Berge cycles in non-uniform hypergraphs

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Submitted: Feb 4, 2020; Accepted: Jul 3, 2020; Published: Jul 24, 2020

Abstract

We consider two extremal problems for set systems without long Berge cycles. First we give Dirac-type minimum degree conditions that force long Berge cycles. Next we give an upper bound for the number of hyperedges in a hypergraph with bounded circumference. Both results are best possible in infinitely many cases.

Mathematics Subject Classifications: 05C65, 05C35, 05C38

1 Introduction

1.1 Classical results on longest cycles in graphs

The *circumference* $c(G)$ of a graph G is the length of its longest cycle. In particular, if a graph has a cycle C which covers all of its vertices, $V(C) = V(G)$, we say it is *hamiltonian*. A classical result of Dirac states that high minimum degree in a graph forces hamiltonicity.

*Research supported in part by the Hungarian National Research, Development and Innovation Office NKFIH grant KH-130371.

†Research is supported in part by NSF grant DMS-1600592 and grants 18-01-00353A and 19-01-00682 of the Russian Foundation for Basic Research.

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Theorem 1 (Dirac [4])**.** *Let $n \geqslant 3$, and let G be an n-vertex graph with minimum degree $\delta(G)$. If $\delta(G) \geqslant n/2$, then G contains a hamiltonian cycle. If G is 2-connected, then $c(G) \geqslant \min\{n, 2\delta(G)\}$.*

Inspired by this theorem, it is common in extremal combinatorics to refer to results in which a minimum degree condition forces some structure as a *Dirac-type condition.* The second part of Theorem 1 cannot be extended to non 2-connected graphs: let $\mathcal{F}_{n,k}$ be the family of graphs in which each block (inclusion maximal 2-connected subgraph) of the graph is a copy of K_{k-1}. Every $F \in \mathcal{F}_{n,k}$ has minimum degree $k-2$, but its longest cycle has length $k-1$.

Theorem 2 (Erdős, Gallai [6])**.** *Let G be an n-vertex graph with no cycle of length k or longer. Then $e(G) \leqslant \frac{n-1}{k-2}\binom{k-1}{2}$.*

So the graphs in $\mathcal{F}_{n,k}$ have the maximum number of edges among the n-vertex graphs with circumference $k-1$. They also maximize the number of cliques of any size:

Theorem 3 (Luo [12])**.** *Let G be an n-vertex graph with no cycle of length k or longer. Then the number of copies of K_r in G is at most $\frac{n-1}{k-2}\binom{k-1}{r}$.*

1.2 Known results on cycles in hypergraphs

A hypergraph $\mathcal{H}$ is a set system. We often refer to the ground set as the set of vertices $V(\mathcal{H})$ of $\mathcal{H}$ and to the sets as the hyperedges $E(\mathcal{H})$ of $\mathcal{H}$. When there is no ambiguity, we may also refer to the hyperedges as edges. In this paper, we prove versions of Theorems 1 and 2 for hypergraphs with no restriction on edge sizes. Namely, we seek long *Berge cycles.*

A **Berge cycle** of length ℓ in a hypergraph is a set of ℓ distinct vertices $\{v_1, \dots, v_\ell\}$ and ℓ distinct edges $\{e_1, \dots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo ℓ. The vertices $\{v_1, \dots, v_\ell\}$ are called **representative vertices** of the Berge cycle.

A **Berge path** of length ℓ in a hypergraph is a set of $\ell+1$ distinct vertices $\{v_1, \dots, v_{\ell+1}\}$ and ℓ distinct edges $\{e_1, \dots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leqslant i \leqslant \ell$. The vertices $\{v_1, \dots, v_{\ell+1}\}$ are called **representative vertices** of the Berge path.

For a hypergraph $\mathcal{H}$, the **2-shadow** of $\mathcal{H}$, denoted $\partial_2\mathcal{H}$, is the graph on the same vertex set such that $xy \in E(\partial_2\mathcal{H})$ if and only if $\{x, y\}$ is contained in an edge of $\mathcal{H}$.

Note that if we require no conditions on multiplicities of edges, then we can arbitrarily add edges of size 1 without creating new Berge cycles or Berge paths. From now on, we only consider *simple* hypergraphs, i.e., those without multiple edges (except if it is stated otherwise).

Bermond, Germa, Heydemann, and Sotteau [1] were among the first to prove Dirac-type results for uniform hypergraphs without long Berge cycles: Let $k > r$ and $\mathcal{H}$ be an r-uniform hypergraph with minimum degree $\delta(\mathcal{H}) \geqslant \binom{k-2}{r-1} + (r-1)$, then $\mathcal{H}$ contains a Berge cycle of length at least k. For large n, generalizations and results for linear hypergraphs are proved by Jiang and Ma [9]. Coulson and Perarnau [3] proved that if $\mathcal{H}$ is an r-uniform hypergraph on n vertices, $r = o(\sqrt{n})$, and $\mathcal{H}$ has minimum degree

$\delta(\mathcal{H}) > \binom{\lfloor (n-1)/2 \rfloor}{r-1}$, then $\mathcal{H}$ contains a Berge hamiltonian cycle. Ma, Hou, and Gao [13, 14] studied r-uniform hypergraphs and improved the result of Bermond, et al. for hamiltonian Berge cycles: if $n \geqslant 2r+4$ and $\delta(\mathcal{H}) > \binom{\lfloor (n-1)/2 \rfloor}{r-1} + \lceil (n-1)/2 \rceil$, then $\mathcal{H}$ has a Hamiltonian Berge cycle. Note that this also covers some small cases of n left open by Coulson and Perarnau.

Our new results differ from these in several aspects. We consider non-uniform hypergraphs, prove exact formulas, prove results for every n (or every $n > 14$), and use only classical tools mentioned above and in Section 3.1.

2 New results

Our first result is a Dirac-type condition that forces hamiltonian Berge cycles.

Theorem 4. *Let $n \geqslant 15$ and let $\mathcal{H}$ be an n-vertex hypergraph such that $\delta(\mathcal{H}) \geqslant 2^{(n-1)/2}+1$ if n is odd, or $\delta(\mathcal{H}) \geqslant 2^{n/2-1}+2$ if n is even. Then $\mathcal{H}$ contains a Berge hamiltonian cycle.*

The following four constructions show that the bounds in Theorem 4 cannot be decreased for any n.

