Proof of Gessel's γ-positivity conjecture

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Abstract

We prove a conjecture of Gessel, which asserts that the joint distribution of descents and inverse descents on permutations has a fascinating refined γ-positivity.

Keywords: descents; inverse descents; Eulerian polynomials; γ-positivity

1 Introduction

Let $\mathfrak{S}_n$ denote the set of all permutations of $[n] := \{1, 2, \ldots, n\}$. A permutation $\pi \in \mathfrak{S}_n$ will be represented here in one line notation as $\pi = \pi_1 \cdots \pi_n$. For a permutation $\pi \in \mathfrak{S}_n$, an index $i \in [n-1]$ is a *descent* of π if $\pi_i > \pi_{i+1}$. Denote by $\mathrm{des}(\pi)$ the number of descents of π. The descent polynomial on $\mathfrak{S}_n$

$$A_n(t) := \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)+1}$$

is known as the classical *Eulerian polynomial* (cf. [14, Section 1.3]) of order n. Foata and Schützenberger [7] proved the following beautiful γ-positivity result, which implies the *symmetry* and *unimodality* (see [2] for definitions) of the Eulerian polynomials.

Theorem 1 (Foata–Schützenberger). *For $n \geqslant 1$,*

$$A_n(t) = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \gamma_{n,i} t^i (1+t)^{n+1-2i}, \tag{1.1}$$

where $\gamma_{n,i}$ are nonnegative integers.

Foata and Strehl [8] later constructed an elegant combinatorial proof of (1.1) via a group action, which has sparked various interesting extensions [1,5,6,10]. For many other γ-positivity results and problems arising in enumerative and geometric combinatorics, we refer the reader to the excellent exposition by Petersen [13]. Regarding the joint distribution of descents and inverse descents on permutations, Gessel (see [1, 2, 12, 15]) has conjectured the following refined γ-positivity:

Conjecture 2 (Gessel 2005)**.** *Let*

$$A_n(s,t) := \sum_{\pi\in\mathfrak{S}_n} s^{\mathrm{des}(\pi^{-1})+1} t^{\mathrm{des}(\pi)+1}.$$

Then, for $n \geqslant 1$

$$A_n(s,t) = \sum_{\substack{i\geqslant 1, j\geqslant 0\\ j+2i\leqslant n+1}} \gamma_{n,i,j} (st)^i (1+st)^j (s+t)^{n+1-j-2i}, \tag{1.2}$$

where $\gamma_{n,i,j}$ are nonnegative integers.

These refined Eulerian polynomials $A_n(s,t)$, that we shall call *double Eulerian polynomials*, were first studied by Carlitz, Roselle and Scoville [4]. Conjecture 2 has received considerable attention since the publication of Brändén [1] in 2008. The existence of integers $\gamma_{n,i,j}$ satisfying (1.2) follows from symmetry properties of $A_n(s,t)$ and Lemma 5. The open problem is nonnegativity. For example, we have:

$$\begin{aligned}
A_1(s,t) &= st,\\
A_2(s,t) &= st(1+st),\\
A_3(s,t) &= st(1+st)^2 + 2(st)^2,\\
A_4(s,t) &= st(1+st)^3 + 7(st)^2(1+st) + (st)^2(s+t),\\
A_5(s,t) &= st(1+st)^4 + 16(st)^2(1+st)^2 + 6(st)^2(1+st)(s+t) + 16(st)^3.
\end{aligned}$$

In this note, we give a proof of Conjecture 2.

Using Eulerian operators and a homogenized multivariate refinement for $A_n(s,t)$, Visontai [15] derived a recurrence for the coefficients $\gamma_{n,i,j}$, from which the nonnegativity does not follow immediately. But surprisingly, we are able to deduce the nonnegativity of $\gamma_{n,i,j}$ from his recurrence. Even more, we characterize completely when the coefficient $\gamma_{n,i,j}$ is positive. Generalizations of Conjecture 2 will also be proved (see Theorems 6 and 10). The question of finding a combinatorial proof of expansion (1.2) is still open.

