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V.

On the Secular Variations and Mutual Relations of the Orbits of the Asteroids.

By SIMON NEWCOMB.

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WHEN we consider the number of members of which the group of asteroids is composed, and the narrow limits within which they are included, it is impossible to avoid the conclusion that their proximity is the result of a common element in the determining reasons which fixed the respective positions of their several orbits, which common element has some special relation to that portion of space in which the orbits of the asteroids are found which it has not to other portions of space.

The object of the present paper is to examine those circumstances of the forms, positions, variations, and general relations of the asteroid orbits which may serve as a test, complete or imperfect, of any hypothesis which may be made respecting the cause from which they originated, or the reason why they are in a group by themselves. It may not, however, be out of place to begin with some general considerations respecting these hypotheses, and the nature of the methods by which they may be tested. Every *a posteriori* test is founded on the supposition that the hypothesis to be tested necessarily or probably implies that certain conditions must be fulfilled by the asteroids or their orbits. The tests consist in observing whether these conditions are fulfilled. The conditions may be divided into two general classes, — those which are rigorous and necessary, and those which are merely probable. The former class consists of those which follow immediately and necessarily from the hypothesis itself; the latter, of those which are deducible from it by the principle of random distribution. The nature of the latter conditions will be clearly seen from the examples which will be given of their deduction from hypotheses.

Two hypotheses worthy of consideration have been promulgated respecting the origin of the asteroids; one, that they are the fragments of a single planet, which was shattered by the operation of some unknown cause; another, that they were formed by the breaking up of a revolving ring of nebulous matter.

The first of these is the celebrated hypothesis of Olbers. It has the advantage of accounting for the phenomena, considered in their more salient aspects, in a very remarkable manner. The normal solar system is still the solar system as we should expect it to be in the absence of any knowledge of the planet or planets revolving between Jupiter and Saturn. The phenomenon (abnormal on this hypothesis) of the place of a single large planet being filled by a collection of small ones of varying brilliancy, large inclinations and eccentricities, and slightly different mean distances, is precisely what might be expected as the result of a force which should break the planet into fragments, each very small in comparison with the original planet. But we shall see hereafter that when we carry the results of this hypothesis to numerical exactness, the observed phenomena are very far from agreeing with these results. Moreover, it is difficult, perhaps impossible, to imagine how any known natural cause, or combination of causes, should produce such a result as the shattering of a planet. But since the limits of our knowledge are not necessarily the limits of possibility, this objection is not fatal, and it is difficult to say what weight ought to be assigned to it.

The second hypothesis, that the asteroids were formed by the breaking up of a ring of nebulous matter, is not at all improbable if the nebular hypothesis is true, but it is subjected to most or all the uncertainties of that hypothesis. The ring must have been considerably inclined to the plane of the ecliptic, or none of its fragments could have fallen into orbits much inclined to that plane; and it could not have revolved in its own plane, else all the fragments would have had nearly the same inclination to the invariable plane of the solar system; and it must have been somewhat eccentric, else all the fragments would have had about the same mean distance. Now it is remarkable that the two last circumstances would cause a tendency in the ring to break into fragments, while a circular ring, revolving in its own plane, would have no such tendency. We should then expect, in case a ring should break up, that its fragments would present some at least of the phenomena presented by the asteroids. But the hypothesis is not equally susceptible with that of Olbers of *a posteriori* tests.

To apply rigorous tests to either of the above-mentioned hypotheses, we need rigorous expressions in terms of the time for the values of the eccentricity, inclination, longitude of perihelion, and longitude of node, of each asteroid used in applying the test. For the probable tests we need the mean and the limiting values of the same elements, and to obtain these the same expressions are necessary.

The subject may be arranged under the following heads : —

§ 1. Computation of the rigorous expressions in terms of the time of the elements of the asteroids.

§ 2. Of the possibility that the orbits of all the asteroids once intersected in a common point.

§ 3. Have the elements of the asteroid orbits ever been materially affected by a resisting medium?

§ 4. Of the relations among the mean distances, eccentricities, and inclinations of the orbits of the asteroids; and between their masses and the velocities with which they must have been projected, if Olbers's hypothesis be true.

§ 5. Of certain observed relations among the asteroids which are the necessary or probable result of known causes, and therefore throw no light on the origin of the asteroids.

§ 1. *Computation of the Rigorous Expressions in Terms of the Time of the Elements of certain of the Asteroid Orbits, or of the Secular Variations of those Elements.*

To obtain the required expressions, we shall start from the expressions given by Laplace in the *Mécanique Céleste*, Liv. II. §§ 55 & 59.

$$\left.\begin{aligned}\frac{dh}{dt} &= \{(0,1)+(0,2)+(0,3)+\&c.\}\, l - [0,1]\, l' - [0,2]\, l'' - \&c. \\ \frac{dl}{dt} &= -\{(0,1)+(0,2)+(0,3)+\&c.\}\, h + [0,1]\, h' + [0,2]\, h'' + \&c.\end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned}\frac{dp}{dt} &= -\{(0,1)+(0,2)+\&c.\}\, q + (0,1)\, q' + (0,2)\, q'' + \&c. \\ \frac{dq}{dt} &= \{(0,1)+(0,2)+\&c.\}\, p - (0,1)\, q' - (0,2)\, q'' - \&c.\end{aligned}\right\} \quad (2)$$

where

$$h = e \sin \pi, \qquad l = e \cos \pi,$$
$$p = i \sin \Omega, \qquad q = i \cos \Omega.$$

The unaccented symbols relate to the disturbed planet, which, in our present investigation, is supposed to be an asteroid; the accented quantities relate to the disturbing planets. The expressions given by Laplace for the quantities (0,1) and [0,1] are

$$(0,1) = -\frac{3\, m' . n\, a^2\, b^{(1)}_{-\frac{1}{2}}}{4\,(1-a^2)^2};$$

$$[0,1] = -\frac{3\, a\, m'\, n\, \{(1+a^2)\, b^{(1)}_{-\frac{1}{2}} + \frac{1}{2}\, a\, b^{(0)}_{-\frac{1}{2}}\}}{2\,(1-a^2)^2};$$

a representing the ratio of the mean distances of the disturbing planet and the asteroid, and the other symbols being used in their usual signification.

These equations may be so simplified as to materially facilitate the computation of the required quantities. The following equations are given by Laplace in § 49.

$$\frac{b^{(0)}_{-\frac{1}{2}}}{(1-\alpha^2)^2} = b^{(0)}_{\frac{3}{2}}; \qquad \frac{-3\, b^{(1)}_{-\frac{1}{2}}}{(1-\alpha^2)^2} = b^{(1)}_{\frac{3}{2}};$$

$$b^{(i)}_s = \frac{(i-1)\,(1+\alpha^2)\, b^{(i-1)}_s - (i+s-2)\,\alpha\, b^{(i-2)}_s}{(i-s)\,\alpha}.$$

By a comparison of these equations with the values of (0.1) and [0.1] given above, the latter reduce to

$$(0,n) = \frac{m^{(n)}\, n}{4}\, \alpha^2\, b^{(1)}_{\frac{3}{2}}; \qquad [0,n] = \frac{m^{(n)}\, n}{4}\, \alpha^2\, b^{(2)}_{\frac{3}{2}}. \tag{3}$$

With these equations we can obtain the numerical values of $(0,n)$ and $[0,n]$ with great ease, provided that we have tables which give the values of $b^{(i)}_s$ for different values of α, such as those published by Runkle in the Smithsonian Contributions to Knowledge.

To integrate the equations (1) and (2), we shall suppose all the accented quantities, l', l'', h', &c., p', &c., q', &c., given in terms of the time. This is admissible, because we neglect the action of the asteroids on the larger planets, and also on each other; we may, therefore, use the expressions for the elements of the larger planets in functions of the time, which are obtained in neglecting the action of the asteroids. These expressions are of the following form:—

$$\left.\begin{array}{l} h' = N'_0 \sin\,(g\,t+\beta) + N'_1 \sin\,(g_1\,t+\beta_1) + N'_2 \sin\,(g_2\,t+\beta_2) + \&c. \\ l' = N'_0 \cos\,(g\,t+\beta) + N'_1 \cos\,(g_1\,t+\beta_1) + N'_2 \cos\,(g_2\,t+\beta_2) + \&c. \\ h'' = N''_0 \sin\,(g\,t+\beta) + N''_1 \sin\,(g_1\,t+\beta_1) + \&c. \\ l'' = N''_0 \sin\,(g\,t+\beta) + N''_1 \sin\,(g_1\,t+\beta_1) + \&c. \\ \qquad \&c. \qquad\qquad\qquad \&c. \end{array}\right\} \tag{4}$$

$$\left.\begin{array}{l} p' = M \sin\gamma + M'_1 \sin\,(k_1\,t+\gamma_1) + M'_2 \sin\,(k_2\,t+\gamma_2) + \&c. \\ q' = M \cos\gamma + M'_1 \cos\,(k_1\,t+\gamma_1) + M'_2 \cos\,(k_2\,t+\gamma_2) + \&c. \\ p'' = M \sin\gamma + M''_1 \sin\,(k_1\,t+\gamma_1) + \&c. \\ \qquad \&c. \qquad\qquad \&c. \end{array}\right\} \tag{5}$$

In the second members of these equations, all the symbols, except t, represent known constants. If we substitute the expressions (4) in (1), and put for brevity

$$\left.\begin{array}{l} E_0 = [0,1]\, N'_0 + [0,2]\, N''_0 + [0,3]\, N'''_0 + \&c. \\ E_1 = [0,1]\, N'_1 + [0,2]\, N''_1 + [0,3]\, N'''_1 + \&c. \\ \qquad \&c. \qquad\qquad \&c. \\ b\ = (0,1) + (0,2) + \&c. \end{array}\right\} \tag{6}$$

the equations (1) reduce to

$$\left.\begin{aligned} \frac{dh}{dt} &= b\,l - E_0 \cos(g\,t + \beta) - E_1 \cos(g_1 t + \beta_1) - \dots \\ \frac{dl}{dt} &= -b\,h + E_0 \sin(g\,t + \beta) + E_1 \sin(g_1 t + \beta_1) + \dots \end{aligned}\right\} \qquad (7)$$

The integrals of these equations may be expressed in the form

$$\left.\begin{aligned} h &= \frac{E_0}{b-g} \sin(g\,t + \beta) + \frac{E_1}{b-g_1} \sin(g_1 t + \beta_1) + \dots + A \sin(b\,t + B) \\ l &= \frac{E_0}{b-g} \cos(g\,t + \beta) + \frac{E_1}{b-g_1} \cos(g_1 t + \beta_1) + \dots + A \cos(b\,t + B) \end{aligned}\right\} \qquad (8)$$

A and B being arbitrary constants, depending on the values of h and l at a given epoch.

The integral values of p and q may be obtained by a similar process. By putting

$$\left.\begin{aligned} I_0 &= (0,1)\,M + (0,2)\,M + (0,3)\,M + \dots = b\,M \\ I_1 &= (0,1)\,M'_1 + (0,2)\,M''_1 + (0,3)\,M'''_1 + \dots \\ &\quad \text{\&c.} \qquad\qquad \text{\&c.} \end{aligned}\right\} \qquad (9)$$

we have from (2) and (5)

$$\left.\begin{aligned} \frac{dp}{dt} &= -b\,q + I_0 \cos\gamma + I_1 \cos(k_1 t + \gamma) + I_2 \cos(k_2 t + \gamma_2) + \dots \\ \frac{dq}{dt} &= b\,p - I_0 \sin\gamma - I_1 \sin(k_1 t + \gamma) - I_2 \sin(k_2 t + \gamma_2) - \dots \end{aligned}\right\} \qquad (10)$$

The integrals of these equations are

$$\left.\begin{aligned} p &= M \sin\gamma + \frac{I_1}{b+k_1} \sin(k_1 t + \gamma) + \frac{I_2}{b+k_2} \sin(k_2 t + \gamma_2) + \dots + K \sin(-b\,t + C) \\ q &= M \cos\gamma + \frac{I_1}{b+k_1} \cos(k_1 t + \gamma_1) + \frac{I_2}{b+k_2} \cos(k_2 t + \gamma_2) + \dots + K \cos(-b\,t + C) \end{aligned}\right\} (11)$$

K and C being arbitrary constants, depending on the values of p and q at a given epoch.