— Let n be odd. Let $\mathcal{H}$ be the n-vertex hypergraph on the ground set $[n]$ with edges $\{A : A \subseteq [(n+1)/2]\} \cup \{B : B \subseteq \{(n+1)/2, \dots n\}\}$. Then $\delta(\mathcal{H}) = 2^{(n-1)/2}$ and $\mathcal{H}$ has no hamiltonian Berge cycle (because it has a cut vertex).

— Let n be even. Let $\mathcal{H}$ be the n-vertex hypergraph on the ground set $[n]$ with edges $\{A : A \subseteq [n/2]\} \cup \{B : B \subseteq \{(n/2+1, \dots n\}\}$ and the set $[n]$. Then $\delta(\mathcal{H}) = 2^{n/2-1}+1$ and $\mathcal{H}$ has no hamiltonian Berge cycle (because it has a cut edge, $[n]$).

— Let n be odd. Let $\mathcal{H}$ be the n-vertex hypergraph on the ground set $[n]$ obtained by taking all edges with at most one vertex in $[(n+1)/2]$. Then $\delta(\mathcal{H}) = 2^{(n-1)/2}$, and $\mathcal{H}$ cannot contain a Berge cycle with two consecutive representative vertices in $[(n+1)/2]$.

— Let n be even. Let $\mathcal{H}$ be the n-vertex hypergraph on the ground set $[n]$ obtained by taking all edges with at most one vertex in $[n/2+1]$ and the edge $[n]$. Then $\delta(\mathcal{H}) = 2^{n/2-1}+1$, and $\mathcal{H}$ cannot contain a Berge cycle with two instances of two consecutive representative vertices in $[n/2+1]$ (because only one edge of $\mathcal{H}$ contains multiple vertices in $[n/2+1]$).

Next, we consider hypergraphs without long Berge paths or cycles.

Theorem 5. *Let $k \geqslant 2$ and let $\mathcal{H}$ be a hypergraph such that $\delta(\mathcal{H}) \geqslant 2^{k-2}+1$. Then $\mathcal{H}$ contains a Berge path with k base vertices.*

A vertex disjoint union of complete hypergraphs of $k-1$ vertices shows that this bound is best possible for $n := |V(\mathcal{H})|$ divisible by $(k-1)$.

We note that Ma, et al. [14] also proved Dirac-type bounds for the existence of long Berge paths in r-uniform hypergraphs, but their results do not imply Theorem 5.

Theorem 6. *Let $k \geqslant 3$ and let $\mathcal{H}$ be a hypergraph such that $\delta(\mathcal{H}) \geqslant 2^{k-2}+2$. Then $\mathcal{H}$ contains a Berge cycle of length at least k.*

The following constructions show that the bound in Theorem 6 is best possible when n is divisible by $(k-1)$ and also when $n \equiv 1 \mod (k-1)$ for $n > (k-1)(2^{k-2}+1)$. In the first case, take a vertex disjoint union of complete hypergraphs with $k-1$ vertices and add one more set, namely $[n]$. In the other case, take $m := (n-1)/(k-1) \geqslant 2^{k-2}+1$ disjoint $(k-1)$-sets $A_1, \dots, A_m$ and an element x such that $[n] = (\cup_{1 \leqslant i \leqslant m} A_i) \cup \{x\}$. Then define $\mathcal{H}$ as the union of complete hypergraphs on the sets A_i's together with the edges of the form $A_i \cup \{x\}$. If we do not insist on connectedness, then $(2^{k-2}+1)$-regular examples can be constructed for *all* $n \geqslant k^2 2^{k-2}$.

Finally, we prove a hypergraph version of Theorem 2.

Theorem 7. *Let $n \geqslant k \geqslant 3$ and let $\mathcal{H}$ be an n-vertex hypergraph with no Berge cycle of length k or longer. Then*

$$e(\mathcal{H}) \leqslant 2 + \frac{n-1}{k-2}\left(2^{k-1}-2\right).$$

The bound in Theorem 7 is best possible when $n \equiv 1 \mod (k-2)$. Take $m := (n-1)/(k-2)$ and disjoint sets $A_1, \dots, A_m$ of size $k-2$. Let x be a new element, and set $[n] = (\cup_{1 \leqslant i \leqslant m} A_i) \cup \{x\}$. Define $\mathcal{H}$ to be the union of all sets A such that there exists an i with $A \setminus \{x\} \subseteq A_i$. Note that the 2-shadow $\partial_2(\mathcal{H})$ is in the family $\mathcal{F}_{n,k}$ defined before Theorem 2.

It would be interesting to find $\max \delta(\mathcal{H})$ for Theorems 5 and 6 for other values of n, and also for the cases when $\mathcal{H}$ is connected or 2-connected respectively. Moreover we also ask to improve the bound for Theorem 7 in the case where $\mathcal{H}$ is 2-connected.

There are many exact results concerning the maximum size of uniform hypergraphs avoiding Berge paths and cycles, see the recent results of Ergemlidze et al. [7] or one by the present authors [8].

3 Dirac type conditions for hamiltonian hypergraphs

In this section, we present a proof for Theorem 4. The proof method relies on reducing the hypergraph to a dense nonhamiltonian graph. In the next three subsections we collect some results about such graphs. Subsections 3.4 and 3.5 contain the proof for hypergraphs.

3.1 Classical tools

Let G be an n-vertex graph. The *hamilton-closure* of G is the unique graph $C(G)$ of order n that can be obtained from G by recursively joining nonadjacent vertices with degree-sum at least n.

Theorem 8 (Bondy, Chvátal [2])**.** *If $C(G)$ is hamiltonian, then so is G.*

A graph G is called *hamiltonian-connected* if for any pair of vertices $x, y \in V(G)$ there is a hamiltonian (x,y)-path. The following corollary can be obtained from Theorem 8 or from the classical result of Pósa [15]: If for every pair of nonadjacent vertices $x, y \in V(G)$ we have $d(x) + d(y) \geqslant |V(G)| + 1$, then G is hamiltonian-connected.

Corollary 9. *If $e(G) \geqslant \binom{n}{2} - 2$ and $n \geqslant 5$ then G is hamiltonian-connected.* $\square$

We will need the following result about the structure of matchings in bipartite graphs. It is a well known fact in the theory of transversal matroids (but one can also give a short, direct proof finding an $M_3 \subseteq M_1 \cup M_2$).

