Pattern avoidance in partial permutations

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Abstract

Motivated by the concept of partial words, we introduce an analogous concept of partial permutations. A *partial permutation of length n with k holes* is a sequence of symbols $\pi = \pi_1\pi_2\cdots\pi_n$ in which each of the symbols from the set $\{1, 2, \ldots, n-k\}$ appears exactly once, while the remaining k symbols of π are "holes".

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We introduce pattern-avoidance in partial permutations and prove that most of the previous results on Wilf equivalence of permutation patterns can be extended to partial permutations with an arbitrary number of holes. We also show that Baxter permutations of a given length k correspond to a Wilf-type equivalence class with respect to partial permutations with $(k-2)$ holes. Lastly, we enumerate the partial permutations of length n with k holes avoiding a given pattern of length at most four, for each $n \geq k \geq 1$.
Keywords: partial permutation, pattern avoidance, Wilf-equivalence, bijection, generating function, Baxter permutation

1 Introduction

Let A be a nonempty set, which we call an *alphabet*. A *word* over A is a finite sequence of elements of A, and the *length* of the word is the number of elements in the sequence. Assume that $\diamond$ is a special symbol not belonging to A. The symbol $\diamond$ will be called *a hole*. A *partial word* over A is a word over the alphabet $A \cup \{\diamond\}$. In the study of partial words, the holes are usually treated as gaps that may be filled by an arbitrary letter of A. The *length* of a partial word is the number of its symbols, including the holes.

The study of partial words was initiated by Berstel and Boasson [6]. Partial words appear in comparing genes [25]; alignment of two sequences can be viewed as a construction of two partial words that are compatible in the sense defined in [6]. Combinatorial aspects of partial words that have been studied include periods in partial words [6, 30], avoidability/unavoidability of sets of partial words [7, 9], squares in partial words [20], and overlap-freeness [21]. For more see the book by Blanchet-Sadri [8].

Let V be a set of symbols not containing $\diamond$. A *partial permutation of* V is a partial word π such that each symbol of V appears in π exactly once, and all the remaining symbols of π are holes. Let $\mathcal{S}_n^k$ denote the set of all partial permutations of the set $[n-k] = \{1, 2, \ldots, n-k\}$ that have exactly k holes. For example, $\mathcal{S}_3^1$ contains the six partial permutations $12\diamond$, $1\diamond 2$, $21\diamond$, $2\diamond 1$, $\diamond 12$, and $\diamond 21$. Obviously, all elements of $\mathcal{S}_n^k$ have length n, and $|\mathcal{S}_n^k| = \binom{n}{k}(n-k)! = n!/k!$. Note that $\mathcal{S}_n^0$ is the familiar symmetric group $\mathcal{S}_n$. For a set $H \subseteq [n]$ of size k, we let $\mathcal{S}_n^H$ denote the set of partial permutations $\pi_1 \cdots \pi_n \in \mathcal{S}_n^k$ such that $\pi_i = \diamond$ if and only if $i \in H$. We remark that our notion of partial permutations is somewhat reminiscent of the notion of insertion encoding of permutations, introduced by Albert et al. [1]. However, the interpretation of holes in the two settings is different.

In this paper, we extend the classical notion of pattern-avoiding permutations to the more general setting of partial permutations. Let us first recall some definitions related to pattern avoidance in permutations. Let $V = \{v_1, \ldots, v_n\}$ with $v_1 < \cdots < v_n$ be any finite subset of $\mathbb{N}$. The *standardization* of a permutation π on V is the permutation $\mathrm{st}(\pi)$ on $[n]$ obtained from π by replacing the letter v_i with the letter i. As an example, $\mathrm{st}(19452) = 15342$. Given $p \in \mathcal{S}_k$ and $\pi \in \mathcal{S}_n$, an *occurrence* of p in π is a subword $\omega = \pi_{i(1)} \cdots \pi_{i(k)}$ of π such that $\mathrm{st}(\omega) = p$; in this context p is called a *pattern*. If there are no occurrences of p in π we also say that π *avoids* p. Two patterns p and q are called *Wilf-equivalent* if for each n, the number of p-avoiding permutations in $\mathcal{S}_n$ is equal to the number of q-avoiding permutations in $\mathcal{S}_n$.

