1 Vector space

Definition 1.1. A vector space V over a field $\mathbb{K}$ is a set V with two operations called addition $+$ and multiplication $\cdot$ such that the following axioms are satisfied:

(1) (i) $u + v \in V$ for all $u, v \in V$. (Addition is closed)

(ii) $u + v = v + u$ for all $u, v \in V$. (Addition is commutative)

(iii) $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$. (Addition is associative)

(iv) There exists an element $0 \in V$, called the zero vector, such that $u + 0 = 0 + u = u$ for all $u \in V$.

(v) For all $u \in V$ there exists an element $-u \in V$, called the additive inverse of u, such that $u + (-u) = 0 = -u + u$.

(2) (i) $\alpha \cdot u \in V$ for all $u \in V$ and $\alpha \in \mathbb{K}$.

(ii) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$ for all $u, v \in V$ and $\alpha \in \mathbb{K}$.

(iii) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$. for all $u \in V$ and $\alpha, \beta \in \mathbb{K}$.

(iv) $(\alpha\beta) \cdot u = \alpha \cdot (\beta \cdot u)$ for all $u \in V$ and $\alpha, \beta \in \mathbb{K}$.

(v) For all $u \in V$ there exists an element $1 \in \mathbb{K}$, called the multiplicative identity of u, such that $1 \cdot u = u \cdot 1 = u$.

Example 1.2. Let $\mathbb{C}$ be the set of complex numbers. Define addition in $\mathbb{C}$ by

$$(a + bi) + (c + di) = (a + c) + (b + d)i \qquad \text{for all} \quad a, b, c, d \in \mathbb{R}, \tag{1}$$

and define scalar multiplication by

$$\alpha \cdot (a + bi) = \alpha a + \alpha bi \qquad \text{for all scalars } \alpha \in \mathbb{R}, \text{and for all } a, b \in \mathbb{R}. \tag{2}$$

Show that $(\mathbb{C}, +, \cdot)$ is a vector space over $\mathbb{R}$.

Solution : Let $u = a + bi,\ v = c + di,\ w = e + fi \in \mathbb{C}$, where $a, b, c, d, e, f \in \mathbb{R}$, we have

(1)

(i) The addition is closed :

$$\begin{aligned} u + v &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \qquad\qquad \text{by (1).} \end{aligned}$$

Since $(a + c)$ and $(b + d)$ are real numbers then $u + v \in \mathbb{C}$.

(ii) The addition is commutative:

$$\begin{aligned} u + v &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \qquad \text{by (1),} \\ &= (c + a) + (d + b)i \qquad \text{because addition on } \mathbb{R} \text{ is commutative,} \\ &= (c + di) + (a + bi) \qquad \text{by (1),} \\ &= v + u \end{aligned}$$

(iii) The addition is associative: we have to prove that $u+(v+w) = (u+v)+w$ for all $u, v, w \in \mathbb{C}$.

The left hand side (L.H.S):

$$\begin{aligned} u + (v + w) &= u + [(c + di) + (e + fi)] \\ &= (a + bi) + [(c + e) + (d + f)i] \qquad \text{by (1),} \\ &= [a + (c + e)] + [b + (d + f)]i \qquad \text{by (1),} \\ &= [(a + c) + e] + [(b + d) + f]i \qquad \text{because addition on } \mathbb{R} \text{ is associative.} \end{aligned}$$

The right hand side (R.H.S):

$$\begin{aligned}(u+v)+w &= [(a+bi)+(c+di)]+w\\ &= [(a+c)+(b+d)i]+(e+fi) \qquad \text{by (1)},\\ &= [(a+c)+e]+[(b+d)+f)]i \qquad \text{by (1)}.\end{aligned}$$

Then L.H.S=R.H.S

(iv) The additive identity : For all $u = a+bi \in \mathbb{C}$, we have

$$\begin{aligned}(a+bi)+(0+0i) &= (a+0)+(b+0)i \qquad \text{by (1)},\\ &= a+bi \qquad \text{because 0 is the additive identity in } \mathbb{R}.\end{aligned}$$

Then the additive identity of $\mathbb{C}$ is $(0+0i)$.

(v) The additive inverse : For all $u = a+bi \in \mathbb{C}$, we have

$$\begin{aligned}(a+bi)+\Big(-a+(-b)i\Big) &= \Big(a+(-a)\Big)+\Big(b+(-b)\Big)i \qquad \text{by (1)},\\ &= 0+0i \qquad \text{because } (-a) \text{ is the additive inverse of } a \text{ in } \mathbb{R}.\end{aligned}$$

Then the additive inverse of $a+bi \in \mathbb{C}$ is $-a+(-b)i$.

(2) Let $u = a+bi, v = c+di \in \mathbb{C}$ and $\alpha \in \mathbb{R}$.

(i) We have to prove that $\alpha \cdot u \in \mathbb{C}$.

$$\begin{aligned}\alpha \cdot u &= \alpha \cdot (a+bi)\\ &= \alpha a + \alpha bi\end{aligned}$$

Since $\alpha a, \alpha b \in \mathbb{R}$, then $\alpha \cdot u \in \mathbb{C}$.

(ii) We have to prove that $\alpha \cdot (u+v) = \alpha \cdot u + \alpha \cdot v$ for all $u, v \in \mathbb{C}$ and $\alpha \in \mathbb{R}$.