Theorem 10. *Let $G[X, Y]$ be a bipartite graph. Suppose that there is a matching M_1 in G joining the vertices of $X_1 \subseteq X$ and $Y_1 \subseteq Y$. Suppose also that we have another matching M_2 with end vertices $X_2 \subseteq X$ and $Y_2 \subseteq Y$ such that $Y_2 \subseteq Y_1$. Then there exists a third matching M_3 from $X_3 \subseteq X$ to $Y_3 \subseteq Y$ such that*

$$Y_3 = Y_1 \quad \textit{and} \quad X_3 \supseteq X_2.$$

Theorem 11 (Erdős [5])**.** *Let n, d be integers with $1 \leqslant d \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$, and set $h(n, d) := \binom{n-d}{2} + d^2$. If G is a nonhamiltonian graph on n vertices with minimum degree $\delta(G) \geqslant d$, then*

$$e(G) \leqslant \max\left\{ h(n, d), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor) \right\} =: e(n, d).$$

3.2 A lemma for nonhamiltonian graphs

The lemma below follows from a result of Voss [16] (and from the even more detailed descriptions by Jung [10] and Jung, Nara [11]). We only state and use a weaker version and for completeness include a short proof. Define five classes of nonhamiltonian graphs.

— Let $n = 2k + 2$, $V = V_1 \cup V_2$, $|V_1| = |V_2| = k + 1$, $(V_1 \cap V_2 = \varnothing)$. We say that $G \in \mathcal{G}_1$ if its edge set is the union of two complete graphs with vertex sets V_1 and V_2 and it contains at most one further edge e_0 (joining V_1 and V_2);

— Let $n = 2k+1$, $V = V_1 \cup V_2$, $|V_1| = |V_2| = k+1$, $V_1 \cap V_2 = \{x_0\}$. We say that $G \in \mathcal{G}_2$ if its edge set is the union of two complete graphs with vertex sets V_1 and V_2;

— Let $n = 2k + 2$, $V = V_1 \cup V_2$, $|V_1| = k + 1$, $|V_2| = k + 2$, $V_1 \cap V_2 = \{x_0\}$. We say that $G \in \mathcal{G}_3$ if its edge set is the union of a complete graph with vertex set V_1 and a 2-connected graph G_2 with vertex set V_2 such that $\deg_G(v) \geqslant k$ for every vertex $v \in V$;

— Let $n = 2k + 1$, $V = V_1 \cup V_2$, $|V_1| = k$, $|V_2| = k + 1$, $(V_1 \cap V_2 = \varnothing)$. We say that $G \in \mathcal{G}_4$ if V_2 is an independent set, and its edge set contains all edges joining V_1 and V_2;

— Let $n = 2k + 2$, $V = V_1 \cup V_2$, $|V_1| = k$, $|V_2| = k + 2$, $(V_1 \cap V_2 = \varnothing)$. We say that $G \in \mathcal{G}_5$ if V_2 contains at most one edge e_0 and $\deg_G(v) \geqslant k$ for every vertex $v \in V$ (so its edge set contains all but at most two edges joining V_1 and V_2).

Lemma 12. *Let $k \geqslant 3$ be an integer, $n \in \{2k+1, 2k+2\}$. Suppose that G is an n-vertex nonhamiltonian graph with $\delta(G) \geqslant k = \lfloor (n-1)/2 \rfloor$, $V := V(G)$. Then $G \in \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_5$.*

Proof. Suppose first that G is not 2-connected. Then there exist two blocks B_1, B_2 of G (i.e., B_i is a maximal 2-connected subgraph or a K_2) which are *endblocks*, i.e., for $i = 1, 2$ there is a vertex $v_i \in B_i$ such that $V(B_i) \setminus \{v_i\}$ does not meet any other block. Then $\{v\} \cup N(v) \subset V(B_i)$ for all $v \in V(B_i) \setminus \{v_i\}$, so an endblock has at least $k+1$ vertices and if $|V(B_i)| = k+1$ then it is a clique. If B_1 and B_2 are disjoint then we get $n = 2k+2$, and $G \in \mathcal{G}_1$. If B_1 and B_2 meet, then G has no other blocks, and $G \in \mathcal{G}_2 \cup \mathcal{G}_3$.

Suppose now that G is 2-connected. By the second part of Dirac's theorem (Theorem 1), the length of a longest cycle C of G is at least $2k$. If $|V(C)| = n-1$, assume $C = v_1 \dots v_{n-1} v_1$ and $v_n \notin V(C)$. Then v_n has at least k neighbors in C, with no two of them appearing consecutively (otherwise we could extend C to a hamiltonian cycle). Without loss of generality, let $N(v_n) = \{v_1, v_3, \dots, v_{2k-1}\}$. If for some $i < j$ such that $v_i, v_j \in N(v_n)$, $v_{i+1}v_{j+1} \in E(G)$, then we obtain the hamiltonian cycle $v_1 v_2 \dots v_i v_n v_j v_{j-1} \dots v_{i+1} v_{j+1} v_{j+2} \dots v_{n-1} v_1$. Therefore the vertices in C of even parity, together with v_n, form an independent set. In case of $n = 2k+1$ we get $G \in \mathcal{G}_4$. If $n = 2k+2$ then in the same way we get that $\{v_{2k+1}\} \cup \{v_2, v_4, \dots, v_{2k-2}\}$ together with v_n is also independent, so the set $\{v_2, \dots, v_{2k-2}\} \cup \{v_{2k}, v_{2k+1}, v_n\}$ contains only the edge $v_{2k}v_{2k+1}$, $G \in \mathcal{G}_5$.

Finally, consider the case that $|V(C)| = n-2$, (i.e., $n = 2k+2$) and let $x, y \notin V(C)$. We claim that $xy \notin E(G)$. Indeed, suppose to the contrary, that $xy \in E(G)$. Without loss of generality, $A := \{v_1, v_3, \dots, v_{2k-3}\} \subseteq N(x)$ or $(A \setminus \{v_{2k-3}\}) \cup \{v_{2k-2}\}) \subseteq N(x)$. Note that for any $v_i \in N(x)$, $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\} \cap N(y) = \varnothing$ (indices are taken modulo $2k$), because we can remove a segment of C with at most 3 vertices and replace it with a segment with at least 4 vertices containing the edge xy. This leads to a contradiction because there is not enough room on the $2k$-cycle C to distribute the at least $k-1$ vertices of $N(y) - \{x\}$.

