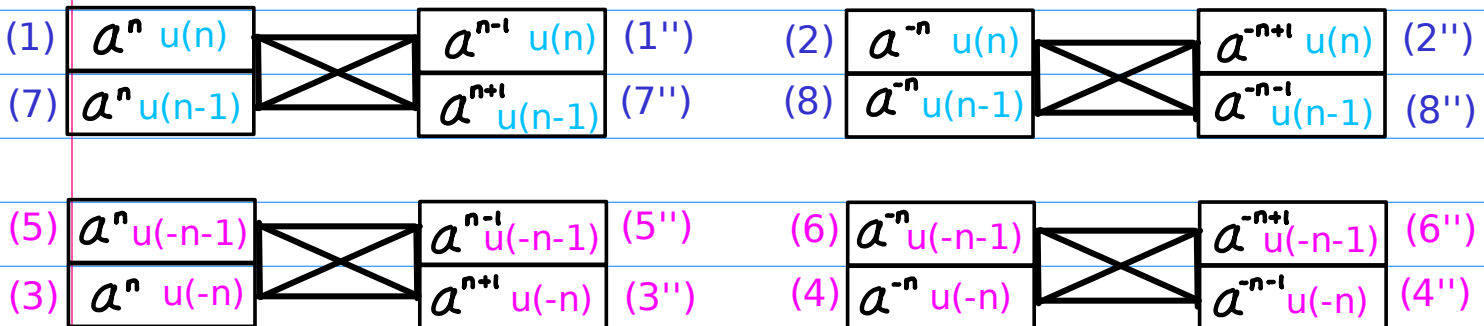
### Exponent Shifts : (SR, id) or (SL, id)



### Range Shifts : (id, SR) or (id, SL)



### Exponent & Range Permutations



### Decomposition

$$(EP, RP) = (EP, id) + (id, RP)$$

EP : Exponent Permutations

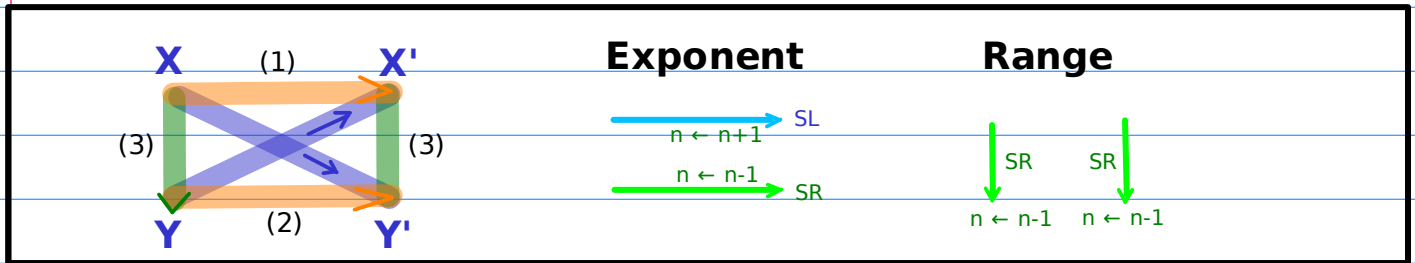
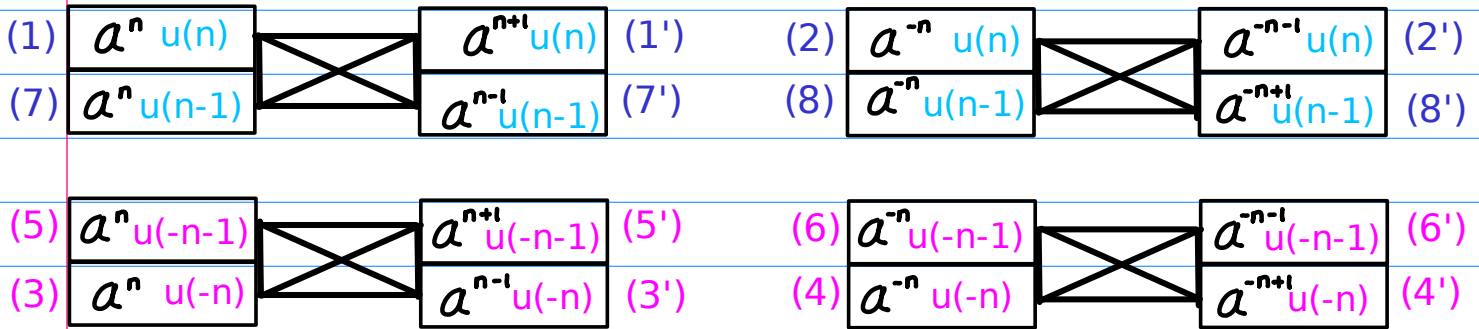
RP : Range Permutations

### A Shifting Shifted Sequence 1

- (1) Exponent Shifting
- (2) Range Shifting

$$(SR, id) + (id, SR) = (SR, SR)$$

$$(SL, id) + (id, SL) = (SL, SL)$$

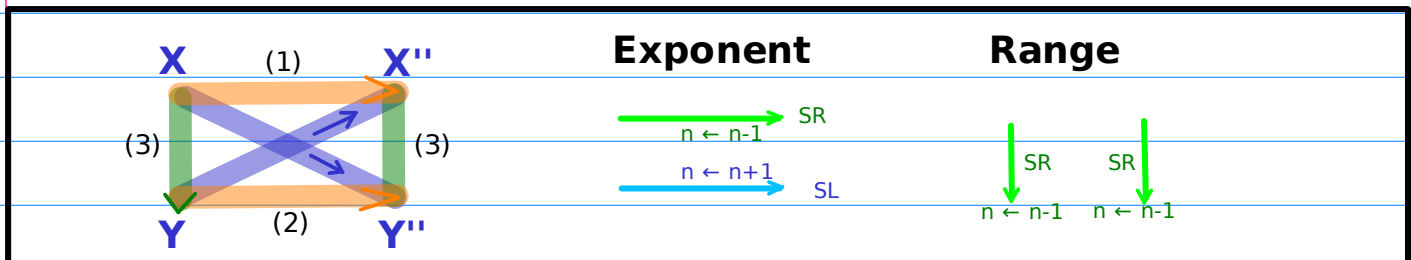
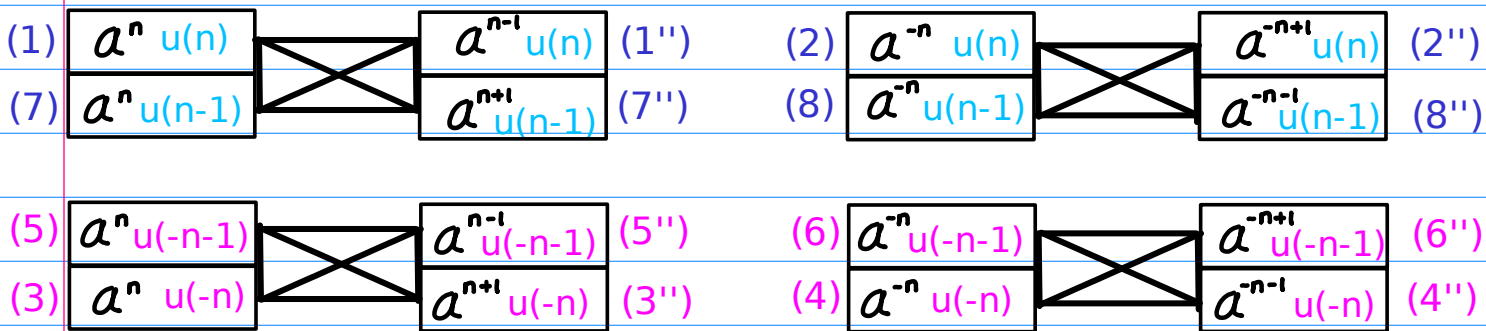


### B Shifting Shifted Sequence 2

- (1) Exponent Shifting
- (2) Range Shifting

$$(SR, id) + (id, SL) = (SR, SL)$$

$$(SL, id) + (id, SR) = (SL, SR)$$



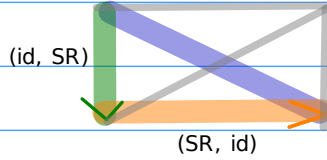
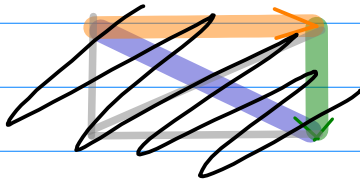
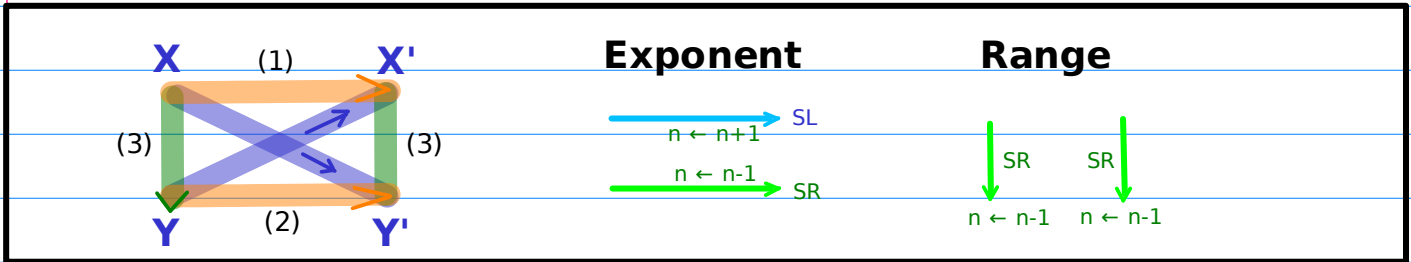
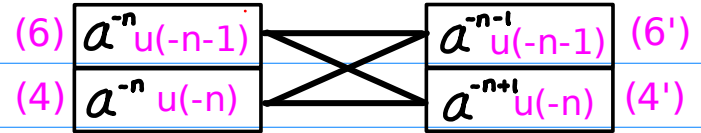
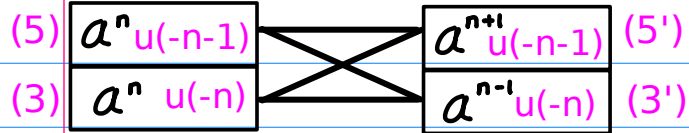
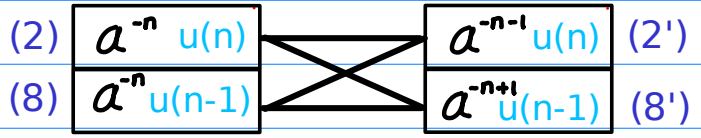
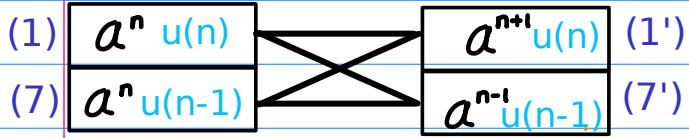
# A Shifting Shifted Sequence 1

(1) Exponent Shifting

(2) Range Shifting

$$(SR, id) + (id, SR) = (SR, SR)$$

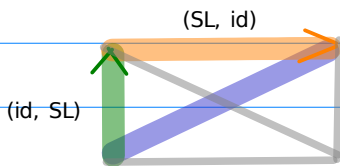
$$(SL, id) + (id, SL) = (SL, SL)$$



(SR, id) shift right exponent

(id, SR) shift right range

(SR, SR)



(SL, id) shift left exponent

(id, SL) shift left range

(SL, SL)

$$(SR, id) + (id, SR) = (SR, SR)$$

$$(SL, id) + (id, SL) = (SL, SL)$$

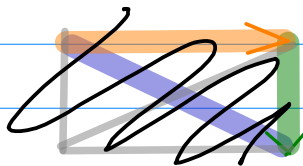
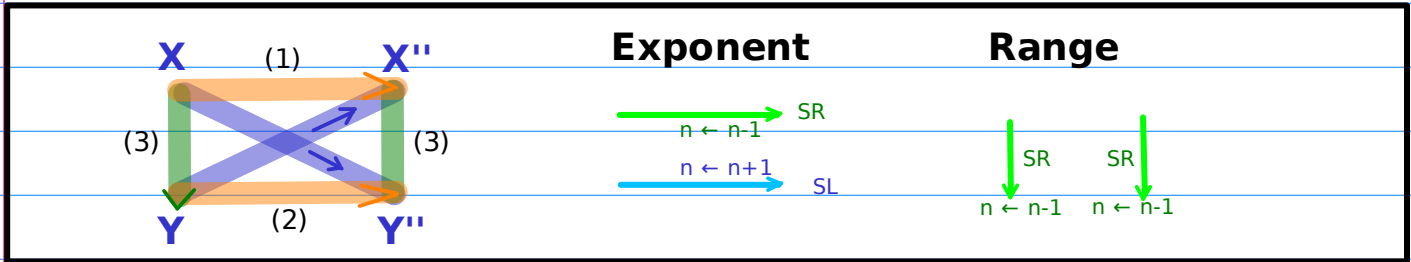
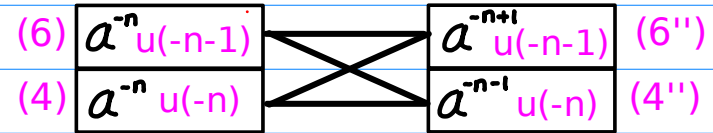
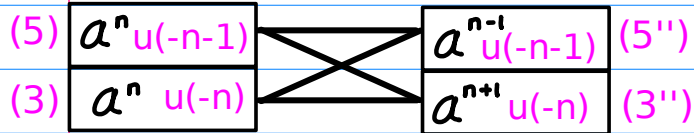
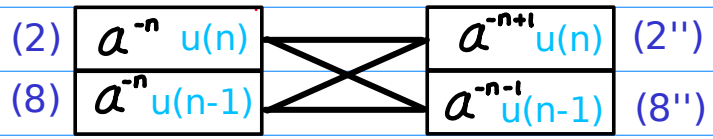
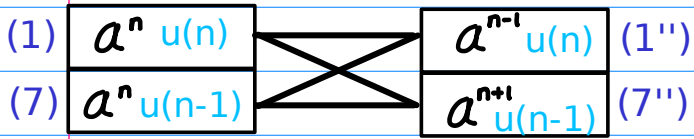
## B Shifting Shifted Sequence 2

(1) Exponent Shifting

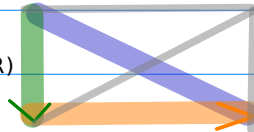
(2) Range Shifting

$$(SR, id) + (id, SL) = (SR, SL)$$

$$(SL, id) + (id, SR) = (SL, SR)$$



(id, SR)



(SL, id)

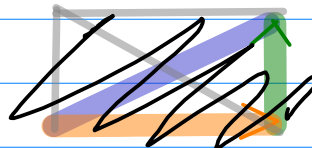
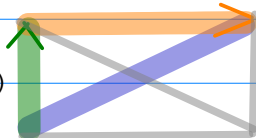
(SL, id) shift right exponent

(id, SR) shift right range

(SL, SR)

(SR, id)

(id, SL)