2 Proof of Conjecture 2

We shall first provide a new direct approach to the following recurrence relation due to Visontai and then apply it to give a proof of Conjecture 2.

Lemma 3 (**Theorem 10** of [15]). *Let $n \geqslant 1$. For all $i \geqslant 1$ and $j \geqslant 0$, we have*

$$\begin{aligned}(n+1)\gamma_{n+1,i,j} &= (n+i(n+2-i-j))\gamma_{n,i,j-1} + (i(i+j)-n)\gamma_{n,i,j} \\ &+ (n+4-2i-j)(n+3-2i-j)\gamma_{n,i-1,j-1} \\ &+ (n+2i+j)(n+3-2i-j)\gamma_{n,i-1,j} \\ &+ (j+1)(2n+2-j)\gamma_{n,i-1,j+1} + (j+1)(j+2)\gamma_{n,i-1,j+2},\end{aligned} \tag{2.1}$$

where $\gamma_{1,1,0}=1$, $\gamma_{1,i,j}=0$ (unless $i=1$ and $j=0$) and $\gamma_{n,i,j}=0$ if $i<1$ or $j<0$.

Proof. We will use the following recurrence of $A_n(s,t)$ computed by Petersen [12, Equation (9)] via the machine of balls in boxes (or 2-D barred permutations):

$$\begin{aligned}nA_n(s,t) &= (n^2st+(n-1)(1-s)(1-t))A_{n-1}(s,t) \\ &+ nst(1-s)\frac{\partial}{\partial s}A_{n-1}(s,t) + nst(1-t)\frac{\partial}{\partial t}A_{n-1}(s,t) \\ &+ st(1-s)(1-t)\frac{\partial^2}{\partial s\partial t}A_{n-1}(s,t).\end{aligned} \tag{2.2}$$

Let $\Gamma_n(X,Y) := \sum_{i,j}\gamma_{n,i,j}X^iY^j$. Observe that decomposition (1.2) is equivalent to the following relationship:

$$A_n(s,t) = (s+t)^{n+1}\Gamma_n(X,Y) \qquad \text{with } X=\frac{st}{(s+t)^2} \text{ and } Y=\frac{1+st}{s+t}.$$

Substituting this into (2.2) and dividing both sides by $(s+t)^{n+1}$, we get

$$\begin{aligned}n\Gamma_n(X,Y) &= \alpha_1\Gamma_{n-1}(X,Y) + \alpha_2\frac{\partial\Gamma_{n-1}(X,Y)}{\partial X} + \alpha_3\frac{\partial\Gamma_{n-1}(X,Y)}{\partial Y} \\ &+ \alpha_4\frac{\partial^2\Gamma_{n-1}(X,Y)}{\partial X^2} + \alpha_5\frac{\partial^2\Gamma_{n-1}(X,Y)}{\partial Y^2} + \alpha_6\frac{\partial^2\Gamma_{n-1}(X,Y)}{\partial X\partial Y},\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}\alpha_1 &= \frac{n^2st+(n-1)(1-s)(1-t)}{s+t} + \frac{n^2st(2-s-t)}{(s+t)^2} + \frac{n(n-1)st(1-s)(1-t)}{(s+t)^3} \\ &= (n-1)(Y-1) + n(n-1)XY + (n^2+n)X,\end{aligned}$$

$$\begin{aligned}\alpha_2 = \frac{nst}{s+t}\left((1-s)\frac{\partial X}{\partial s} + (1-t)\frac{\partial X}{\partial t}\right) + \frac{nst(1-s)(1-t)}{(s+t)^2}\left(\frac{\partial X}{\partial t} + \frac{\partial X}{\partial s}\right) + \\ + \frac{st(1-s)(1-t)}{s+t}\frac{\partial^2 X}{\partial s\partial t} = (n-1)XY + (6-4n)X^2Y + X - 6X^2,\end{aligned}$$