If $xy \notin E(G)$ then without loss of generality let $N(x) = \{v_1, v_3, \dots v_{2k-1}\}$. Then the set $\{x\} \cup \{v_2, \dots, v_{2k}\}$ is an independent set. If $yv_i \in E(G)$ for some $i \in \{2, 4, \dots, 2k\}$, then because y has k neighbors in C and no two of them appear consecutively, $N(y) = \{v_2, v_4, \dots, v_{2k}\}$, and we obtain a hamiltonian cycle by replacing the segment $v_1 v_2 v_3 v_4$ of C with the path $v_1 x v_3 v_2 y v_4$. Therefore $V_2 := \{v_2, v_4, \dots, v_{2k}\} \cup \{x, y\}$ is an independent set of size $k+2$, and so $G \in \mathcal{G}_5$. □

3.3 A maximality property of the graphs in $\mathcal{G}_1 \cup \dots \cup \mathcal{G}_5$

Let $G \in \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_5$ be a graph. Delete a set of edges $\mathcal{A}$ from $E(G)$ where $|\mathcal{A}| \leqslant 1$ for $G \in \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$ and $|\mathcal{A}| \leqslant 2$ for $G \in \mathcal{G}_1 \cup \mathcal{G}_5$. Then add a set of new edges $\mathcal{B}$ as defined below:

— For $G \in \mathcal{G}_1$, $|\mathcal{B}| = 2$ and it consists of any two disjoint pairs joining V_1 and V_2;
— for $G \in \mathcal{G}_2 \cup \mathcal{G}_3$, $|\mathcal{B}| = 1$ and it consists of any pair x_1x_2 joining $V_1 \setminus \{x_0\}$ and $V_2 \setminus \{x_0\}$ (here $x_1 \in V_1$ and $x_2 \in V_2$);
— for $G \in \mathcal{G}_4$, $|\mathcal{B}| = 1$ and it consists of any pair contained in V_2;
— and for $G \in \mathcal{G}_5$, $|\mathcal{B}| = 2$ and it consists of any two distinct pairs contained in V_2.

Lemma 13. *If $k \geqslant 6$, then the graph $(E(G) \setminus \mathcal{A}) \cup \mathcal{B}$ defined by the above process is hamiltonian, except if $G \in \mathcal{G}_3$, x_0 has exactly two neighbors x_2 and y_2 in V_2, $\mathcal{A} = \{x_0y_2\}$, $\mathcal{B} = \{x_1x_2\}$, and $G[V_2 \setminus \{x_0\}]$ is either a K_{k+1} or misses only the edge x_2y_2.*

Proof. If $G \in \mathcal{G}_1$ and we add two disjoint edges x_1x_2 and y_1y_2 joining V_1 and V_2 ($x_1, y_1 \in V_1$) then to form a hamiltonian cycle we need an x_1, y_1 path P_1, and a x_2, y_2 path P_2 of length k, $V(P_i) = V_i$ and $E(P_i) \subset E(G) \setminus \mathcal{A}$. Such paths exist because the graph $G[V_i] \setminus \mathcal{A}$ has at least $\binom{k+1}{2} - 2$ edges, so it satisfies the condition of Corollary 9.

If $G \in \mathcal{G}_2 \cup \mathcal{G}_3$ and we add an edge x_1x_2 joining $V_1 \setminus \{x_0\}$ and $V_2 \setminus \{x_0\}$ then we need paths P_1, P_2 of length $|V_i| - 1$ joining x_i to x_0, $V(P_i) = V_i$ and $E(P_i) \subset E(G) \setminus \mathcal{A}$. If $G[V_i] \setminus \mathcal{A}$ satisfies the condition of Corollary 9 then we can find P_i. The only missing case is when $|V_2| = k + 2$ (so $G \in \mathcal{G}_3$). Let G_2 be the graph on $|V_2| + 1$ vertices obtained from $G[V_2] \setminus \mathcal{A}$ by adding a new vertex x_2' and two edges x_0x_2' and x_2x_2'. If G_2 has a hamiltonian cycle C then it should contain x_0x_2' and x_2x_2' so the rest of the edges of C can serve as P_2 we are looking for. Consider the hamilton-closure $C(G_2)$ and apply Theorem 8 to G_2. Since the degrees of $V_2 \setminus \{x_0\}$ in G_2 are at least $k-1$ and $2(k-1) \geqslant k+3 = |V(G_2)|$, $C(G_2)$ is a complete graph on $V_2 \setminus \{x_0\}$. So $C(G_2)$ is hamiltonian unless the only neighbors of x_0 in G_2 are x_2 and x_2'. Hence $N_G(x_0) \cap V_2 = \{x_2, y_2\}$ and $\mathcal{A} = \{x_0y_2\}$.

The last case is when $G \in \mathcal{G}_5$, $|\mathcal{A}| = 2$, $\mathcal{B} = \{e_1, e_2\}$ (two distinct edges inside V_2). (The proofs of the other cases, especially when $G \in \mathcal{G}_4$ are easier). We create a graph H_0 from G as follows: Delete the edge e_0 (if it exists), delete the edges of $\mathcal{A}$ joining V_1 and V_2, add two new vertices z_1, z_2 to V_1 and join z_i to the endpoints of e_i. We obtain the graph H by adding all possible $\binom{k+2}{2}$ pairs from $V_1 \cup \{z_1, z_2\}$ to H_0.

If H is hamiltonian then its hamiltonian cycle must use only edges of H_0 (because V_2 is an independent set of size $k + 2$ in H). If the graph H_0 is hamiltonian then its hamiltonian cycle must use the two edges of the degree 2 vertex z_i, so $(G \setminus (\{e_0\} \cup \mathcal{A})) \cup \mathcal{B}$ is hamiltonian as well. So it is sufficient to show that H has a hamiltonian cycle.