The left hand side (L.H.S) :

$$\begin{aligned}\alpha \cdot (u+v) &= \alpha \cdot [(a+bi)+(c+di)] \\ &= \alpha \cdot [(a+c)+(b+d)i] \qquad \text{by (1)} \\ &= \alpha(a+c)+\alpha(b+d)i \qquad \text{by (2)} \\ &= (\alpha a+\alpha c)+(\alpha b+\alpha d)i \qquad \text{because multiplication distributes over addition in } \mathbb{R}.\end{aligned}$$

The right hand side (R.H.S) :

$$\begin{aligned}\alpha \cdot u+\alpha \cdot v &= \alpha \cdot (a+bi)+\alpha \cdot (c+di) \\ &= (\alpha a+\alpha bi)+(\alpha c+\alpha di) \qquad \text{by (2),} \\ &= (\alpha a+\alpha c)+(\alpha b+\alpha d)i \qquad \text{by (1),}\end{aligned}$$

Then L.H.S=R.H.S

(iii) We have to prove that $(\alpha+\beta)\cdot u = \alpha\cdot u+\beta\cdot u$ for all $u \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$.

The L.H.S :

$$\begin{aligned}(\alpha+\beta)\cdot u &= (\alpha+\beta)\cdot(a+bi) \\ &= (\alpha+\beta)a+(\alpha+\beta)bi \qquad \text{by (2),} \\ &= (\alpha a+\beta a)+(\alpha b+\beta b)i \qquad \text{because multiplication distributes over addition in } \mathbb{R}.\end{aligned}$$

The R.H.S :

$$\begin{aligned}\alpha\cdot u+\beta\cdot u &= \alpha\cdot(a+bi)+\beta\cdot(a+bi) \\ &= (\alpha a+\alpha bi)+(\beta a+\beta bi) \qquad \text{by (2),} \\ &= (\alpha a+\beta a)+(\alpha b+\beta b)i \qquad \text{by (1).}\end{aligned}$$

Then L.H.S=R.H.S

(iv) We have to prove that $(\alpha\ \beta) \cdot u = \alpha \cdot (\beta \cdot u)$ for all $u \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$.
The L.H.S :

$$\begin{aligned}(\alpha\beta) \cdot u &= (\alpha\beta) \cdot (a + bi)\\ &= (\alpha\beta)a + (\alpha\beta)b\ i \qquad \text{by (2),}\\ &= \alpha\beta a + \alpha\beta b\ i \qquad \text{because multiplication is associative in } \mathbb{R}.\end{aligned}$$

The R.H.S :

$$\begin{aligned}\alpha \cdot (\beta \cdot u) &= \alpha \cdot [\beta \cdot (a + bi)]\\ &= \alpha \cdot [\beta a + \beta b\ i] \qquad \text{by (2),}\\ &= \alpha\beta a + \alpha\beta b\ i \qquad \text{by (2).}\end{aligned}$$

Then L.H.S=R.H.S

(v) The multiplicative identity : we have to show that $1 \cdot u = u$ for all $u = a + bi \in \mathbb{C}$. (Note that, 1 represents scalar from the field $\mathbb{R}$ and NOT from the set $\mathbb{C}$).

$$\begin{aligned}1 \cdot u &= 1 \cdot (a + bi)\\ &= 1a + 1bi \qquad \text{by (2),}\\ &= a + bi\\ &= u\end{aligned}$$

We have proved that all axioms hold in $\mathbb{C}$. Hence, $(\mathbb{C}, +, \cdot)$ is a vector space over $\mathbb{R}$.

Example 1.3. Let $M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$ be the set of all two by two matrices with entries in $\mathbb{R}$. For $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in M_{2\times 2}$ and $\alpha \in \mathbb{R}$, addition and scalar multiplication of matrices defined by

$$A + B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{pmatrix} \tag{3}$$

$$\alpha \cdot A = \alpha \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix}. \tag{4}$$

Prove that $(M_{2\times 2}, +, \cdot)$ is a vector space over $\mathbb{R}$.

Solution: Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \in M_{2\times 2}$.

(1)

(i)

$$A + B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{pmatrix} \quad \text{by (3)}.$$

Since $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are real numbers, then $a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4 \in \mathbb{R}$. Hence, $A + B \in M_{2\times 2}(\mathbb{R})$.

(ii) We have to show that $A+B=B+A$ for all $A, B \in M_{2\times 2}$.

$$\begin{aligned}
A+B &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \\
&= \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix} \quad \text{by (3),} \\
&= \begin{pmatrix} b_1+a_1 & b_2+a_2 \\ b_3+a_3 & b_4+a_4 \end{pmatrix} \quad \text{because addition on } \mathbb{R} \text{ is commutative} \\
&= \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} + \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \quad \text{by (3),} \\
&= B+A
\end{aligned}$$

(iii) We have to show that $A+(B+C)=(A+B)+C$ for all $A, B, C \in M_{2\times 2}$.

The L.H.S:

$$\begin{aligned}
A+(B+C) &= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \left[\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}\right] \\
&= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1+c_1 & b_2+c_2 \\ b_3+c_3 & b_4+c_4 \end{pmatrix} \quad \text{by (3),} \\
&= \begin{pmatrix} a_1+(b_1+c_1) & a_2+(b_2+c_2) \\ a_3+(b_3+c_3) & a_4+(b_4+c_4) \end{pmatrix} \quad \text{by (3),} \\
&= \begin{pmatrix} (a_1+b_1)+c_1 & (a_2+b_2)+c_2 \\ (a_3+b_3)+c_3 & (a_4+b_4)+c_4 \end{pmatrix} \quad \text{because addition on } \mathbb{R} \text{ is associative.}
\end{aligned}$$

The R.H.S:

$$(A+B)+C = \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right] + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix} + \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \qquad \text{by (3),}$$

$$= \begin{pmatrix} (a_1+b_1)+c_1 & (a_2+b_2)+c_2 \\ (a_3+b_3)+c_3 & (a_4+b_4)+c_4 \end{pmatrix} \qquad \text{by (3).}$$

Then L.H.S= R.H.S

(iv) For all $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in M_{2\times 2}$, we have

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

Then the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the additive identity.