Let $\pi \in \mathcal{S}_n^k$ be a partial permutation and let $i(1) < \cdots < i(n-k)$ be the indices of the non-hole elements of π. A permutation $\sigma \in \mathcal{S}_n$ is an *extension* of π if

$$\mathrm{st}(\sigma_{i(1)} \cdots \sigma_{i(n-k)}) = \pi_{i(1)} \cdots \pi_{i(n-k)}.$$

For example, the partial permutation 2◇1 has three extensions, namely 312, 321 and 231. In general, the number of extensions of $\pi \in \mathcal{S}_n^k$ is $\binom{n}{k}k! = n!/(n-k)!$.

We say that $\pi \in \mathcal{S}_n^k$ *avoids the pattern* $p \in \mathcal{S}_\ell$ if each extension of π avoids p. For example, $\pi = 32\diamond154$ avoids 1234, but π does not avoid 123: the permutation 325164 is an extension of π and it contains two occurrences of 123. Let $\mathcal{S}_n^k(p)$ be the set of all the partial permutations in $\mathcal{S}_n^k$ that avoid p, and let $s_n^k(p) = |\mathcal{S}_n^k(p)|$. Similarly, if $H \subseteq [n]$ is a set of indices, then $\mathcal{S}_n^H(p)$ is the set of p-avoiding permutations in $\mathcal{S}_n^H$, and $s_n^H(p)$ is its cardinality.

We say that two patterns p and q are *k-Wilf-equivalent* if $s_n^k(p) = s_n^k(q)$ for all n. Notice that 0-Wilf equivalence coincides with the standard notion of Wilf equivalence. We also say that two patterns p and q are *$\star$-Wilf-equivalent* if p and q are k-Wilf-equivalent for all $k \geq 0$. Two patterns p and q are *strongly k-Wilf-equivalent* if $s_n^H(p) = s_n^H(q)$ for each n and for each k-element subset $H \subseteq [n]$. Finally, p and q are *strongly $\star$-Wilf-equivalent* if they are strongly k-Wilf-equivalent for all $k \geq 0$.

We note that although strong k-Wilf equivalence implies k-Wilf equivalence, and strong $\star$-Wilf equivalence implies $\star$-Wilf equivalence, the converse implications are not true. For the smallest example illustrating this, consider the patterns $p = 1342$ and $q = 2431$. A partial permutation avoids p if and only if its reverse avoids q, and thus p and q are $\star$-Wilf-equivalent. However, p and q are not strongly 1-Wilf-equivalent, and hence not strongly $\star$-Wilf-equivalent either. To see this, we fix $H = \{2\}$ and easily check that $s_5^H(p) = 13$ while $s_5^H(q) = 14$.

1.1 Our Results

The main goal of this paper is to establish criteria for k-Wilf equivalence and $\star$-Wilf equivalence of permutation patterns. We are able to show that most pairs of Wilf-equivalent patterns that were discovered so far are in fact $\star$-Wilf-equivalent. The only exception is the pair of patterns $p = 2413$ and $q = 1342$. Although these patterns are known to be Wilf-equivalent [33], they are neither 1-Wilf-equivalent nor 2-Wilf equivalent (see Section 7). Let us remark that 2413 and 1342 are the only two known Wilf-equivalent patterns whose Wilf-equivalence does not follow from the stronger concept of shape-Wilf equivalence. The results of this paper confirm that these two Wilf-equivalent patterns have a special place among the known Wilf-equivalent pairs.

Many of our arguments rely on properties of partial 01-fillings of Ferrers diagrams. These fillings are introduced in Section 2, where we also establish the link between partial fillings and partial permutations. In particular, we introduce the notion of *shape-$\star$-Wilf equivalence*. The shape-$\star$-Wilf equivalence refines the concept of shape-Wilf equivalence, which has been often used as a tool in the study of permutation patterns [3, 4, 33]. We will show that previous results on shape-Wilf equivalence remain valid in the more refined setting of shape-$\star$-Wilf equivalence as well.