To give a clear idea of a geometrical construction of these equations, suppose a sphere described around the sun as a centre, and let the radius of this sphere be taken for unity. Let the point in which the pole of the ecliptic (or other plane of reference) cuts the surface of the sphere be taken as an origin of co-ordinates. From this origin draw on the sphere a radius vector equal to M, and making an angle equal to $\gamma - 90°$ with the axis of x, from which we suppose the longitude to be reckoned. From the end of the first radius vector, draw another equal to $\frac{I_1}{b+k_1}$, and making an angle equal to $\gamma_1 - 90°$ with the axis of x, and so continue through all the terms of the second member of (11). The end of the last radius vector, K, will be on the point in which the pole of the orbit of the asteroid intersects the sphere at the origin of time. If, now, we suppose the radius vector $\frac{I_1}{b+k_1}$ to move around the end of M with a uniform angular motion equal to k_1, $\frac{I_2}{b+k_2}$ to move around with an angular motion of k_2, and so on to K; the end of K will continually be on the point in which the pole of the orbit of the asteroid intersects the surface of the sphere.

M and γ are the same for all the planets and asteroids; the end of M may therefore be taken as an origin around which the poles of the orbits of all the planets and asteroids move. This point is evidently the mean position of the pole of each separate planet and asteroid during all time,* and the plane of which it is the pole may be regarded as the mean position of the plane of the orbit of each heavenly body during all time. This plane is the invariable plane of maximum areas; as may be seen from the fact that the constant M vanishes when we take that plane as the plane of reference.†

This plane ought, also, to be the probable mean position of the orbits of the several asteroids at any one time, in so far as the positions of the separate orbits are independent of each other, and to the same degree, the nodes on this plane ought to be distributed at random. We shall return to this subject when considering the distribution of the nodes and perihelia in longitude.

In the above investigations and constructions, quantities of the third order with respect to the eccentricities and inclinations have been neglected, and we have therefore made no distinction between the distance from the origin of the poles of the orbits on the surface of the sphere, and of the points in which these poles intersect the tangent plane, or the secant plane which passes through the point, and is parallel to the tangent plane; or between i, $\sin i$, and $\tan i$.

In the above construction, the distance of the final point, or pole of the orbit of the asteroid, from the pole of the ecliptic, will be equal to the inclination, and its longitude increased by 90° will be equal to the longitude of the node. If, then, one of the radii vectores $\frac{I}{b+k}$ . . . . . K is longer than the sum of all the others, it is evident that the amount by which it exceeds that sum will be an inferior limit of the inclination, and that the mean motion of the node will be equal to the coefficient of t in the angle which corresponds to the longest radius vector.

Let us now apply the formulæ given above to the numerical computation of the elements of the asteroids in terms of the time. Many of the required quantities being

* In other words, if we mark on the plane tangent to the sphere, and parallel to the invariable plane, the points in which the pole of the asteroid orbit intersects it at equidistant intervals of time, to infinity, the point of tangency will be the centre of gravity of all these points.

† *Méc. Céleste*, Liv. II. No. 62.

functions of the mean distances only of the asteroids, it will be convenient to tabulate them for different values of that element; and by means of such tables, the equations required can be obtained for any asteroid whatever, of which both the eccentricity and inclination are small, with very little labor. In the computation I use the following values of the masses of the disturbing planets.

Venus $\frac{1}{390000}$, Earth $\frac{1}{354936}$, Mars $\frac{1}{2860637}$, Jupiter $\frac{1}{1050}$,
Saturn $\frac{1}{3502}$, Uranus $\frac{1}{21000}$, Neptune $\frac{1}{17000}$.

The direct action of Mercury is neglected, it being entirely insensible.

Making use of these values of the masses, and of the usual values of the mean distances; calling (0,1), [0,1], the coefficients between an asteroid and Venus, (0,2) [0,2], between an asteroid and the Earth, and so on, we have the following values of those coefficients in seconds, the unit of time being 365¼ days.

a	(0,1)	(0,2)	(0,3)	(0,4)	(0,5)	(0,6)	(0,7)
2.2	0″.102	0″.268	0″.184	31″.076	1″.156	0″.022	0″.007
2.4	0.073	0.183	0.104	38.359	1.345	0.025	0.008
2.6	0.053	0.131	0.066	47.372	1.550	0.028	0.009
2.8	0.040	0.096	0.044	58.727	1.775	0.032	0.010
3.0	0.031	0.073	0.032	73.315	2.021	0.036	0.011
3.2	0.025	0.057	0.023	92.495	2.291	0.039	0.012
a	[0,1]	[0,2]	[0,3]	[0,4]	[0,5]	[0,6]	[0,7]
2.2	0″.041	0″.148	0″.148	16″.036	0″.331	0″.003	0″.001
2.4	0.027	0.093	0.078	21.485	0.419	0.004	0.001
2.6	0.018	0.062	0.046	28.587	0.523	0.005	0.001
2.8	0.013	0.042	0.029	37.930	0.644	0.006	0.001
3.0	0.009	0.030	0.019	50.381	0.784	0.007	0.001
3.2	0.007	0.022	0.013	67.277	0.947	0.008	0.002

We next require the numerical values of the quantities represented in (4) and (5) by N and M. These are obtained by the simultaneous integration, for the larger planets, of the systems of equations (1) and (2), a process so laborious, that only three or four astronomers have ever attempted to carry it through. It seems to have been done most completely by Le Verrier, before the discovery of Neptune, as he then gave the differential coefficients of the several quantities with respect to the masses; and he has since taken into account the action of that planet, but has not given these differential coefficients, nor has he considered in what way the action of Neptune might be modified by that of the planets inside of Jupiter. Still, it is quite possible, that the inaccuracies thus introduced are no greater than those which proceed from the neglect of terms of the third order, and we shall therefore use throughout the values given by

Le Verrier in the second volume of his *Annales de l'Observatoire.* They are condensed into the following tables. The quantities omitted are presumed to be insensible. Some of them are known to be so; the remainder are certainly very small, and no data for their computation are given by Le Verrier or any other writer.

n	N^{I}_{n}	N^{II}_{n}	N^{III}_{n}	N^{IV}_{n}	N^{V}_{n}	N^{VI}_{n}
0	+.00048	+.00053	+.0007	+.003057	+.00274	+.031
1	+.01679	+.01661	+.0191	+.042675	+.03334	—.048
2	—.00038	+.00237	+.0152	—.015509	+.04831	—.002
3	+.01689	+.01062	+.0017	—.000020	—.00002	....
4	—.02383	—.01892	—.0033	+.000012	+.00001	....
5	—.01301	+.01178	+.0292	—.000001	—.00001	....
6	+.01534	—.01691	+.0730	—.000001	—.00001	....
7	....	....	....	+.000095	+.00011	....

n	M^{I}_{n}	M^{II}_{n}	M^{III}_{n}	M^{IV}_{n}	M^{V}_{n}	M^{VI}_{n}	M^{VII}_{n}
1	+.0025	+.0023	+.0017	+.001159	+.00093	—.0176	....
2	+.0003	+.0027	+.0093	—.006306	+.01580	....	....
3	+.0209	+.0146	+.0030	—.000040	—.00005	....	....
4	+.0099	+.0083	+.0018	—.000011	—.00002	....	....
5	—.0075	+.0054	+.0485	—.000002	—.00002	....	....
6	+.0244	—.0243	+.0338	....	....	....	....
7	....	....	....	—.001514	—.00145	....	+.0111

From these data we find the following values of E_0, E_1, &c., I_0, I_1, &c., and b.

a	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7
2.2	+0″.0502	+0″.7012	—0″.2301	+0″.0022	—0″.0041	+0″.0055	+0″.0089	+0″.0016
2.4	+0.0670	+0.9341	—0.3116	+0.0011	—0.0024	+0.0030	+0.0045	+0.0021
2.6	+0.0890	+1.2394	—0.4173	+0.0005	—0.0014	+0.0018	+0.0025	+0.0028
2.8	+0.1179	+1.6413	—0.5568	—0.0001	—0.0007	+0.0011	+0.0016	+0.0037
3.0	+0.1564	+2.1768	—0.7431	—0.0005	—0.0002	+0.0007	+0.0010	+0.0049
3.2	+0.20 5	+2.9030	—0.9974	—0.0010	+0.0002	+0.0005	+0.0006	+0.0065

a	I_1	I_2	I_3	I_4	I_5	I_6	I_7	b
2.2	+0″.0379	—0″.1753	+0″.0052	+0″.0031	+0″.0095	+0″.0023	—0″.0486	32″.815
2.4	+0.0460	—0.2192	+0.0029	+0.0020	+0.0054	+0.0008	—0.0599	40.097
2.6	+0.0564	—0.2733	+0.0012	+0.0012	+0.0034	+0.0003	—0.0738	49.209
2.8	+0.0695	—0.3456	—0.0001	+0.0006	+0.0022	+0.0001	—0.0913	60.724
3.0	+0.0865	—0.4209	—0.0012	+0.0001	+0.0015	+0.0001	—0.1138	75.519
3.2	+0.1089	—0.5467	—0.0024	—0.0003	+0.0010	0.0000	—0.1432	94.942

In the computations of the quantities E and I, we have a test of their accuracy, because they ought to fulfil the conditions.

$$E_0 + E_1 + \dots = [0,1]\,(N'_0 + N'_1 + \dots) + [0,2]\,(N''_0 + N''_1 + \dots) + \&c.$$

$$I_1 + I_2 + I_3 + \dots = (0,1)\,(M'_1 + M'_2 + \dots) + (0,2)\,(M''_1 + M''_2 + \dots) + \&c.$$

We omit the terms dependent on M_0 in the values of I, because in the final result they will bring out $k_0 = M$, as in the equation (9).

We shall now put $\varepsilon_n = \frac{N_n}{b - g_n}$, $x_n = \frac{M_n}{b + k_n}$; and to obtain the numerical values of these quantities we shall use the following values of g, g_1, &c., k_1, k_2, &c., which are deduced from the data given in the work of Le Verrier above referred to, and are partially corrected for the terms of third order.

$g = 2.901$ $\quad g_4 = 7.575$ $\quad k = 0.0$ $\quad k_4 = -7.086$

$g_1 = 3.808$ $\quad g_5 = 17.153$ $\quad k_1 = -3.166$ $\quad k_5 = -17.468$

$g_2 = 22.828$ $\quad g_6 = 17.863$ $\quad k_2 = -26\ 568$ $\quad k_6 = -18.568$

$g_3 = 5.299$ $\quad g_7 = 0.693$ $\quad k_3 = -4.795$ $\quad k_7 = -0.756$

We thus obtain the following values of ε and $\varkappa$ for every .05 in the value of a between the limits of the mean distances of the asteroids.