$$\begin{aligned}\alpha_3 = \frac{nst}{s+t}\left((1-s)\frac{\partial Y}{\partial s} + (1-t)\frac{\partial Y}{\partial t}\right) + \frac{nst(1-s)(1-t)}{(s+t)^2}\left(\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial s}\right) + \\ + \frac{st(1-s)(1-t)}{s+t}\frac{\partial^2 Y}{\partial s\partial t} = (-2n+2)XY^2 + 2nX - 2XY,\end{aligned}$$

$$\alpha_4 = \frac{st(1-s)(1-t)}{s+t}\frac{\partial X}{\partial s}\frac{\partial X}{\partial t} = 4X^3(Y-1) - X^2(Y-1),$$

$$\alpha_5 = \frac{st(1-s)(1-t)}{s+t}\frac{\partial Y}{\partial s}\frac{\partial Y}{\partial t} = -X(Y-1) + XY^2(Y-1)$$

and

$$\alpha_6 = \frac{st(1-s)(1-t)}{s+t}\left(\frac{\partial X}{\partial s}\frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial s}\frac{\partial X}{\partial t}\right) = -XY(Y-1) + 4X^2Y(Y-1).$$

Extracting the coefficient of X^iY^j in both sides of (2.3), we get (2.1) after shifting the index n by one. □

The same manipulations above can be applied to deduce the recurrence relations for the γ-coefficients of two extensions of $A_n(s,t)$ in the next section. It does not follow immediately from recurrence relation (2.1) that $\gamma_{n+1,i,j}$ is a nonnegative integer: the left-hand side has the multiplicative factor $(n+1)$ and the coefficient $i(i+j)-n$ in the right-hand side may assume negative values. Nevertheless, the crucial observation that $\gamma_{n,i,j} \neq 0$ only if when $i(i+j) \geqslant n$ will lead to the nonnegativity of $\gamma_{n,i,j}$, as stated in the following.

Theorem 4. *For $n \geqslant 1$, the coefficients $\gamma_{n,i,j}$ are nonnegative. Moreover, the coefficient $\gamma_{n,i,j}$ is positive if and only if $i \geqslant 1, j \geqslant 0$, $2i+j \leqslant n+1$ and $i(i+j) \geqslant n$.*

Proof. We will prove this result by induction on n using recurrence relation (2.1) for the coefficient $\gamma_{n,i,j}$.

Clearly, the result is true for $n \leqslant 5$ by the first formulae produced in the introduction. Suppose that this result is true for some n with $n \geqslant 5$. We need to show the result for $n+1$. We can assume that $i \geqslant 1, j \geqslant 0$ and $2i+j \leqslant n+2$, otherwise $\gamma_{n+1,i,j} = 0$. There are three cases to be considered.

Case 1: If $i(i+j) = n$, then the inductive hypothesis implies that all $\gamma_{n,i,j-1}$, $\gamma_{n,i-1,j-1}$, $\gamma_{n,i-1,j}$, $\gamma_{n,i-1,j+1}$, $\gamma_{n,i-1,j+1}$, $\gamma_{n,i-1,j+2}$ are 0 (except $\gamma_{n,i,j}$ may not be zero), since now

$$i(i+j-1) < n, \quad (i-1)(i+j-2) < n, \quad (i-1)(i+j+1) < n.$$

Thus, $\gamma_{n+1,i,j} = 0$ if $i(i+j) = n$.

Case 2: If $i(i+j) < n$, then the inductive hypothesis implies that all $\gamma_{n,i,j-1}$, $\gamma_{n,i-1,j-1}$, $\gamma_{n,i-1,j}$, $\gamma_{n,i-1,j+1}$, $\gamma_{n,i-1,j+1}$ and $\gamma_{n,i-1,j+2}$, including $\gamma_{n,i,j}$, are 0, which forces $\gamma_{n+1,i,j} = 0$.