Our first main result is Theorem 4.4 in Section 4, which states that a permutation pattern of the form $123\cdots\ell X$ is strongly $\star$-Wilf-equivalent to the pattern $\ell(\ell-1)\cdots 321X$, where $X = x_{\ell+1}x_{\ell+2}\cdots x_m$ is any permutation of $\{\ell+1,\dots,m\}$. This theorem is a strengthening of a result of Backelin, West and Xin [4], who show that patterns of this form are Wilf-equivalent. Our proof is based on a different argument than the original proof of Backelin, West and Xin. The main ingredient of our proof is an involution on a set of fillings of Ferrers diagrams, discovered by Krattenthaler [24]. We adapt this involution to partial fillings and use it to obtain a bijective proof of our result.

Our next main result is Theorem 5.1 in Section 5, which states that for any permutation X of the set $\{4,5,\dots,k\}$, the two patterns $312X$ and $231X$ are strongly $\star$-Wilf-equivalent. This is also a refinement of an earlier result involving Wilf equivalence, due to Stankova and West [34]. As in the previous case, the refined version requires a different proof than the weaker version.

In Section 6, we study the k-Wilf equivalence of patterns whose length is small in terms of k. It is not hard to see that all patterns of length ℓ are k-Wilf equivalent whenever $\ell \le k+1$, because $s_n^k(p) = 0$ for every such p and every $n \ge \ell$. The shortest patterns that exhibit nontrivial behavior with respect to k-Wilf equivalence are the patterns of length $k+2$. For these patterns, we show that k-Wilf equivalence yields a new characterization of Baxter permutations: a pattern p of length $k+2$ is a Baxter permutation if and only if $s_n^k(p) = \binom{n}{k}$. For any non-Baxter permutation q of length $k+2$, $s_n^k(q)$ is strictly smaller than $\binom{n}{k}$ and is in fact a polynomial in n of degree at most $k-1$.

In Section 7, we focus on explicit enumeration of $s_n^k(p)$ for small patterns p. We obtain explicit closed-form formulas for $s_n^k(p)$ for every p of length at most four and every $k \ge 1$.

1.2 A note on monotone patterns

Before we present our main results, let us illustrate the above definitions on the example of the monotone pattern $12\cdots\ell$. Let $\pi \in \mathcal{S}_n^k$, and let $\pi' \in \mathcal{S}_{n-k}$ be the permutation obtained from π by deleting all the holes. Note that π avoids the pattern $12\cdots\ell$ if and only if π' avoids $12\cdots(\ell-k)$. Thus,

$$s_n^k(12\cdots\ell) = \binom{n}{k} s_n^0(12\cdots(\ell-k)), \tag{1}$$

where $\binom{n}{k}$ counts the possibilities of placing k holes. For instance, if $\ell = k+3$ then $s_n^k(12\cdots\ell) = \binom{n}{k} s_n^0(123)$, and it is well known that $s_n^0(123) = C_n$, the n-th Catalan number. For general ℓ, Regev [29] found an asymptotic formula for $s_n^0(12\cdots\ell)$, which can be used to obtain an asymptotic formula for $s_n^k(12\cdots\ell)$ as n tends to infinity.

2 Partial fillings

In this section, we introduce the necessary definitions related to partial fillings of Ferrers diagrams. These notions will later be useful in our proofs of $\star$-Wilf equivalence of patterns.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ be a non-increasing sequence of k nonnegative integers. A *Ferrers diagram with shape* λ is a bottom-justified array D of cells arranged into k columns, such that the j-th column from the left has exactly λ_j cells. Note that our definition of Ferrers diagram is slightly more general than usual, in that we allow columns with no cells. If each column of D has at least one cell, then we call D a *proper Ferrers diagram*. Note that every row of a Ferrers diagram D has nonzero length (while we allow columns of zero height). If all the columns of D have zero height—in other words, D has no rows—then D is called *degenerate*.

For the sake of consistency, we assume throughout this paper that the rows of each diagram and each matrix are numbered from bottom to top, with the bottom row having number 1. Similarly, the columns are numbered from left to right, with column 1 being the leftmost column.

By *cell* (i, j) of a Ferrers diagram D we mean the cell of D that is the intersection of i-th row and j-th column of the diagram. We assume that the cell (i, j) is a unit square whose corners are lattice points with coordinates $(i-1, j-1)$, $(i, j-1)$, $(i-1, j)$ and (i, j). The point $(0, 0)$ is the bottom-left corner of the whole diagram. We say a diagram D *contains* a lattice point (i, j) if either $j = 0$ and the first column of D has height at least i, or $j > 0$ and the j-th column of D has height at least i. A point (i, j) is a *boundary point* of the diagram D if D contains the point (i, j) but does not contain the cell $(i+1, j+1)$ (see Figure 1). Note that a Ferrers diagram with r rows and c columns has $r + c + 1$ boundary points.