(v) For all $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in M_{2\times 2}$, we have $(-A) = \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix} \in M_{2\times 2}$, where

$$A+(-A) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} -a_1 & -a_2 \\ -a_3 & -a_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the matrix $(-A)$ is the additive inverse for the matrix A.

(2)

(i) We have to show that $\alpha \cdot A \in M_{2\times 2}(\mathbb{R})$ for all $A \in M_{2\times 2}$ and $\alpha \in \mathbb{R}$.

$\alpha \cdot A = \alpha \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix}$ by (4).

Since $\alpha, a_1, a_2, a_3, a_4$ are real numbers then $\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4 \in \mathbb{R}$.

Hence, $\alpha \cdot A \in M_{2\times 2}(\mathbb{R})$.

(ii) We have to show that $\alpha \cdot (A+B) = \alpha \cdot A + \alpha \cdot B$ for all $A, B \in M_{2\times 2}$ and $\alpha \in \mathbb{R}$.

The L.H.S:

$$\alpha \cdot (A+B) = \alpha \cdot \left[\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\right]$$

$$= \alpha \cdot \begin{pmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{pmatrix} \qquad \text{by (3),}$$

$$= \begin{pmatrix} \alpha(a_1+b_1) & \alpha(a_2+b_2) \\ \alpha(a_3+b_3) & \alpha(a_4+b_4) \end{pmatrix} \qquad \text{by (4),}$$

$$= \begin{pmatrix} \alpha a_1+\alpha b_1 & \alpha a_2+\alpha b_2 \\ \alpha a_3+\alpha b_3 & \alpha a_4+\alpha b_4 \end{pmatrix} \qquad \text{because multiplication distributes over addition in } \mathbb{R}.$$

The R.H.S:

$$\alpha \cdot A + \alpha \cdot B = \alpha \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \alpha \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix} + \begin{pmatrix} \alpha b_1 & \alpha b_2 \\ \alpha b_3 & \alpha b_4 \end{pmatrix} \qquad \text{by (4),}$$

$$= \begin{pmatrix} \alpha a_1+\alpha b_1 & \alpha a_2+\alpha b_2 \\ \alpha a_3+\alpha b_3 & \alpha a_4+\alpha b_4 \end{pmatrix} \qquad \text{by (3).}$$

Then L.H.S= R.H.S

(iii) We have to show that $(\alpha+\beta) \cdot A = \alpha \cdot A + \beta \cdot A$ for all $A \in M_{2\times 2}$ and $\alpha, \beta \in \mathbb{R}$.

The L.H.S:

$$(\alpha+\beta)\cdot A = (\alpha+\beta)\cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha+\beta)a_1 & (\alpha+\beta)a_2 \\ (\alpha+\beta)a_3 & (\alpha+\beta)a_4 \end{pmatrix} \qquad \text{by (4),}$$

$$= \begin{pmatrix} \alpha a_1+\beta a_1 & \alpha a_2+\beta a_2 \\ \alpha a_3+\beta a_3 & \alpha a_4+\beta a_4 \end{pmatrix} \qquad \text{because multiplication distributes over addition in } \mathbb{R}.$$

The R.H.S:

$$\alpha\cdot A+\beta\cdot A = \alpha\cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} + \beta\cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{pmatrix} + \begin{pmatrix} \beta a_1 & \beta a_2 \\ \beta a_3 & \beta a_4 \end{pmatrix} \qquad \text{by (4),}$$

$$= \begin{pmatrix} \alpha a_1+\beta a_1 & \alpha a_2+\beta a_2 \\ \alpha a_3+\beta a_3 & \alpha a_4+\beta a_4 \end{pmatrix} \qquad \text{by (3).}$$

Then L.H.S = R.H.S

(iv) We have to show that $(\alpha\beta)\cdot A = \alpha\cdot(\beta\cdot A)$ for all l $A \in M_{2\times 2}$ and $\alpha, \beta \in \mathbb{R}$.

The L.H.S:

$$(\alpha\beta)\cdot A = (\alpha\beta)\cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha\beta)a_1 & (\alpha\beta)a_2 \\ (\alpha\beta)a_3 & (\alpha\beta)a_4 \end{pmatrix} \qquad \text{by (4),}$$

$$= \begin{pmatrix} \alpha(\beta a_1) & \alpha(\beta a_2) \\ \alpha(\beta a_3) & \alpha(\beta a_4) \end{pmatrix} \qquad \text{because multiplication on } \mathbb{R} \text{ is associative.}$$

The R.H.S:

$$\alpha \cdot (\beta \cdot A) = \alpha \cdot \left[\beta \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\right]$$

$$= \alpha \cdot \begin{pmatrix} \beta a_1 & \beta a_2 \\ \beta a_3 & \beta a_4 \end{pmatrix} \qquad \text{by (4),}$$

$$= \begin{pmatrix} \alpha(\beta a_1) & \alpha(\beta a_2) \\ \alpha(\beta a_3) & \alpha(\beta a_4) \end{pmatrix} \qquad \text{by (4).}$$

Then L.H.S= R.H.S

(v) For all $A \in M_{2\times 2}$, we have $1 \in \mathbb{R}$ such that

$$1 \cdot A = 1 \cdot \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 1a_1 & 1a_2 \\ 1a_3 & 1a_4 \end{pmatrix} = A.$$

Then $1 \in \mathbb{R}$ is the multiplicative identity .

Example 1.4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$. For $x, y \in V$ and $\alpha \in \mathbb{R}$, we define addition and scalar multiplication as following

$$x \oplus y = xy,$$

$$\alpha \otimes x = x^{\alpha}.$$

Show that $(V, \oplus, \otimes)$ is a vector space over $\mathbb{R}$.