Case 3: If $i(i+j) \geqslant n+1$, then we need further to distinguish two subcases. Subcase I: $2i+j \leqslant n+1$. In this case, the expression $(i(i+j)-n)\gamma_{n,i,j}$ in the right-hand side of (2.1) is positive by the inductive hypothesis, and so $\gamma_{n+1,i,j} > 0$. Subcase II: $2i+j = n+2$. In this case, as

$$i(i+j-1) = i(n+1-i) \geqslant n \quad \text{and} \quad 2i+j-1 = n+1,$$

we have $(n + i(n+2-i-j))\gamma_{n,i,j-1} > 0$ (again by the inductive hypothesis) in the right-hand side of (2.1), and therefore $\gamma_{n+1,i,j} > 0$. This ends the proof by induction. □

For the sake of completeness, we provide a proof of the following fundamental result regarding the basis

$$\mathcal{B}_n := \{(st)^i(1+st)^j(s+t)^{n-j-2i} : i, j \geqslant 0, j+2i \leqslant n\}.$$

Lemma 5. *The set $\mathcal{B}_n$ is a basis (over $\mathbb{Z}$) for polynomials $A(s,t) = \sum_{k,l\geqslant 0}^{n} a_{k,l}s^k t^l \in \mathbb{Z}[s,t]$ with symmetries*

$$a_{k,l} = a_{l,k} \quad \text{and} \quad a_{k,l} = a_{n-k,n-l} \quad \text{for all } k,l \geqslant 0. \tag{2.4}$$

Proof. Clearly, all polynomials in $\mathcal{B}_n$ satisfy the symmetries (2.4), as well as their linear combinations. It remains to show that each polynomial with symmetries (2.4) can be expanded uniquely in $\mathcal{B}_n$.

Let $b_{n,i,j} := (st)^i(1+st)^j(s+t)^{n-j-2i}$. We order the polynomials in $\mathcal{B}_n$ as:

$$b_{n,i,j} \text{ is before } b_{n,u,v} \text{ if } i < u \text{ or } i = u \text{ but } j < v$$

so that the term $s^{n-i}t^{i+j}$ does not appear in any polynomial after $b_{n,i,j}$ in this order. Let $\mathcal{A}_n$ be the set of all polynomials in $\mathbb{Z}[s,t]$ with symmetries (2.4). We say a polynomial $A(s,t) \in \mathcal{A}_n$ has *complexity*

$$(\lfloor (n+2)/2 \rfloor - i)(\lfloor (n+3)/2 \rfloor - i) - j$$

if it contains the term $s^{n-i}t^{i+j}$ but does not contain any term $s^k t^l$ satisfying $k > n-i$ or $k = n-i$ but $l < i+j$. For such a polynomial $A(s,t)$, consider the polynomial $A(s,t) - a_{n-i,i+j}b_{n,i,j}$ (obviously in $\mathcal{A}_n$). The complexity of this new polynomial reduces at least by one, since the term $s^{n-i}t^{i+j}$ vanishes. Therefore, by induction on the complexity, we can show that each polynomial from $\mathcal{A}_n$ can be expanded uniquely in $\mathcal{B}_n$. □

3 Generalizations

Fix a positive integer $k \leqslant n$. Define the *generalized double Eulerian polynomial* $A_n^{(k)}(s,t)$ by the identity

$$\sum_{i,j\geqslant 0} \binom{ij+n-k}{n} s^i t^j = \frac{A_n^{(k)}(s,t)}{(1-s)^{n+1}(1-t)^{n+1}}. \tag{3.1}$$

The generalized double Eulerian polynomials first arise implicitly in [11]. Gessel [15] (see also [9]) further noticed that the generalized double Eulerian polynomials have the following nice interpretation

$$A_n^{(k)}(s,t) = \sum_{\pi \in \mathfrak{S}_n} s^{\mathrm{des}(\pi^{-1})+1} t^{\mathrm{des}(\pi\sigma)+1},$$

where $\sigma \in \mathfrak{S}_n$ is any fixed permutation with $\mathrm{des}(\sigma) = k-1$. Note that $A_n^{(1)}(s,t) = A_n(s,t)$. This suggests the following more general form of γ-positivity, first appeared as Conjecture 10.2 (also due to Gessel) in [1].