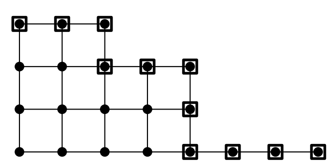

Figure 1: A Ferrers diagram with shape $(3, 3, 2, 2, 0, 0, 0)$. The black dots represent the points. The black dots in squares are the boundary points.

A *01-filling* of a Ferrers diagram assigns to each cell the value 0 or 1. A 01-filling is called a *transversal filling* (or just a *transversal*) if each row and each column has exactly one 1-cell. A 01-filling is *sparse* if each row and each column has at most one 1-cell. A permutation $p = p_1 p_2 \cdots p_\ell \in \mathcal{S}_\ell$ can be represented by a *permutation matrix* which is a 01-matrix of size $\ell \times \ell$, whose cell (i, j) is equal to 1 if and only if $p_j = i$. If there is no risk of confusion, we abuse terminology by identifying a permutation pattern p with the corresponding permutation matrix. Note that a permutation matrix is a transversal of a diagram with square shape.

Let P be permutation matrix of size $n \times n$, and let F be a sparse filling of a Ferrers diagram. We say that F *contains* P if F has a (not necessarily contiguous) square submatrix of size $n \times n$ which induces in F a subfilling equal to P. This notion of containment generalizes usual permutation containment.

We now extend the notion of partial permutations to partial fillings of diagrams. Let D be a Ferrers diagram with k columns. Let H be a subset of the set of columns of D. Let ϕ be a function that assigns to every cell of D one of the three symbols 0, 1 and $\diamond$, such that every cell in a column belonging to H is filled with $\diamond$, while every cell in a column not belonging to H is filled with 0 or 1. The pair $F = (\phi, H)$, will be referred to as *a partial 01-filling (or a partial filling) of the diagram* D. See Figure 2. The columns from the set H will be called *the* $\diamond$*-columns* of F, while the remaining columns will be called *the standard columns*. Observe that if the diagram D has columns of height zero, then ϕ itself is not sufficient to determine the filling F, because it does not allow us to determine whether the zero-height columns are $\diamond$-columns or standard columns. For our purposes, it is necessary to distinguish between partial fillings that differ only by the status of their zero-height columns.

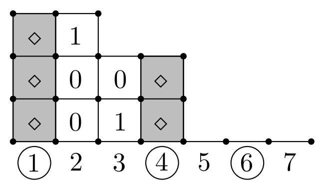

Figure 2: A partial filling with $\diamond$-columns 1, 4 and 6.

We say that a partial 01-filling is a *partial transversal filling* (or simply a *partial transversal*) if every row and every standard column has exactly one 1-cell. We say that a partial 01-filling is *sparse* if every row and every standard column has at most one 1-cell. A *partial 01-matrix* is a partial filling of a (possibly degenerate) rectangular diagram. In this paper, we only deal with transversal and sparse partial fillings.

There is a natural correspondence between partial permutations and transversal partial 01-matrices. Let $\pi \in \mathcal{S}_n^k$ be a partial permutation. A *partial permutation matrix representing* π is a partial 01-matrix P with $n-k$ rows and n columns, with the following properties:

- If $\pi_j = \diamond$, then the j-th column of P is a $\diamond$-column.
- If π_j is equal to a number i, then the j-th column is a standard column. Also, the cell in column j and row i is filled with 1, and the remaining cells in column j are filled with 0's.

If there is no risk of confusion, we will make no distinction between a partial permutation and the corresponding partial permutation matrix.

To define pattern-avoidance for partial fillings, we first introduce the concept of substitution into a $\diamond$-column, which is analogous to substituting a number for a hole in a partial permutation. The idea is to insert a new row with a 1-cell in the $\diamond$-column; this increases the height of the diagram by one. Let us now describe the substitution formally.

