Differentiation of Discrete Functions

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Based on
Introduction to Matrix Algebra, Autar Kaw
https://ma.mathforcollege.com

Outline

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Direct Fit Polynomials

given $n+1$ data points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$$

one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

to find the first derivative

$$P_n'(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \cdots + (n-1) a_{n-1} x^{n-2} + n a_n x^{n-1}$$

similarly other derivatives can be found

Direct Fit Polynomials

given $(n+1)$ data points

$$(x_{0,}y_0),(x_{1,}y_1),\ldots,(x_{n-1,}y_{n-1}),(x_{n,}y_n)$$

one can fit a n^{th} order polynomial given by

$$P_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

to find the first derivative

$$P'_n(x) = \frac{dP_n(x)}{dx} = a_1 + 2a_2 x + \cdots + (n-1)a_{n-1}x^{n-2} + n a_n x^{n-1}$$

similarly other derivatives can be found

Lagrange Polynomials (1)

given $(n+1)$ data points

$$(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$$

one can fit a n^{th} order Lagrange polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial
that approximates the function $y = f(x)$
given at $(n+1)$ data points
as $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$

Lagrange Polynomials (2-1)

one can fit a n^{th} order Lagrange polynomial given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial for the function $y = f(x)$

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n = \sum_{i=0}^{n} L_i(x) f(x_i)$$

given at $(n+1)$ data points as $(x_{0,} y_0), (x_{1,} y_1), \ldots, (x_{n-1,} y_{n-1}), (x_{n,} y_n)$, and

$$L_i(x) = \prod_{j=0,\ j \neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

Lagrange Polynomials (2-2)

given a set of $(n+1)$ nodes $\{x_0, x_1, \ldots, x_n\}$,
which must all be distinct, $x_j \neq x_i$ for indices $j \neq i$,

the **Lagrange basis** for polynomials of degree $\leq n$ for those nodes
is the set of polynomials $\{L_0(x), L_1(x), \ldots, L_n(x)\}$

$$L_i(x) = \prod_{j=0,\ j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

$$= \frac{x - x_0}{x_i - x_0} \cdot \frac{x - x_1}{x_i - x_1} \cdots \cdot \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot 1 \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdot \frac{x - x_{i+2}}{x_i - x_{i+2}} \cdots \cdot \frac{x - x_n}{x_i - x_n}$$

https://en.wikipedia.org/wiki/Lagrange_polynomial

Lagrange Polynomials (2-3)

each **Lagrange basis** of degree n take values
$L_i(x_j) = 0$ if $j \neq i$ and $L_i(x_i) = 1$.

Using the **Kronecker delta** this can be written $L_i(x_j) = \delta_{ij}$.

Each **basis** polynomial can be explicitly described by the product:

$$L_i(x) = \prod_{j=0,\ j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

$$= \frac{x - x_0}{x_i - x_0} \cdot \frac{x - x_1}{x_i - x_1} \cdots\cdot \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot 1 \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdot \frac{x - x_{i+2}}{x_i - x_{i+2}} \cdots\cdot \frac{x - x_n}{x_i - x_n}$$

https://en.wikipedia.org/wiki/Lagrange_polynomial

Lagrange Polynomials (2-4)

$$L_i(x_j) = \prod_{k=0,\ k\neq i}^{n} \frac{x_j - x_k}{x_i - x_k}$$

$$= \frac{x_j - x_0}{x_i - x_0} \cdot \frac{x_j - x_1}{x_i - x_1} \cdots\cdot \frac{x_j - x_{i-1}}{x_i - x_{i-1}} \cdot 1 \cdot \frac{x_j - x_{i+1}}{x_i - x_{i+1}} \cdot \frac{x_j - x_{i+2}}{x_i - x_{i+2}} \cdots\cdot \frac{x_j - x_n}{x_i - x_n}$$

$$= 0 \qquad \because \quad {}^{\exists} j = k \neq i$$

$$L_i(x_i) = \prod_{k=0,\ k\neq i}^{n} \frac{x_i - x_k}{x_i - x_k}$$

$$= \frac{x_i - x_0}{x_i - x_0} \cdot \frac{x_i - x_1}{x_i - x_1} \cdots\cdot \frac{x_i - x_{i-1}}{x_i - x_{i-1}} \cdot 1 \cdot \frac{x_i - x_{i+1}}{x_i - x_{i+1}} \cdot \frac{x_i - x_{i+2}}{x_i - x_{i+2}} \cdots\cdot \frac{x_i - x_n}{x_i - x_n}$$

$$= 1$$

https://en.wikipedia.org/wiki/Lagrange_polynomial

Lagrange Polynomials (3)

hen to find the first derivative, one can differentiate $f_n(x)$ for other derivatives.

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

For example, the second order Lagrange polynomial
passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$f_2(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

$$L_0(x) = \prod_{j=0,\ j \neq i}^{2} \frac{x - x_j}{x_0 - x_j}, \qquad L_1(x) = \prod_{j=0,\ j \neq i}^{2} \frac{x - x_j}{x_1 - x_j}, \qquad L_2(x) = \prod_{j=0,\ j \neq i}^{2} \frac{x - x_j}{x_2 - x_j}$$

Lagrange Polynomials (4)

the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$L_0(x) = \prod_{j=0,\ j\neq i}^{2} \frac{x - x_j}{x_0 - x_j}, \qquad L_1(x) = \prod_{j=0,\ j\neq i}^{2} \frac{x - x_j}{x_1 - x_j}, \qquad L_2(x) = \prod_{j=0,\ j\neq i}^{2} \frac{x - x_j}{x_2 - x_j}$$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2}, \qquad L_1(x) = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2}, \qquad L_2(x) = \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1}$$

$$f_2(x) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} f(x_0) + \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} f(x_2)$$

Lagrange Polynomials (5)

the second order Lagrange polynomial passing through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$f_2(x) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} f(x_0) + \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} f(x_2)$$

$$\frac{d}{dx}(x - x_1)(x - x_2) = (x - x_2) + (x - x_1) = 2x - (x_1 + x_2)$$

$$\frac{d}{dx} f_2(x) = \frac{2x - (x_1 + x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - (x_0 + x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{2x - (x_0 + x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

$$\frac{d^2}{dx^2} f_2(x) = \frac{2f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{2f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{2f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

Tangent Lines

- as $h \to 0$, $\mathrm{Q} \to \mathrm{P}$
  and the secant line $\to$ the tangent line
- the slope of the tangent line

$$m_{tangent} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$
$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$