Let A be the graph on $V(H)$ consisting of the edges of $\mathcal{A}$ joining V_1 and V_2 together with the (at most) two missing pairs $E(K(V_1, V_2)) \setminus E(G)$. We will again apply Theorem 8 to H, so consider the hamilton-closure $C(H)$. The degree $\deg_H(x)$ of an $x \in V_1$ is $(2k+3) - \deg_A(x)$. The degree $\deg_H(y)$ of a $y \in V_2$ is at least $|V_1| - \deg_A(y) = k - \deg_A(y)$. Since $\deg_A(x) + \deg_A(y) \leqslant |E(A)| + 1 \leqslant 5$ we get for $k \geqslant 6$ that

$$\deg_H(x) + \deg_H(y) \geqslant (3k + 3) - (\deg_A(x) + \deg_A(y)) \geqslant 3k - 2 \geqslant 2k + 4 = |V(H)|.$$

So $C(H)$ contains the complete bipartite graph $K(V_1, V_2) = K_{k,k+2}$. Then it is really a simple task to find a hamiltonian cycle in $C(H)$ and therefore $(E(G) \setminus \mathcal{A}) \cup \mathcal{B}$ is hamiltonian. □

3.4 Proof of Theorem 4, reducing the hypergraph to a dense graph

Fix $\mathcal{H}$ to be an n vertex hypergraph satisfying the minimum degree condition. We will find a hamiltonian Berge cycle in $\mathcal{H}$.

Recall that $H = \partial_2(\mathcal{H})$ denotes the 2-shadow of $\mathcal{H}$, a graph on $V = V(\mathcal{H})$. Define a bipartite graph $B := B[E(\mathcal{H}), E(H)]$ with parts $E(\mathcal{H})$ and $E(H)$ and with edges $\{h, xy\}$ where a hyperedge $h \in E(\mathcal{H})$ is joined to the graph edge $xy \in E(H)$ if $\{x, y\} \subseteq h$. In the case of $\{x, y\} \in \mathcal{H}$ we consider the edge $xy \in E(H)$ and $\{x, y\} \in E(\mathcal{H})$ as two distinct objects of B, so B is indeed a bipartite graph (with $|E(\mathcal{H})| + |E(H)|$ vertices and no loops). Let M be a maximum matching of B. So M can be considered as a partial injection of maximum size, i.e., a bijection ϕ between two subsets $\mathcal{M} \subseteq E(\mathcal{H})$ and $\mathcal{E} \subseteq E(H)$ such that $|\mathcal{M}| = |\mathcal{E}|$, $\phi(m) \subseteq m$ for $m \in \mathcal{M}$ (and $\phi(m_1) \neq \phi(m_2)$ for $m_1 \neq m_2$). Consider the subgraph $G = (V, \mathcal{E})$ of H. Then G does not have a hamiltonian cycle, otherwise by replacing the edges of a hamiltonian cycle with their corresponding matched hyperedges in M, we obtain a hamiltonian Berge cycle in $\mathcal{H}$ (with representative vertices in the same order). In this subsection we are going to prove that

$$\delta(G) \geqslant \lfloor (n-1)/2 \rfloor := k. \tag{1}$$

Since G has no hamiltonian cycle and $k \geqslant 7$, if (1) holds, then by Lemma 12, $G \in \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_5$. We will consider this case and prove the remainder of Theorem 4 in the next subsection.

Let $\mathcal{H}_2 := E(\mathcal{H}) \cap \partial_2(\mathcal{H})$, the set of 2-element edges of $\mathcal{H}$. We may assume that among all maximum sized matchings of B the matching M maximizes $|\mathcal{M} \cap \mathcal{H}_2|$.

Claim 14. *$\mathcal{H}_2 \subseteq \mathcal{M}$, $\partial_2(\mathcal{M}) = E(H)$, and every $m \in E(\mathcal{H}) \setminus \mathcal{M}$ induces a complete graph in G.*

Proof. If $m \in E(\mathcal{H})$ contains an edge $e \in E(H) \setminus E(G)$ then one can enlarge the matching M by adding $\{m, e\}$ to it, if it is possible. Since M is maximal, it cannot be enlarged, so $m \in \mathcal{M}$. This implies the second and the third statements. We also obtained that if $\{x, y\} \in E(\mathcal{H})$ then $xy \in E(G)$, so $\phi(m) = xy$ for some $m \in \mathcal{M}$. In case of $|m| > 2$ we can replace the pair $\{m, xy\}$ by the pair $\{\{x, y\}, xy\}$ in M and the new matching covers more edges from $\mathcal{H}_2$ than M does (in the graph B). So $|m| = 2$, all members of $\mathcal{H}_2$ must belong to $\mathcal{M}$. $\square$

To continue the proof of Theorem 4, let $d := \delta(G)$, $v \in V$ such that $D := N_G(v)$, $|D| = d$. Since G is not hamiltonian, Theorem 1 gives $d \leqslant k$. Let $\mathcal{H}_v = \{e \in \mathcal{H} : v \in e\}$ denote the edges of H incident to v, ($\deg_{\mathcal{H}}(v) = |\mathcal{H}_v|$), and split it into two parts, $\mathcal{H}_v = \mathcal{D} \cup \mathcal{L}$ where $\mathcal{D} := \{e \in E(\mathcal{H}) : v \in e \subseteq \{v\} \cup D\}$ and $\mathcal{L} := \mathcal{H}_v \setminus \mathcal{D}$. Split $\mathcal{D}$ further into three parts according to the sizes of its edges, $\mathcal{D} = \mathcal{D}^- \cup \mathcal{D}_2 \cup \mathcal{D}_3$ where $\mathcal{D}_i := \{e \in \mathcal{D} : |e| = i\}$ (for $i = 2, 3$) and $\mathcal{D}^- := \mathcal{D} \setminus (\mathcal{D}_2 \cup \mathcal{D}_3)$. Since $\mathcal{D}$ can have at most 2^d members and we handle $\mathcal{D}_2$ and $\mathcal{D}_3$ separately we get

$$|\mathcal{D}| \leqslant 2^d - d - \binom{d}{2} + |\mathcal{D}_2| + |\mathcal{D}_3|. \tag{2}$$