Example 1.5. Is the set $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b > 0 \right\}$ with the usual addition and scalar multiplication of matrices define a vector space over $\mathbb{R}$?

Solution: Let $\alpha = -2 \in \mathbb{R}$, then $\alpha \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2a \\ -2b \end{bmatrix} \notin V$.

Since $a, b > 0$ then $-2a, -2b < 0$.

Proposition 1.6. *Let V be a vector space over $\mathbb{K}$, then we have*

(1) The additive identity, $0 \in V$, is unique.

(2) The additive inverse, $(-u) \in V$, for $u \in V$ is unique.

(3) For all $u \in V$ we have $0 \cdot u = 0$.

(4) For all $u \in V$ we have $(-1) \cdot u = -u$.

(5) For all $u, v, w \in V$, if $u + v = u + w$ then $v = w$.

(6) For all $u, v \in V$, the equation $u + x = v$ has a unique solution $x = v - u \in V$.

(7) For all $u \in V$, we have $-(-u) = u$.

2 Subspace

In this section we suppose that $(V, +, \cdot)$ is a vector space over $\mathbb{K}$.

Definition 2.1. A non-empty subset U of V is called a subspace of V if $(U, +, \cdot)$ is a vector space over $\mathbb{K}$.

Proposition 2.2. *A non-empty subset U of a vector space V over $\mathbb{K}$ is a subspace of V if and only if the following conditions are satisfied:*

(1) $0 \in U$.

(2) For all $u, v \in U$, we have $u + v \in U$.

(3) For all $u \in U$ and $\alpha \in \mathbb{K}$, we have $\alpha \cdot u \in U$.

Remark 2.3. Every vector space V has two subspaces namely V and $\{0\}$, which are called trivial subspaces. Any other subspace of V is called a proper subspace of V.

Example 2.4. Show that which of these sets are subspace of $\mathbb{R}^3$

(1) $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$.

(2) $U = \{(x, y, 1) \mid x, y \in \mathbb{R}\}$.

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Proposition 2.5. *If W_1 and W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.*

Proof. We have to satisfy the three conditions in Proposition 2.2.

(1) Since W_1 and W_2 are subspaces of V, then $0 \in W_1$ and $0 \in W_2$.

Hence,

$$0 \in W_1 \cap W_2.$$

(2) Let $u, v \in W_1 \cap W_2$, then $u, v \in W_1$ and $u, v \in W_2$.

Since W_1 and W_2 are subspaces of V, then $u + v \in W_1$ and $u + v \in W_2$.

Hence,

$$u + v \in W_1 \cap W_2.$$

(3) Let $\alpha \in \mathbb{K}$ and $u \in W_1 \cap W_2$, then $u \in W_1$ and $u \in W_2$.

Since W_1 and W_2 are subspaces of V then $\alpha \cdot u \in W_1$ and $\alpha \cdot u \in W_2$.

Hence,

$$\alpha \cdot u \in W_1 \cap W_2.$$

□

Example 2.6. Show that if W_1 and W_2 are subspaces of a vector space V, then $W_1 \cup W_2$ is NOT a subspace of V.

To prove this, we have $W_1 = \{(a, 0) \mid a \in \mathbb{R}\}$ and $W_2 = \{(0, b) \mid b \in \mathbb{R}\}$ are both subspaces of $\mathbb{R}^2$. But $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$ because $(1, 0) \in W_1 \cup W_2$ and $(0, 1) \in W_1 \cup W_2$ while $(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2$.

Proposition 2.7. *Let $W_1, W_2, \cdots, W_n$ are subspaces of a vector space V over a field $\mathbb{K}$, then we have*

(1) $W_1 \cap W_2 \cap \cdots \cap W_n$ is a subspace of V.

(2) $W_1 + W_2 + \cdots + W_n = \{w_1 + w_2 + \cdots + w_n \mid w_i \in W_i, i = 1, 2, \cdots n\}$ is a subspace of V.

Proof. □

3 Linear Combinations and Span

Definition 3.1. Let $v_1, v_2, \cdots, v_n$ be vectors in a vector space V over $\mathbb{K}$. A linear combination of these vectors is any expression of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$.

Example 3.2. Consider the vector space $\mathbb{R}^2$. The vector $v = (-7, -13)$ is a linear combination of $v_1 = (-2, 1)$ and $v_2 = (1, 5)$, where

$$v = 2v_1 + (-3)v_2.$$

Example 3.3. Consider the vector space $\mathbb{R}^2$. The vector $v = (1, -3)$ is a linear combination of $v_1 = (0, 1)$, $v_2 = (2, -1)$, $v_3 = (1, -2)$ and $v_4 = (0, 3)$ where

$$v = (-2)v_1 + (0)v_2 + 1v_3 + (\frac{1}{3})v_4.$$

Sometimes we cannot write a vector v in a vector space V as a linear combination of $v_1, v_2, \cdots, v_n \in V$, as explained in this example.

Example 3.4. Let $v_1 = (2, 5, 3), v_2 = (1, 1, 1)$, and $v = (4, 2, 0)$. Because there exist no scalars $\alpha_1, \alpha_2 \in \mathbb{K}$ such that $v = \alpha_1 v_1 + \alpha_2 v_2$ then v is not a linear combination of v_1 and v_2.

Definition 3.5. Let V be a vector space over $\mathbb{K}$, and let $S = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. We say that S spans V, or S generates V, if every vector v in V can be written as a linear combination of vectors in S. That is, for all $v \in V$, we have

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$.

Example 3.6. Show that the set $S = \{(1,0),(0,1)\}$ spans the vector space $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$.