TABLE I.—*For Eccentricities and Perihelia.*

a	ε	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7
2.20	+.001679	+.024174	—.023044	+.000079	—.000162	+.000352	+.000597	+.000049
2.25	.001710	.024567	.021326	.000064	.000133	.000269	.000448	.000050
2.30	.001741	.024960	.019984	.000052	·000109	.000209	.000343	.000051
2.35	.001772	.025352	.018913	.000041	.000089	.000164	.000262	000052
2.40	.001802	.025743	.018043	.000033	.000073	.000131	.000203	.000053
2.45	.001832	.026133	.017329	.000026	.000060	.000105	.000159	.000054
2.50	.001862	.026523	.016735	.000020	.000050	.000085	.000126	.000055
2.55	.001893	.026912	.016237	.000015	.000041	.000069	.000100	.000056
2.60	.001922	.027299	.015817	.000011	.000034	.000056	.000081	.000057
2.65	.001951	.027685	.015463	.000007	.000027	.000046	.000066	.000058
2.70	.001980	.028070	.015162	.000004	.000022	.000038	.000054	.000059
2.75	.002009	.028454	.014906	+.000001	.000017	.000031	.000045	.000060
2.80	.002038	.028837	.014688	—.000001	.000014	.000026	.000037	.000061
2.85	.002067	.029218	.014502	.000003	.000011	.000022	.000030	.000062
2.90	.002196	.029598	.014346	.000004	.000008	.000018	.000025	.000063
2.95	.002225	.029977	.014214	.000006	.000006	.000015	.000021	.000064
3.00	.002153	.030355	.014104	.000007	.000004	.000012	.000017	.000065
3.05	.002181	.030732	.014017	.000008	—.000002	.000010	.000014	.000066
3.10	.002309	.031107	.013935	.000009	.000000	.000008	.000012	.000067
3.15	.002237	.031481	.013875	.000010	+.000001	.000007	.000010	.000068
3.20	+.002265	+.031854	—.013831	—.000011	+.000002	+.000006	+.000008	+.000069

TABLE II.—*For Inclinations and Nodes.* TABLE III.

a	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	$\varkappa_5$	$\varkappa_6$	$\varkappa_7$	b
2.20	+.001277	—.028054	+000185.	+.000122	+.000616	+.000165	—.001516	32″.815
2.25	.001269	.023416	000150.	.000103	.000475	.000110	.001518	34.488
2.30	.001261	.020246	000123.	.000086	.000373	.000074	.001520	36.256
2.35	.001254	.017946	000101.	.000072	.000296	.000052	.001522	38.123
2.40	.001247	.016200	000082.	.000060	.000237	.000037	.001523	40.097
2.45	.001241	.014831	000065.	.000049	.000191	.000027	.001523	42.184
2.50	.001235	.013729	000051.	.000040	.000156	.000020	.001524	44.394
2.55	.001230	.012824	000039.	.000033	.000129	.000014	.001524	46.732
2.60	+.001225	—.012070	+000028.	+.000028	+.000106	+.000010	—.001524	49.209

a	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	$\varkappa_5$	$\varkappa_6$	$\varkappa_7$	b
2.65	+.001220	—.011432	+.000018	+.000023	+.000087	+.000007	—.001524	51″.833
2.70	.001216	.010886	.000009	.000018	.000071	.000005	.001524	54.619
2.75	.001212	.010413	+.000003	.000014	.000059	.000004	.001524	57.577
2.80	.001208	.010002	—.000001	.000011	.000050	.000003	.001523	60.724
2.85	.001204	.009640	.000003	.000008	.000043	.000002	.001523	64.074
2.90	.001201	.009320	.000006	.000006	.000037	.000002	.001523	67.641
2.95	.001198	.009036	.000009	.000004	.000031	.000001	.001523	71.450
3.00	.001196	.008782	.000012	.000002	.000026	.000001	.001522	75.519
3.05	.001193	.008555	.000015	+.000001	.000022	+.000001	.001522	79.875
3.10	.001191	.008350	.000019	—.000001	.000018	.000000	.001522	84.543
3.15	.001189	.008164	.000023	.000002	.000015	.000000	.001521	89.555
3.20	+.001187	—.007996	—.000027	—.000004	—.000013	.000000	—.001521	94.942

The following are the values of β, β_1, &c., γ, γ_1, &c., and M, supposing the time to be reckoned from 1800, and the longitudes from the equinox of 1850.0. $M = .027460$.

$\beta = 98^\circ\ 30'\ 28''$	$\gamma = 106^\circ\ 50'\ 15''$
$\beta_1 = 26\ 32\ 23$	$\gamma_1 = 135\ 41\ 7$
$\beta_2 = 127\ 17\ 9$	$\gamma_2 = 126\ 42\ 20$
$\beta_3 = 86\ 27\ 45$	$\gamma_3 = 23\ 20\ 25$
$\beta_4 = 36\ 18\ 43$	$\gamma_4 = 262\ 48\ 16$
$\beta_5 = 335\ 28\ 27$	$\gamma_5 = 297\ 28\ 24$
$\beta_6 = 315\ 11\ 1$	$\gamma_6 = 73\ 53\ 49$
$\beta_7 = 74\ 2\ 14$	$\gamma_7 = 202\ 15\ 43$

If now, we put for brevity,

$$(0) = \beta + g\,t$$
$$(1) = \beta_1 + g_1\,t \qquad [1] = \gamma_1 + k_1\,t$$
$$(2) = \beta_2 + g_2\,t \qquad [2] = \gamma_2 + k_2\,t$$
$$\vdots \qquad\qquad \vdots$$
$$(7) = \beta_7 + g_7\,t \qquad [7] = \gamma_7 + k_7\,t$$

we shall have, for the elements h, l, p, and q of any asteroid in terms of the time,

$$\left.\begin{aligned} h &= \varepsilon \sin (0) + \varepsilon_1 \sin (1) + \varepsilon_2 \sin (2) + \ldots + \varepsilon_7 \sin (7) + A \sin (B + b\,t) \\ l &= \varepsilon \cos (0) + \varepsilon_1 \cos (1) + \varepsilon_2 \cos (2) + \ldots + \varepsilon_7 \cos (7) + A \cos (B + b\,t) \\ p &= M \sin \gamma + \varkappa_1 \sin [1] + \varkappa_2 \sin [2] + \ldots + \varkappa_7 \sin [7] + K \sin (C - b\,t) \\ q &= M \cos \gamma + \varkappa_1 \cos [1] + \varkappa_2 \cos [2] + \ldots + \varkappa_7 \cos [7] + K \cos (C - b\,t) \end{aligned}\right\} \qquad (12)$$

A, B, $\varkappa$, and C being fixed by the values of h, l, p, and q at a given epoch. The quantities ε, ε_1, &c., $\varkappa_1$, $\varkappa_2$, &c., M, and b, are taken from Tables 1, 2, and 3, by entering with the mean distance of the asteroid as the argument.

Let us now apply these data to those asteroids the elements of which are determined with sufficient accuracy, and the eccentricities and inclinations of which are sufficiently small. The latter class may be presumed to include all those for which each of these elements is less than 11°. The uncertainty in the present mean values of the elements is such, that no advantage will result from any attempt to carry the results to more than four places of decimals; which are, moreover, quite sufficient for our present purposes. The following are the assumed values of the longitude of perihelion, longitude of node, eccentricities, and inclinations, for January 1, 1850, of such of the asteroids fulfilling the above conditions as I have to the present time (March, 1860) been able to obtain accurate elements.

Name.	Symbol.	π	☊	e	i
Ceres	(1)	149° 12'.5	80° 49'.0	0.0803	10° 36'.5
Vesta	(4)	250 23.1	103 19.5	0.0902	7 8.1
Astræa	(5)	134 35.6	141 24.8	0.1900	5 19.6
Flora	(8)	32 57.2	110 18.3	0.1567	5 53.2
Metis	(9)	70 51.2	68 30.0	0.1234	5 35.9
Hygea	(10)	227 44.0	287 39.7	0.1005	3 47.2
Parthenope	(11)	315 57.1	125 2.6	0.0988	4 36.9
Irene	(14)	179 15.0	86 38.8	0.1652	9 7.1
Psyche	(16)	12 24.1	150 32.0	0.1363	3 3.9
Thetis	(17)	259 12.8	125 26.5	0.1268	5 35.6
Fortuna	(19)	30 8.7	211 24.9	0.1578	1 32.2
Massilia	(20)	98 16.0	206 40.2	0.1438	0 41.0
Lutetia	(21)	326 58.4	80 26.8	0.1620	3 5.2
Themis	(24)	137 37.6	36 14.3	0.1177	0 49.1
Proserpina	(26)	235 4.2	45 53.2	0.0875	3 35.7
Euterpe	(27)	87 25.4	93 36.8	0.1730	1 35.5
Bellona	(28)	122 16.1	144 43.3	0.1544	9 22.5
Amphitrite	(29)	56 52.4	356 26.8	0.0725	6 8.0
Urania	(30)	30 47.5	308 13.0	0.1268	2 6.0
Pomona	(32)	194 12.0	220 48.9	0.0824	5 29.0
Circe	(34)	149 48.3	184 47.3	0.1083	5 26.5
Fides	(37)	65 56.1	8 10.7	0.1749	3 7.2
Leda	(38)	100 29.5	296 27.8	0.1556	6 58.5
Lætitia	(39)	1 48.0	157 20.5	0.1110	10 20.8
Harmonia	(40)	1 6.3	93 29.5	0.0462	4 15.8

From these elements, and from the preceding tables, we obtain the following expressions for h, l, p, and q, in terms of the time, for the above asteroids.

Ceres. (1)

$$h = .0020 \sin(0) + .0286 \sin(1) - .0148 \sin(2) + .1102 \sin(158^\circ\, 51' + 58''.58\, t)$$
$$l = .0020 \cos(0) + .0286 \cos(1) - .0148 \cos(2) + .1102 \cos(158^\circ\, 51' + 58''.58\, t)$$
$$p = M \sin\gamma + .0012 \sin[1] - .0103 \sin[2] + .0001 \sin[5] - .0015 \sin[7] + .1652 \sin(80^\circ\, 11' - 58''.58\, t)$$
$$q = M \cos\gamma + .0012 \cos[1] - .0103 \cos[2] + .0001 \cos[5] - .0015 \cos[7] + .1652 \cos(80^\circ\, 11' - 58''.58\, t)$$

Vesta. (4)

$$h = .0018 \sin(0) + .0254 \sin(1) - .0188 \sin(2) - .0001 \sin(4) + .0002 \sin(5) + .0003 \sin(6) + .1048 \sin(231^\circ\, 29' + 38''.56\, t)$$
$$= .0018 \cos(0) + .0254 \cos(1) - .0188 \cos(2) - .0001 \cos(4) + .0002 \cos(5) + .0003 \cos(6) + .1048 \cos(231^\circ\, 29' + 38''.56\, t)$$

$p = M \sin \gamma + .0013 \sin [1] - .0176 \sin [2] + .0001 \sin [3] + .0001 \sin [4] + .0003 \sin [5] - .0015 \sin [7] + .1123 \sin (107° 5' - 38''.56\ t)$

$q = M \cos \gamma + .0013 \cos [1] - .0176 \cos [2] + .0001 \cos [3] + .0001 \cos [4] + .0003 \cos [5] - .0015 \cos [7] + .1123 \sin (107° 5' - 38''.56\ t)$

Astræa. ⑤

$h = .0019 \sin (0) + .0271 \sin (1) - .0160 \sin (2) + .0001 \sin (6) + .2146 \sin (140° 31' + 48.''03\ t)$