Recall that the matching M in the bipartite graph B can be considered as a bijection $\phi : \mathcal{M} \to \mathcal{E}$, where $\mathcal{M} \subseteq E(\mathcal{H})$ and $\mathcal{E} \subseteq E(H)$. Define another matching M_2 in B by an injection $\phi_2 : \mathcal{D}_2 \cup \mathcal{D}_3 \to E(G)$ as follows. If $m \in \mathcal{M} \cap (\mathcal{D}_2 \cup \mathcal{D}_3)$ then $\phi_2(m) := \phi(m)$. In particular, since $\mathcal{D}_2 \subseteq \mathcal{M}$, if $\{v, x\} \in \mathcal{D}_2$ then $\phi_2(\{v, x\}) = vx$. If $m = \{v, x, y\} \in \mathcal{D}_3 \setminus \mathcal{M}$ then let $\phi_2(m) := xy$. Since $\phi_2(\mathcal{D}_2 \cup \mathcal{D}_3) \subseteq E(G)$ we can apply Theorem 10 to the matchings M and M_2 in B with $X_1 := \mathcal{M}$, $Y_1 := E(G)$, and $X_2 := \mathcal{D}_2 \cup \mathcal{D}_3$. So there exists a subfamily $\mathcal{L}_3 \subseteq \mathcal{H} \setminus (\mathcal{D}_2 \cup \mathcal{D}_3)$ and a bijection $\phi_3 : (\mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{L}_3) \to E(G)$. The matching M' defined by ϕ_3 is also a largest matching of B. Every $m \in \mathcal{L}$ has an element

$x \notin D$, so $vx \notin E(G)$. If m is not matched in M', then we add $\{m, vx\}$ to M' to get a larger matching. Hence $m \in \mathcal{L}_3$. These yield

$$|\mathcal{L}| \leqslant |\mathcal{L}_3| = e(G) - |\mathcal{D}_2| - |\mathcal{D}_3|. \tag{3}$$

Summing up (2) and (3), then using the lower bound for $|\mathcal{H}_v|$ and the upper bound of Theorem 11 for $e(G)$ we obtain

$$2^k + 1 \leqslant \deg_{\mathcal{H}}(v) \leqslant 2^d - \binom{d+1}{2} + e(n,d)$$

The inequality $2^k + 1 \leqslant 2^d - \binom{d+1}{2} + e(n,d)$ does not hold for $n \geqslant 15$ and $d < k$, e.g., for $(n,k,d) = (16,7,6)$, the right hand side is only $64 - 21 + 85 = 128$. This completes the proof of $d = k$.

3.5 Proof of Theorem 4, the end

We may assume that $G \in \mathcal{G}_1 \cup \dots \cup \mathcal{G}_5$ by Lemma 12, ϕ is a bijection $\phi : \mathcal{M} \to E(G)$ with $\phi(m) \subseteq m$ where $\mathcal{M} \subseteq E(\mathcal{H})$, and Claim 14 holds. Let $\mathcal{L}_v$ denote the set of edges $m \in \mathcal{H}$ containing an edge vy of $E(H) \setminus E(G)$. Note that $\mathcal{L}_v \subseteq \mathcal{M}$. If $\deg_G(v) = k$, then the family $\mathcal{L}_v$ is non-empty, otherwise $\deg_{\mathcal{H}}(v) \leqslant 2^k$.

Call a graph F with vertex set V a *Berge graph of* $\mathcal{H}$ if $E(F) \subseteq E(H)$, and there exists a subhypergraph $\mathcal{F} \subseteq \mathcal{H}$, and a bijection $\psi : \mathcal{F} \to E(F)$ such that $\psi(m) \subseteq m$ for each $m \in \mathcal{F}$. We are looking for a Berge graph of $\mathcal{H}$ having a hamiltonian cycle. In particular, the graph G is a Berge graph of $\mathcal{H}$ and it is almost hamiltonian. We will show that a slight change to G yields a hamiltonian Berge graph of $\mathcal{H}$.

If $G \in \mathcal{G}_2 \cup \mathcal{G}_3$ then choose any $v \in V_1 \setminus \{x_0\}$ and let $m \in \mathcal{L}_v$. There exists an edge $vy \in (E(H) \setminus E(G))$ contained in m. Then $y \in V_2 \setminus \{x_0\}$. The graph $(E(G) \setminus \{\phi(m)\}) \cup \{vy\}$ is a Berge graph of $\mathcal{H}$ (we map m to the edge vy instead of $\phi(m)$). According to Lemma 13 (with $\mathcal{A} := \{\phi(m)\}$ and $\mathcal{B} := \{vy\}$) it is hamiltonian except if we run into the only exceptional case: x_0 has exactly two G-neighbors x_2 and y_2 in V_2, $vy = vx_2$, and $\phi(m) = x_0y_2$. In this case m contains $\{x_0, v, x_2, y_2\}$ so it can be avoided by choosing $y := y_2$ instead of $y := x_2$.

If $G \in \mathcal{G}_4$ then we argue in a very similar way. Choose any $v \in V_2$ and let $m \in \mathcal{L}_v$ containing an edge $vy \in (E(H) \setminus E(G))$. Then $y \in V_2$ and the graph $(E(G) \setminus \{\phi(m)\}) \cup \{vy\}$ is a Berge graph of $\mathcal{H}$ that is hamiltonian by Lemma 13 with $\mathcal{A} := \{\phi(m)\}$ and $\mathcal{B} := \{vy\}$. From now on we may suppose that $n = 2k+2$ so $|\mathcal{L}_v| \geqslant 2$ for $\deg_G(v) = k$.

If $G \in \mathcal{G}_1$ then define $\mathcal{M}_{1,2}$ as the members of $\mathcal{M}$ meeting both V_1 and V_2. The minimum degree condition on $\mathcal{H}$ implies that $|\mathcal{M}_{1,2}| \geqslant 2$. Since $\mathcal{M}_{1,2}$ can have at most one member of size 2, we can choose an m_1, $|m_1| \geqslant 3$. By symmetry we may suppose that $|m_1 \cap V_1| \geqslant 2$ and let $x_2 \in V_2 \cap m_1$. Choose an element $y \in V_2$, $y \notin e_0$, $y \neq x_2$. Since $|\mathcal{L}_y| \geqslant 2$ we can choose an $m_2 \in \mathcal{M}_{1,2}$ such that $m_1 \neq m_2$ and $y \in m_2$. Take any pair $\{y_1, y\} \subseteq m_2$ with $y_1 \in V_1$. Then one can choose an $x_1 \in m_1 \cap V_1$ so that $x_1 \neq x_2$. So the pairs $\{x_1, x_2\} \subseteq m_1$ and $\{y_1, y\} \subseteq m_2$ are disjoint. Lemma 13 with $\mathcal{A} := \{\phi(m_1), \phi(m_2)\}$

and $\mathcal{B} := \{x_1x_2, y_1y\}$ implies that the graph $(E(G) \setminus \mathcal{A}) \cup \mathcal{B}$ is a hamiltonian Berge graph of $\mathcal{H}$.