Let F be a partial filling of a Ferrers diagram with m columns. Assume that the j-th column of F is a $\diamond$-column. Let h be the height of the j-th column. A *substitution* into the j-th column is an operation consisting of the following steps:

1. Choose a number i, with $1 \leq i \leq h+1$.

2. Insert a new row into the diagram, between rows $i-1$ and i. The newly inserted row must not be longer than the $(i-1)$-th row, and it must not be shorter than the i-th row, so that the new diagram is still a Ferrers diagram. If $i = 1$, we also assume that the new row has length m, so that the number of columns does not increase during the substitution.

3. Fill all the cells in column j with the symbol 0, except for the cell in the newly inserted row, which is filled with 1. Remove column j from the set of $\diamond$-columns.

4. Fill all the remaining cells of the new row with 0 if they belong to a standard column, and with $\diamond$ if they belong to a $\diamond$-column.

Figure 3 illustrates an example of substitution.

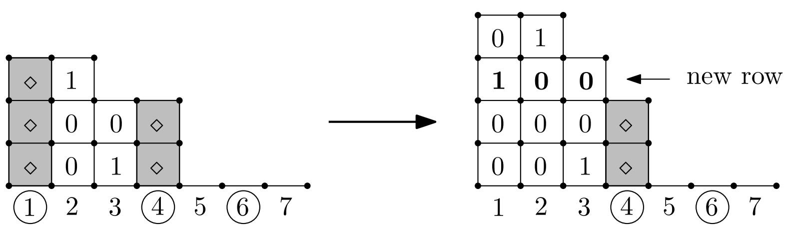

Figure 3: A substitution into the first column of a partial filling, involving an insertion of a new row between the second and third rows of the original partial filling.

Note that a substitution into a partial filling increases the number of rows by 1. A substitution into a transversal (resp. sparse) partial filling produces a new transversal (resp. sparse) partial filling. A partial filling F with k $\diamond$-columns can be transformed into a (non-partial) filling F' by a sequence of k substitutions; we then say that F' is an *extension* of F.

Let P be a permutation matrix. We say that a partial filling F of a Ferrers diagram *avoids* P if every extension of F avoids P. Note that a partial permutation $\pi \in S_k^n$ avoids a permutation p, if and only if the partial permutation matrix representing π avoids the permutation matrix representing p.

We remark that a related, but different, notion of avoidance has been studied by Linusson [26]: he defines that a 01 matrix is *extendably τ-avoiding* if it can be the upper left corner of a τ-avoiding permutation matrix.

3 A generalization of a Wilf-equivalence by Babson and West

We say that two permutation matrices P and Q are *shape-$\star$-Wilf-equivalent*, if for every Ferrers diagram D there is a bijection between P-avoiding and Q-avoiding partial transversals of D that preserves the set of $\diamond$-columns. Observe that if two permutations are shape-$\star$-Wilf-equivalent, then they are also strongly $\star$-Wilf-equivalent, because a partial permutation matrix is a special case of a partial filling of a Ferrers diagram.

The notion of shape-$\star$-Wilf-equivalence is motivated by the following proposition, which extends an analogous result of Babson and West [3] for shape-Wilf-equivalence of non-partial permutations.

Proposition 3.1. *Let P and Q be shape-$\star$-Wilf-equivalent permutations, let X be an arbitrary permutation. Then the two permutations $\left(\begin{smallmatrix} 0 & X \\ P & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & X \\ Q & 0 \end{smallmatrix}\right)$ are strongly $\star$-Wilf-equivalent.*

Let us remark that Proposition 3.1 can in fact be stated and proven in the following alternative form: if P and Q are shape-$\star$-Wilf-equivalent patterns and X is any pattern, then $\left(\begin{smallmatrix} 0 & X \\ Q & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & X \\ P & 0 \end{smallmatrix}\right)$ are shape-$\star$-Wilf-equivalent as well. While this alternative statement appears stronger, it cannot be used to obtain any new pairs of strongly $\star$-Wilf-equivalent patterns. Since strong $\star$-Wilf equivalence is the main focus of this paper, we have chosen to state the proposition in the simpler form, to make the proof shorter. The stronger statement can be proven by an obvious modification of the argument.