Solution: We have to show that for all $v = (a,b) \in \mathbb{R}^2$ there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $v = \alpha_1(1,0) + \alpha_2(0,1)$.

$$\begin{aligned}(a,b) &= \alpha_1(1,0) + \alpha_2(0,1)\\ &= (\alpha_1,0) + (0,\alpha_2)\\ &= (\alpha_1,\alpha_2)\end{aligned}$$

Then $\alpha_1 = a$ and $\alpha_2 = b$. So, any vector $v = (a,b) \in \mathbb{R}^2$ can be written in the form $(a,b) = a(1,0) + b(0,1)$. Thus S spans $\mathbb{R}^2$.

Example 3.7. Let $S = \left\{v_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, v_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$, and

$V = \left\{v = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \mid a,b,c,d \in \mathbb{R}\right\}$

(1) Does S spans V ?

(2) Define a vector space U such that S spans U.

(3) Find a set that spans V.

Solution: (1) If S spans V then for all $v \in V$, there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\begin{aligned}\begin{bmatrix}a & b\\ c & d\end{bmatrix} &= \alpha_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \alpha_2\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}\alpha_1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \alpha_2\end{bmatrix}\\ &= \begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}\end{aligned}$$

So, $\alpha_1 = a$ and $\alpha_2 = d$. But if b or c is non-zero then v cannot be written as a linear combination of the vectors in S. Hence, S not spans V.

(2) From (1), we can see that if $U = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ then S spans U.

(3) The set that spans V is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Example 3.8. Show that the set $S = \{(0,1,1),(1,0,1),(1,1,0)\}$ spans $\mathbb{R}^3$ and write the vector $(2,4,8)$ as a linear combination of vectors in S.

Solution:

A vector in $\mathbb{R}^3$ has the form $v = (x, y, z)$.

Hence we need to show that, for some scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, every such v can be written as

$$\begin{aligned}(x, y, z) &= \alpha_1(0,1,1) + \alpha_2(1,0,1) + \alpha_3(1,1,0) \\ &= (\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2)\end{aligned}$$

This give us system of equations

$$\begin{aligned}x &= \alpha_2 + \alpha_3 \\ y &= \alpha_1 + \alpha_3 \\ z &= \alpha_1 + \alpha_2\end{aligned}$$

This system of equations can be written in matrix form

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We can write it as $A\alpha = b$. Since $det(A) = 2$ then this system has a solution.

Now, to write $(2,4,8)$ as a linear combination of vectors in S, we find that

$$A^{-1} = \begin{bmatrix} -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{bmatrix}$$

Then

$$\alpha = A^{-1}b$$

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix}-0.5 & 0.5 & 0.5\\ 0.5 & -0.5 & 0.5\\ 0.5 & 0.5 & -0.5\end{bmatrix} \cdot \begin{bmatrix}2\\ 4\\ 8\end{bmatrix}$$

So, $\alpha_1 = 5, \alpha_2 = 3, \alpha_3 = -1$, and

$$(2, 4, 8) = 5(0, 1, 1) + 3(1, 0, 1) + (-1)(1, 1, 0).$$

4 Linear independence

Definition 4.1. Let V be a vector space over a field $\mathbb{K}$. A subset $\{v_1, v_2, \cdots, v_n\}$ in V is linearly dependent over $\mathbb{K}$ if there exists scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$, (not all zero), such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0.$$

Definition 4.2. Let V be a vector space over a field $\mathbb{K}$. A subset $\{v_1, v_2, \cdots, v_n\}$ in V is linearly independent over $\mathbb{K}$ if $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

Example 4.3. Show that the set $\{(1, 0, 1), (1, -1, 1), (2, -1, 2), (0, 0, 1)\}$ is linearly dependent over $\mathbb{R}$.

Solution: We have to show that there exists $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ not all zero such that

$$\alpha_1(1, 0, 1) + \alpha_2(1, -1, 1) + \alpha_3(2, -1, 2) + \alpha_4(0, 0, 1) = (0, 0, 0)$$

We have the following system of equations

$$\begin{aligned} \alpha_1 + \alpha_2 + 2\alpha_3 &= 0 \\ -\alpha_2 - \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 &= 0 \end{aligned}$$

Put the first equation in the last equation, we get $\alpha_4 = 0$.

From the second equation, we have $\alpha_2 = -\alpha_3$. Let $\alpha_2 = 1$ then $\alpha_3 = -1$ and $\alpha_1 = 1$. Hence, $(1, 0, 1) + (1, -1, 1) + (-1)(2, -1, 2) + (0)(0, 0, 1) = (0, 0, 0)$.

Example 4.4. Show that the set $\{(1, 0, 1), (0, 0, 1)\}$ is linearly independent over $\mathbb{R}$.

Solution:

$$\begin{aligned} \alpha_1(1, 0, 1) + \alpha_2(0, 0, 1) &= (0, 0, 0) \\ (\alpha_1, 0, \alpha_1) + (0, 0, \alpha_2) &= (0, 0, 0) \\ (\alpha_1, 0, \alpha_1 + \alpha_2) &= (0, 0, 0) \end{aligned}$$

So, $\alpha_1 = 0$, $\alpha_1 + \alpha_2 = 0$ then $\alpha_2 = 0$. Then it is linearly independent over $\mathbb{R}$.

Example 4.5. Show that the set $S = \{i, i + 1\}$ is linearly dependent over $\mathbb{C}$, but it is linearly independent over $\mathbb{R}$.

Solution: Since $(-1 + i)i + (1)(1 + i) = 0$, so, S is linearly dependent over $\mathbb{C}$.

Let $\alpha(i) + \beta(1 + i) = 0$, where $\alpha, \beta \in \mathbb{R}$

Then

$$\begin{aligned} \alpha i + \beta + \beta i &= 0 + 0i \\ \beta + (\alpha + \beta)i &= 0 + 0i \end{aligned}$$

So, $\beta = 0$, $\alpha + \beta = 0$ and then $\alpha = 0$. Hence, S is linearly independent over $\mathbb{R}$.