$l = .0019 \cos (0) + .0271 \cos (1) - .0160 \cos (2) + .0001 \cos (6) + .2146 \cos (140° 31' + 48.''03\ t)$

$p = M \sin \gamma + .0012 \sin [1] - .0125 \sin [2] + .0001 \sin [5] - .0015 \sin [7] + .0830 \sin (151° 38' - 48''.03 t)$

$q = M \cos \gamma + .0012 \cos [1] - .0125 \cos [2] + .0001 \cos [5] - .0015 \cos [7] + .0830 \cos (151° 38' - 48''.03 t)$

Flora. ⑧

$h = .0017 \sin (0) + .0242 \sin (1) - .0230 \sin (2) - .0002 \sin (4) + .0003 \sin (5) + .0006 \sin (6) + .1323 \sin (43° 27' + 32''.86\ t)$

$l = .0017 \cos (0) + .0242 \cos (1) - .0230 \cos (2) - .0002 \cos (4) + .0003 \cos (5) + .0006 \cos (6) + .1323 \cos (43° 27' + 32''.86\ t)$

$p = M \sin \gamma + .0013 \sin [1] - .0280 \sin [2] + .0002 \sin [3] + .0006 \sin [5] + .0002 \sin [6] - .0015 \sin [7] + .1021 \sin (116° 37' - 32''.86\ t)$

$q = M \cos \gamma + .0013 \cos [1] - .0280 \cos [2] + .0002 \cos [3] + .0006 \cos [5] + .0002 \cos [6] - .0015 \cos [7] + .1021 \cos (116° 37' - 32''.86\ t)$

Metis. ⑨

$h = .0018 \sin (0) + .0256 \sin (1) - .0183 \sin (2) - .0001 \sin (4) + .0001 \sin (5) + .0002 \sin (6) + .1187 \sin (86° 19' + 39''.52\ t)$

$l = .0018 \cos (0) + .0256 \cos (1) - .0183 \cos (2) - .0001 \cos (4) - .0001 \cos (5) + .0002 \cos (6) + .1187 \sin (86° 19' + 39''.52\ t)$

$p = M \sin \gamma + .0013 \sin [1] - .0168 \sin [2] + .0003 \sin [5] - .0015 \sin [7] + .0837 \sin (67° 10' - 39''.52 t)$

$q = M \cos \gamma + .0013 \cos [1] - .0168 \cos [2] + .0003 \cos [5] - .0015 \cos [7] + .0837 \cos (67° 10' - 39''.52 t)$

Hygea. ⑩

$h = .0022 \sin (0) + .0315 \sin (1) - .0139 \sin (2) + .1307 \sin (216° 7' + 88''.95\ t)$

$l = .0022 \cos (0) + .0315 \cos (1) - .0139 \cos (2) + .1307 \cos (216° 7' + 88''.95\ t)$

$p = M \sin \gamma + .0012 \sin [1] - .0082 \sin [2] - .0015 \sin [7] + .0869 \sin (286° 6' - 88''.95\ t)$

$q = M \cos \gamma + .0012 \cos [1] - .0082 \cos [2] - .0015 \cos [7] + .0869 \cos (286° 6' - 88''.95\ t)$

Parthenope. ⑪

$h = .0018 \sin (0) + .0262 \sin (1) - .0173 \sin (2) + .0001 \sin (5) + .0002 \sin (6) + .0775 \sin (298° 2' + 42''.30\ t)$

$l = .0018 \cos (0) + .0262 \cos (1) - .0173 \cos (2) + .0001 \cos (5) + .0002 \cos (6) + .0775 \cos (298° 2' + 42''.30\ t)$

$p = M \sin \gamma + .0012 \sin [1] - .0148 \sin [2] + .0002 \sin [5] - .0015 \sin [7] + .0694 \sin (134° 7' - 42''.30 t)$

$q = M \cos \gamma + .0012 \cos [1] - .0148 \cos [2] + .0002 \cos [5] - .0015 \cos [7] + .0694 \cos (134° 7' - 42''.30 t)$

Irene. ⑭

$h = .0019 \sin(0) + .0272 \sin(1) - .0159 \sin(2) + .0001 \sin(6) + .1990 \sin(178° 57' + 48''.68\, t)$

$l = .0019 \cos(0) + .0272 \cos(1) - .0159 \cos(2) + .0001 \cos(6) + .1990 \cos(178° 57' + 48''.68\, t)$

$p = M \sin\gamma + .0012 \sin[1] - .0123 \sin[2] + .0001 \sin[5] - .0015 \sin[7] + .1407 \sin(86° 57' - 48.''68\, t)$

$q = M \cos\gamma + .0012 \cos[1] - .0123 \cos[2] + .0001 \cos[5] - .0015 \cos[7] + .1407 \cos(86° 57' - 48.''68\, t)$

Psyche. ⑯

$h = .0022 \sin(0) + .0298 \sin(1) - .0143 \sin(2) + .1013 \sin(13° 37' + 69.''35\, t)$

$l = .0022 \cos(0) + .0299 \cos(1) - .0143 \cos(2) + .1013 \cos(13° 37' + 69.''35\, t)$

$p = M \sin\gamma + .0012 \sin[1] - .0092 \sin[2] - .0015 \sin[7] + .0453 \sin(173° 21' - 69''.35\, t)$

$q = M \cos\gamma + .0012 \cos[1] - .0092 \cos[2] - .0015 \cos[7] + .0453 \cos(173° 21' - 69''.35\, t)$

Thetis. ⑰

$h = .0018 \sin(0) + .0263 \sin(1) - .0171 \sin(2) + .0001 \sin(5) + .0002 \sin(6) + .1368 \sin(244° 30' + 43''.22\, t)$

$l = .0018 \cos(0) + .0263 \cos(1) - .0171 \cos(2) + .0001 \cos(5) + .0002 \cos(6) + .1368 \cos(244° 30' + 43''.22\, t)$

$p = M \sin\gamma + .0012 \sin[1] - .0143 \sin[2] + .0002 \sin[5] - .0015 \sin[7] + .0858 \sin(132° 57' - 43''.22\, t)$

$q = M \cos\gamma + .0012 \cos[1] - .0143 \cos[2] + .0002 \cos[5] - .0015 \cos[7] + .0858 \cos(132° 57' - 43''.22\, t)$

Fortuna. ⑲

$h = .0018 \sin(0) + .0261 \sin(1) - .0174 \sin[2] + .0001 \sin(5) + .0002 \sin(6) + .1302 \sin(37° 23' + 41''.84\, t)$

$l = .0018 \cos(0) + .0261 \cos(1) - .0174 \cos[2] + .0001 \cos(5) + .0002 \cos(6) + .1302 \sin(37° 23' + 41''.84\, t)$

$p = M \sin\gamma + .0012 \sin[1] - .0151 \sin[2] + .0002 \sin[5] - .0015 \sin[7] + .0383 \sin(230° 34' - 41''.84\, t)$

$q = M \cos\gamma + .0012 \cos[1] - .0151 \cos[2] + .0002 \cos[5] - .0015 \cos[7] + .0383 \cos(230° 34' - 41''.84\, t)$

Massilia. ⑳

$h = .0018 \sin(0) + .0258 \sin(1) - .0179 \sin(2) + .0001 \sin(5) + .0002 \sin(6) + .1535 \sin(110° 11' + 40''.49\, t)$

$l = .0018 \cos(0) + .0258 \cos(1) - .0179 \cos(2) + .0001 \cos(5) + .0002 \cos(6) + .1535 \sin(110° 11' + 40''.19\, t)$

$p = M \sin\gamma + .0012 \sin[1] - .0159 \sin[2] + .0002 \sin[5] - .0015 \sin[6] + .0238 \sin(237° 58' - 40''.49\, t)$

$q = M \cos\gamma + .0012 \cos[1] - .0159 \cos[2] + .0002 \cos[5] - .0015 \cos[6] + .0238 \cos(237° 58' - 40''.49\, t)$

Lutetia. ㉑

$h = .0018 \sin(0) + .0260 \sin(1) - .0176 \sin(2) + .0001 \sin(5) + .0002 \sin(6) + .1342 \sin(318° 48' + 41''.56\, t)$

$l = .0018 \cos(0) + .0260 \cos(1) - .0176 \cos(2) + .0001 \cos(5) + .0002 \cos(6) + .1342 \cos(318° 48' + 41''.56\, t)$

$p = M \sin\gamma + .0012 \sin[1] - .0152 \sin[2] + .0002 \sin[5] - .0015 \sin[6] + .0384 \sin(79° 54' - 41''.56\, t)$

$q = M \cos\gamma + .0012 \cos[1] - .0152 \cos[2] + .0002 \cos[5] - .0015 \cos[6] + .0384 \cos(79° 54' - 41''.56\, t)$

Themis. (24)

$h = .0022 \sin (0) + .0315 \sin (1) - .0139 \sin (2) + .1441 \sin (147° 40' + 89''.55\, t)$
$l = .0022 \cos (0) + .0315 \cos (1) - .0139 \cos (2) + .1441 \cos (147° 40' + 89''.55\, t)$
$p = M \sin \gamma + .0012 \sin [1] - .0082 \sin [2] - 0015 \sin [7] + .0187 \sin (318° 53' - 89''.55\, t)$
$q = M \cos \gamma + .0012 \cos [1] - .0082 \cos [2] - 0015 \cos [7] + .0187 \cos (318° 53' - 89''.55\, t)$

Proserpina. (26)

$h = .0019 \sin (0) + .0277 \sin (1) - .0155 \sin (2) + .1115 \sin (220° 23' + 52''.15\, t)$
$l = .0019 \cos (0) + .0277 \cos (1) - .0155 \cos (2) + .1115 \cos (220° 23' + 52''.15\, t)$
$p = M \sin \gamma + .0012 \sin [1] - .0114 \sin [2] + .0001 \sin [5] - .0015 \sin [7] + .0514 \sin (31° 35' - 52''.15\, t)$
$q = M \cos \gamma + .0012 \cos [1] - .0114 \cos [2] + .0001 \cos [5] - .0015 \cos [7] + .0514 \cos (31° 35' - 52''.15\, t)$

Euterpe. (27)

$h = .0018 \sin (0) + .0253 \sin (1) - .0189 \sin (2) - .0001 \sin (4) + .0002 \sin (5) + .0003 \sin (6)$
$\quad + .1772 \sin (98° 2' + 38''.02\, t)$
$l = .0018 \cos (0) + .0253 \cos (1) - .0189 \cos (2) - .0001 \cos (4) + .0002 \cos (5) + .0003 \cos (6)$
$\quad + .1772 \cos (98° 2' + 38''.02\, t)$
$p = M \sin \gamma + .0013 \sin [1] - .0181 \sin [2] + .0001 \sin [3] + .0003 \sin [5] - .0015 \sin [7]$
$\quad + .0157 \sin (110° 14' - 38''.02\, t)$
$q = M \cos \gamma + .0013 \cos [1] - .0181 \cos [2] + .0001 \cos [3] + .0003 \cos [5] - .0015 \cos [7]$
$\quad + .0157 \cos (110° 14' - 38''.02\, t)$

Bellona. (28)

$h = .0020 \sin (0) + .0286 \sin (1) - .0148 \sin (2) + .1665 \sin (131° 37' + 59''.13\, t)$
$l = .0020 \cos (0) + .0286 \cos (1) - .0148 \cos (2) + .1665 \cos (131° 37' + 59''.13\, t)$
$p = M \sin \gamma + .0012 \sin [1] - .0102 \sin [2] + .0001 \sin [5] - .0015 \sin [7] + .1517 \sin (151° 11' - 59''.13\, t)$
$q = M \cos \gamma + .0012 \cos [1] - .0102 \cos [2] + .0001 \cos [5] - .0015 \cos [7] + .1517 \cos (151° 11' - 59''.13\, t)$