(SR, id) shift left exponent

(id, SL) shift left range

(SR, SL)

$$(SL, id) + (id, SR) = (SL, SR)$$

$$(SR, id) + (id, SL) = (SR, SL)$$

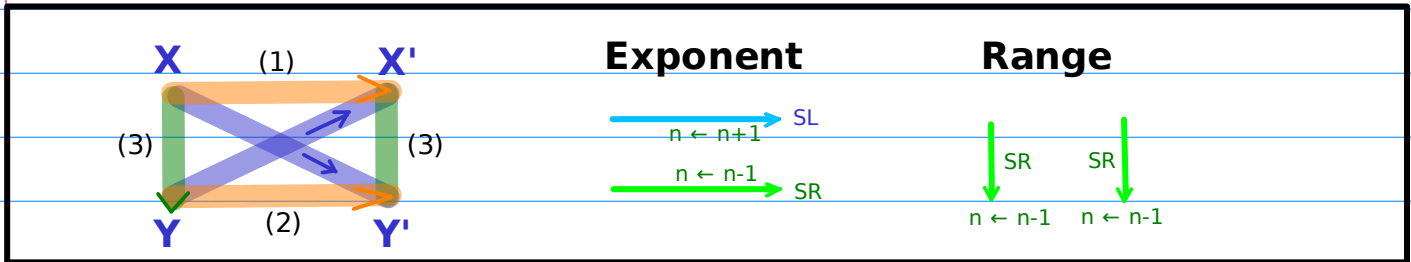




# Butterfly Relations

A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$



(\*) unit starting

(1)  $\xrightarrow{*a \leftarrow (SL, id)}$  (1') SL

$/z \leftarrow (SL, SL)$

C.R. starting

(7)  $\xrightarrow{/a \rightarrow (SR, id)}$  (7') SR

$*z \Rightarrow (SR, SR)$

$/a \rightarrow (SR, id)$

(\*) C.R. starting

(5)  $\xrightarrow{*a \leftarrow (SL, id)}$  (5') SL

$/z \leftarrow (SL, SL)$

unit starting

(3)  $\xrightarrow{/a \rightarrow (SR, id)}$  (3') SR

$*z \Rightarrow (SR, SR)$

$/a \rightarrow (SR, id)$

(\*) unit starting

(2)  $\xrightarrow{/a \leftarrow (SL, id)}$  (2') SL

$/z \leftarrow (SL, SL)$

C.R. starting

(8)  $\xrightarrow{*a \rightarrow (SR, id)}$  (8') SR

$*z \Rightarrow (SR, SR)$

$*a \rightarrow (SR, id)$

(\*) C.R. starting

(6)  $\xrightarrow{/a \leftarrow (SL, id)}$  (6') SL

$/z \leftarrow (SL, SL)$

unit starting

(4)  $\xrightarrow{*a \rightarrow (SR, id)}$  (4') SR

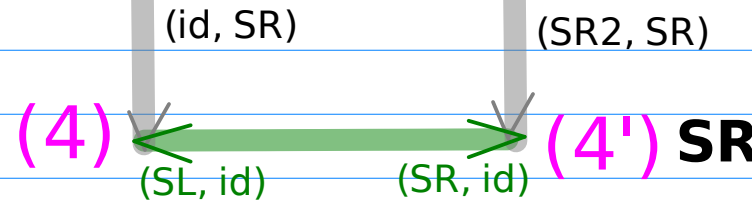
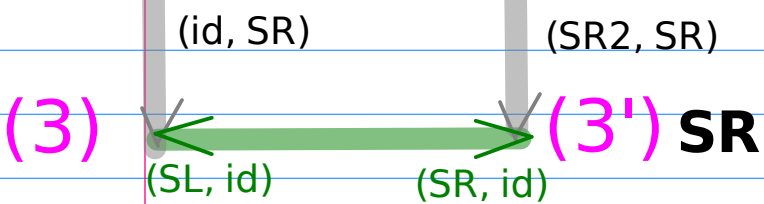
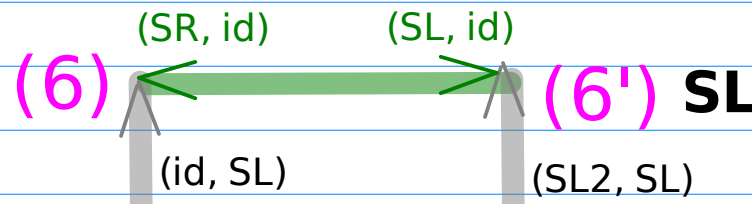
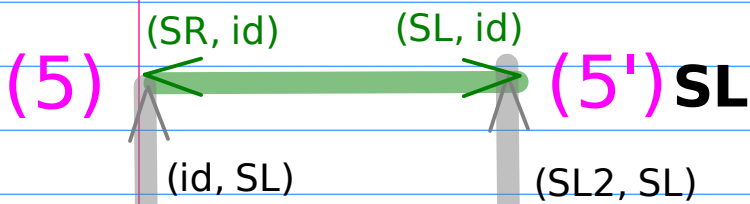
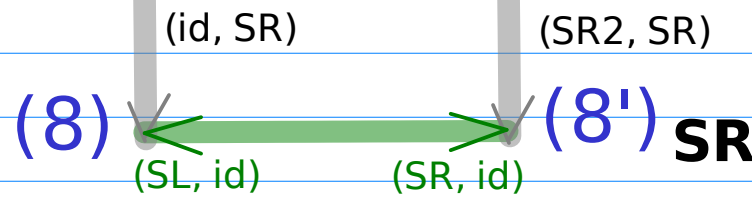
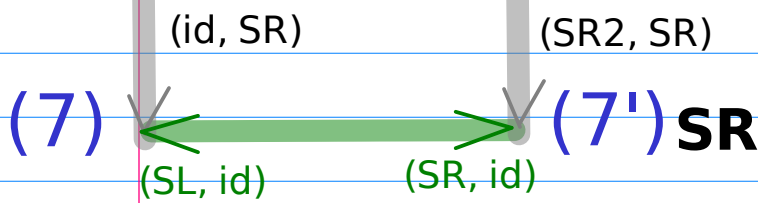
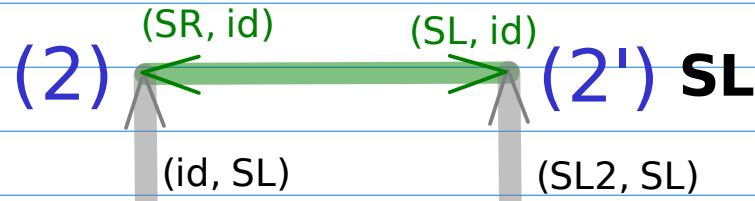
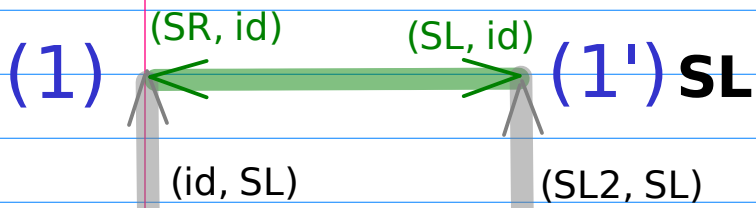
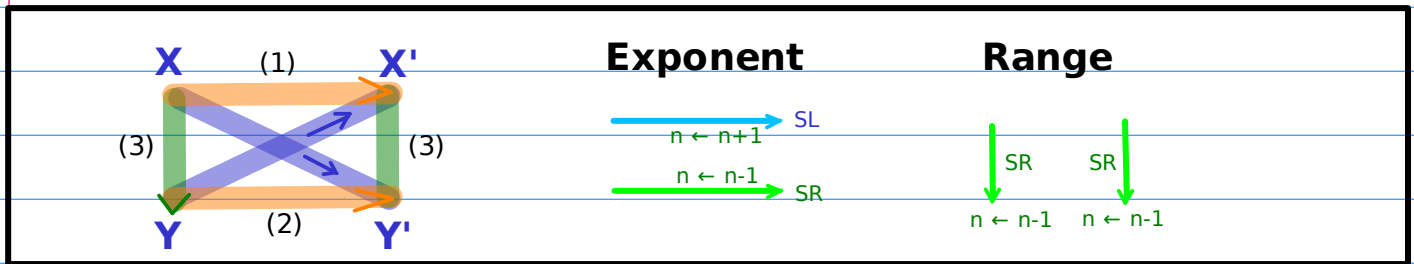
$*z \Rightarrow (SR, SR)$

$*a \rightarrow (SR, id)$

# Butterfly Relations

A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

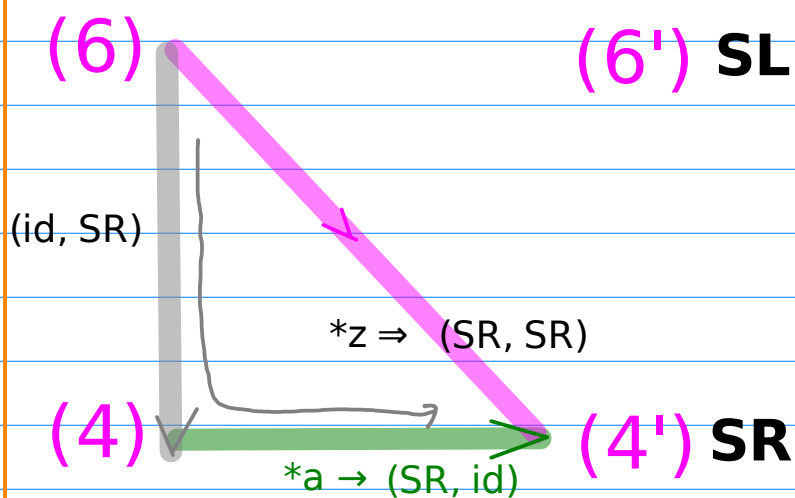
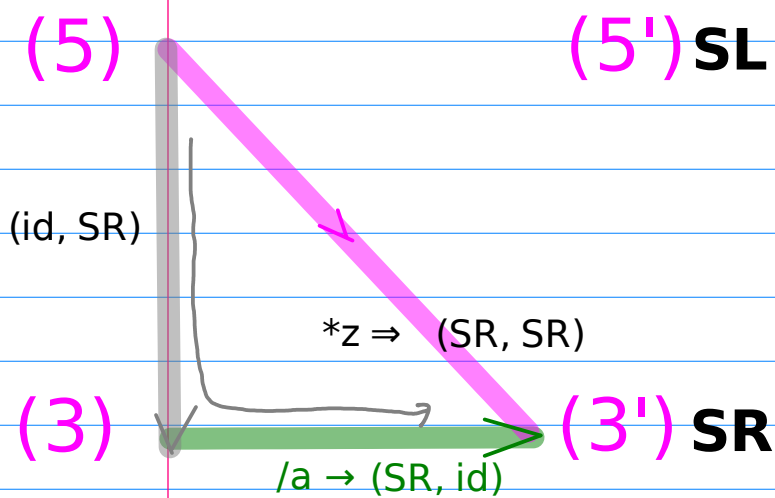
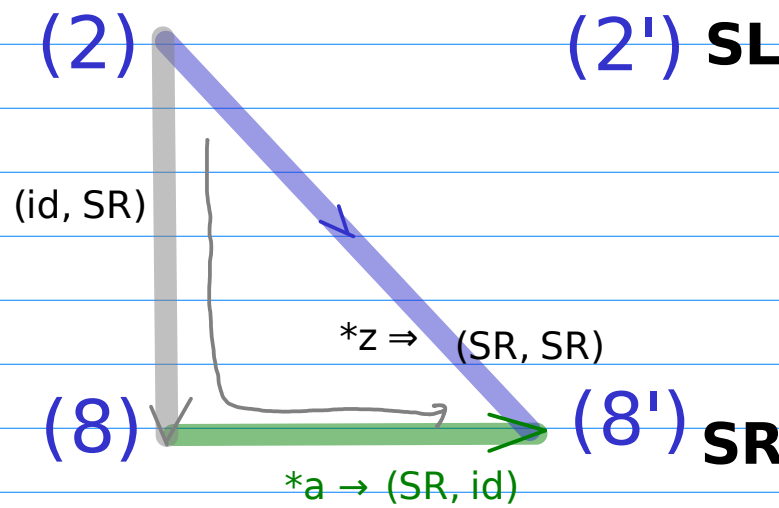
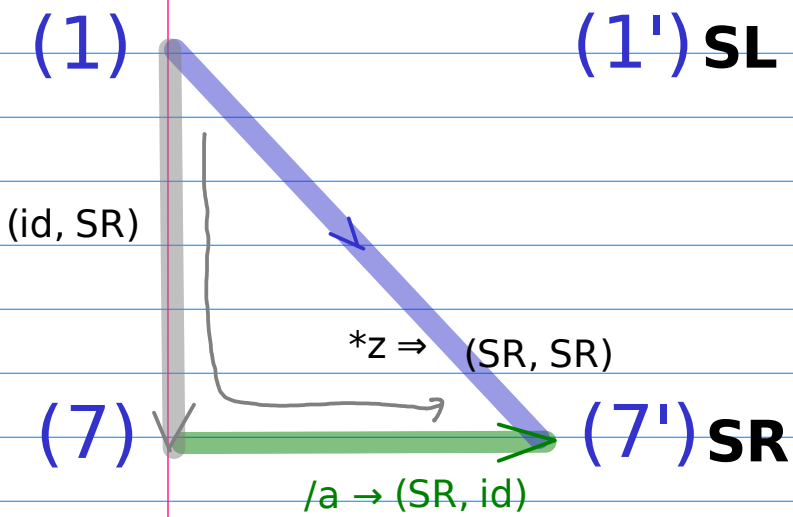
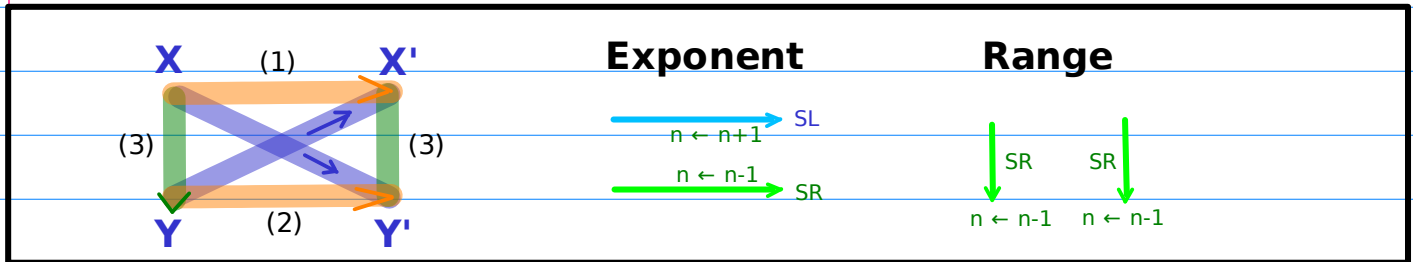


# Decomposition of Exp and Rng Shifts (1)

A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

$$(id, SR) + (SR, id) = (SR, SR)$$

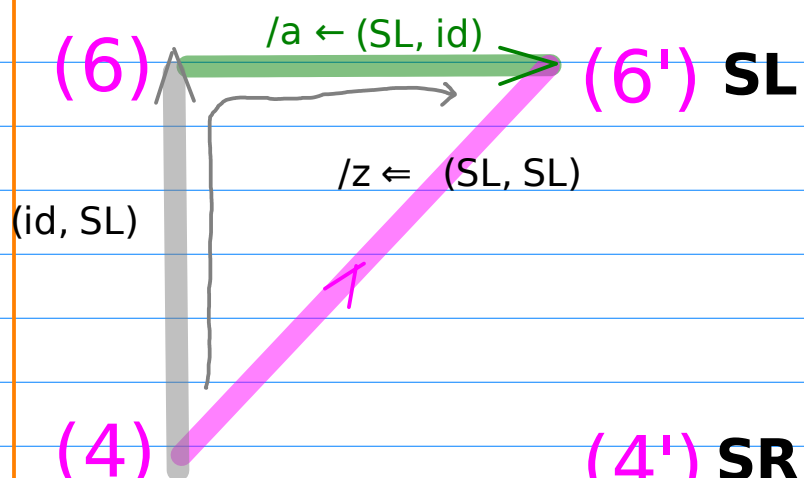
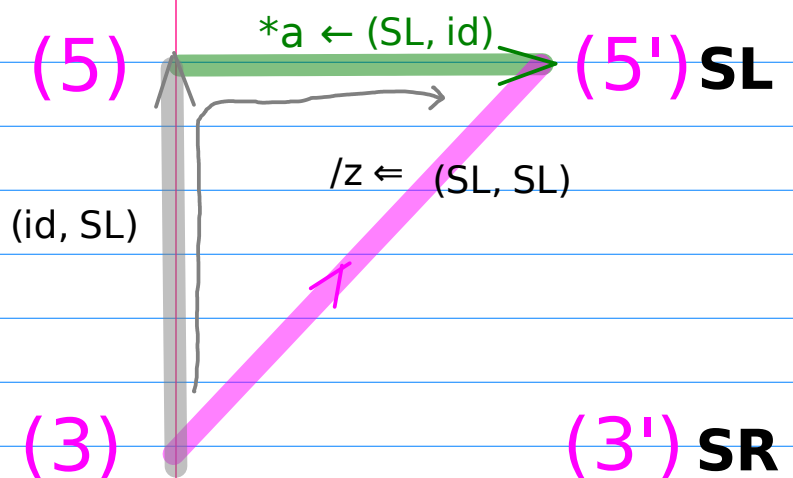
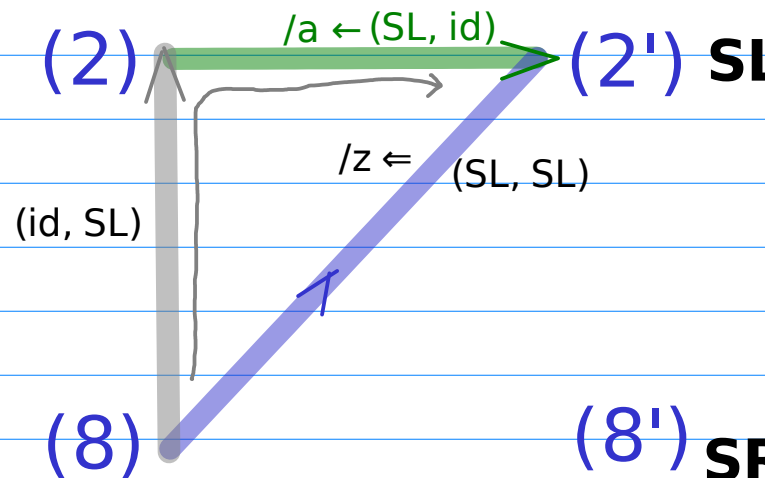
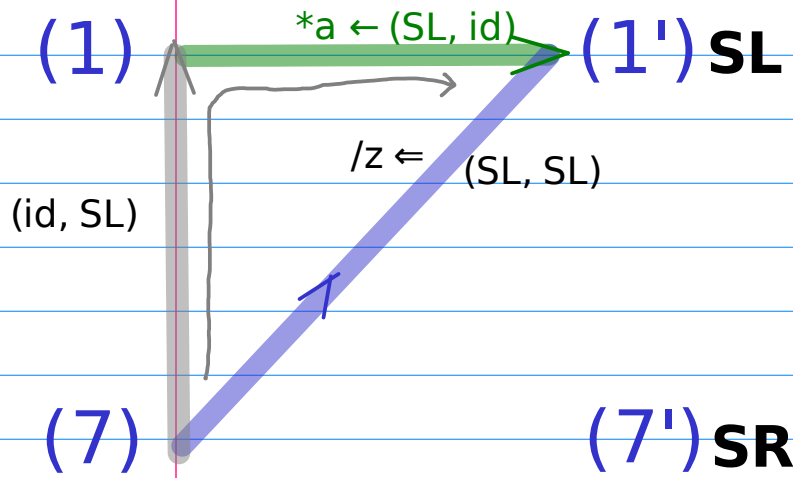
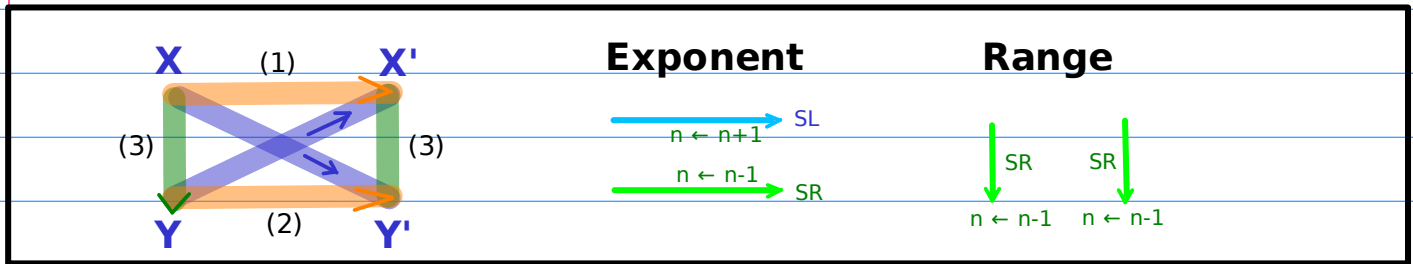