Theorem 4.6. *If $A = (a_{ij}) \in M_{n\times n}(\mathbb{K})$, and $C_j = \{a_{1j}, a_{2j}, \cdots, a_{nj}\}$, $j = 1, 2, \cdots, n$ are the n columns of A then $\{C_1, C_2, \cdots, C_n\}$ is linearly dependent over $\mathbb{K}$ if and only if $detA = 0$.*

Corollary 4.7. *The n rows of a matrix $A \in M_{n\times n}(\mathbb{K})$ are linearly dependent over $\mathbb{K}$ if and only if $detA = 0$.*

5 Basis and dimension

Definition 5.1. Let V be a vector space over $\mathbb{K}$. A subset $S = \{v_1, v_2, \cdots, v_n\}$ is called a basis for V if

(i) V is spanned by S, that is, for every $v \in V$ there exists scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{K}$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$.

(ii) The set S is linearly independent over $\mathbb{K}$.

Example 5.2. Show that the set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for the vector space $\mathbb{R}^3$.

Solution: (i) we have to show that S spans $\mathbb{R}^3$. That is, f or all $v = (x, y, z) \in \mathbb{R}^3$, we have to find scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^3$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$$(x, y, z) = \alpha_1(1, 0, 0) + \alpha_2(0, 1, 0) + \alpha_3(0, 0, 1)$$
$$(x, y, z) = (\alpha_1, \alpha_2, \alpha_3)$$

So, $(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$ and, hence, $\mathbb{R}^3$ is generated by S.

(ii) To show that S is linearly independent, Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Since $det(A) \neq 0$ then S is linearly independent.

Finally, we get S is a basis for $\mathbb{R}^3$.

Example 5.3. Let $e_1 = (1, 0, 0, \cdots, 0), e_2 = (0, 1, 0, 0, \cdots, 0), \cdots, e_n = (0, 0, \cdots, 1)$. Then $B = \{e_1, e_2, \cdots, e_n\}$ is a basis for $\mathbb{R}^n$. This basis called the standard basis for $\mathbb{R}^n$.

Theorem 5.4. *Let V be a vector space over a field $\mathbb{K}$, and $S = \{v_1, v_2, \cdots, v_n\}$ be a basis of V containing n vectors.Then any subset containing more than n vectors in V is linearly dependent.*

Definition 5.5. Let V be a vector space with a basis $S = \{v_1, v_2, \cdots, v_n\}$ has n vectors. Then, we say n is the dimension of V and we write $\dim(V) = n$.

Theorem 5.6. *Any vector space V has a basis. All bases for V are of the same dimension.*

Example 5.7. The following vector spaces over $\mathbb{R}$ have dimensions :

(1) $\dim(\mathbb{R}^n) = n$.

(2) $\dim \mathbb{R} = 1$.

(3) $\dim \mathbb{C} = 2$.

(4) $\dim M_{n,n}(\mathbb{R}) = n^2$.

Theorem 5.8. *Let V be a vector space such that $dim(V) = n$. Let $S = \{v_1, v_2, \cdots, v_n\}$ be a subset of V. Then we have*

(1) If S spans V, then S is also linearly independent hence a basis for V.

(2) If S is linearly independent, then S also spans V hence is a basis for V.

Example 5.9. Show that S is not a basis for $\mathbb{R}^3$ where $S = \{(6,4,1),(3,-5,1),(8,13,6),(0,6,9)\}$.

Solution: Since $\dim(\mathbb{R}^3) = 3$, then any basis for $\mathbb{R}^3$ must have 3 vectors, while here S has four.

Example 5.10. Show that $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for $M_{2,2}(\mathbb{R})$.

Solution: Since S has four vectors and $\dim(M_{2,2}(\mathbb{R}) = 4$ then, by Theorem 5.8, we have to show that either S spans V or S is linearly independent.

6 Dot and cross products

Definition 6.1. Let $v = (a_1, a_2, \cdots, a_n)$ be a vector in a vector space V. The length (or norm or magnitude) of v is

$$\|v\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

Example 6.2. Suppose that the vector $v = (2,-1,4,1)$, then the length of v is

$$\|v\| = \sqrt{2^2 + (-1)^2 + 4^2 + 1^2} = \sqrt{22}.$$

Definition 6.3. Let $u = (a_1, a_2, \cdots, a_n)$ and $v = (b_1, b_2, \cdots, b_n)$ are vectors in a vector space V. The dot product of u and v is defined by

$$u \cdot v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

.

Definition 6.4. The angle θ between two vectors u and v is determined by the formula

$$u \cdot v = \|u\| \|v\| \cos\theta$$

Example 6.5. Let $u = (1, 3, 0)$ and $v = (-2, 1, 5)$. The dot product of u and v is

$$u \cdot v = 1(-2) + 3(1) + 0(5) = 1,$$

and the angle between them is

$$\cos\theta = \frac{u \cdot v}{\|u\| \|v\|} = \frac{1}{\sqrt{10}\,\sqrt{30}}$$

So,

$$\theta = \cos^{-1}\big(\frac{1}{\sqrt{10}\,\sqrt{30}}\big).$$

Some properties of the dot product : Let u, v and w are vectors in a vector space V over $\mathbb{K}$. The dot product has the following properties:

(1) $v \cdot v = \|v\|^2$

(2) $u \cdot v = v \cdot u$

(3) $u \cdot (v + w) = u \cdot v + u \cdot w$

(4) $(\alpha u) \cdot v = u \cdot (\alpha v) = \alpha(u \cdot v)$, where $\alpha \in \mathbb{K}$.

(5) If $u \cdot v > 0$ then the angle formed by the vectors $(0 < \theta < 90)$.