Theorem 6 (Generalization of **Conj. 2**). *For $n \geqslant 1$ and $1 \leqslant k \leqslant n$, we have*

$$A_n^{(k)}(s,t) = \sum_{\substack{i\geqslant 1, j\geqslant 0\\ j+2i\leqslant n+1}} \gamma_{n,i,j}^{(k)} (st)^i (1+st)^j (s+t)^{n+1-j-2i}, \tag{3.2}$$

where $\gamma_{n,i,j}^{(k)}$ are nonnegative integers.

For $s = 1$ or $t = 1$, expansion (3.2) reduces to the classical result (1.1) with $\gamma_{n,i} = \sum_{j\geqslant 0} \gamma_{n,i,j}^{(k)}$. Thus, the coefficients $\gamma_{n,i,j}^{(k)}$ are refinements of the triangle $\gamma_{n,i}$. We decompose the proof of Theorem 6 into the following three lemmas.

Lemma 7. *The generalized double Eulerian polynomial $A_n^{(k)}(s,t)$ satisfies the recurrence relation*

$$\begin{aligned} nA_n^{(k)}(s,t) &= (n^2 st + (n-k)(1-s)(1-t))A_{n-1}^{(k)}(s,t) \\ &\quad + nst(1-s)\frac{\partial}{\partial s}A_{n-1}^{(k)}(s,t) + nst(1-t)\frac{\partial}{\partial t}A_{n-1}^{(k)}(s,t) \\ &\quad + st(1-s)(1-t)\frac{\partial^2}{\partial s \partial t}A_{n-1}^{(k)}(s,t), \end{aligned} \tag{3.3}$$

where

$$A_k^{(k)}(s,t) = \sum_{\pi\in\mathfrak{S}_k} s^{\operatorname{des}(\pi^{-1})+1} t^{k-\operatorname{des}(\pi)} = t^{k+1} A_k(s, 1/t). \tag{3.4}$$

Proof. In the special case when $\sigma = k(k-1)\cdots 21$, we have $\operatorname{des}(\pi\sigma) = k-1-\operatorname{des}(\pi)$ for each $\pi \in \mathfrak{S}_k$ and (3.4) follows. For simplicity, the left-hand side of (3.1) is denoted as $F_n^{(k)}(s,t)$. Since

$$\begin{aligned} n\binom{ij+n-k}{n} &= (ij+n-k)\frac{(ij+n-k-1)!}{(n-1)!(ij-k)!} \\ &= ij\binom{ij+n-k-1}{n-1} + (n-k)\binom{ij+n-k-1}{n-1}, \end{aligned}$$

we have

$$\begin{aligned} nF_n^{(k)}(s,t) &= \sum_{i,j\geqslant 0} n\binom{ij+n-k}{n} s^i t^j \\ &= \sum_{i,j\geqslant 0} ij\binom{ij+n-k-1}{n-1} s^i t^j + \sum_{i,j\geqslant 0} (n-k)\binom{ij+n-k-1}{n-1} s^i t^j \\ &= st\frac{\partial^2}{\partial s\partial t}F_{n-1}^{(k)}(s,t) + (n-k)F_{n-1}^{(k)}(s,t). \end{aligned}$$

Substituting $F_n^{(k)}(s,t) = A_n^{(k)}(s,t)(1-s)^{-n-1}(1-t)^{-n-1}$ into the above relation, we get (3.3) after simplification. □