# Decomposition of Exp and Rng Shifts (2)

A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

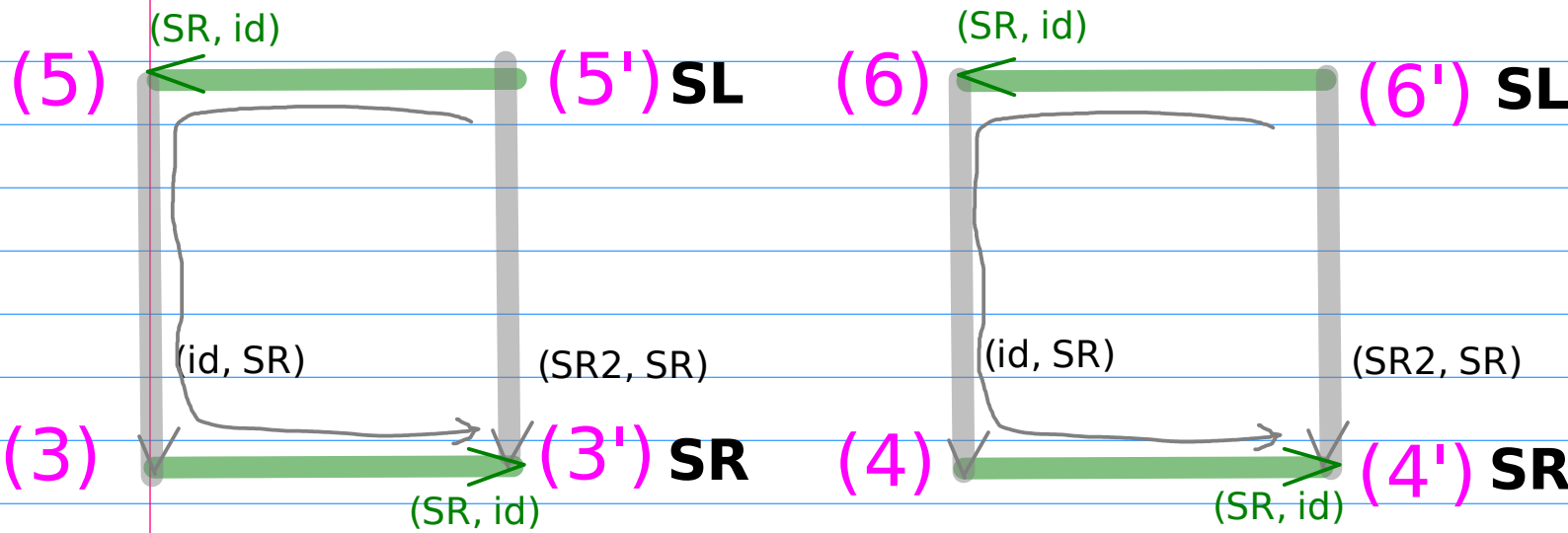
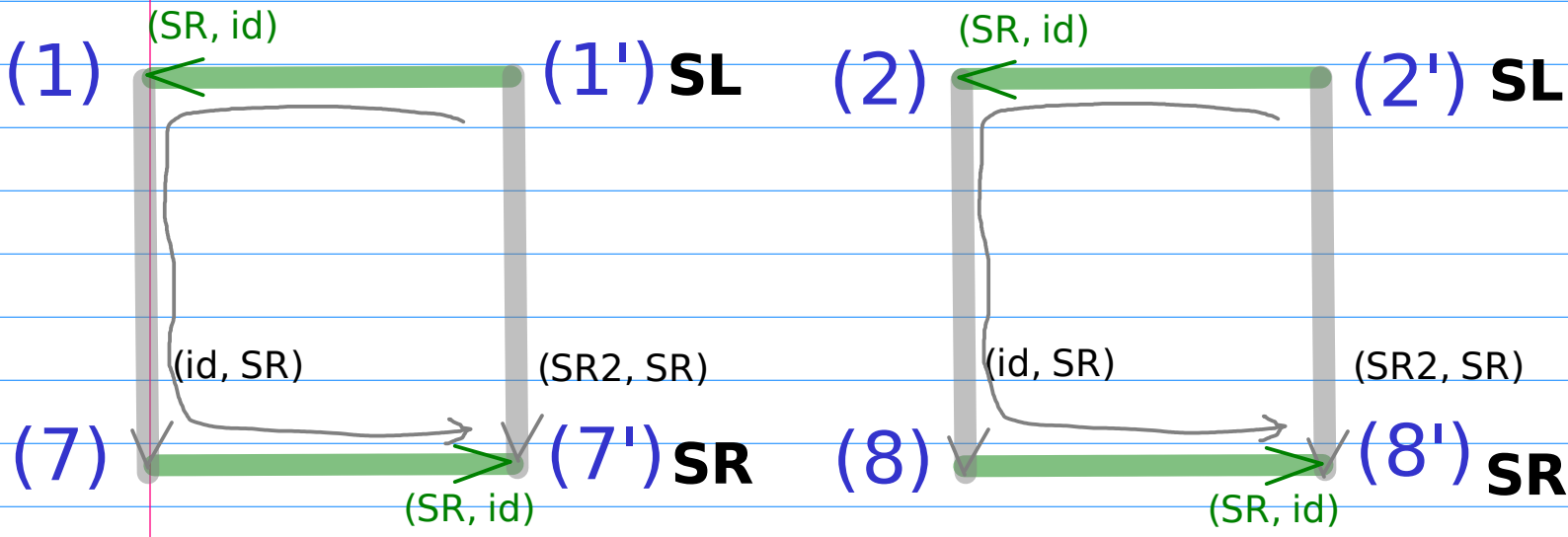
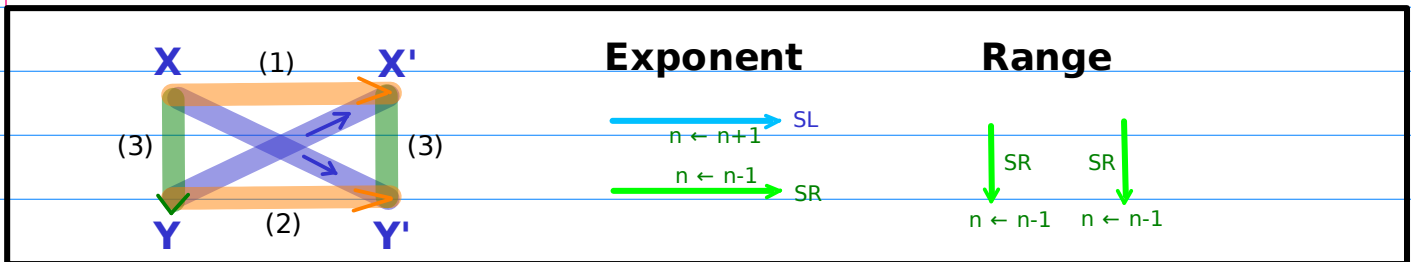
$$(id, SL) + (SL, id) = (SL, SL)$$



# Decomposition of Exp and Rng Shifts (3)

A Shifting Shifted Sequence 1  
 Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

$$(SR, id) + (id, SR) + (SR, id) = (SR2, SR)$$





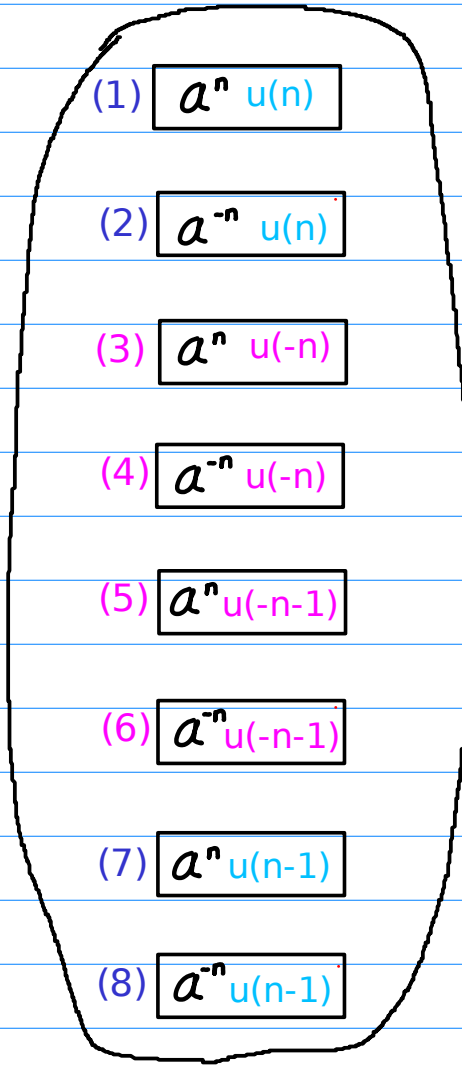
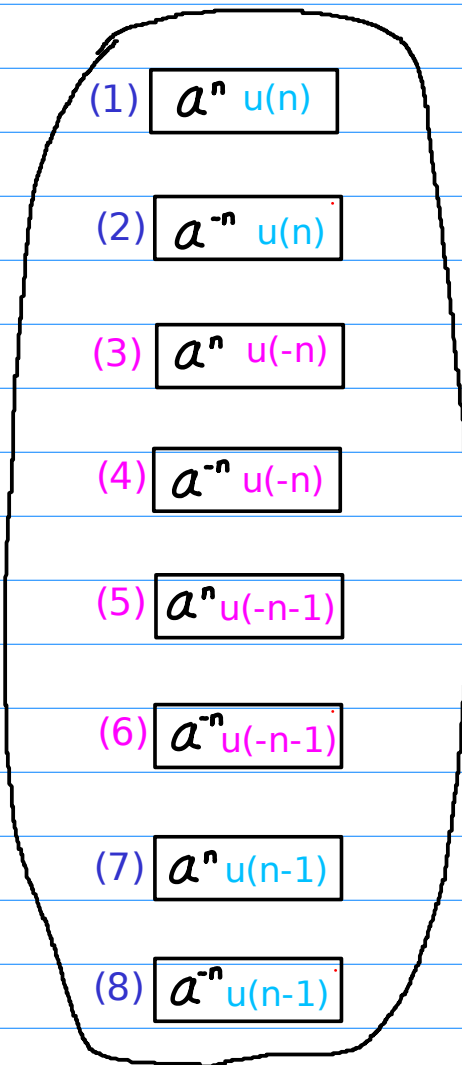


# Intra-Permutations over (1) ~ (8)

Unshifted Sequence  $x$



Unshifted Sequence  $x$



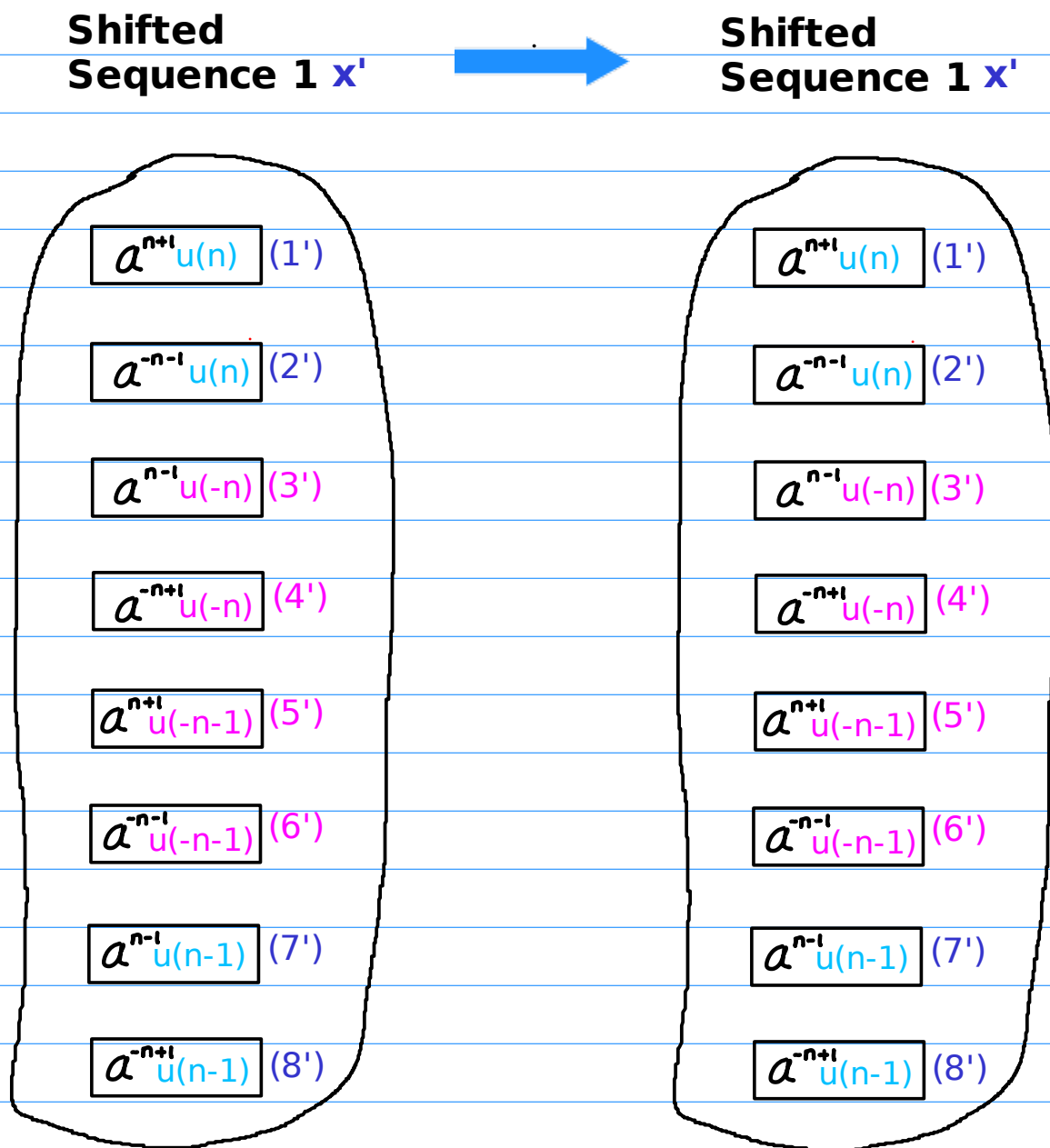
Inter-permutations over unshifted sequence and shifted sequence