Amphitrite. (29)

$h = .0019 \sin (0) + .0269 \sin (1) - .0162 \sin (2) + .0001 \sin (6) + .0603 \sin (83° 44' + 46''.97\, t)$
$l = .0019 \cos (0) + .0269 \cos (1) - .0162 \cos (2) + .0001 \cos (6) + .0603 \cos (83° 41' + 46''.97\, t)$
$p = M \sin \gamma + .0012 \sin [1] - .0128 \sin [2] + .0001 \sin [5] - .0015 \sin [7] + .1088 \sin$
$\quad (347° 59' - 46''.97\, t)$
$q = M \cos \gamma + .0012 \cos [1] - .0128 \cos [2] + .0001 \cos [5] - .0015 \cos [7] + .1088 \cos$
$\quad (347° 59' - 47''.97\, t)$

Urania. (30)

$h = .0018 \sin (0) + .0256 \sin (1) - .0186 \sin (2) - .0001 \sin (4) + .0002 \sin (5) + .0003 \sin (6)$
$\quad + .1005 \sin (41° 26' + 38''.67\, t)$
$l = .0018 \cos (0) + .0256 \cos (1) - .0186 \cos (2) - .0001 \cos (4) + .0002 \cos (5) + .0003 \cos (6)$
$\quad + .1005 \cos (41° 26' + 38''.67\, t)$
$p = M \sin \gamma + .0013 \sin [1] - .0174 \sin [2] + .0001 \sin [3] + .0003 \sin [5] - .0015 \sin [7]$
$\quad + .0466 \sin (295° 24' - 38''.67\, t)$
$q = M \cos \gamma + .0013 \cos [1] - .0174 \cos [2] + .0001 \cos [3] + .0003 \cos [5] - .0015 \cos [7]$
$\quad + .0466 \cos (295° 24' - 38''.67\, t)$

Pomona. (32)

$$h = .0019 \sin(0) + .0272 \sin(1) - .0159 \sin(2) + .0001 \sin(6) + .1157 \sin(189^\circ\ 52' + 48''.66\ t)$$
$$l = .0019 \cos(0) + .0272 \cos(1) - .0159 \cos(2) + .0001 \cos(6) + .1157 \cos(189^\circ\ 52' + 48''.66\ t)$$
$$p = M \sin\gamma + .0012 \sin[1] - .0123 \sin[2] + .0001 \sin[5] + .0015 \sin[7] + .1080 \sin(228^\circ\ 37' - 48''.66\ t)$$
$$q = M \cos\gamma + .0012 \cos[1] - .0123 \cos[2] + .0001 \cos[5] + .0015 \cos[7] + .1080 \cos(228^\circ\ 37' - 48''.66\ t)$$

Circe. (34)

$$h = .0020 \sin(0) + .0280 \sin(1) - .0152 \sin(2) + .1379 \sin(156^\circ\ 58' + 53''.97\ t)$$
$$l = .0020 \cos(0) + .0280 \cos(1) - .0152 \cos(2) + .1379 \cos(156^\circ\ 58' + 53''.97\ t)$$
$$p = M \sin\gamma + .0012 \sin[1] - .0110 \sin[2] + .0001 \sin[5] - .0015 \sin[7] + .0975 \sin(196^\circ\ 39' - 53''.97\ t)$$
$$q = M \cos\gamma + .0012 \cos[1] - .0110 \cos[2] + .0001 \cos[5] - .0015 \cos[7] + .0975 \cos(196^\circ\ 39' - 53''.97\ t)$$

Fides. (37)

$$h = .0019 \sin(0) + .0277 \sin(1) - .0155 \sin(2) + .1625 \sin(75^\circ\ 58' + 51''.55\ t)$$
$$l = .0019 \cos(0) + .0277 \cos(1) - .0155 \cos(2) + .1625 \cos(75^\circ\ 58' + 51''.55\ t)$$
$$p = M \sin\gamma + .0012 \sin[1] - .0115 \sin[2] + .0001 \sin[5] - .0015 \sin[7] + .0555 \sin(349^\circ\ 37' - 51''.55\ t)$$
$$q = M \cos\gamma + .0012 \cos[1] - .0115 \cos[2] + .0001 \cos[5] - .0015 \cos[7] + .0555 \cos(349^\circ\ 37' - 51''.55\ t)$$

Leda. (38)

$$h = .0020 \sin(0) + .0284 \sin(1) - .0150 \sin(2) + .1630 \sin(111^\circ\ 43' + 56''.97\ t)$$
$$l = .0020 \cos(0) + .0284 \cos(1) - .0150 \cos(2) + .1630 \cos(111^\circ\ 43' + 56''.97\ t)$$
$$p = M \sin\gamma + .0012 \sin[1] - .0105 \sin[2] + .0001 \sin[3] + .0015 \sin[7] + .1393 \sin(294^\circ\ 5' - 56''.97\ t)$$
$$q = M \cos\gamma + .0012 \cos[1] - .0105 \cos[2] + .0001 \cos[3] + .0015 \cos[7] + .1393 \cos(294^\circ\ 5' - 56''.97\ t)$$

Lætitia. (39)

$$h = .0020 \sin(0) + .0286 \sin(1) - .0148 \sin(2) + .0766 \sin(359^\circ\ 42' + 58''.80\ t)$$
$$l = .0020 \cos(0) + .0286 \cos(1) - .0148 \cos(2) + .0766 \cos(359^\circ\ 42' + 58''.80\ t)$$
$$p = M \sin\gamma + .0012 \sin[1] - .0102 \sin[2] + .0001 \sin[5] - .0015 \sin[7] + .1720 \sin(163^\circ\ 59' - 58''.80\ t)$$
$$q = M \cos\gamma + .0012 \cos[1] - .0102 \cos[2] + .0001 \cos[5] - .0015 \cos[7] + .1720 \cos(163^\circ\ 59' - 58''.80\ t)$$

Harmonia. (40)

$$h = .0017 \sin(0) + .0247 \sin(1) - .0208 \sin(2) - .0001 \sin(4) + .0003 \sin(5) + .0005 \sin(6) + .0126 \sin(25^\circ\ 27' + 35''.09\ t)$$
$$l = .0017 \cos(0) + .0247 \cos(1) - .0208 \cos(2) - .0001 \cos(4) + .0003 \cos(5) + .0005 \cos(6) + .0126 \cos(25^\circ\ 27' + 35''.09\ t)$$
$$p = M \sin\gamma + .0013 \sin[1] - .0223 \sin[2] + .0001 \sin[3] + .0001 \sin[4] + .0005 \sin[5] + .0001 \sin[6] - .0015 \sin[7] + .0656 \sin(99^\circ\ 41' - 35''.09\ t)$$
$$q = M \cos\gamma + .0013 \cos[1] - .0223 \cos[2] + .0001 \cos[3] + .0001 \cos[4] + .0005 \cos[5] + .0001 \cos[6] - .0015 \cos[7] + .0656 \cos(99^\circ\ 41' - 35''.09\ t)$$

From the preceding expressions, we easily deduce the following conclusions: —

1. Harmonia is the only asteroid, among those whose elements are well determined,

the orbit of which can ever approach indefinitely near the circular form. Doris may possibly be found to be an additional asteroid in this class.

2. Euterpe is the only known asteroid the orbit of which can ever approach indefinitely near the invariable plane of the planetary system.

3. The perihelion of each asteroid (Harmonia, Doris, and Euterpe excepted) revolves nearly in the same time as its node, the time of revolution varying from about 15,000 to 40,000 years.

4. The following are the greatest and least values which can be attained by the eccentricities and inclinations to the invariable plane of the orbits of the asteroids included in the preceding tables: —

Asteroid.	Greatest Eccentricity.	Least Eccentricity.	Greatest Inclination.	Least Inclination.
			° ′	° ′
Ceres	0.1556	0.0648	10 13	8 43
Vesta	0.1514	0.0582	7 38	5 13
Astræa	0.2598	0.1696	5 38	3 53
Flora	0.1823	0.0823	7 40	4 2
Metis	0.1648	0.0726	5 56	3 39
Hygea	0.1783	0.0831	5 37	4 22
Parthenope	0.1231	0.0319	5 0	2 58
Irene	0.2441	0.1539	8 56	7 12
Psyche	0.1476	0.0550	3 17	1 55
Thetis	0.1823	0 0913	5 54	3 56
Fortuna	0.1758	0.0846	3 14	1 10
Massilia	0.1993	0.1077	2 26	0 17
Lutetia	0.1799	0.0885	3 12	1 8
Themis	0.1917	0.0965	1 42	0 27
Proserpina	0.1566	0.0664	3 46	2 8
Euterpe	0.2238	0.1306	2 7	0 00
Bellona	0.2119	0.1211	9 26	7 57
Amphitrite	0.1054	0.0152	7 7	5 20
Urania	0.1471	0.0539	3 51	1 29
Pomona	0.1608	0.0706	7 3	5 20
Circe	0.1831	0.0927	6 23	4 48
Fides	0.2076	0.1174	4 0	2 22
Leda	0.2084	0.1176	8 45	7 13
Lætitia	0.1220	0.0312	10 36	9 7
Harmonia	0.0607	0.0000	5 15	2 16

§ 2.

We now have the necessary data for investigating the questions referred to in the beginning of this paper. Since it is deducible from Olbers's hypothesis that the orbits of all the asteroids once intersected each other in a common point, we may first find whether it is possible or probable that there ever was such intersection. I have considered this question in Gould's Astronomical Journal, No. 129, with reference to the asteroids Vesta and Hygea, and there shown that, although the aphelion distance of Vesta sometimes exceeds the perihelion distance of Hygea, an intersection of their orbits is not possible so long as the equations (12) of the preceding section will give nearly true values of the elements by attributing any values whatever to the angles (1), (2), (3), &c. In order that any conclusion that one may draw respecting this question may rest on as broad a foundation as possible, we shall now consider the same question with respect to other asteroids.

In the paper referred to, it is shown that an intersection of the orbits is not possible unless their elements and secular coefficients are such as to fulfil the condition

$$(k'h - k h')^2 + (k'l - k l')^2 > (k' - k)^2,$$

k' representing the semi-parameter of the outer orbit, and k that of the other. Substituting for h and l their expressions in the preceding section, and putting $B + b\,t = N$, $B' + b'\,t = N'$, this inequality will take the form

$$\begin{aligned} &- 2\,k k'\,A A' \cos (N' - N) - 2\,\Sigma_n\,k\,A'\,(k'\varepsilon_n - k'\,\varepsilon_n) \cos (N' - (n)) \\ &+ 2\,\Sigma_n k'\,A'\,(k'\varepsilon_n - k\,\varepsilon'_n) \cos (N - n) + 2\,\Sigma_m\,\Sigma_n\,(k'\varepsilon_m - k\,\varepsilon'_m)\,(k'\varepsilon_n - k\,\varepsilon'_n) \cos ((m) - (n)) \\ &> (k' - k)^2 - \Sigma\,(k'\varepsilon - k\,\varepsilon')^2 - k'^2\,A^2 - k^2\,A'^2. \end{aligned} \qquad (13)$$

If A and A' are both small, the eccentricities of the orbits will likewise be small, and k and k' will be subject to only very slight variations. If we suppose k and k' constant, the maximum value of the first member of the above inequality will be, after transposing the last three terms,

$$\{k'\,A + k\,A' + \Sigma \pm (k'\varepsilon - k\,\varepsilon')\}^2,$$

and by finding the value of this expression, using the mean values of the parameters k' and k, and so taking the doubtful signs that the value of this expression shall be the greatest possible, we may at once find whether an intersection of the orbits is possible, by observing whether the condition $k'\,A + k\,A' \pm \Sigma\,(k'\varepsilon - k\,\varepsilon') > k' - k$ is fulfilled.