(6) If $u \cdot v < 0$ then the angle formed by the vectors, $(90 < \theta \leq 180)$.

(7) If $u \cdot v = 0$ then the angle formed by the vectors is 90 degrees.

Definition 6.6. Let u and v are vectors in a vector space V. If

$$u \cdot v = 0$$

then we say that u and v are **orthogonal**.

Definition 6.7. A subset $S = \{v_1, v_2, \cdots, v_n\}$ of a vector space V form an orthogonal set if all vectors in S are orthogonal to each other, $v_i \cdot v_j = 0$ for $i \neq j$. In addition, if all vectors in an orthogonal set S has length one, $\|v_i\| = 1$, then S is called an orthonormal set.

Theorem 6.8. *Any orthogonal set is linearly independent .*

Gram-Schmidt process : If $B = \{v_1, v_2, \cdots, v_n\}$ is a basis for a vector space V. Then we can define an orthogonal basis $W = \{w_1, w_2, \cdots, w_n\}$ for V by using the following steps:

$$
\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 \\
w_3 &= v_3 - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 \\
&\vdots \\
w_n &= v_n - \frac{w_1 \cdot v_n}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_n}{w_2 \cdot w_2} w_2 - \cdots - \frac{w_{n-1} \cdot v_n}{w_{n-1} \cdot w_{n-1}} w_{n-1}
\end{aligned}
$$

In addition, the set

$$\left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \cdots, \frac{w_n}{\|w_n\|} \right\}$$

is an orthonormal basis for V.

Example 6.9. Let $S = \{v_1 = (1, 1, 0), v_2 = (1, 1, 1), v_3 = (3, 1, 1)\}$ be a basis for $\mathbb{R}^3$. We will use Gram-Schmidt process to find orthogonal and orthonormal bases for $\mathbb{R}^3$.

$$
\begin{aligned}
w_1 &= v_1 = (1,1,0) \\
w_2 &= v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 \\
&= (1,1,1) - \frac{(1,1,0)\cdot(1,1,1)}{(1,1,0)\cdot(1,1,0)}\,(1,1,0) \\
&= (1,1,1) - \frac{1+1+0}{1+1+0}\,(1,1,0) \\
&= (0,0,1) \\
w_3 &= v_3 - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 \\
&= (3,1,1) - \frac{(1,1,0)\cdot(3,1,1)}{(1,1,0)\cdot(1,1,0)}\,(1,1,0) - \frac{(0,0,1)\cdot(3,1,1)}{(0,0,1)\cdot(0,0,1)}\,(0,0,1) \\
&= (3,1,1) - \frac{4}{2}(1,1,0) - \frac{1}{1}(0,0,1) \\
&= (3,1,1) - (2,2,0) - (0,0,1) \\
&= (1,-1,0)
\end{aligned}
$$

Then $W = \{w_1, w_2, w_3\} = \{(1,1,0), (0,0,1), (1,-1,0)\}$ is an orthogonal basis for $\mathbb{R}^3$.

Since $\|w_1\| = \sqrt{2}, \|w_2\| = 1, \|w_3\| = \sqrt{2}$ then the set

$$
U = \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|} \right\} = \left\{ \frac{1}{\sqrt{2}}(1,1,0), (0,0,1), \frac{1}{\sqrt{2}}(1,-1,0) \right\}
$$

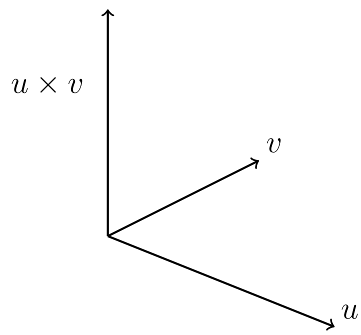
is an orthonormal basis for $\mathbb{R}^3$.

Definition 6.10. Let $u = (a_1, a_2, a_3), v = (b_1, b_2, b_3) \in \mathbb{R}^3$ then we define the **cross product** of u and v as following

$$u \times v = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(a_2b_3 - b_2a_3) - j(a_1b_3 - b_1a_3) + k(a_1b_2 - b_1a_2).$$

That is, $\quad u \times v = (a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - b_1a_2).$

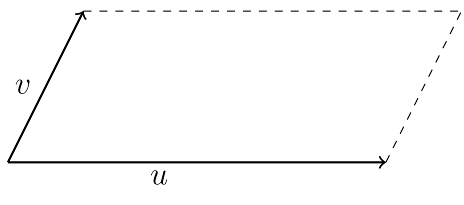
Geometrically, the cross product of vectors u and v represents a vector that is orthogonal to both of u and v.



Definition 6.11. The angle θ between two vectors u and v is determined by the formula

$$\|u \times v\| = \|u\| \; \|v\| \; \sin\theta.$$

Note that, the length of $u \times v$ represents the area of the parallelogram that spanned by u and v.



Example 6.12. Find the area of the parallelogram that spanned by the vectors $u = (1, 3, 2)$ and $v = (-2, 1, 0)$.

Solution :

$$u \times v = (-2, -4, 7)$$

$$\|u \times v\| = \sqrt{4 + 16 + 49} = \sqrt{69}$$

7 Eigenvalues and eigenvectors

Definition 7.1. Let A be an $n \times n$ matrix. If there is a number $\lambda \in \mathbb{C}$ and a vector $x \neq 0$ such that $Ax = \lambda x$, then we say that λ is an eigenvalue for A, and x is called an eigenvector for A with eigenvalue λ.

Example 7.2. If

$$A = \begin{pmatrix} 1 & 3 \\ 6 & -2 \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

then

$$Ax = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4x \ .$$

So, $\lambda = 4$ is an eigenvalue of A, and x is an eigenvector for A with this eigenvalue.