Intra-permutations over unshifted sequence

Intra-permutations over shifted sequence



# Intra-Permutations over (1') ~ (8')

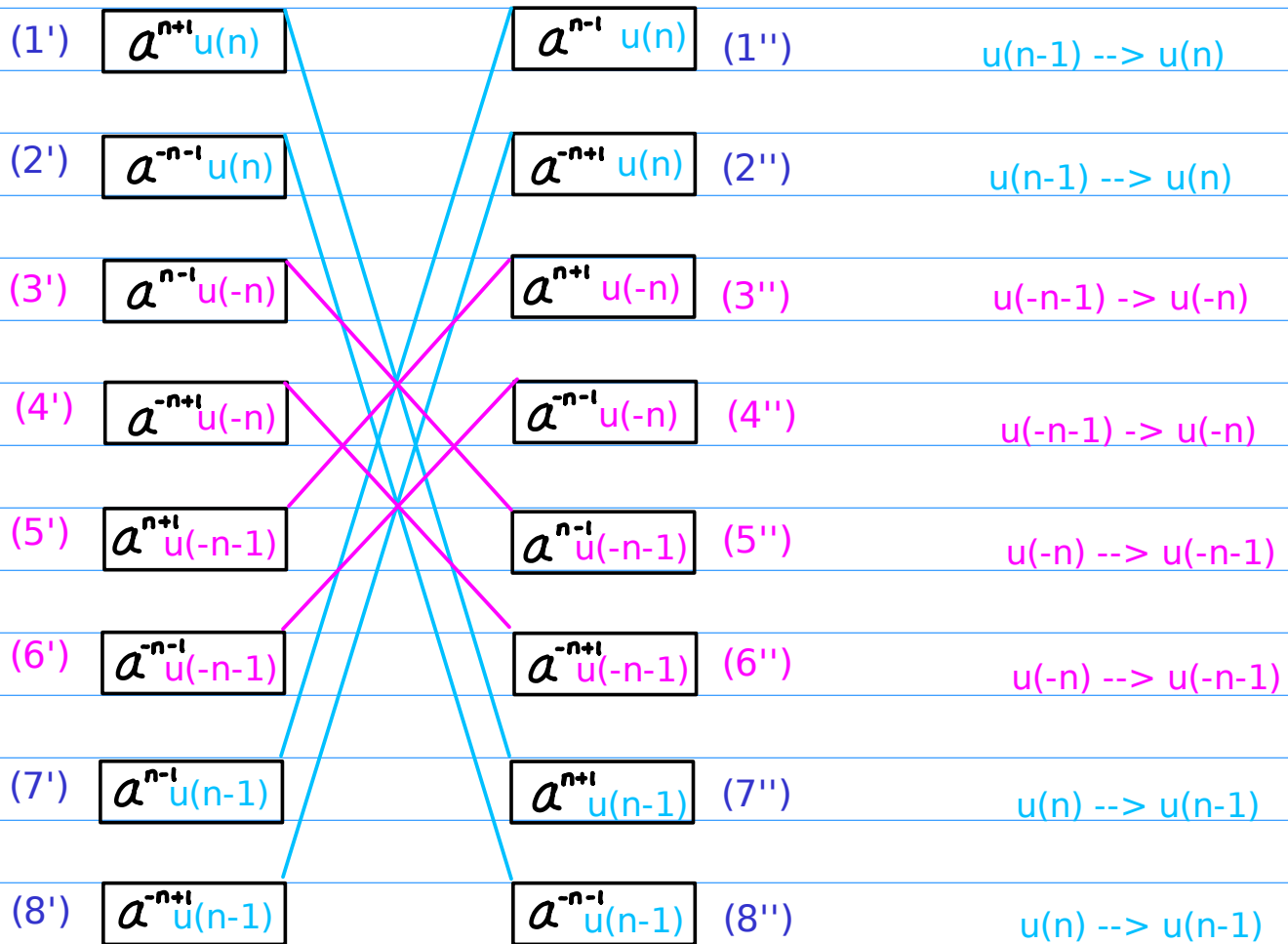


Inter-permutations over unshifted sequence and shifted sequence

Intra-permutations over unshifted sequence

Intra-permutations over shifted sequence

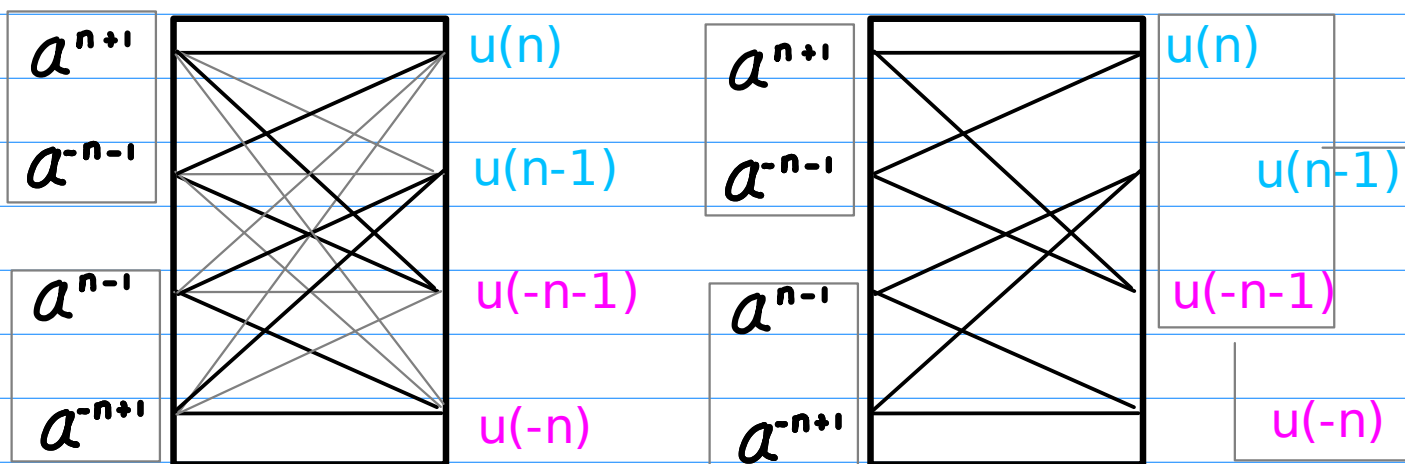
## Shifted Sequence 1 $x'$





$$a^n \times R(n)$$

$a^{n+1}$	$a^{-n-1}$	×	$u(n)$	$u(-n-1)$
$a^{n-1}$	$a^{-n+1}$	×	$u(n-1)$	$u(-n)$




$n \leftarrow n+2$ or $n \leftarrow n-2$
---


(1') $a^{n+1}u(n)$	$a^{-n-1}u(n)$	(2')
(3') $a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	(4')
(5') $a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	(6')
(7') $a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	(8')

(1'') $a^{n-1}u(n)$	$a^{-n+1}u(n)$	(2'')
(3'') $a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	(4'')
(5'') $a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	(6'')
(7'') $a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	(8'')



permutation over (1) ~ (8)    **(x)**        **(x)**

<b>A. Flipping</b>	(1) - (4)	(5) - (8)
Base Inverting	(2) - (3)	(6) - (7)
Range Flipping	(3) - (2)	(7) - (6)
	(4) - (1)	(8) - (5)
<b>B. Range Shifting</b>	(1) - (7)	(5) - (3)
Range Flipping	(2) - (8)	(6) - (4)
Range Complementing	(3) - (5)	(7) - (1)
	(4) - (6)	(8) - (2)
<b>C. Complementary Inverting</b>	(1) - (6)	(5) - (2)
Base Inverting	(6) - (1)	(6) - (1)
Range Complementing	(2) - (5)	(7) - (4)
	(5) - (2)	(8) - (3)

permutation over (1') ~ (8')    **(x')**        **(x')**

<b>D. Flipping2</b>	(1') - (4')	(5') - (8')
Base Inverting	(2') - (3')	(6') - (7')
Shifted Range Flipping	(3') - (2')	(7') - (6')
	(4') - (1')	(8') - (5')
<b>E. Shifting2</b>	(1') - (7')	(5') - (3')
Shifted Range Flipping	(2') - (8')	(6') - (4')
Range Complementing	(3') - (5')	(7') - (1')
	(4') - (6')	(8') - (2')
<b>F. Complementary Inverting</b>	(1') - (6')	(5') - (2')
Base Inverting	(6') - (1')	(6') - (1')
Range Complementing	(2') - (5')	(7') - (4')
	(5') - (2')	(8') - (3')

**Shifted Range Flipping = Exponent Shifting2 + Range Flipping**

**Shifting2 = Shifted Range Flipping + Range Complementing**  
**= Exponent Shifting2 + Range Flipping + Range Complementing**  
**= Exponent Shifting2 + Range (Flipping + Complementing)**  
**= Exponent Shifting2 + Range Shifting**

## Permutation over (1) ~ (8)

Permutations	A	B	C
Base Inverting	X		X
Range Flipping	X	X	
Range Complementing		X	X

## Permutation over (1') ~ (8')

Permutations	D	E	F
Base Inverting	X		X
Shifted Range Flipping	X	X	
Range Complementing		X	X

**Shifted Range Flipping = Exponent Shifting<sub>2</sub> + Range Flipping**

**Shifting<sub>2</sub> = Shifted Range Flipping + Range Complementing**  
**= Exponent Shifting<sub>2</sub> + Range Flipping + Range Complementing**  
**= Exponent Shifting<sub>2</sub> + Range (Flipping + Complementing)**  
**= Exponent Shifting<sub>2</sub> + Range Shifting**

**Over (1) ~ (8)**

**Base Inverting**

$$a^n \longleftrightarrow a^{-n}$$

**Range Flipping**

$$R(n) \longleftrightarrow R(-n)$$

**Range Complementing**

$$R(n) \longleftrightarrow \overline{R(n)}$$

**Over (1') ~ (8')**

**Base Inverting**

$$a^n \longleftrightarrow a^{-n}$$

**Shifted Range Flipping**

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

**Range Complementing**

$$R(n) \longleftrightarrow \overline{R(n)}$$



**A.I Flipping**  
**Base Inverting**  
**Range Flipping**

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

**D.I Flipping2**  
**Base Inverting**  
**Shifted Range Flipping**

$$a^n \longleftrightarrow a^{-n}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

**B.I Range Shifting**  
**Range Flipping**  
**Range Complementing**

$$R(n) \longleftrightarrow R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

**E.I Shifting2**  
**Shifted Range Flipping**  
**Range Complementing**

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

**C.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

**F.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

### G.I Flipping2

**Base Inverting  
Shifted Range Flipping**

$$\begin{aligned} a^n &\longleftrightarrow a^{-n} \\ a^n R(n) &\longleftrightarrow a^{sh2(n)} R(-n) \\ a^n R(n) &\longleftrightarrow a^{-sh2(n)} R(-n) \end{aligned}$$

### H.I Shifting2

**Shifted Range Flipping  
Range Complementing**

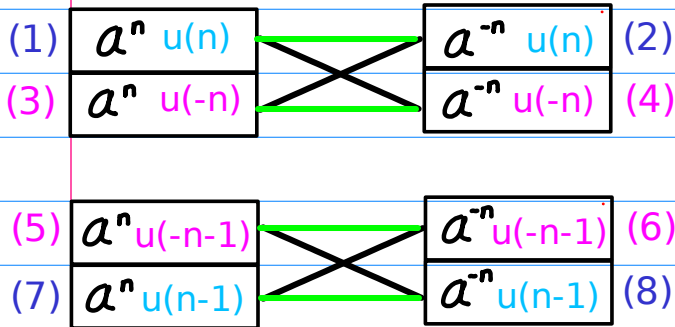
$$\begin{aligned} a^n R(n) &\longleftrightarrow a^{sh2(n)} R(-n) \\ R(n) &\longleftrightarrow \overline{R(n)} \\ a^n R(n) &\longleftrightarrow a^{sh2(n)} \overline{R(-n)} \end{aligned}$$

### I.I Complementary Inverting

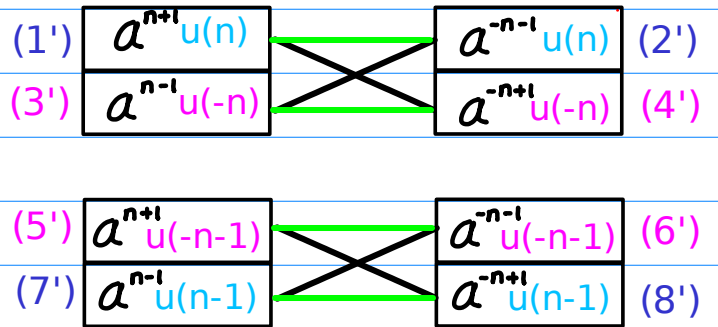
**Base Inverting  
Range Complementing**

$$\begin{aligned} a^n &\longleftrightarrow a^{-n} \\ R(n) &\longleftrightarrow \overline{R(n)} \\ a^n R(n) &\longleftrightarrow a^{-n} \overline{R(n)} \end{aligned}$$

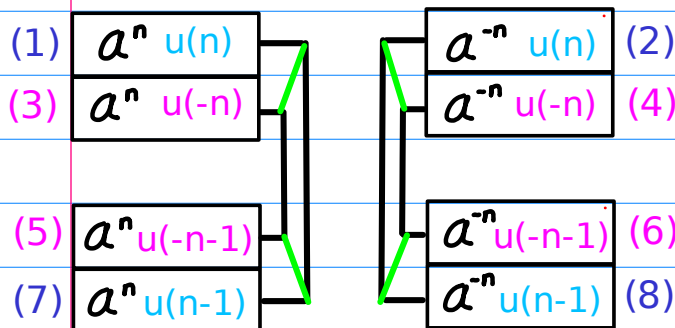
**A.I Flipping**  
**Base Inverting**  
**Range Flipping**



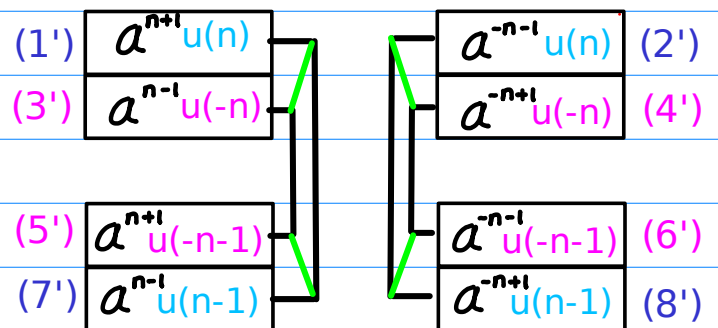
**D.I Flipping2**  
**Base Inverting**  
**Shifted Range Flipping**



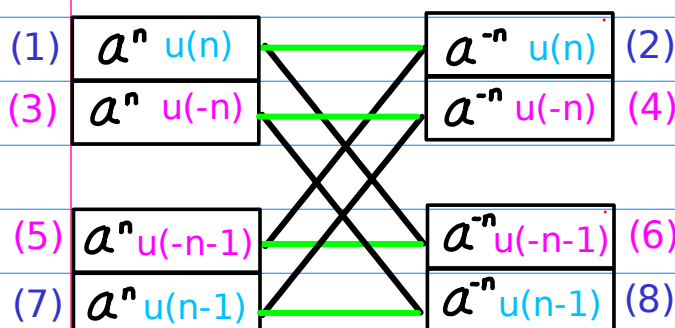
**B.I Range Shifting**  
**Range Flipping**  
**Range Complementing**



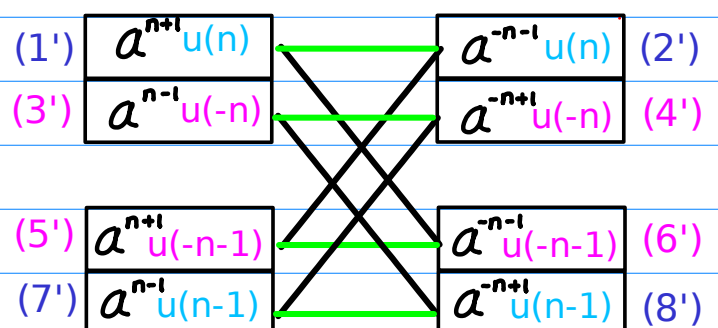
**E.I Shifting2**  
**Shifted Range Flipping**  
**Range Complementing**