If A and A' are both large, the first member of the above inequality can attain its greatest magnitude only when we have very nearly $N^1 - N = 180°$. If this condition is fulfilled, the eccentricity of the one orbit will be at its maximum when that of the other is at its minimum. The second member of (13) will attain its least magnitude when the eccentricity of the outer planet is at its maximum, and that of the inner one at its minimum; that is when we have

$$\left.\begin{aligned} (0) = (1) = (3) = (5) = (6) = (7) = N' \\ (2) = (4) = N = 180° - N' \end{aligned}\right\} \qquad (14)$$

But, since $k'\,\varepsilon - k\,\varepsilon'$ has in general the same sign as ε, it is evident from an inspection of (13), that these conditions will also make its first member attain its least magnitude;

and, as the change is in each member of the same order of magnitude, the planets having large eccentricities can be treated without great error in nearly the same manner as those for which this element is small. We may, however, obtain a more simple inequality than (13), which will have the advantage of enabling us to ascertain how near any number of asteroids could ever have come to a common point of intersection. Suppose the conditions (14) to be fulfilled. Then suppose any one of the angles (0), (1), (2), to vary, and let it differ by the quantity α from its first value. Then the second member of (13) will, in consequence of the change in the eccentricities of the planets, change its value by a quantity very nearly of the form $\mu \cos \alpha$, and the change in the first member will evidently be of the same form, and may be represented by $\mu' \cos \alpha$. μ' and μ are one half the changes which the first and second members of (13) respectively undergo by a change of 180° in the value of the angle. Hence the state of the orbits most favorable to (13) will be found either when $\alpha = 0$, or when $\alpha = 180°$, according to the relative magnitudes of μ and μ'. The perihelion of the outer planet will then have the same longitude as the aphelion of the inner one; and since in the state supposed the eccentricities of the orbits are equal to $(A' \pm \varepsilon \pm \varepsilon_1 \pm \varepsilon_2 \ldots)$ and $(A \mp \varepsilon \mp \varepsilon_1 \mp \varepsilon_2 \ldots)$, it follows that the condition of possibility of intersection will be

$$a\,(1 + A \pm \varepsilon \pm \varepsilon_1 \pm \varepsilon_2 \ldots \pm \varepsilon_7) > a'\,(1 - A \mp \varepsilon' \mp \varepsilon_1' \ldots \mp \varepsilon'_7)\,; \tag{15}$$

the corresponding ε's having opposite signs in the two members, and being so taken as most to favor the condition. From Table I. it is evident that ε, ε_1, and ε_7 should be taken negatively (without regard to their signs in the Table); while ε_3, ε_4, &c. should be taken positively; and ε_2 is doubtful, but would generally have to be taken positively. ε', ε'_1, and ε_7 must therefore be regarded as positive, and the other ε's as negative.

From this, and from the preceding numerical expressions for the secular variations, we deduce the following system of values of the eccentricities, and consequent perihelion and aphelion distances of certain of the asteroids which lie near the extreme limits of the zone, as being those most favorable to the intersection of their orbits in a common point.

Outer Asteroids.			Inner Asteroids.		
	Eccentricities.	Perihelion Distances.		Eccentricities.	Aphelion Distances.
Hygea	.1505	2.68	Vesta	.0968	2.58
Themis	.1639	2.63	Flora	.1304	2.49
Psyche	.1190	2.58	Metis	.1101	2.65
Lætitia	.1024	2.49	Urania	.0925	2.58
			Harmonia	.0077	2.29

From these values it follows that while the coefficients ε, ε_1, &c. and A have the values assigned to them in the preceding investigation, the orbit of Hygea could never have intersected with that of Vesta, Flora, Metis, Urania, or Harmonia; the orbit of Themis could never have intersected with that of Vesta, Flora, Metis, or Harmonia; and that of Psyche could never have intersected with that of Flora or Harmonia.

It is also to be remarked that the angles (0), (1), (2), &c. are entirely independent of the circumstances of the explosion, being functions of the time alone, and that therefore the chances are tens of thousands to one against their all having had, at the time of the explosion, values near those which have been here assumed for them.

The question now arises whether this result can be considered conclusive against Olbers's hypothesis. Three possible sources of inaccuracy are to be considered: —

1. The effect of the quantities of the third order, neglected in the analysis and computations.

2. The effects of the mutual attraction of the asteroids.

3. The possible action, on the asteroids, of other forces than that of gravitation.

The quantities of the third order would not probably affect the eccentricities by a greater amount than two or three units in the third place of decimals. If so, they would not materially change the character of the preceding result.

Two asteroids might, by their mutual attraction, change each other's mean distance very materially, provided that they passed sufficiently near each other. Two asteroids, taken at random, might be expected to pass within 10,000 miles of each other about once in four hundred millions (400,000,000) of years, and therefore *some* two asteroids might be expected to pass this near each other about once in 250,000 years. If the magnitude of the largest asteroids is, as we might judge from their brilliancy, the hundred-thousandth part that of the earth, an approach of another asteroid within ten thousand miles of it would be sufficient to cause a material change in the mean motion of the latter. It does not, therefore, seem possible absolutely to disprove Olbers's hypothesis, by an attempt at rigorous computations of the secular variations of the asteroids.

The only force besides the attraction of gravitation which it will be worth while to consider, is the resistance of a medium. If, as Encke supposes, the celestial spaces are pervaded by a very rare resisting medium, it does not seem at all improbable that, during the millions of years which may have been occupied by the asteroids in moving through it, their orbits may have been so altered by its action as to invalidate our conclusions, drawn from reasoning in which the effect of this action has been neglected. Our next question will then be that of the third section.

§ 3.

Have the Orbits of the Asteroids ever been materially affected by a resisting Medium?

It is highly probable that all the asteroids are of nearly the same density; it is at least highly improbable that there exists any relation between the density and the magnitude, by virtue of which the smaller asteroids are more dense than the larger ones. If, then, these bodies are retarded by the action of a resisting medium, we may expect to see its effect more manifest on the small ones than on the large ones; the resistance being proportional to the superficial area, while the inertia is proportional to the mass, and probably to the volume. If, then, the large and small asteroids were originally arranged indiscriminately with respect to their distance from the sun, the effect of a resisting medium would be manifested in a tendency among the smaller asteroids to be nearer the sun than the larger ones. If we represent the mass of an asteroid by m, and the number of asteroids by n, the condition of indiscriminate arrangement would be

$$\frac{\Sigma\, m\, a}{\Sigma\, m} = \frac{\Sigma\, a}{n} \pm \beta\,;$$

in which β is a quantity of the order of magnitude of the chance errors of distribution; which diminishes inversely as the square root of n; and therefore vanishes when n is infinite. A tendency in the smaller asteroids to be near the sun will be manifested by the quantity $\frac{\Sigma\, m\, a}{\Sigma\, m} - \frac{\Sigma\, a}{n}$ being greater than any probable value of β, and *vice versa*, a tendency in the larger asteroids to be near the sun will be manifested by the same expression being negative, and greater than any probable value of β.

Unfortunately, however, if we apply this method to the actual case now in question, we shall fall into error from a cause which seems unavoidable. In fact, an asteroid near the sun will be more easily discovered than a more distant one of the same magnitude, owing to its greater brilliancy; and this circumstance is of itself sufficient to cause a tendency in the smaller known asteroids to be near the sun, though no such tendency should exist in the whole group, known and unknown. We could eliminate the effects of this cause, provided that we knew the general law which connects any assumed magnitude with the number of asteroids of that magnitude. But such a law can be derived only from observation, and the discussion of the observations will be subject to the same difficulty with the application of the test. Still, we may make some deductions respecting the effect of a resisting medium by considering its different effects on bodies of different magnitudes.

Judging from its brilliancy, Atalanta is the smallest known asteroid. Its brilliancy is only about $\frac{1}{150}$ as great as would be that of Vesta at the same distance. Its mean distance is 2.748, — considerably greater than the average. Supposing that by the resistance of a medium it was brought from the farthest limit of the zone of the asteroids to the position in which we find it, it would have been caused to approach the sun by the amount .407. The brilliancy of different asteroids at equal distances being proportional to the squares of their diameters, while the effect of the resisting medium is inversely proportional to their diameters, the effect of the medium on Vesta would be only $\frac{1}{12}$ as great as on Atalanta; the mean distance of the latter would, therefore, have been diminished by 0.034; and this we may regard as the extreme limit of the possible change in the mean distance of Vesta from this cause.

Hygea is smaller than Vesta, and Themis smaller than either Vesta or Flora. Hence, if these asteroids have been affected by a medium, the former positions of their orbits were more unfavorable for a common point of intersection than their present ones; hence our conclusions respecting the possibility of a common point of intersection are not invalidated by our not having taken into account the action of this possible cause. Moreover, one effect of the medium would be to increase the eccentricities of all the asteroids, and for this reason the former forms of the orbits were less favorable to intersection if this cause has acted.

§ 4.

Of the Relations between the Masses of the Asteroids and certain Elements of their Orbits.

On any probable hypotheses that we can make respecting the cause of an explosion of a planet, the smaller fragments ought, on the whole, to be thrown off with a greater velocity than the larger ones. Moreover, when, as in the case of the asteroids, each fragment is very small compared with the original mass, it seems at least highly probable that the velocities of those thrown in any one direction would be nearly the same, on the whole, as the velocities of those thrown in a direction at right angles to that of the first.

Thus we have two probable tests of Olbers's hypothesis. To apply them, we shall first deduce certain relations between the velocity with which a fragment would be thrown, and the elements of the orbit in which it would afterwards move.

For this purpose, take, for the axis of X, a line passing through the sun and the position of the planet at the time of the explosion, let the axis of Y be in the plane of the orbit of the planet, and that of Z perpendicular to it. Represent by a_0 the distance of the planet from the sun, and by ξ and ζ the velocities of the projected fragments in the direction of the axes of X and Z respectively; and by η the velocity in the direction of the axis of Y relatively to that of a planet moving in a circular orbit at the distance a_0. ξ, η, and ζ will then be equal to the velocities of projection of the fragment in the directions of the corresponding axes, plus the velocities of the planet in the same direction over and above those due to a circular orbit.

The only elements which we can determine are the eccentricity, inclination, and the amount by which the mean distance differs from a_0. In determining these, we may, in a rough approximation like the present, neglect all quantities of the second order with respect to the velocities of projection ξ, η, and ζ. Represent by δa the difference between a_0 and a, the latter being the mean distance of the fragment after its projection, by v_0 the velocity of the planet in a circular orbit, and by v the actual velocity of the fragment after projection. We then have

$$\frac{1}{a} = \frac{2\,k^2 - a_0 v^2}{a_0\,k^2}; \quad v^2 = (v_0 + \eta)^2 + \xi^2 + \zeta^2.$$

We then obtain by suitable reductions, neglecting quantities of the second order, and observing that $v_0 = \frac{k}{a_0^{\frac{1}{2}}}$,

$$\delta a = \frac{2\,a^{\frac{3}{2}}\,\eta}{k}, \tag{16}$$

k here representing the Gaussian constant.