Our proof of Proposition 3.1 is based on the same idea as the original argument of Babson and West [3]. Before we state the proof, we need some preparation. Let M be a partial matrix with r rows and c columns. Let i and j be numbers satisfying $0 \le i \le r$ and $0 \le j \le c$. Let $M(>i, >j)$ be the submatrix of M formed by the cells (i', j') satisfying $i' > i$ and $j' > j$. In other words, $M(>i, >j)$ is formed by the cells to the right and above the point (i, j). The matrix $M(>r, >j)$ is assumed to be the degenerate matrix with 0 rows and $c - j$ columns, while $M(>i, >c)$ is assumed to be the empty matrix for any value of i. When the matrix $M(>i, >j)$ intersects a $\diamond$-column of M, we assume that the column is also a $\diamond$-column of $M(>i, >j)$, and similarly for standard columns.

We will also use the analogous notation $M(\le i, \le j)$ to denote the submatrix of M formed by the cells to the left and below the point (i, j).

Note that if M is a partial permutation matrix, then $M(>i, >j)$ and $M(\le i, \le j)$ are sparse partial matrices.

Let X be any nonempty permutation matrix, and M be a partial permutation matrix. We say that a point (i, j) of M is *dominated by X in M* if the partial matrix $M(>i, >j)$ contains X. Similarly, we say that a cell of M is dominated by X, if the top-right corner of the cell is dominated by X. Note that if a point (i, j) is dominated by X in M, then all the cells and points in $M(\le i, \le j)$ are dominated by X as well. In particular, the points dominated by X form a (not necessarily proper) Ferrers diagram.

Let $k \equiv k(M) \ge 0$ be the largest integer such that the point $(0, k)$ is dominated by X. If no such integer exists, set $k = 0$. Observe that all the cells of M dominated by X

appear in the leftmost k columns of M. Let $M(X)$ be the partial subfilling of M induced by the points dominated by X; formally $M(X)$ is defined as follows:

- $M(X)$ has k columns, some of which might have height zero,
- the cells of $M(X)$ are exactly the cells of M dominated by X,
- a column j of $M(X)$ is a $\diamond$-column, if and only if j is a $\diamond$-column of M.

Our proof of Proposition 3.1 is based on the next lemma.

Lemma 3.2. *Let M be a partial permutation matrix, and let P and X be permutation matrices. Then M contains $\left(\begin{smallmatrix} 0 & X \\ P & 0 \end{smallmatrix}\right)$ if and only if $M(X)$ contains P.*

Proof. Assume that M contains $\left(\begin{smallmatrix} 0 & X \\ P & 0 \end{smallmatrix}\right)$. It is easy to see that M must then contain a point (i,j) such that the matrix $M(>i,>j)$ contains X while the matrix $M(\leq i, \leq j)$ contains P. By definition, the point (i,j) is dominated by X in M, and hence all the points of $M(\leq i, \leq j)$ are dominated by X as well. Thus, $M(\leq i, \leq j)$ is a (possibly degenerate) submatrix of $M(X)$, which implies that $M(X)$ contains P.

The converse implication is proved by an analogous argument. □

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. Let P and Q be two shape-$\star$-Wilf-equivalent matrices, and let f be the bijection that maps P-avoiding partial transversals to Q-avoiding partial transversals of the same diagram and with the same $\diamond$-columns. Let M be a partial permutation matrix avoiding $\left(\begin{smallmatrix} 0 & X \\ P & 0 \end{smallmatrix}\right)$.

By Lemma 3.2, $M(X)$ is a sparse partial filling avoiding P. Let F denote the partial filling $M(X)$. Consider the transversal partial filling F^- obtained from F by removing all the rows and all the standard columns that contain no 1-cell. Clearly F^- is a P-avoiding partial transversal. Use the bijection f to map the partial filling F^- to a Q-avoiding partial transversal G^- of the same shape as F^-. By reinserting the zero rows and zero standard columns into G^-, we obtain a sparse Q-avoiding filling G of the same shape as F. Let us transform the partial matrix M into a partial matrix N by replacing the cells of $M(X)$ with the cells of G, while the values of all the remaining cells of M remain the same.

We claim that the matrix N avoids $\left(\begin{smallmatrix} 0 & X \\ Q & 0 \end{smallmatrix}\right)$. By Lemma 3.2, this is equivalent to claiming that $N(X)$ avoids Q. We will in fact show that $N(X)$ is exactly the filling G. To show this, it is enough to show, for any point (i,j), that $M(X)$ contains (i,j) if and only if $N(X)$ contains (i,j). This will imply that $M(X)$ and $N(X)$ have the same shape, and hence $G = N(X)$.