**C.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**

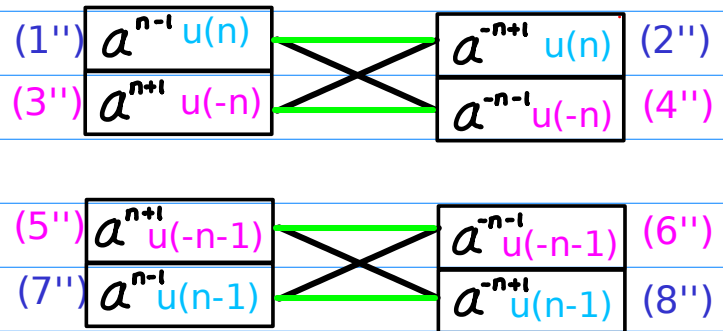


**F.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**



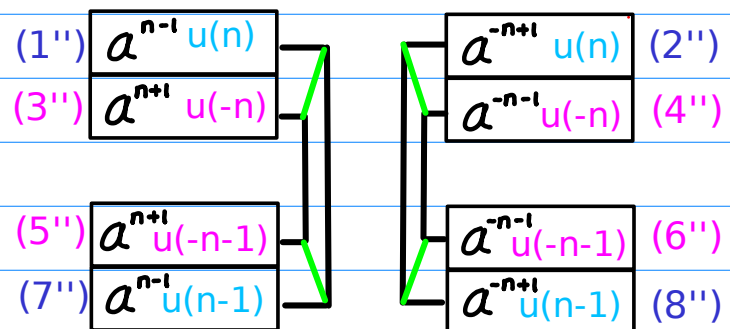
### G.I Flipping2

Base Inverting  
Shifted Range Flipping



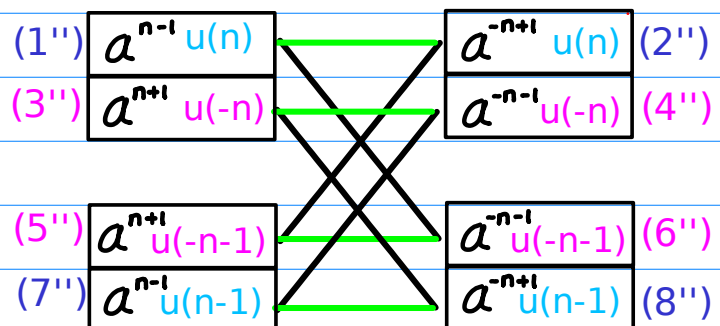
### H.I Shifting2

Shifted Range Flipping  
Range Complementing



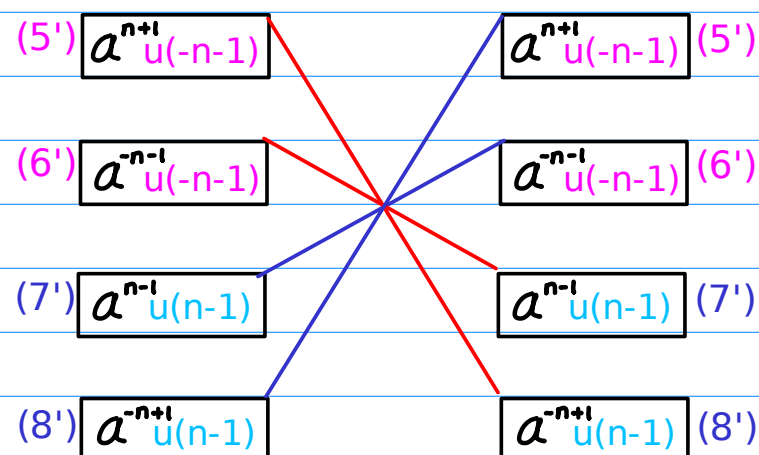
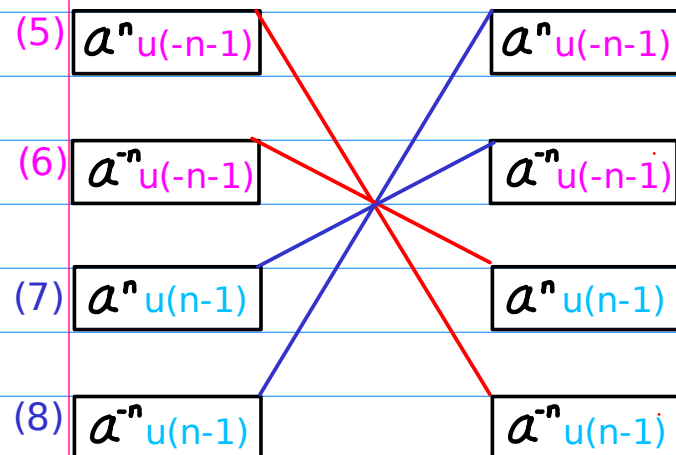
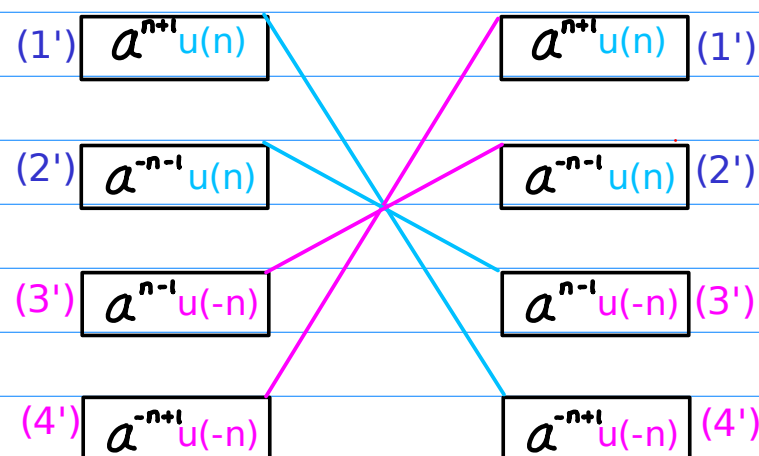
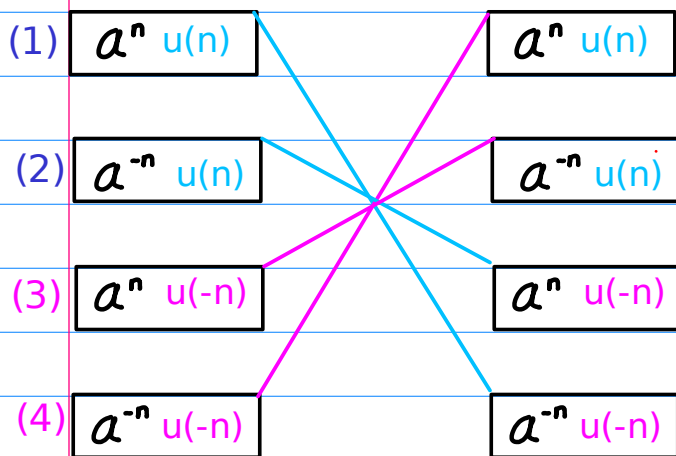
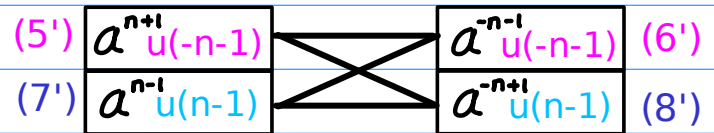
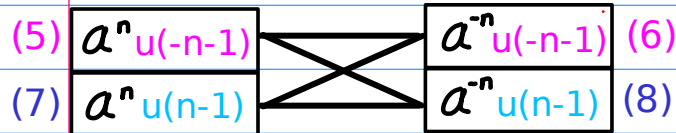
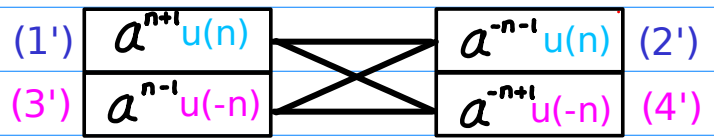
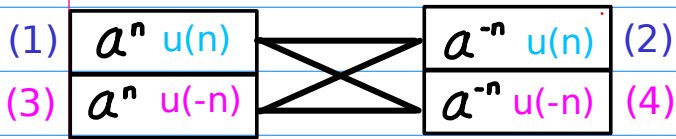
### I.I Complementary Inverting

Base Inverting  
Range Complementing



**A.I Flipping**  
**Base Inverting**  
**Range Flipping**

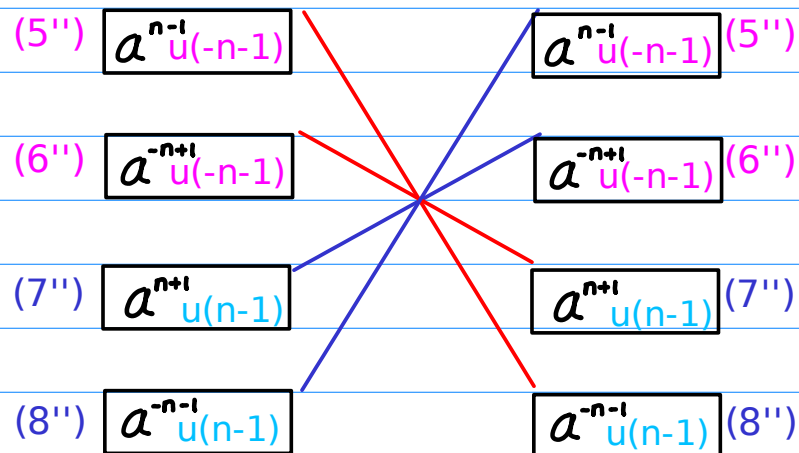
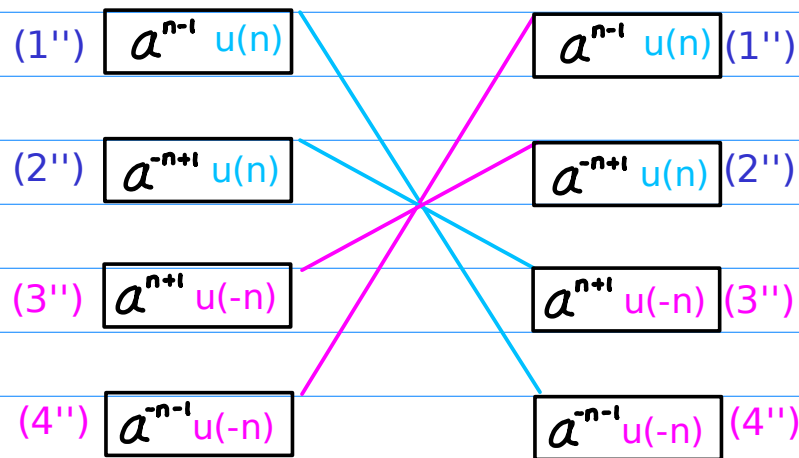
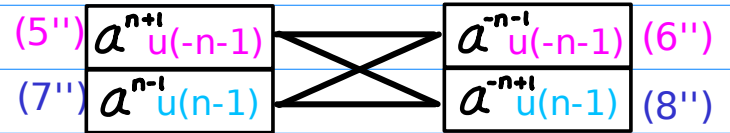
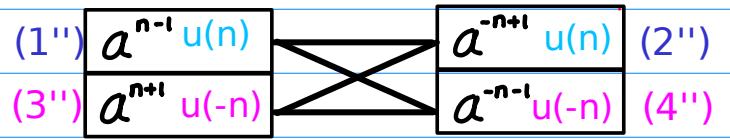
**D.I Flipping2**  
**Base Inverting**  
**Shifted Range Flipping**



(1) - (4)      (5) - (8)  
 (2) - (3)      (6) - (7)  
 (3) - (2)      (7) - (6)  
 (4) - (1)      (8) - (5)

(1') - (4')      (5') - (8')  
 (2') - (3')      (6') - (7')  
 (3') - (2')      (7') - (6')  
 (4') - (1')      (8') - (5')

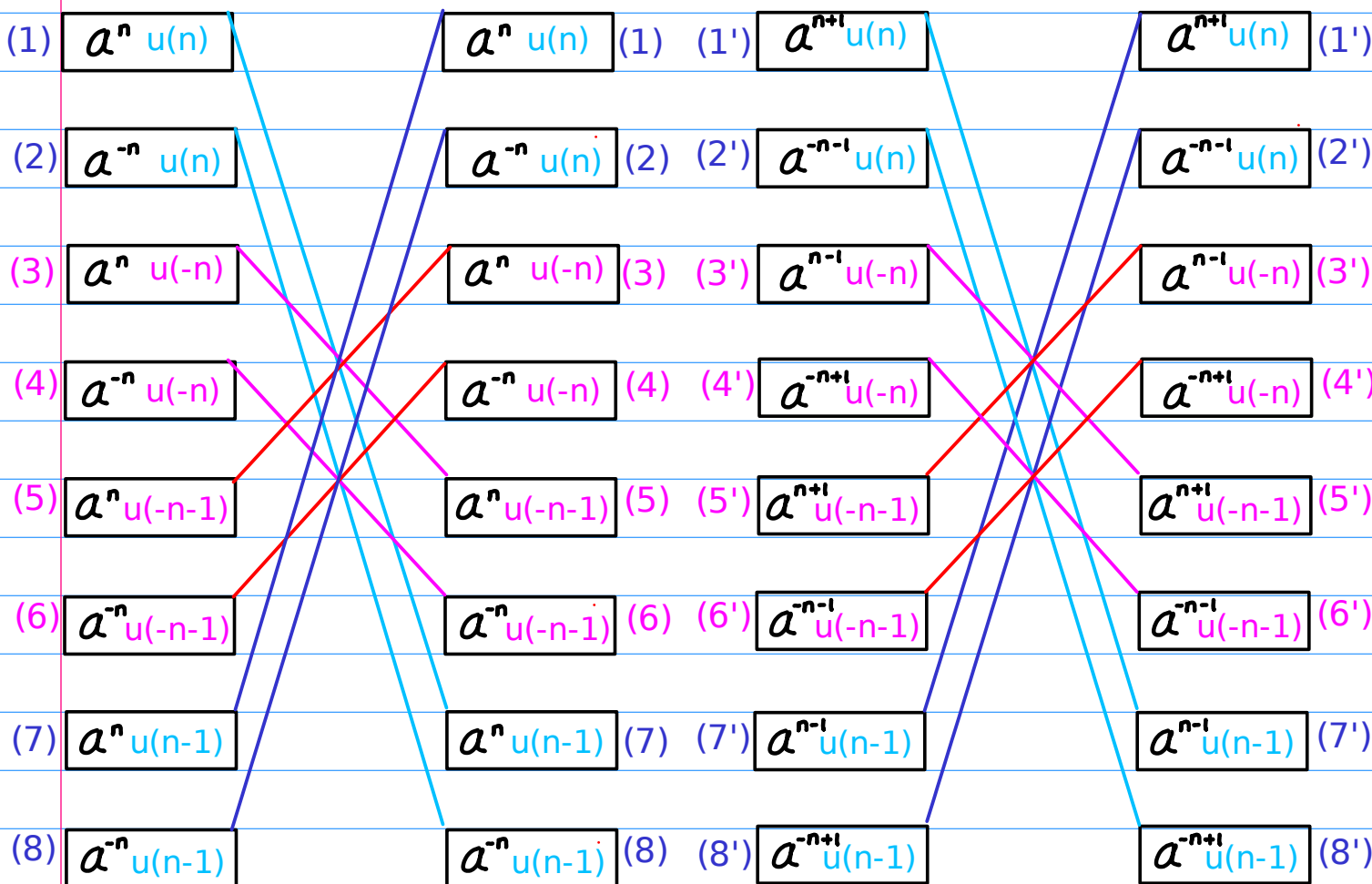
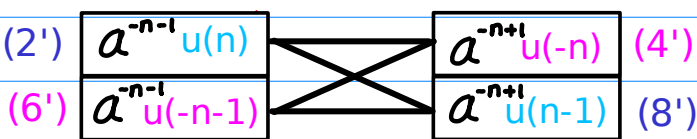
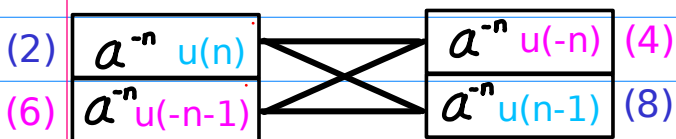
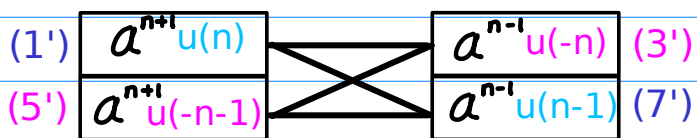
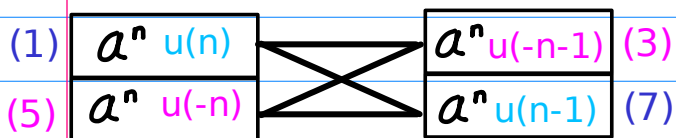
# G.I Flipping2 Base Inverting Shifted Range Flipping