Lemma 8. *Fix a positive integer k. Let $n \geqslant k$. Then, for all $i \geqslant 1$ and $j \geqslant 0$*

$$\begin{aligned}(n+1)\gamma^{(k)}_{n+1,i,j} = (n+1-k+i(n+2-i-j))\gamma^{(k)}_{n,i,j-1} \\ + (i(i+j)-(n+1-k))\gamma^{(k)}_{n,i,j} \\ + (n+4-2i-j)(n+3-2i-j)\gamma^{(k)}_{n,i-1,j-1} \\ + (n+2i+j)(n+3-2i-j)\gamma^{(k)}_{n,i-1,j} \\ + (j+1)(2n+2-j)\gamma^{(k)}_{n,i-1,j+1} + (j+1)(j+2)\gamma^{(k)}_{n,i-1,j+2},\end{aligned} \tag{3.5}$$

where $\gamma^{(k)}_{k,i,j} = \gamma_{k,i,k+1-2i-j}$.

Proof. Follows from Lemma 7 by the same manipulations as the proof of Lemma 3. Note that $\gamma^{(k)}_{k,i,j} = \gamma_{k,i,k+1-2i-j}$ is equivalent to (3.4). □

If we sum up both side of (3.5) for all possible j, then we go back to the recurrence relation for $\gamma_{n,i}$ [7, Remarque 5.3]:

$$\gamma_{n+1,i} = i\gamma_{n,i} + 2(n+3-2i)\gamma_{n,i-1}.$$

Note that in recurrence (3.5) the integer $i(i+j)-(n+1-k)$ may assume negative value. The nonnegativity of coefficients $\gamma^{(k)}_{n,i,j}$ is confirmed by the following lemma based on Theorem 4.

Lemma 9. *Fix a positive integer k. Let $n \geqslant k$. Then, for $i \geqslant 1, j \geqslant 0$ and $2i+j \leqslant n+1$:*

(i) all the coefficients $\gamma^{(k)}_{n,i,j}$ are nonnegative;

(ii) the coefficient $\gamma^{(k)}_{n,i,j}$ vanishes if $i(i+j) < n+1-k$.

We will prove this result by induction on n (for $n \geqslant k$) using recurrence relation (3.5) for the generalized coefficients $\gamma^{(k)}_{n,i,j}$.

Proof. As $\gamma^{(k)}_{k,i,j} = \gamma_{k,i,k+1-2i-j}$, the two statements are true for $n = k$ by Theorem 4. Suppose that this result is true for some n with $n \geqslant k$. We need to show the two statements for $n+1$. It suffices to show statement (ii) in view of recurrence (3.5). We distinguish two cases.

Case 1: If $i(i+j) = n+1-k$, then the inductive hypothesis implies that all $\gamma^{(k)}_{n,i,j-1}$, $\gamma^{(k)}_{n,i-1,j-1}$, $\gamma^{(k)}_{n,i-1,j}$, $\gamma^{(k)}_{n,i-1,j+1}$, $\gamma^{(k)}_{n,i-1,j+1}$, $\gamma^{(k)}_{n,i-1,j+2}$ vanish (except $\gamma^{(k)}_{n,i,j}$ may be positive), since now

$$\max\{i(i+j-1), (i-1)(i+j-2), (i-1)(i+j+1)\} < n+1-k.$$

Thus, $\gamma^{(k)}_{n+1,i,j} = 0$ if $i(i+j) = n+1-k$.

Case 2: If $i(i+j) < n+1-k$, then the inductive hypothesis implies that all $\gamma^{(k)}_{n,i,j-1}$, $\gamma^{(k)}_{n,i-1,j-1}$, $\gamma^{(k)}_{n,i-1,j}$, $\gamma^{(k)}_{n,i-1,j+1}$, $\gamma^{(k)}_{n,i-1,j+1}$ and $\gamma^{(k)}_{n,i-1,j+2}$, including $\gamma^{(k)}_{n,i,j}$, vanish, which forces $\gamma^{(k)}_{n+1,i,j} = 0$.