Representing for the present by p the parameter of the orbit, we have

$$p = \frac{a_0^2\,(v^2 - \xi^2)}{k^2}$$

$$\frac{p}{a} = \frac{a_0}{k^2}\left(2\,v^2 - 2\,\xi^2 + \frac{a_0^2\,v^2\,\xi^2}{k^2} - \frac{a_0^2\,v^4}{k^2}\right)$$

$$e^2 = 1 - \frac{p}{a} = \frac{1}{k^2}\left(k^2 - 2\,a_0\,v^2 + 2\,a_0\,\xi^2 - \frac{a_0^2\,v^2\,\xi^2}{k^2} + \frac{a_0^2\,v^4}{k^2}\right).$$

But by the conditions of circular motion

$$-\,k^2 + 2\,a_0\,v_0^2 - \frac{a_0^2\,v_0^4}{k^2} = 0.$$

Wherefore

$$e^2 = \frac{1}{k^2}\left(2\,a_0\,(v_0^2 - v^2) + 2\,a_0\,\xi^2 - \frac{a_0^2\,v_0^2\,\xi^2}{k^2} + \frac{a_0^2\,(v^4 - v_0^4)}{k^2}\right).$$

Developing the expressions within the parenthesis to quantities of the second order inclusive, and observing that $\frac{v_0^2}{k^2} = \frac{1}{a_0}$, we find

$$e^2 = \frac{a_0\,(\xi^2 + 4\,\eta^2)}{k^2}; \quad e = \frac{\sqrt{a\,(\xi^2 + 4\,\eta^2)}}{k}. \tag{17}$$

For the inclination we easily find

$$i = a_0^{\frac{1}{2}}\,\frac{\zeta}{k}. \tag{18}$$

If the eccentricity of the planet before the explosion were small, the mean values of ξ, η, and ζ would be very nearly the mean velocity of projection of the fragments.

Representing by λ, λ', and λ'' the angles which the direction of projection of any fragment makes with the axes of co-ordinates, by α the velocity of projection, and by ξ_0, η_0, and ζ_0 the velocities of the planet in the direction of the three axes relatively to the velocity due to a circular orbit, we have

$$\xi = \xi_0 + \alpha \cos \lambda$$
$$\eta = \eta_0 + \alpha \cos \lambda'$$
$$\zeta = \zeta_0 + \alpha \cos \lambda''.$$

Since by hypothesis $\cos \lambda$ has all values at random between $+1$ and -1, and ξ_0 and η_0 are small compared with α, any small positive value of ξ_0 will diminish the absolute numerical values of ξ for the several fragments for which $\alpha \cos \lambda$ is negative nearly as much as it will increase those values for the fragments for which $\alpha \cos \lambda$ is positive, and *vice versa.* A similar remark will of course apply to η and ζ. On the whole, however, the mean values of ξ, η, and ζ, taken without regard to their signs, will be increased by a quantity of the order of magnitude of $\frac{\xi_0^2}{\alpha}$, $\frac{\eta_0^2}{\alpha}$, $\frac{\zeta_0^2}{\alpha}$, respectively.

A comparison of (17) and (18) shows that the mean value of the eccentricities of the fragments ought to be nearly $\sqrt{5}$ times as great as that of the inclinations; and that the mean value of $\frac{\delta a}{a}$ should be about twice as great as that of the inclinations, or a little less than that of the eccentricities.

From the equations (16), (17), and (18), we easily obtain expressions for ξ, η, and ζ in terms of the elements. They are,

$$\xi = k \sqrt{\frac{a^2 e^2 - 2 \delta a^2}{a_0^3}}; \; \eta = k \frac{\delta a}{2 a_0^{\frac{3}{2}}}; \; \zeta = k \frac{i}{a_0^{\frac{1}{2}}}. \tag{19}$$

To apply these equations rigorously, we should know the values of the eccentricities and inclinations of the several fragments immediately after the explosion. But from the equation (15) of § 2, it appears that the eccentricities of the fragments must immediately after the explosion have been quite near their mean values; moreover, the eccentricities and inclinations are subject to but comparatively slight variations, as they will very rarely approach either of the limits given on page 138. We shall, therefore, use the mean values of those elements for the asteroids whose secular variations are given in § 1, and the present values for those which are not there included.

The mean of the perihelion distances of the outer asteroids given in the Table in § 2 is 2.60, and the mean of the aphelion distances of the inner ones is 2.52. The mean of these may be taken as the most probable value of a_0, which we shall therefore put equal to 2.56.

Symbol.	e_0	δa	$\frac{a^{\frac{1}{2}}}{k}\xi$	$\frac{a^{\frac{1}{2}}\eta}{k}$	$\frac{a^{\frac{1}{2}}\zeta}{k}$	$\frac{a^{\frac{1}{2}}v}{k}$	v	M	m_n
(1)	0.114	+0.21	0.08	0.04	0.17	0.19	2.3	251	48
(2)	0.239	+0.21	0.23	0.04	0.57	0.61	7.2	166	46
(3)	0.256	+0.11	0.25	0.02	0.22	0.34	4.0	66	45
(4)	0.108	—0.20	0.27	—0.04	0.12	0.15	1.8	234	47
(5)	0.218	+0.02	0.22	0.00	0.08	0.23	2.7	17	26
(6)	0.202	—0.13	0.19	—0.03	0.26	0.33	3.9	44	42
(7)	0.231	—0.17	0.22	—0.03	0.10	0.25	3.0	40	40
(8)	0.136	—0.36	0.09 $\sqrt{-1}$	—0.08	0.10	(0.13)	(1.5)	19	30
(9)	0.122	—0.17	0.11	—0.03	0.11	0.16	1.9	30	38
(10)	0.134	+0.59	0.15 $\sqrt{-1}$	0.10	0.09	(0.14)	(1.8)	55	44
(11)	0.084	—0.11	0.07	—0.02	0.07	0.10	1.2	20	31
(12)	0.219	—0.23	0.20	—0.05	0.15	0.25	3.0	14	22
(13)	0.085	+0.02	0.08	0.00	0.30	0.31	3.7	18	27
(14)	0.201	+0.03	0.20	0.01	0.14	0.24	2.8	22	33
(15)	0.187	+0.08	0.18	0.02	0.21	0.28	3.3	43	41
(16)	0.106	+0.36	0.06 $\sqrt{-1}$	0.06	0.05	(0.08)	(1.0)	29	35
(17)	0.140	—0.09	0.14	—0.02	0.09	0.16	1.9	14	24
(18)	0.217	—0.26	0.18	—0.06	0.18	0.27	3.2	14	23
(19)	0.134	—0.12	0.13	—0.02	0.04	0.14	1.7	16	25
(20)	0.157	—0.15	0.15	—0.03	0.03	0.15	1.8	22	32
(21)	0.138	—0.12	0.13	—0.02	0.04	0.14	1.7	12	19
(22)	0.104	—0.06	0.10	—0.01	0.24	0.26	3.1	8	9
(23)	0.232	+0.07	0.23	0.01	0.18	0.29	3.4	7	8
(24)	0.147	+0.59	0.14 $\sqrt{-1}$	0.10	0.03	(0.10)	(1.4)	10	16
(25)	0.253	—0.16	0.25	—0.03	0.37	0.44	5.2	7	7
(26)	0.115	+0.10	0.11	0.02	0.08	0.14	1.7	10	14
(27)	0.183	—0.21	0.16	—0.04	0.03	0.17	2.0	11	18
(28)	0.170	+0.22	0.15	0.04	0.15	0.22	2.6	29	36
(29)	0.068	—0.01	0.07	—0.00	0.11	0.13	1.5	36	39
(30)	0.105	—0.20	0.07	—0.04	0.05	0.09	1.1	10	13
(31)	0.216	+0.60	0.14	0.10	0.46	0.49	5.8	11	17
(32)	0.119	+0.03	0.12	0.01	0.11	0.13	1.5	7	4
(33)	0.337	+0.31	0.32	0.06	0.03	0.33	3.9	8	10
(34)	0.131	+0.13	0.12	0.03	0.10	0.16	1.9	5	3
(35)	0.221	+0.42	0.17	0.07	0.14	0.23	2.7		
(36)	0.298	+0.19	0.29	0.04	0.33	0.44	5.2	2	1
(37)	0.165	+0.08	0.16	0.02	0.06	0.17	2.0	12	21
(38)	0.166	+0.18	0.15	0.03	0.14	0.21	2.5	10	15
(39)	0.092	+0.21	0.05	0.04	0.17	0.18	2.1	46	43
(40)	0.035	—0.29	0.11 $\sqrt{-1}$	—0.07	0.07	(0.09)	(1.1)	19	29
(41)	0.202	—0.16	0.19	—0.03	0.28	0.34	4.0		
(42)	0.223	—0.13	0.21	—0.03	0.15	0.26	3.1	7	5
(43)	0.168	—0.36	0.05	—0.08	0.06	0.13	1.5	7	6
(44)	0.147	—0.13	0.14	—0.03	0.07	0.15	1.8	8	11
(45)	0.085	+0.18	0.05	0.03	0.12	0.14	1.6	9	12
(46)	0.169	+0.22	0.17	0.00	0.04	0.17	2.0	2	2
(47)	0.128	+0.32	0.04	0.06	0.09	0.12	1.4	30	37
(48)	0.077	+0.55	0.18 $\sqrt{-1}$	0.09	0.11	(0.14)	(1.7)	12	20
(49)	0.238	+0.53	0.16	0.09	0.06	0.19	2.0	18	28
(50)	0.287	+0.09	0.28	0.02	0.05	0.29	3.4	23	34
(51)	0.063	—0.18	0.05 $\sqrt{-1}$	—0.04	0.18	(0.18)	(2.1)		
(52)	0.102	+0.54	0.15 $\sqrt{-1}$	0.09	0.13	(0.16)	(2.0)		
(53)	0.180	+0.05	0.18	0.01	0.09	0.20	2.4		
(54)	0.188	+0.16	0.18	0.03	0.20	0.27	3.2		
(55)	0.134	+0.22	0.11	0.04	0.13	0.17	2.0		
(56)	0.203	—0.03	0.20	—0.01	0.13	0.24	2.8		
(57)	0.106	+0.59	0.17 $\sqrt{-1}$	0.10	0.26	(0.29)	(3.4)		

The computation of ξ, η, and ζ, with the data necessary thereto, are given in the preceding table.

In this I have also included the quantities necessary for finding whether there exists any relation between the magnitudes of the asteroids, and the quantities ξ, η, and ζ, or the velocities with which they were projected. The second column gives the value of the eccentricity used in the computation; and the third the difference between the actual mean distance of the asteroid and 2.56. Then follow the values of $a^{\frac{1}{2}}\,\xi$, $a^{\frac{1}{2}}\,\eta$, and $a^{\frac{1}{2}}\,\zeta$, as given by the formulæ (19). The seventh column contains the values of $a^{\frac{1}{2}}\,v$, which is proportional to the probable absolute velocity of projection of the asteroid, v being equal to $\sqrt{\xi^2 + \eta^2 + \zeta^2}$. To express these velocities in ordinary astronomical units, they must be multiplied by $\frac{\text{Gaussian } k}{\sqrt{2.56}}$. If we take the earth's mean distance, and the solar day, as the units of space and time respectively, this factor will be .0107; and if for these units we take the English mile and the solar second, the factor will be 11.95. The eighth column gives the absolute velocity of projection of the asteroid, in miles per second. These numbers are not to be regarded as absolutely accurate, being separately subjected to possible errors of 4 or 5 in the last place. But from what precedes, it will be seen that the mean of a considerable number of them will be nearly exact. When ξ is imaginary, its value has been supposed zero in computing v, and the latter has been put between parentheses.