- |             |             |
|-------------|-------------|
| (1') - (4') | (5') - (8') |
| (2') - (3') | (6') - (7') |
| (3') - (2') | (7') - (6') |
| (4') - (1') | (8') - (5') |

**B.I Range Shifting  
Range Flipping  
Range Complementing**

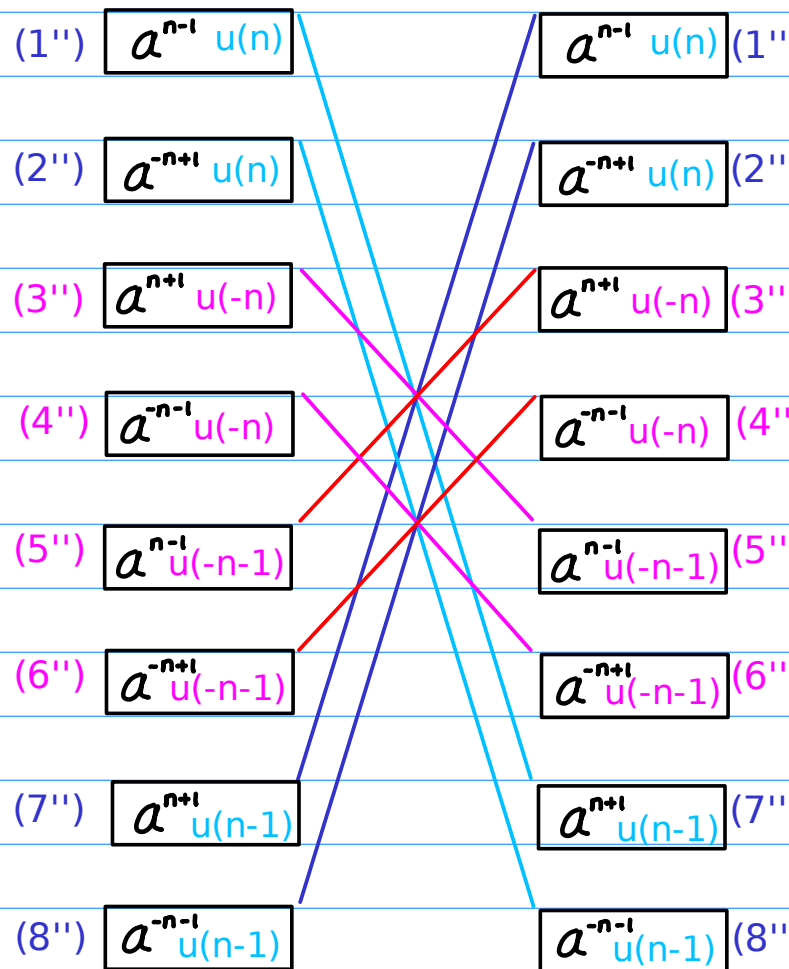
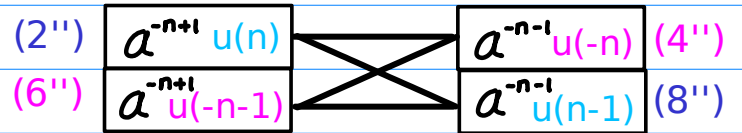
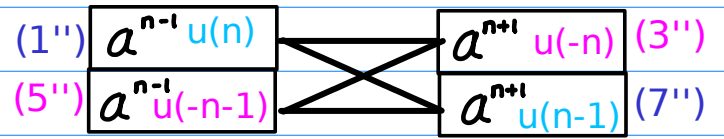
**E.I Shifting2  
Shifted Range Flipping  
Range Complementing**



(1) - (7)      (5) - (3)  
 (2) - (8)      (6) - (4)  
 (3) - (5)      (7) - (1)  
 (4) - (6)      (8) - (2)

(1') - (7')      (5') - (3')  
 (2') - (8')      (6') - (4')  
 (3') - (5')      (7') - (1')  
 (4') - (6')      (8') - (2')

## H.I Shifting2 Shifted Range Flipping Range Complementing

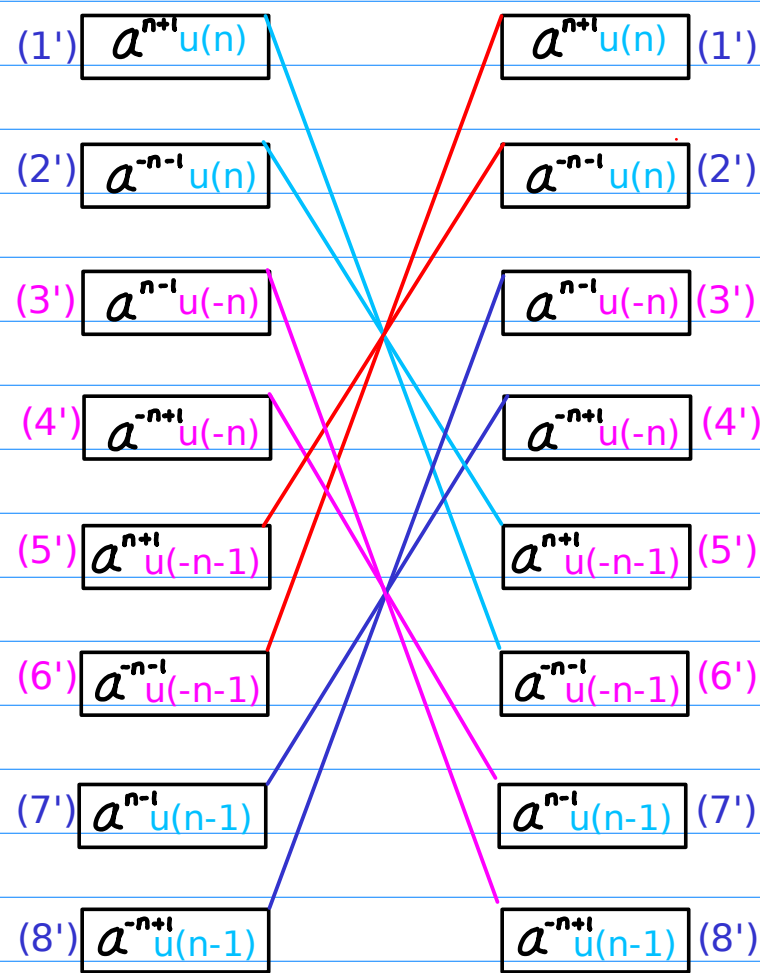
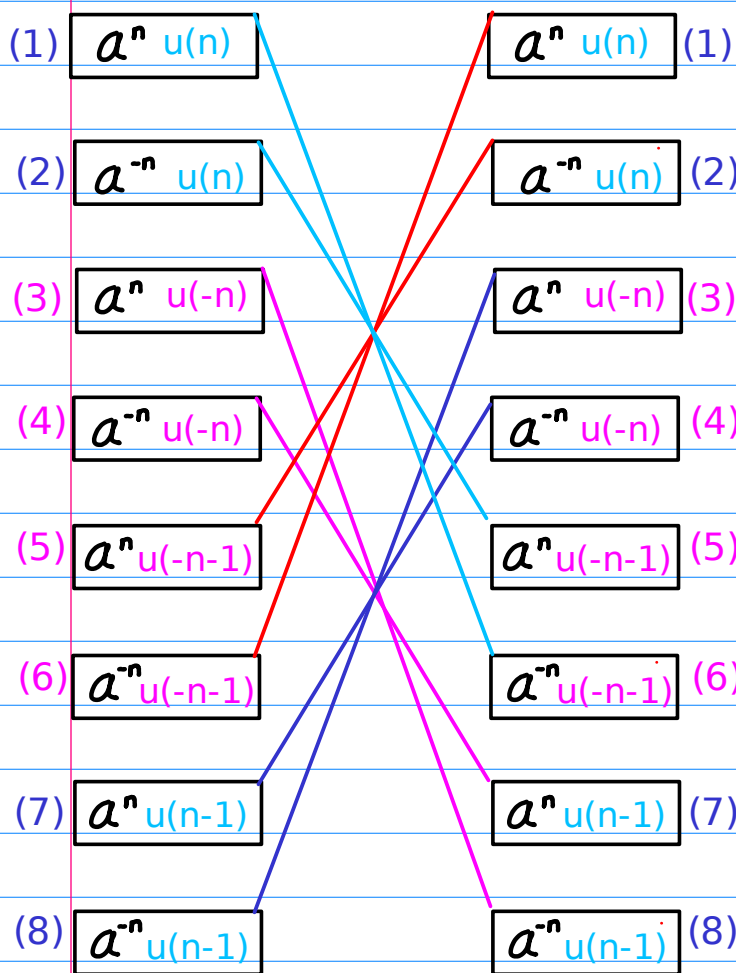
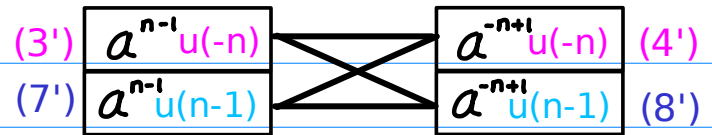
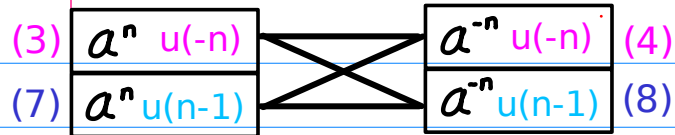
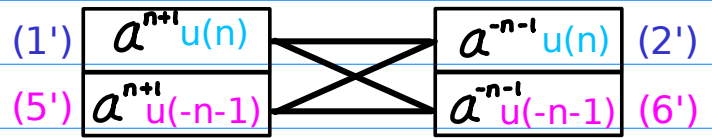
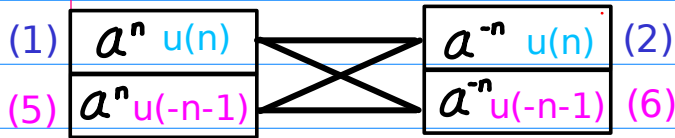


(1') - (7')      (5') - (3')  
 (2') - (8')      (6') - (4')  
 (3') - (5')      (7') - (1')  
 (4') - (6')      (8') - (2')



### C.I Complementary Inverting Base Inverting Range Complementing

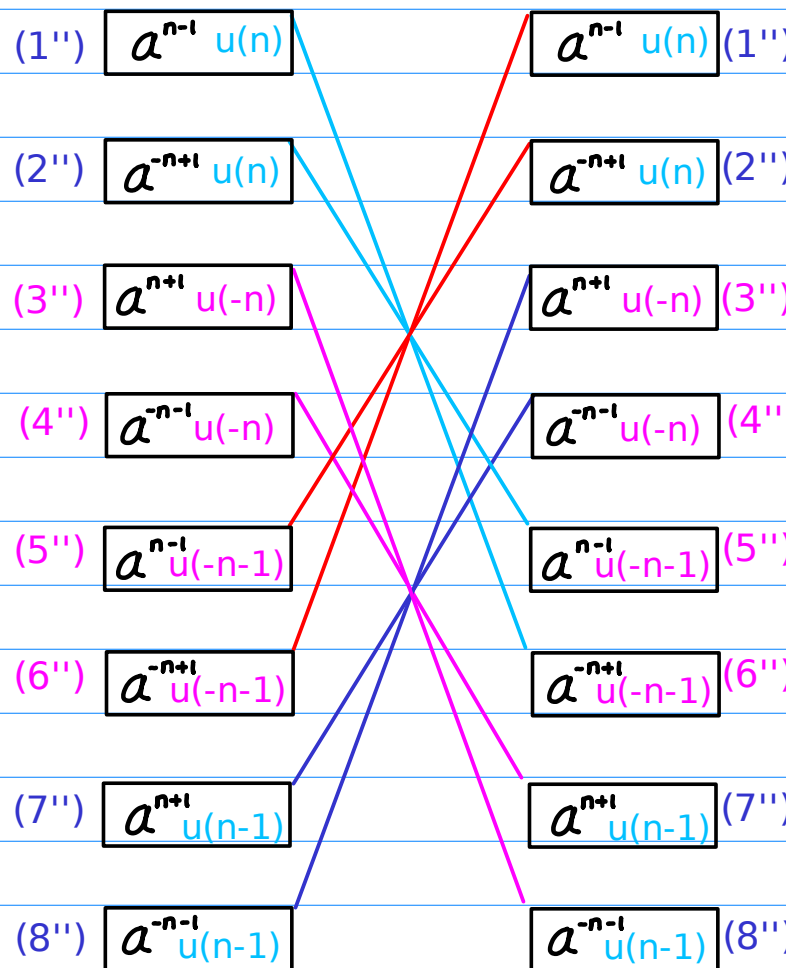
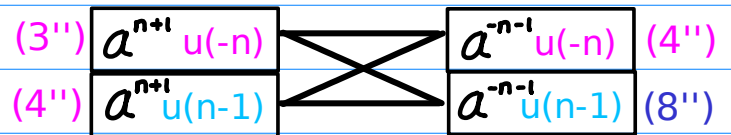
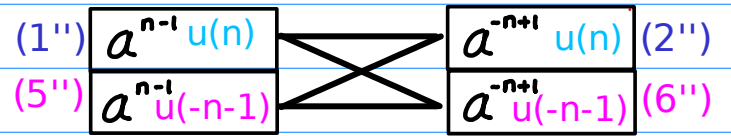
### F.I Complementary Inverting Base Inverting Range Complementing



- (1) - (6)
- (6) - (1)
- (2) - (5)
- (5) - (2)
- (5) - (2)
- (3) - (4)
- (4) - (3)
- (7) - (8)
- (8) - (7)

- (1') - (6')
- (6') - (1')
- (2') - (5')
- (5') - (2')
- (3') - (4')
- (4') - (3')
- (7') - (8')
- (8') - (7')

# I.I Complementary Inverting Base Inverting Range Complementing



- |             |             |
|-------------|-------------|
| (1') - (6') | (5') - (2') |
| (6') - (1') | (6') - (1') |
| (2') - (5') | (7') - (4') |
| (5') - (2') | (8') - (3') |

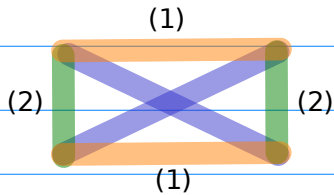


## A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{|c|} \hline (1) \quad a^n u(n) \\ \hline (3) \quad a^n u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \quad (2) \\ \hline a^{-n} u(-n) \quad (4) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5) \quad a^n u(-n-1) \\ \hline (7) \quad a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(-n-1) \quad (6) \\ \hline a^{-n} u(n-1) \quad (8) \\ \hline \end{array}$$

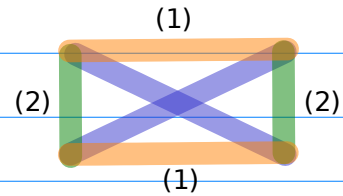


## D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

$$\begin{array}{|c|} \hline (1') \quad a^{n+1} u(n) \\ \hline (3') \quad a^{n+1} u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \quad (2') \\ \hline a^{-n+1} u(-n) \quad (4') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5') \quad a^{n+1} u(-n-1) \\ \hline (7') \quad a^{n+1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(-n-1) \quad (6') \\ \hline a^{-n+1} u(n-1) \quad (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-n} R(-n) \end{array}$$

$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n) \end{array}$$

$b^n$	$b^{-n}$
$a^n$	$a^{-n}$
$a^{-n}$	$a^n$

$b^n$	$b^{-sh2(n)}$
$a^{(n+1)}$	$a^{-(n-1)}$
$a^{-(n+1)}$	$a^{(n-1)}$
$a^{(n-1)}$	$a^{-(n+1)}$
$a^{-(n-1)}$	$a^{(n+1)}$

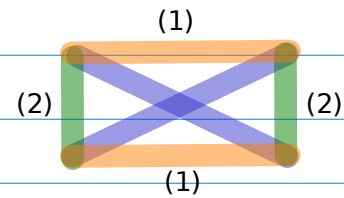
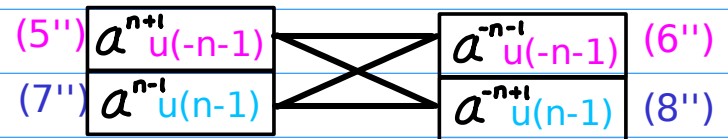
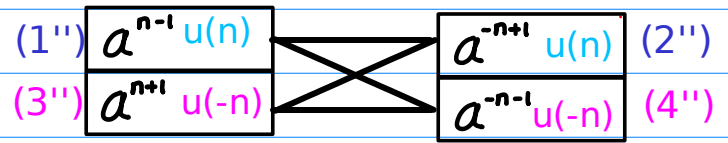
$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