If $G \in \mathcal{G}_5$ then $|\mathcal{L}_v| \geqslant 2$ for any $v \in V_2 \setminus e_0$ and for all members m of $\mathcal{L}_v$ we have $|m \cap V_2| \geqslant 2$. Fix $v \in V_2 \setminus e_0$ and let m_1 be an arbitrary member of $\mathcal{L}_v$. Choose a pair $\{v, v'\} \subseteq m_1 \cap V_2$. Fix another vertex $u \in V_2 \setminus (e_0 \cup \{v, v'\})$ and let m_2 be an arbitrary member of $\mathcal{L}_u$. Choose a pair $\{u, u'\} \subseteq m_2 \cap V_2$. Then $u \notin \{v, v'\}$ so the pairs $\{u, u'\}$ and $\{v, v'\}$ are distinct. Again, apply Lemma 13 with $\mathcal{A} := \{\phi(m_1), \phi(m_2)\}$ and $\mathcal{B} := \{uu', vv'\}$. This completes the proof of Theorem 4. □

Remark 15. We can also show that all extremal examples are slight modifications of the four types of the sharpness examples described after Theorem 4.

4 Dirac-type conditions for long Berge cycles

In this section we prove Theorem 5 for Berge paths and Theorem 6 for Berge cycles. In fact we prove the two statements simultaneously.

Proof of Theorems 5 and 6. Suppose that $\delta(\mathcal{H}) \geqslant 2^{k-2} + 1$, $k \geqslant 3$ and that $\mathcal{H}$ has no Berge cycle of length k or longer. We will show that it contains a Berge path of length $k-1$ (thus establishing Theorem 5) and then that $\delta(\mathcal{H}) = 2^{k-2} + 1$ (which completes the proof of Theorem 6).

Choose a longest Berge path in $\mathcal{H}$ according the following rules. We say that a Berge path with edges $\{e_1, \ldots, e_s\}$ is *better* than a Berge path with edges $\{f_1, \ldots, f_t\}$ if

a) $s > t$ or

b) $s = t$ and $\sum |e_i| < \sum |f_j|$.

Consider a best Berge path $\mathcal{P}$ in $\mathcal{H}$. Let the base vertices of the path be $v_1, v_2, \ldots, v_p$. Let $e_1, \ldots, e_{p-1}$ be the edges of the path ($v_i, v_{i+1} \in e_i$). First, we show that $p \geqslant k-1$. (In fact, $p \geqslant k$ follows but that will be proved later).

Indeed, let $\mathcal{H}^{(p)}$ be the hypergraph consisting of the edges of $\mathcal{H}$ containing v_p, contained in $\{v_1, \ldots, v_p\}$ and also the edges of the path, i.e.,

$$E(\mathcal{H}^{(p)}) := \{e \in E(\mathcal{H}) : v_p \in e \subseteq \{v_1, \ldots, v_p\}\} \cup \{e_1, \ldots, e_{p-1}\}.$$

Then for $p \leqslant k-2$ (and $k \geqslant 3$) we have

$$|E(\mathcal{H}^{(p)})| \leqslant 2^{p-1} + (p-1) \leqslant 2^{k-2} < \delta(\mathcal{H}) \leqslant \deg_{\mathcal{H}}(v_p).$$

So there exists an edge f in $E(\mathcal{H}) \setminus E(\mathcal{H}^{(p)})$ containing v_p. Then $e_1, \ldots, e_{p-1}, f$ form a Berge path longer than $\mathcal{P}$, a contradiction.

Now we have $p \geqslant k-1$, so we can define $W := \{v_1, \ldots, v_{k-1}\}$. Let $\mathcal{P}_1$ be the subhypergraph consisting of the first $k-1$ edges of $\mathcal{P}$, $E(\mathcal{P}_1) := \{e_1, \ldots, e_{k-1}\}$ (if $p = k-1$ we take $\mathcal{P}_1 := \mathcal{P}$). Let $\mathcal{H}_1$ be the subhypergraph of $\mathcal{H}$ consisting of the edges incident to v_1.

Claim 16. *Every edge $f \in E(\mathcal{H}_1) \setminus E(\mathcal{P}_1)$ is contained in $W := \{v_1, \ldots, v_{k-1}\}$.*

Proof. First, we show that every edge $f \in E(\mathcal{H}_1) \setminus E(\mathcal{P}_1)$ avoids $\{v_k, \ldots, v_p\}$. Otherwise, if there exists an edge $f \in E(\mathcal{H}_1) \setminus E(\mathcal{P}_1)$ such that $f \cap \{v_k, \ldots, v_p\} \neq \varnothing$, then suppose that v_i has the minimum index ($k \leqslant i \leqslant p$) such that v_i is a vertex of such an f. Then $e_1, \ldots, e_{i-1}$ and f are forming a Berge cycle of length i, since these edges are all distinct and $v_1, v_i \in f$. Finally, suppose that there is an edge $f \in E(\mathcal{H}_1) \setminus E(\mathcal{P}_1)$ such that $v \in f$, $v \notin W$. Then $v \notin \{v_1, \ldots, v_p\}$ so the path $f, e_1, \ldots, v_p$ is longer than $\mathcal{P}$, a contradiction. □

Let $\mathcal{K}$ be the family of all 2^{k-2} subsets of W that contain v_1. We claim there is a one-to-one mapping φ from $\mathcal{H}_1 \setminus e_{k-1}$ to $\mathcal{K}$. The existence of such a φ implies

$$\delta(\mathcal{H}) \leqslant \deg_{\mathcal{H}}(v_1) \leqslant 2^{k-2} + 1. \tag{4}$$

If an edge e of $\mathcal{H}_1$ satisfies $e \subseteq W$, then let $\varphi(e) = e$. Otherwise, let $\mathcal{A} \subseteq \mathcal{H}_1$ be the set of the edges of $\mathcal{H} \setminus \{e_{k-1}\}$ that contain both v_1 and some vertex outside of W. By Claim 16, each $e \in \mathcal{A}$ must be some edge e_i in $\mathcal{P}_1$. Hence it remains to show that all elements of $\mathcal{A}$ can be mapped to distinct elements of $\mathcal{K}$ that are not edges of $\mathcal{H}$.