The column marked M gives a series of numbers proportional to the superficial area of the asteroids, as deduced from their brilliancy. The square roots of these numbers will therefore be proportional to the diameters of the asteroids. In obtaining the values of M, I have used the table of apparent magnitudes published by Mr. Pogson in the Monthly Notices of the Royal Astronomical Society, for January, 1859.

The column headed m_n gives a series of numbers showing the order of magnitude of the forty-eight asteroids contained in Mr. Pogson's table.

It will be observed that a considerable number of the values of ξ are imaginary. These values pertain to those orbits of which, with the tabular eccentricity, the perihelion distance is greater, or the aphelion distance less, than 2.56. The real values of ξ are on the whole very nearly the same as those of ζ; if we regard the imaginary values as zero, ξ will on the whole be sensibly less than ζ. Still their agreement is quite remarkable, and this favors Olbers's hypothesis, since, as before remarked, it is what might naturally be expected if this hypothesis were true. But the values of η are far less than those of ξ and ζ, which indicates that those fragments which were projected in the direction of the line of motion of the planet were thrown with much less velocity than the others. To the smallness of η is alone due the fact, that the eccentricities are not as many times larger than the inclinations as would be required on the hypothesis of explosion. Now this smallness of η may be accounted for on Olbers's hypothesis, if we reflect that all the asteroids for which $a\frac{1}{2}\,\eta$ is greater than .120 would be thrown without the limit of the zone in which they are now found; and that owing to their consequent frequent approach to Jupiter when in their aphelion, or to Mars when in their perihelion, their orbits might be entirely deranged.

We may now determine whether there exists between the masses of the asteroids, and the velocities with which, on Olbers's hypothesis, they were thrown, any relation in virtue of which the smaller asteroids were thrown with greater velocity than the larger ones, or *vice versa.* This question would be solved with most theoretical rigor as follows: — If n asteroids be numbered in the order of their magnitude, and also in the order of their velocity of projection, if m_n represents the number of any asteroid in the order of magnitude, and ν_n in the order of velocity of projection, then will the condition of no relation between these two classes of numbers be, when n is large,

$$\frac{4\,\Sigma\, m_n\, \nu_n}{n+1} = n^2 \pm \beta,$$

β being a small quantity of the order of magnitude of the chance errors of distribution, or of $n^{\frac{3}{2}}$. But a more simple method will give a result practically quite as good. If we take the forty-eight asteroids of which both the magnitudes and velocities are given, we find that the average velocity of projection of that half of which the magnitude is greatest is .209, or about 2.49 miles per second; and of that half of which the magnitude is least, .217, or about 2.59 miles per second. This difference is much less than that which might result from the chance inequalities; hence no relation like that sought for exists between the masses of the asteroids, and the velocities with which they were projected, if Olbers's hypothesis be true.

The velocities η are, some positive, others negative, which indicates that the fragments must have been projected both backward and forward with respect to the direction of the planet's motion. The signs of ξ and ζ it is not possible to determine, since the eccentricity and inclination of a fragment would have the same value, whether these quantities were positive or negative.

In finding the values of ζ, no allowance could be made for the latitude of the planet at the time of the explosion. This must have been very small relative to the invariable plane, else the inferior limits of the inclinations of none of the asteroids could have been nearly zero.

§ 5.

Of certain observed Relations among the Orbits of the Asteroids, which are the Result, in whole or in part, of known Causes.

It has frequently been noticed by writers on the distribution of the asteroids, that the perihelia and nodes of these bodies are very unequally distributed in longitude. For about two thirds of the asteroids, these elements are found in the first semicircle of longitude.

These inequalities of distribution proceed principally from the fact that some of the principal terms in the expressions for h, l, p, and q, given in § 1, have common angles for all the asteroids, and that the coefficients of each of these angles have the same sign for the different asteroids. Thus in the expressions (12) the terms

$$\varepsilon \sin (0) + \varepsilon_1 \sin (1) + \varepsilon_2 \sin (2) + \ldots . + \varepsilon_7 \sin (7),$$

and

$$\varepsilon \cos (0) + \varepsilon_1 \cos (1) + \varepsilon_2 \cos (2) + \ldots . + \varepsilon_7 \cos (7),$$

are common to all the asteroids, the different ε's all having the same sign when they are of appreciable magnitude. The average value of the first of these expressions is about $+.0011$; and of the second, about .0314. These common terms, therefore, cause a tendency in the perihelia to be near the longitude of which the tangent is $\frac{11}{314}$, or very nearly 0°. About 33 of the 57 known asteroids have their perihelia within 90° of this point of longitude. This is but one more than the probable number which we should expect as the effect of the above-mentioned tendency. The perihelia are distributed in the four quadrants as shown in the second column of the following table. The third column shows the probable number, taking into account the above-mentioned tendency.

1	22	16
2	15	13
3	9	12
4	11	16

The excess of the number in the first quadrant over that in the fourth, and of the number in the second quadrant over that in the third, proceeds from the unequal distribution of the angles B around the circle; and this again is merely a chance temporary accumulation of those angles in the first two quadrants, which, from their expressions in § 1, will evidently not be permanent, but will wear away in the course of a few thousand years, and which did not exist a few thousand years ago.

The expressions for p and q of all the asteroids contain the common terms

$$M \sin \gamma + k_1 \sin [1] + k_2 \sin [2] + \dots + k_7 \sin [7],$$

and

$$M \cos \gamma + k_1 \cos [1] + k_2 \cos [2] + \dots + k_7 \cos [7].$$

The mean value of the first of these is about +.0180, and that of the second very small. Hence a common tendency exists among the nodes of the asteroids to be in 90° of longitude. The second and third columns of the following table exhibit the real and the probable distribution.

1	15	15
2	20	16
3	13	13
4	9	13

The excess of the number in the third quadrant over that in the fourth proceeds from a cause similar to that which produces the excess in the perihelia, above referred to.

In the general expressions for the eccentricity, inclination, longitude of perihelion, and longitude of node of Jupiter, the principal terms are

$$h_{IV} = + .0031 \sin (0) + .0427 \sin (1) - .0155 \sin (2)$$
$$l_{IV} = + .0031 \cos (0) + .0427 \cos (1) - .0155 \cos (2).$$
$$p_{IV} = M \sin \gamma + .0012 \sin [1] - .0063 \sin [2] - .0015 \sin [7]$$
$$q_{IV} = M \cos \gamma + .0012 \cos [1] - .0063 \cos [2] - .0015 \cos [7].$$

The comparison of the coefficients in these terms with the values of the corresponding ε's and $\varkappa$'s given in Tables I. and II. of § 1, show that the corresponding quantities have the same signs, and that the different ratios of their magnitudes do not differ very materially from each other. The cause of this relation is, moreover, evident from an examination of the process by which the values of ε and $\varkappa$ were obtained. It follows from it that the general law of grouping of the nodes and perihelia of the asteroids may be expressed by saying that there is always a tendency in the perihelia of the asteroids to coincide in longitude with the perihelion of Jupiter, and in their nodes to coincide in longitude with the node of Jupiter. Sometimes, however, this tendency may be more than compensated by the circumstance of a number of the angles $B + bt$ and $C - bt$ having values nearly 180° different from the longitude of the perihelion, or longitude of the node of Jupiter. It will be most manifest in those asteroids which have small eccentricities and inclinations; thus the perihelion of Harmonia will very rarely be as much as 90° distant from that of Jupiter. For a similar reason the mutual inclination of the orbits of Euterpe and Jupiter will always be quite small, it being represented very nearly by $\sqrt{(p' - p)^2 + (q' - q)^2}$.

The fact that the orbit of every asteroid, or nearly every one, is interlinked with the orbits of one or more other asteroids, so that if they were material we should by removing one carry off all the others with it, has sometimes been adduced as indicating a connection of some sort between these bodies. Let us examine the conditions of such interlinking. Suppose that one orbit of any pair is revolved around its node on the other orbit, as an axis, till the planes of the two orbits coincide. If their elements fulfil the condition (13),

$$(k\,l' - k'l)^2 + (k\,h' - k'h)^2 > (k' - k)^2,$$

the orbits will then intersect in two points. If, on this supposition, they do not intersect, it is evident that they cannot interlink; hence the preceding condition is one which must be fulfilled to render it possible for them to interlink. It is also necessary that, in the position supposed, these points should fall on opposite sides of the line of nodes. Now, in view of the small differences of mean distances, and considerable eccentricities of the asteroids, it cannot be regarded as at all singular that a large number of pairs should fulfil these conditions. As the orbits pass through their secular variations, some pairs which now interlink will cease to do so, and others which now do not interlink will do so. A change of this kind in some pair of orbits my be expected to occur in nearly every century. Hence the fact of interlinking does not indicate any relation among the asteroids other than their being found together in a continuous zone; and can throw no light whatever on the question of their origin.

In looking over a table of the elements of the asteroids, it is quite noticeable that the inclinations have a much wider range than the eccentricities. Thus, while there is but a single asteroid the eccentricity of which is less than .06, there are ten or twelve whose inclination is smaller than this quantity. Again, several of the inclinations considerably exceed the superior limit of the eccentricities. This may be seen by the following table, which exhibits the distribution in magnitude of the eccentricities and inclinations used in the preceding section: —

	e's	i's
From .00 to .05	1	8
.05 to .10	7	15
.10 to .15	20	16
.15 to .20	10	7
.20 to .25	14	4
.25 to .30	4	3
Above .30	1	4

It will also be observed, that there is a relative deficiency in the number of the above-mentioned elements having small values. The latter fact is easily accounted for. In the first place, if we consider the planet at the origin of its orbital motion, we see that, in order that its orbit may be *very nearly* circular, two independent improbable conditions must be fulfilled; — firstly, that the direction of its motion shall be *very nearly* at right angles to the line passing through the planet and the sun; secondly, that its velocity should be *very nearly* equal to $\frac{k}{\sqrt{r}}$, r representing the distance of the planet from the sun. If we regard the probabilities of these separate circumstances as small quantities of the first order, the probability of their concurrence will be a small quantity of the second order. The probability that the eccentricity does not exceed a small quantity, σ, will therefore be proportional to σ^2 so long as σ does not exceed a certain narrow limit.

The same reasoning can be applied to the inclinations. In order that the inclination of the orbit of a planet to a plane taken at random shall be very small, it is requisite both that the planet should be very near this plane, and that the line of direction of its motion should be very nearly in this plane.

To show the same result in a general form, we observe that h and l represent the negatives of the co-ordinates of the centre of the ellipse in which the planet is moving, when the mean distance of the planet is taken for unity. If now we project the positions of these points on the plane of the ecliptic, we might expect to find those near the sun distributed nearly at random. If we draw a circle with a small radius, ϱ, another with a radius 2ϱ, &c., around the sun as a centre, and if the centres of the orbits are equally distributed, the space between the first and second circles will contain three times as many centres as the inner one, the space between the second and third five times as many, and so on. It will be perceived that there is really a deficiency of small eccentricities, and a superabundance of small inclinations, though neither irregularity is greater than what *might* result from chance deviations in distribution.

It has been suggested by an acute astronomer that the excess of small inclinations proceeds from the fact that observers generally look for asteroids very near the plane of the ecliptic. Considerable weight is given to this supposition by the circumstance that most of the asteroids have been near their node at the time of their discovery.

It seems highly probable, from this circumstance, that the mean inclination of the whole number of asteroids, known and unknown, is very much greater than that of the known ones. If so, the fact furnishes an additional argument against the hypothesis of explosion, since ζ must then be much greater than ξ.