Let (i,j) be a point of M not belonging to $M(X)$. Since (i,j) is not in $M(X)$, we see that $M(>i,>j)$ is the same matrix as $N(>i,>j)$, and this means that (i,j) is not dominated by X in N, hence (i,j) is not in $N(X)$.

Now assume that (i,j) is a point of $M(X)$. Let (i',j') be a boundary point of $M(X)$ such that $i' \geq i$ and $j' \geq j$. Then the matrix $M(>i',>j')$ is equal to the matrix

$N(>i', >j')$, showing that (i', j') belongs to $N(X)$, and hence (i, j) belongs to $N(X)$ as well. We conclude that $N(X)$ and $M(X)$ have the same shape. This means that $N(X)$ avoids Q, and hence N avoids $\left(\begin{smallmatrix} 0 & X \\ Q & 0 \end{smallmatrix}\right)$.

Since we have shown that $M(X)$ and $N(X)$ have the same shape, it is also easy to see that the above-described transformation $M \mapsto N$ can be inverted, showing that the transformation is a bijection between partial permutation matrices avoiding $\left(\begin{smallmatrix} 0 & X \\ P & 0 \end{smallmatrix}\right)$ and those avoiding $\left(\begin{smallmatrix} 0 & X \\ Q & 0 \end{smallmatrix}\right)$. The bijection clearly preserves the position of $\diamond$-columns, and shows that $\left(\begin{smallmatrix} 0 & X \\ P & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & X \\ Q & 0 \end{smallmatrix}\right)$ are strongly $\star$-Wilf equivalent. □

4 Strong $\star$-Wilf-equivalence of $12 \cdots \ell X$ and $\ell \cdots 21X$

We will use Proposition 3.1 as the main tool to prove strong $\star$-Wilf equivalence. To apply the proposition, we need to find pairs of shape-$\star$-Wilf-equivalent patterns. A family of such pairs is provided by the next proposition, which extends previous results of Backelin, West and Xin [4].

Proposition 4.1. *Let $I_\ell = 12 \cdots \ell$ be the identity permutation of order ℓ, and let $J_\ell = \ell(\ell-1) \cdots 21$ be the anti-identity permutation of order ℓ. The permutations I_ℓ and J_ℓ are shape-$\star$-Wilf-equivalent.*

Before stating the proof, we introduce some notation and terminology. Let F be a sparse partial filling of a Ferrers diagram, and let (i, j) be a boundary point of F. Let $h(F, j)$ denote the number of $\diamond$-columns among the first j columns of F. Let $I(F, i, j)$ denote the largest integer ℓ such that the partial matrix $F(\leq i, \leq j)$ contains I_ℓ. Similarly, let $J(F, i, j)$ denote the largest ℓ such that $F(\leq i, \leq j)$ contains J_ℓ.

We let F_0 denote the (non-partial) sparse filling obtained by replacing all the symbols $\diamond$ in F by zeros.

Let us state without proof the following simple observation.

Observation 4.2. Let F be a sparse partial filling.

1. F contains a permutation matrix P if and only if F has a boundary point (i, j) such that $F(\leq i, \leq j)$ contains P.

2. For any boundary point (i, j), we have $I(F, i, j) = h(F, j) + I(F_0, i, j)$ and $J(F, i, j) = h(F, j) + J(F_0, i, j)$.

The key to the proof of Proposition 4.1 is the following result, which is a direct consequence of more general results of Krattenthaler [24, Theorems 1–3] obtained using the theory of growth diagrams.

Fact 4.3. *Let D be a Ferrers diagram. There is a bijective mapping κ from the set of all (non-partial) sparse fillings of D onto itself, with the following properties.*

1. *For any boundary point (i, j) of D, and for any sparse filling F, we have $I(F, i, j) = J(\kappa(F), i, j)$ and $J(F, i, j) = I(\kappa(F), i, j)$.*

2. *The mapping κ preserves the number of 1-cells in each row and column. In other words, if a row (or column) of a sparse filling F has no 1-cell, then the same row (or column) of $\kappa(F)$ has no 1-cell either.*

In Krattenthaler's paper, the results are stated in terms of proper Ferrers diagrams. However, the bijection obviously extends to Ferrers diagrams with zero-height columns as well. This is because adding zero-height columns to a (non-partial) filling does not affect pattern containment.