Thus, the proof is completed by induction. □

3.1 Type B analog

Consider the *Type B Coxeter group* $\mathfrak{B}_n$, whose elements are regarded as signed permutations of $[n]$. The *type B descent* statistic of $\pi \in \mathfrak{B}_n$, denoted $\mathrm{des}_B(\pi)$, is defined as

$$\mathrm{des}_B(\pi) := \#\{i \in [n-1] : \pi_i > \pi_{i+1}\} + \chi(\pi_1 < 0).$$

This type B descent statistic was introduced by Brenti [3]. Gessel [15] noted that the *Type B double Eulerian polynomials* $B_n(s,t) := \sum_{\sigma\in\mathfrak{B}_n} s^{\mathrm{des}_B(\pi^{-1})} t^{\mathrm{des}_B(\pi)}$ has the generating function

$$\sum_{i,j\geqslant 0} \binom{2ij+i+j+n}{n} s^i t^j = \frac{B_n(s,t)}{(1-s)^{n+1}(1-t)^{n+1}}. \tag{3.6}$$

We have the following type B analog of Conjecture 2, which also implies the γ-positivity of $B_n(1,t)$, a result known previously [5, 6].

Theorem 10 (Type B analog of **Conj. 2**)**.** *For $n \geqslant 1$,*

$$B_n(s,t) = \sum_{\substack{i,j\geqslant 0\\ j+2i\leqslant n}} \widetilde{\gamma}_{n,i,j} (st)^i (1+st)^j (s+t)^{n-j-2i},$$

where $\widetilde{\gamma}_{n,i,j}$ are nonnegative integers. Moreover, $\widetilde{\gamma}_{n,i,j}$ is positive if and only if $i, j \geqslant 0$, $2i + j \leqslant n$ and $2i(i + j + 1) + j \geqslant n$.

For instance, the first few expansions of $B_n(s,t)$ are

$$\begin{aligned}
B_1(s,t) &= 1 + st,\\
B_2(s,t) &= (1 + st)^2 + 4st,\\
B_3(s,t) &= (1 + st)^3 + 16st(1 + st) + 4st(s + t),\\
B_4(s,t) &= (1 + st)^4 + 41st(1 + st)^2 + 30st(s + t)(1 + st) + st(s + t)^2 + 80(st)^2.
\end{aligned}$$

We have the following recursion for the type B γ-coefficients $\widetilde{\gamma}_{n,i,j}$.

Lemma 11. *Let $n \geqslant 2$. For all $i, j \geqslant 0$, we have*

$$\begin{aligned}
n\widetilde{\gamma}_{n,i,j} = {} & (2n - j + 2i(n - i - j))\widetilde{\gamma}_{n-1,i,j-1} + (2i(i + j + 1) + j + 1 - n)\widetilde{\gamma}_{n-1,i,j}\\
& + 2(n + 2 - 2i - j)(n + 1 - 2i - j)\widetilde{\gamma}_{n-1,i-1,j-1}\\
& + 2(n + 2i + j)(n + 1 - 2i - j)\widetilde{\gamma}_{n-1,i-1,j}\\
& + (j + 1)(4n - 2j)\widetilde{\gamma}_{n-1,i-1,j+1} + 2(j + 1)(j + 2)\widetilde{\gamma}_{n-1,i-1,j+2}.
\end{aligned} \tag{3.7}$$

Proof. By (3.6) using similar approach as Lemma 3. All the computations are routine and tedious which we omit. □

Proof of Theorem 10. The proof is essentially the same as that of Theorem 4 by induction on n, but using recursion (3.7) for $\widetilde{\gamma}_{n,i,j}$ instead. □

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