We can write the equation $Ax = \lambda x$ as a linear system. Since $\lambda x = \lambda I x$, (where $I = I_n$ is the identity matrix), we have that

$$Ax = \lambda x \Longleftrightarrow Ax - \lambda x = 0 \Longleftrightarrow (A - \lambda I)x = 0$$

This linear system has a non-trivial solution $x \neq 0$ if and only if

$$det(A - \lambda I) = 0, \quad (why?).$$

Definition 7.3. The characteristic equation of a square matrix A is the equation

$$det(A - \lambda I) = 0.$$

Theorem 7.4. *The eigenvalues of a square matrix A are the solutions of the characteristic equation*

$$det(A - \lambda I) = 0.$$

How to find the eigenvalues and the eigenvectors:

To find the eigenvalues of a matrix A, we have to find the solution of the characteristic equation $det(A - \lambda I) = 0$, then to find the eigenvectors for A with eigen value λ we have to solve the linear system $(A - \lambda I)x = 0$, as explained in this example.

Example 7.5. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}.$$

Solution: We have to find $A - \lambda I$.

$$A - \lambda I = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 - \lambda & 3 \\ 3 & -6 - \lambda \end{pmatrix}$$

Now, we have to find the solution to the characteristic equation $det(A - \lambda I) = 0$.

$$\begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) - 3.3 = \lambda^2 + 4\lambda - 21 = 0$$

Then

$$\lambda^2 + 4\lambda - 21 = (\lambda + 7)(\lambda - 3) = 0$$

So, the eigenvalues of A are

$$\lambda_1 = -7 \quad \text{and} \quad \lambda_2 = 3$$

To find the eigenvector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for $\lambda_1 = -7$, we have to solve the following system

$$(A - \lambda_1 I)x = 0$$

$$\begin{pmatrix} 2-(-7) & 3 \\ 3 & -6-(-7) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using Gauss elimination, $(R_1 \to \frac{1}{9}R_1,\ R_2 \to -3R_1 + R_2)$, we get

$$\begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We have only one equation with two variables $x_1 + \frac{1}{3}x_2 = 0$, then $x_1 = \frac{-1}{3}x_2$.

Assume $x_2 = c_1$, gives us $x = \begin{pmatrix} \frac{-1}{3}c_1 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} \frac{-1}{3} \\ 1 \end{pmatrix}$, where $c_1 \in \mathbb{R}$.

Similarly, we can show that the eigenvector for $\lambda_2 = 3$ is $x = c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where $c_2 \in \mathbb{R}$.

8 Linear transformation on vector spaces

Definition 8.1. Let V and W are vector spaces over a field $\mathbb{K}$. A linear transformation T from V into W is a mapping $T : V \to W$ such that

(i) $T(u + v) = T(u) + T(v)$

(ii) $T(\alpha u) = \alpha T(u)$

for all $u, v \in V$ and $\alpha \in \mathbb{K}$. If $T : V \to V$ then we say that T is a linear transformation on V.

Example 8.2. Show that $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 + a_2, a_2 - a_3)$ is a linear transformation.

Solution:

(i) Let $u = (a_1, a_2, a_3), v = (b_1, b_2, b_3) \in \mathbb{R}^3$. Then

$u + v = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$, and

$$\begin{aligned} T(u+v) &= T(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= (a_1 + b_1 + a_2 + b_2, a_2 + b_2 - a_3 - b_3) \\ &= (a_1 + a_2 + b_1 + b_2, a_2 - a_3 + b_2 - b_3) \\ &= (a_1 + a_2, a_2 - a_3) + (b_1 + b_2, b_2 - b_3) \\ &= T(u) + T(v) \end{aligned}$$

(ii) Let $\alpha \in \mathbb{K}$, then $\alpha u = (\alpha a_1, \alpha a_2, \alpha a_3)$.

$$\begin{aligned} T(\alpha u) &= T(\alpha a_1, \alpha a_2, \alpha a_3) \\ &= (\alpha a_1 + \alpha a_2, \alpha a_2 - \alpha a_3) \\ &= \alpha(a_1 + a_2, a_2 - a_3) \\ &= \alpha T(u) \end{aligned}$$

Then T is a linear transformation.

Example 8.3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(a_1, a_2, a_3) = (a_1 - 1, a_2)$. Is T a linear transformation?

Solution: Let $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3) \in \mathbb{R}^3$. Then

$$u + v = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

$$\begin{aligned} T(u + v) &= T(a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= (a_1 + b_1 - 1, a_2 + b_2) \end{aligned}$$

On the other hand,

$$\begin{aligned} T(u) + T(v) &= T(a_1, a_2, a_3) + T(b_1, b_2, b_3) \\ &= (a_1 - 1, a_2) + (b_1 - 1, b_2) \\ &= (a_1 + b_1 - 2, a_2 + b_2) \end{aligned}$$

So, $T(u + v) \neq T(u) + T(v)$, and hence, T is NOT a linear transformation.

Example 8.4. Let $M \in M_{m,m}(\ \mathbb{K}\ )$ and $N \in M_{n,n}(\ \mathbb{K}\ )$. Define

$T : M_{m,n}(\ \mathbb{K}\ ) \to M_{m,n}(\ \mathbb{K}\ )$ by $T(A) = MAN$ for all $A \in M_{m,n}(\ \mathbb{K}\ )$.

Show that T is a linear transformation.

Solution: Let $A, B \in M_{m,n}(\ \mathbb{K}\ )$ and $\alpha \in \mathbb{K}$.

(i)

$$\begin{aligned} T(A+B) &= M(A+B)N \\ &= MAN + MBN \\ &= T(A) + T(B) \end{aligned}$$

(ii) $T(\alpha A) = M(\alpha A)N = \alpha(MAN) = \alpha T(A)$

Then T is a linear transformation.