Observe that if $e_i \in \mathcal{A}$ then $\{v_i, v_{i+1}\} \notin \mathcal{H}$. Otherwise, we get a better path by replacing e_i by $\{v_i, v_{i+1}\}$. Also, for $1 \leqslant i \leqslant k-2$, $e_i \in \mathcal{A}$ implies $v_1 \in e_i$ and $\{v_i, v_{i+1}\} \subset e_i$. Since $e_i \not\subset W$ we get $|e_i| \geqslant 4$ for $i \geqslant 2$. We also obtain that in case of $i \geqslant 3$, $e_i \in \mathcal{A}$ we have $\{v_1, v_i, v_{i+1}\} \notin \mathcal{P}$, and moreover $\{v_1, v_i, v_{i+1}\} \notin \mathcal{H}$ since otherwise we get a better path by replacing e_i by $\{v_1, v_i, v_{i+1}\}$. For $3 \leqslant i \leqslant k-2$ (and $e_i \in \mathcal{A}$) define $\varphi(e_i)$ as $\{v_1, v_i, v_{i+1}\}$.

If $e_2 \in \mathcal{A}$ and $\{v_1, v_2, v_3\} \notin \mathcal{H}$ then we proceed as above, $\varphi(e_2) := \{v_1, v_2, v_3\}$. Otherwise, if $e_2 \in \mathcal{A}$ (so $|e_2| \geqslant 4$) and $\{v_1, v_2, v_3\} \in \mathcal{H}$ then $\{v_1, v_2, v_3\} \in \mathcal{P}$ too (otherwise, we get a better path by replacing e_2 by $\{v_1, v_2, v_3\}$). We get $e_1 = \{v_1, v_2, v_3\}$ (and $e_1 \subset e_2$). We claim that $\{v_1, v_3\} \notin \mathcal{H}$. Otherwise we rearrange the base vertices of the path $\mathcal{P}$ by exchanging v_1 and v_2 (and get the order $v_2, v_1, v_3, \ldots, v_p$) and observe that the Berge path $\{v_2, v_1, v_3\}, \{v_1, v_3\}, e_3, \ldots, e_{p-1}$ is better than $\mathcal{P}$, a contradiction. So in this case $\varphi(e_2) := \{v_1, v_3\}$. Finally, if $e_1 \in \mathcal{A}$ then $\varphi(e_1) := \{v_1, v_2\}$, and the definition of φ is complete.

We have shown that $\deg_{\mathcal{H}}(v_1) \leqslant |\mathcal{H}_1 \setminus \{e_{k-1}\}| + 1 \leqslant 2^{k-2} + 1$. Equality holds, so $v_1 \in e_{k-1}$. In particular e_{k-1} must exist, so $\mathcal{P}$ was a Berge path of length at least $k-1$. □

Our method works for multihypergraphs as well. If the maximum multiplicity of an edge is μ, then the corresponding necessary bounds on the minimum degrees are $\mu 2^{k-2}+1$ or $\mu 2^{k-2}+2$, respectively. Indeed, suppose that $\delta(\mathcal{F}) \geqslant \mu 2^{k-2}+1$, $k \geqslant 3$ and that $\mathcal{F}$ has no Berge cycle of length k or longer. Let $\mathcal{H}$ be the simple hypergraph obtained from $\mathcal{F}$ by keeping one copy from the multiple edges. We have $\delta(\mathcal{H}) \geqslant 2^{k-2}+1$. Then Theorems 5 implies that $\mathcal{H}$ (and $\mathcal{F}$ as well) contain a Berge path with k base vertices.

As in the proof of Theorem 6, consider a best Berge path $\mathcal{P}$ in $\mathcal{H}$ with base vertices $v_1, v_2, \ldots, v_p$ and edges $e_1, \ldots, e_{p-1}$. We have $p \geqslant k$. Then (4) gives $\deg_{\mathcal{H}}(v_1) = 2^{k-2}+1$ and we get $\deg_{\mathcal{H}}(v_1) = |\mathcal{H}_1 \setminus \{e_{k-1}\}| + 1$. Since we also obtained $\{v_1, v_{k-1}, v_k\} \subset e_{k-1}$, the multiplicity of e_{k-1} could not exceed 1. So $\delta(\mathcal{F})$ could not exceed $\mu 2^{k-2}+1$.

5 Maximum number of edges

Proof of Theorem 7. Suppose that among all n-vertex hypergraphs with $c(\mathcal{H}) < k$ and $e(\mathcal{H})$ edges our $\mathcal{H}$ is chosen so that $\sum_{e\in E(\mathcal{H})} |e|$ is minimized.

We claim that $\mathcal{H}$ is a downset, that is, for any $e \in E(\mathcal{H})$ and $e' \subset e$, $e' \in E(\mathcal{H})$. Indeed, if there exists a set e' and an edge e such that $e' \subset e$ where $e' \notin E(\mathcal{H})$ and $e \in E(\mathcal{H})$, then the hypergraph obtained by replacing e with e' also does not contain a Berge cyle of length k or longer. This contradicts the choice of $\mathcal{H}$.

Let $H = \partial_2\mathcal{H}$ be the 2-shadow of $\mathcal{H}$. Suppose that H contains a cycle $C = v_1v_2\dots v_\ell v_1$. Every edge v_iv_{i+1} of C is contained in a edge of $\mathcal{H}$. But since $\mathcal{H}$ is a downset, the edge $\{v_i, v_{i+1}\}$ is also contained in $E(\mathcal{H})$. Therefore $\mathcal{H}$ also contains a (Berge) cycle of length ℓ. Hence the graph H contains no cycles of length at least k.

Let $e_r(\mathcal{H})$ be the number of edges of $\mathcal{H}$ of size r. In H, every edge e of $\mathcal{H}$ is represented by a clique of order $|e|$, and so $e_r(\mathcal{H})$ is at most the number of cliques of size r in H. Since $c(H) < k$, each edge contains at most $k-1$ vertices. By Theorem 3,

$$e(\mathcal{H}) = e_0(\mathcal{H})+e_1(\mathcal{H})+\sum_{r=2}^{k-1} e_r(\mathcal{H}) \leqslant 1+n+\sum_{r=2}^{k-1}\frac{n-1}{k-2}\binom{k-1}{r} = 2+\frac{n-1}{k-2}\left(2^{k-1}-2\right). \quad \square$$

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