## G.I Flipping2

### (1) Base Inverting

### (2) Shifted Range Flipping



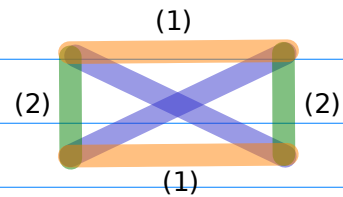
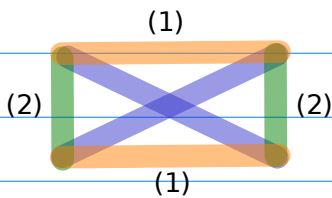
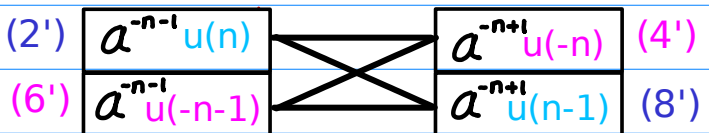
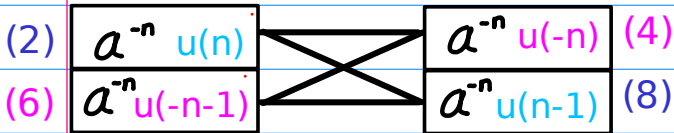
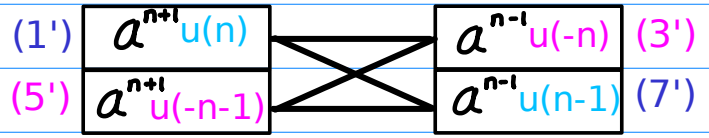
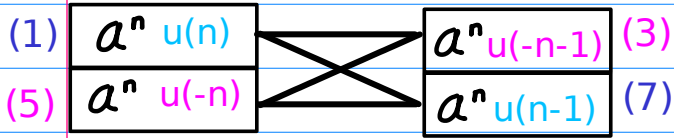
$$\begin{aligned}
 a^n &\stackrel{(1)}{\longleftrightarrow} a^{-n} \\
 a^n R(n) &\stackrel{(2)}{\longleftrightarrow} a^{sh2(n)} R(-n) \\
 a^n R(n) &\longleftrightarrow a^{-sh2(n)} R(-n)
 \end{aligned}$$

$b^n$	$b^{-sh2(n)}$
$a^{(n+1)}$	$a^{-(n-1)}$
$a^{-(n+1)}$	$a^{(n-1)}$
$a^{(n-1)}$	$a^{-(n+1)}$
$a^{-(n-1)}$	$a^{(n+1)}$

$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

**B.I Range Shifting**  
**(1) Range Complementing**  
**(2) Range Flipping**

**E.I Shifting2**  
**(1) Shifted Range Flipping**  
**(2) Range Complementing**



$$R(n) \xleftrightarrow{(1)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} \overline{R(-n)}$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$b^n$	$b^{sh2(n)}$
$a^{(n+1)}$	$a^{(n-1)}$
$a^{-(n+1)}$	$a^{-(n-1)}$
$a^{(n-1)}$	$a^{(n+1)}$
$a^{-(n-1)}$	$a^{-(n+1)}$

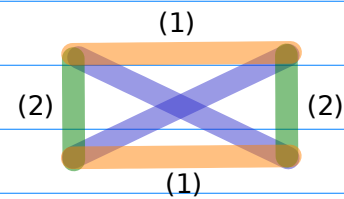
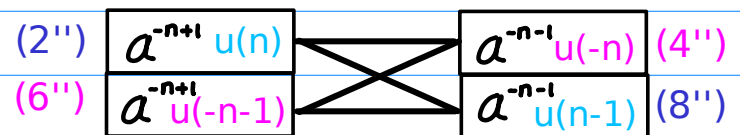
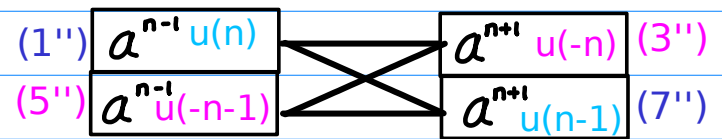
$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$

$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$

## H.I Shifting2

(1) Shifted Range Flipping

(2) Range Complementing



$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} \overline{R(-n)}$$

$$b^n \quad b^{sh2(n)}$$

$$a^{(n+1)} \quad a^{(n-1)}$$

$$a^{-(n+1)} \quad a^{-(n-1)}$$

$$a^{(n-1)} \quad a^{(n+1)}$$

$$a^{-(n-1)} \quad a^{-(n+1)}$$

$$R(n) \quad \overline{R(-n)}$$

$$u(n) \quad u(n-1)$$

$$u(n-1) \quad u(n)$$

$$u(-n) \quad u(-n-1)$$

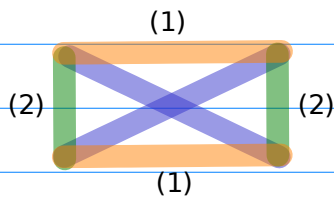
$$u(-n-1) \quad u(-n)$$

## C.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

$$\begin{array}{|c|} \hline (1) \quad a^n u(n) \\ \hline (5) \quad a^n u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \quad (2) \\ \hline a^{-n} u(-n-1) \quad (6) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3) \quad a^n u(-n) \\ \hline (7) \quad a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(-n) \quad (4) \\ \hline a^{-n} u(n-1) \quad (8) \\ \hline \end{array}$$



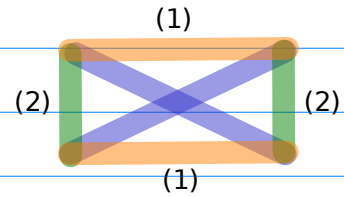
$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

## F.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

$$\begin{array}{|c|} \hline (1') \quad a^{n+1} u(n) \\ \hline (5') \quad a^{n+1} u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \quad (2') \\ \hline a^{-n-1} u(-n-1) \quad (6') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3') \quad a^{n-1} u(-n) \\ \hline (7') \quad a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n+1} u(-n) \quad (4') \\ \hline a^{-n+1} u(n-1) \quad (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

$b^n$	$b^{-n}$
$a^n$	$a^{-n}$
$a^{-n}$	$a^n$

$b^n$	$b^{-n}$
$a^{(n+1)}$	$a^{-(n+1)}$
$a^{-(n+1)}$	$a^{(n+1)}$
$a^{(n-1)}$	$a^{-(n-1)}$
$a^{-(n-1)}$	$a^{(n-1)}$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

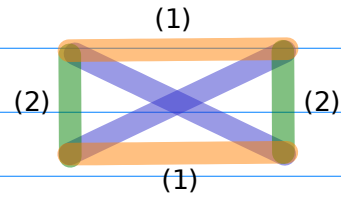
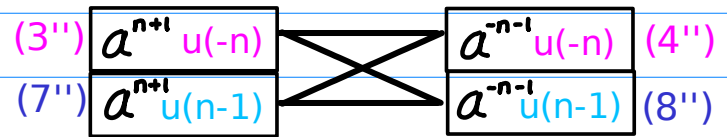
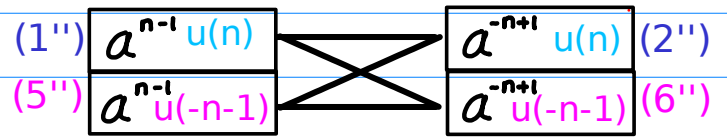
$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$



# I.I Complementary Inverting

## (1) Base Inverting

## (2) Range Complementing



$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \xleftrightarrow{\quad} & a^{-n} \overline{R(n)}
 \end{array}$$

$b^n$	$b^{-n}$
$a^{(n+1)}$	$a^{-(n+1)}$
$a^{-(n+1)}$	$a^{(n+1)}$
$a^{(n-1)}$	$a^{-(n-1)}$
$a^{-(n-1)}$	$a^{(n-1)}$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

## A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-n} R(-n)
 \end{array}$$

## D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

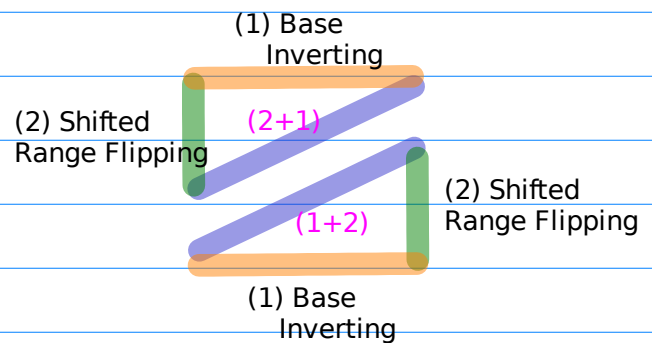
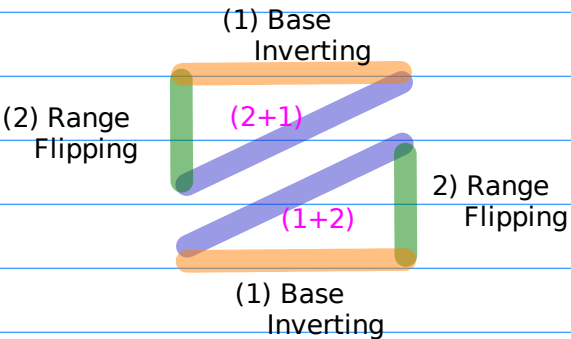
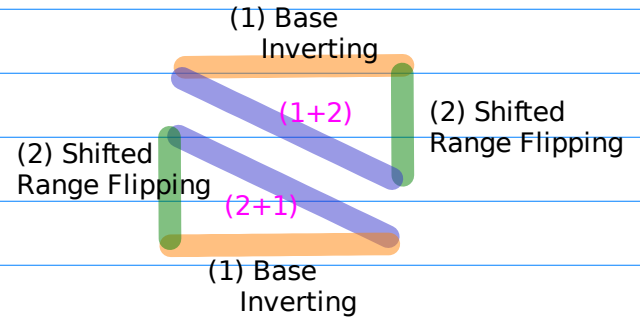
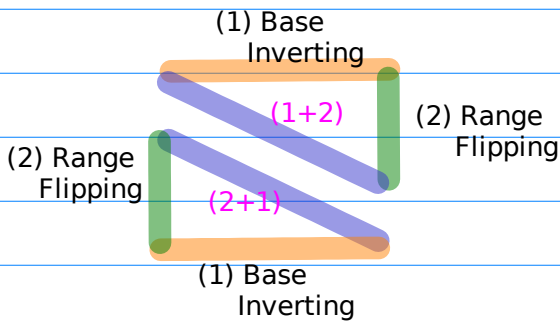
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n R(-n) \\
 & \xrightarrow{(1)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 & \xrightarrow{(1)} & a^{-sh2(n)} R(-n)
 \end{array}$$



## D.I Flipping2

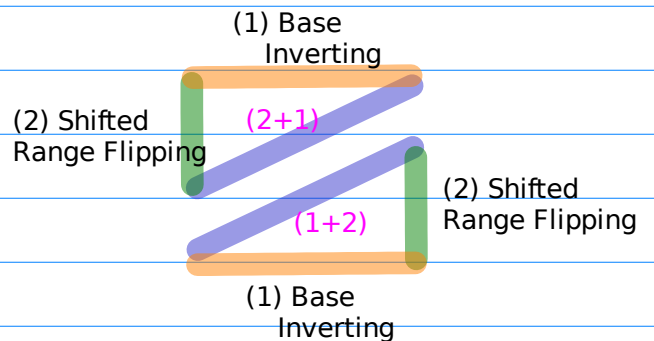
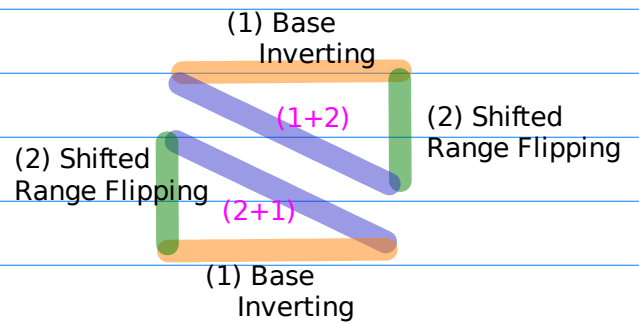
(1) Base Inverting

(2) Shifted Range Flipping

$$\begin{aligned}
 a^n &\xleftrightarrow{(1)} a^{-n} \\
 a^n R(n) &\xleftrightarrow{(2)} a^{sh2(n)} R(-n) \\
 a^n R(n) &\xleftrightarrow{\quad} a^{-sh2(n)} R(-n)
 \end{aligned}$$

$$\begin{aligned}
 a^n R(n) &\xrightarrow{(1)} a^{-n} R(n) \\
 &\xrightarrow{(2)} a^{-sh2(n)} R(-n)
 \end{aligned}$$

$$\begin{aligned}
 a^n R(n) &\xrightarrow{(2)} a^{sh2(n)} R(-n) \\
 &\xrightarrow{(1)} a^{-sh2(n)} R(-n)
 \end{aligned}$$



**B.I Range Shifting**  
**(1) Range Complementing**  
**(2) Range Flipping**

**E.I Shifting2**  
**(1) Shifted Range Flipping**  
**(2) Range Complementing**

$$R(n) \xleftrightarrow{(1)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

$$\xrightarrow{(2)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

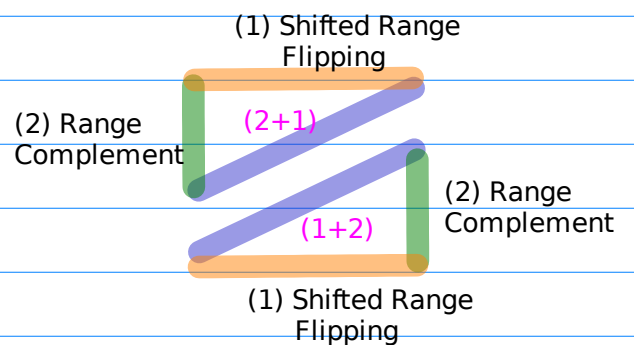
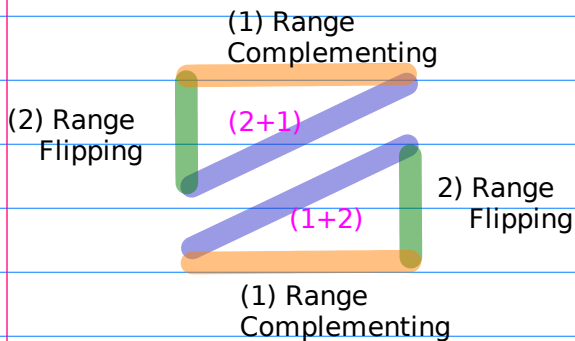
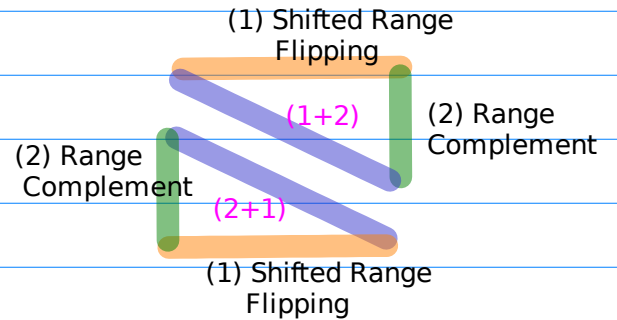
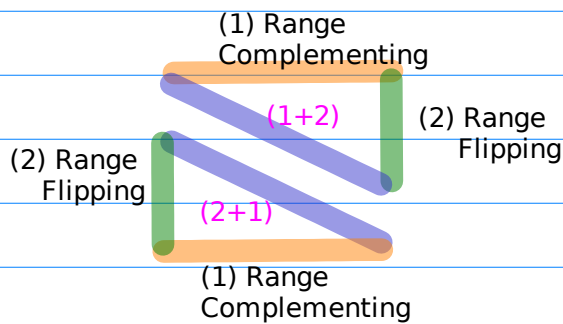
$$\xrightarrow{(2)} a^{-sh2(n)} R(-n)$$

$$a^n rng(n) \xrightarrow{(2)} a^n \overline{R(n)}$$

$$\xrightarrow{(1)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(2)} a^{sh2(n)} R(-n)$$

$$\xrightarrow{(1)} a^{-sh2(n)} R(-n)$$



## E.I Shifting2

(1) Shifted Range Flipping

(2) Range Complementing

$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

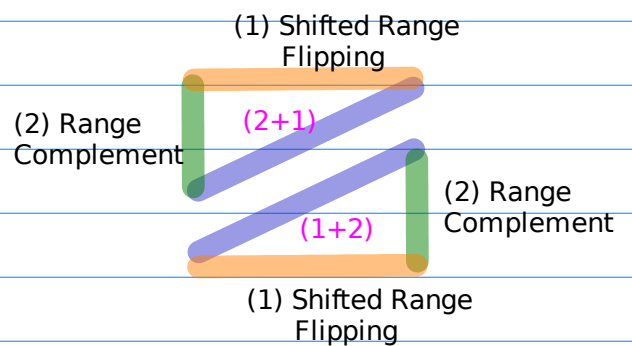
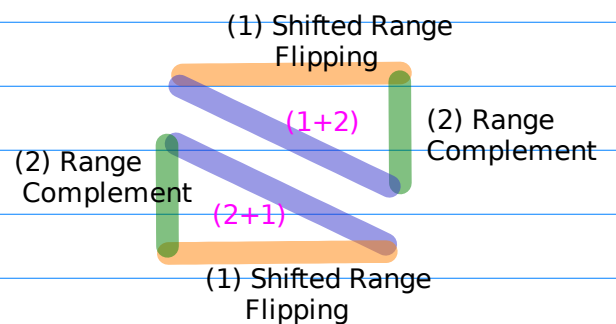
$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} \overline{R(-n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

$$\xrightarrow{(2)} a^{-sh2(n)} R(-n)$$

$$a^n R(n) \xrightarrow{(2)} a^{sh2(n)} R(-n)$$

$$\xrightarrow{(1)} a^{-sh2(n)} R(-n)$$



## C.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

## F.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

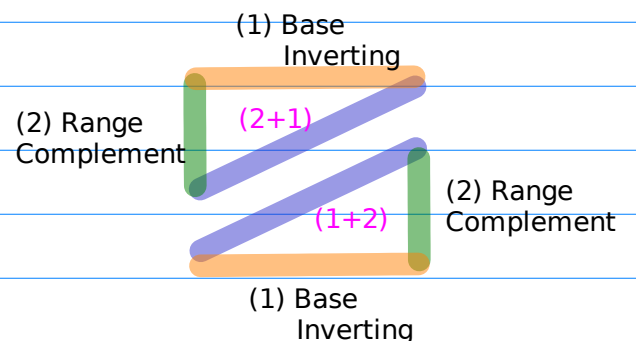
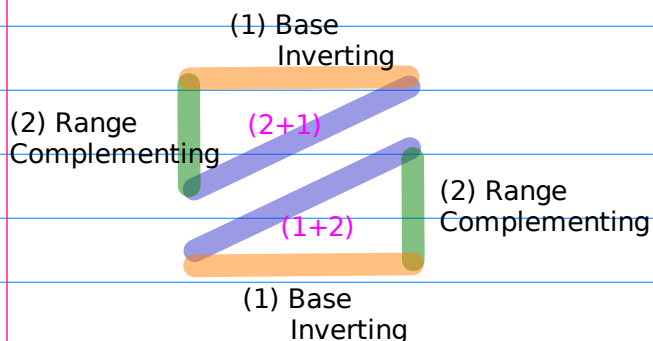
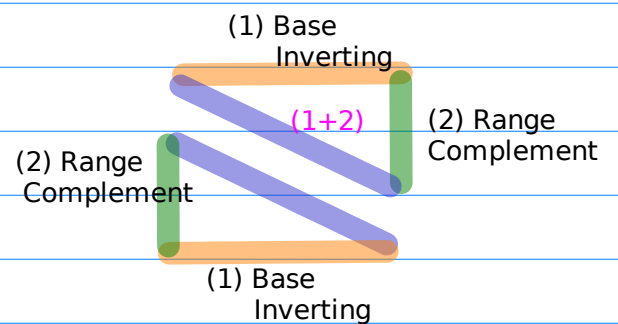
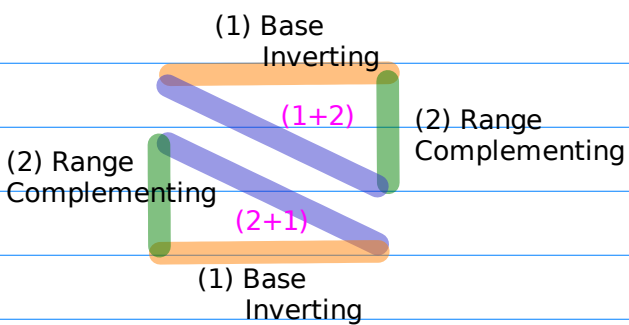
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$



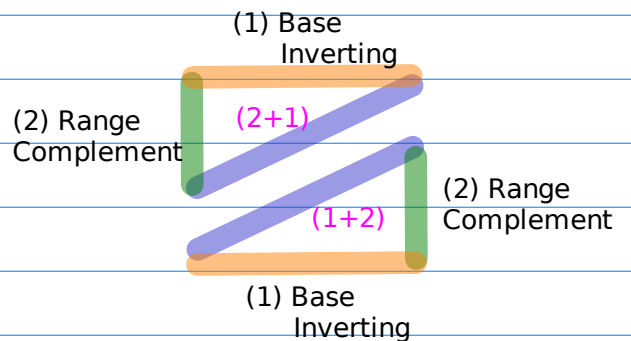
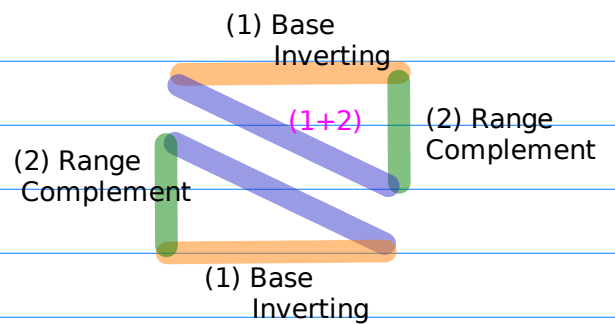
# F.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \xleftrightarrow{\quad} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$

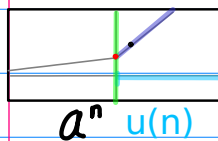




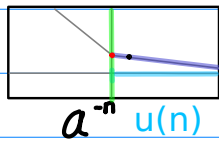


## A.II Flipping Base Inverting Range Flipping

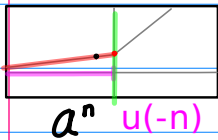
(1) 0000



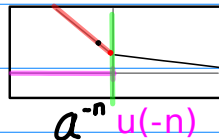
(2) 0001



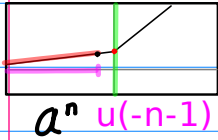
(3) 0010



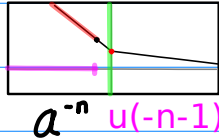
(4) 0011



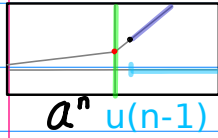
(5) 0100



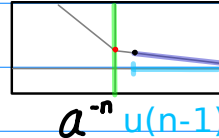
(6) 0101



(7) 0110

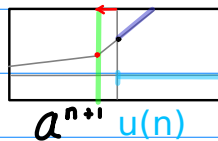


(8) 0111

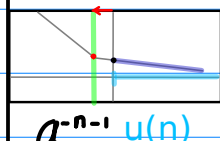


## D.II Flipping2 Base Inverting Shifted Range Flipping

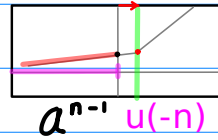
(1') 1000



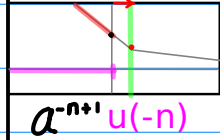
(2') 1001



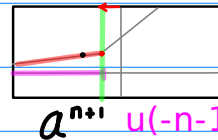
(3') 1010



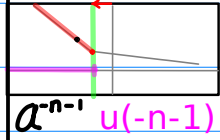
(4') 1011



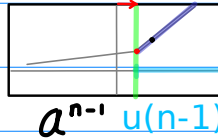
(5') 1100



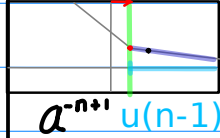
(6') 1101



(7') 1110



(8') 1111



Shifted Range Flipping  
= Exponent Shifting2  
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

$$a^n \longleftrightarrow a^{-n}$$

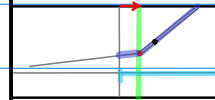
$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

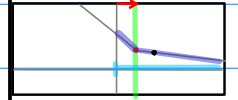
# G.II Flipping2 Base Inverting Shifted Range Flipping

(1'') 1000



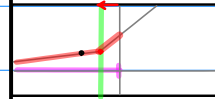
$$a^{n-1}u(n)$$

(2'') 1001



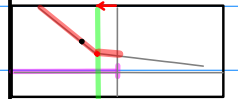
$$a^{-n+1}u(n)$$

(3'') 1010



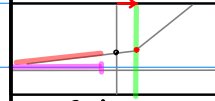
$$a^{n+1}u(-n)$$

(4'') 1011



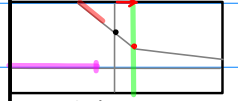
$$a^{-n-1}u(-n)$$

(5'') 1100



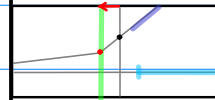
$$a^{n-1}u(-n-1)$$

(6'') 1101



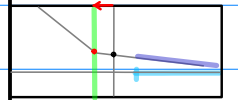
$$a^{-n+1}u(-n-1)$$

(7'') 1110



$$a^{n+1}u(n-1)$$

(8'') 1111



$$a^{-n-1}u(n-1)$$

⋮  
⋮

Shifted Range Flipping  
= Exponent Shifting  
+ Range Flipping

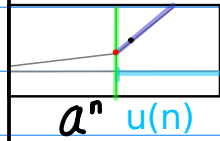
$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

$$a^n \longleftrightarrow a^{-n}$$

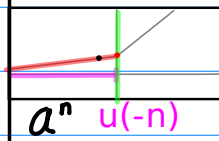
$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

## B.II Range Shifting Range Flipping Range Complementing

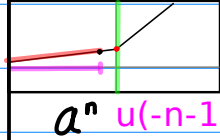
(1) 0000



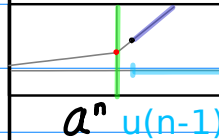
(3) 0010



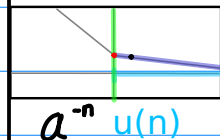
(5) 0100



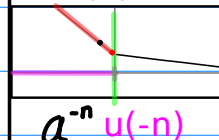
(7) 0110



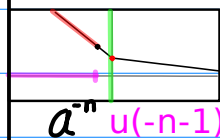
(2) 0001



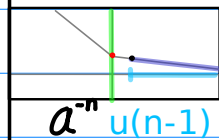
(4) 0011



(6) 0101



(8) 0111



Range Shifting  
= Range Flipping  
+ Range Complementing

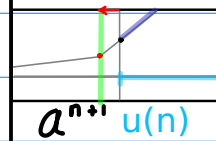
$$R(n) \longleftrightarrow R(-n)$$

$$R(n) \longleftrightarrow R(-n)$$

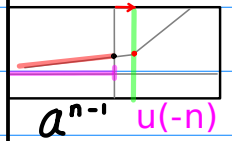
$$R(n) \longleftrightarrow \overline{R(n)}$$

## E.II Shifting2 Shifted Range Flipping Range Complementing

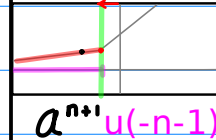
(1') 1000



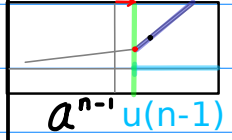
(3') 1010



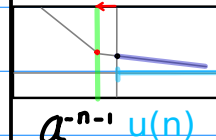
(5') 1100



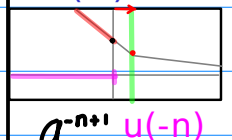
(7') 1110



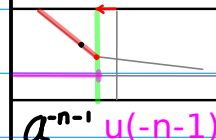
(2') 1001



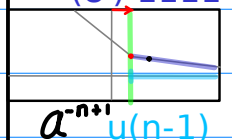
(4') 1011



(6') 1101



(8') 1111



Shifted Range Flipping  
= Exponent Shifting2  
+ Range Flipping

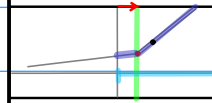
$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

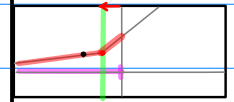
## H.II Shifting2 Shifted Range Flipping Range Complementing

(1'') 1000



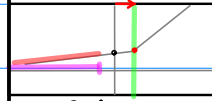
$$a^{n-1} u(n)$$

(3'') 1010



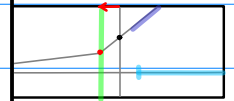
$$a^{n+1} u(-n)$$

(5'') 1100



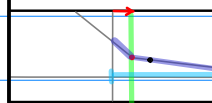
$$a^{n-1} u(-n-1)$$

(7'') 1110



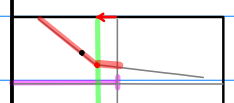
$$a^{n+1} u(n-1)$$

(2'') 1001



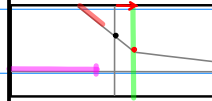
$$a^{-n+1} u(n)$$

(4'') 1011



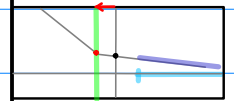
$$a^{-n-1} u(-n)$$

(6'') 1101



$$a^{-n+1} u(-n-1)$$

(8'') 1111



$$a^{-n-1} u(n-1)$$

⋮  
⋮

Shifted Range Flipping  
= Exponent Shifting2  
+ Range Flipping

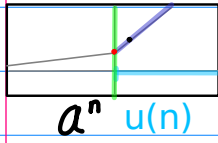
$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

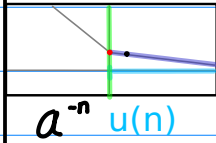
$$R(n) \longleftrightarrow \overline{R(n)}$$

## C.II Complementary Inverting Base Inverting Range Complementing

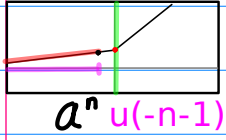
(1) 0000



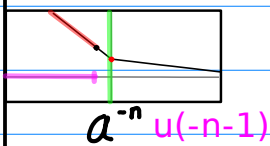
(2) 0001



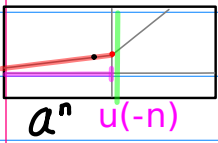
(5) 0100



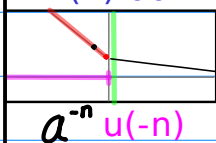
(6) 0101



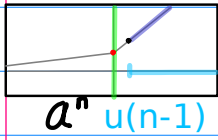
(3) 0010



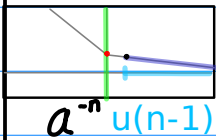
(4) 0011



(7) 0110



(8) 0111



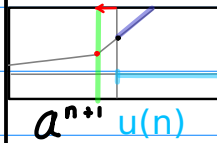
$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

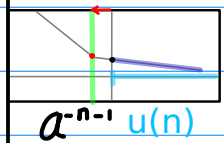
$$R(n) \longleftrightarrow \overline{R(n)}$$

## F.II Complementary Inverting Base Inverting Range Complementing

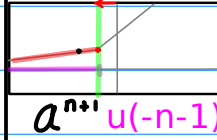
(1') 1000



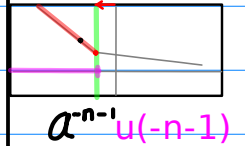
(2') 1001



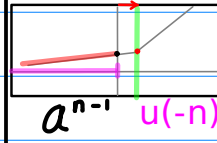
(5') 1100



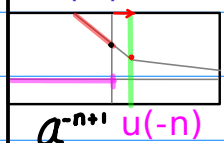
(6') 1101



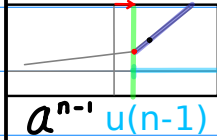
(3') 1010



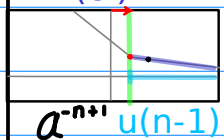
(4') 1011



(7') 1110



(8') 1111



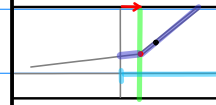
$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

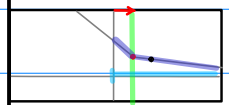
# I.II Complementary Inverting Base Inverting Range Complementing

(1'') 1000



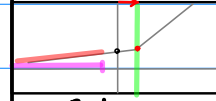
$$a^{n-1} u(n)$$

(2'') 1001



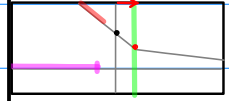
$$a^{-n+1} u(n)$$

(5'') 1100



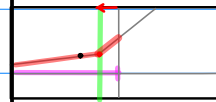
$$a^{n-1} u(-n-1)$$

(6'') 1101



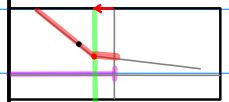
$$a^{-n+1} u(-n-1)$$

(3'') 1010



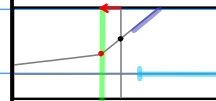
$$a^{n+1} u(-n)$$

(4'') 1011



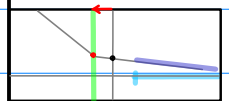
$$a^{-n-1} u(-n)$$

(7'') 1110



$$a^{n+1} u(n-1)$$

(8'') 1111



$$a^{-n-1} u(n-1)$$

Shifted Range Flipping  
= Exponent Shifting  
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$