From the previous theorem, we easily obtain the proof of the main proposition in this section.

Proof of Proposition 4.1. Let D be a Ferrers diagram. Let F be an I_ℓ-avoiding partial transversal of D. Let F_0 be the sparse filling obtained by replacing all the $\diamond$ symbols of F by zeros. Define $G_0 = \kappa(F_0)$, where κ is the bijection from Fact 4.3. Note that all the $\diamond$-columns of F are filled with zeros both in F_0 and G_0. Let G be the sparse partial filling obtained from G_0 by replacing zeros with $\diamond$ in all such columns. Then G is a sparse partial filling with the same set of $\diamond$-columns as F.

We see that for any boundary point (i, j) of the diagram D, $h(F, j) = h(G, j)$. By the properties of κ, we further obtain $I(F_0, i, j) = J(G_0, i, j)$. In view of Observation 4.2, this implies that G is a J_ℓ-avoiding filling. It is clear that this construction can be inverted, thus giving the required bijection between I_ℓ-avoiding and J_ℓ-avoiding transversal partial fillings of D. □

Combining Proposition 3.1 with Proposition 4.1, we get directly the main result of this section.

Theorem 4.4. *For any $\ell \leq m$, and for any permutation X of $\{\ell+1, \dots, m\}$, the permutation pattern $123\cdots(\ell-1)\ell X$ is strongly $\star$-Wilf-equivalent to the pattern $\ell(\ell-1)\cdots 21X$.*

Notice that Theorem 4.4 implies, among other things, that all the patterns of size three are strongly $\star$-Wilf-equivalent.

5 Strong $\star$-Wilf-equivalence of $312X$ and $231X$

We will now focus on the two patterns 312 and 231. The main result of this section is the following theorem.

Theorem 5.1. *The patterns 312 and 231 are shape-$\star$-Wilf-equivalent. By Proposition 3.1, this means that for any permutation X of the set $\{4, 5, \dots, m\}$, the two permutations $312X$ and $231X$ are strongly $\star$-Wilf-equivalent.*

Theorem 5.1 generalizes a result of Stankova and West [34], who have shown that 312 and 231 are shape-Wilf equivalent. The original proof of Stankova and West [34] is rather complicated, and does not seem to admit a straightforward generalization to the setting of shape-$\star$-Wilf-equivalence. Our proof of Theorem 5.1 is different from the argument of

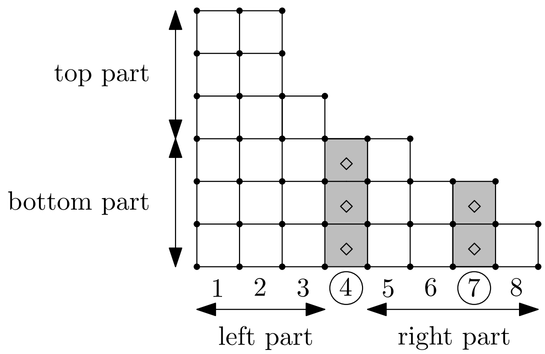

Figure 4: An example of a Ferrers diagram with two $\diamond$-columns. The left, right, top, and bottom parts are shown.

Stankova and West, and it is based on a bijection of Jelínek [22], obtained in the context of pattern-avoiding ordered matchings.

Let us begin by giving a description of 312-avoiding and 231-avoiding partial transversals. We first introduce some terminology. Let D be a Ferrers diagram with a prescribed set of $\diamond$-columns. If j is the index of the leftmost $\diamond$-column of D, we say that the columns $1, 2, \dots, j-1$ form *the left part of* D, and the columns to the right of column j form *the right part of* D. We also say that the rows that intersect column j form the *bottom part of* D and the remaining rows form *the top part of* D. See Figure 4.

If D has no $\diamond$-column, then the left part and the top part is the whole diagram D, while the right part and the bottom part are empty.

The intersection of the left part and the top part of D will be referred to as *the top-left part of* D. The top-right, bottom-left and bottom-right parts are defined analogously. Note that the top-right part contains no cells of D, the top-left and bottom-right parts form a Ferrers subdiagram of D, and the bottom-left part is a rectangle.

Observation 5.2. A partial transversal F of a Ferrers diagram avoids the pattern 312 if and only if it satisfies the following conditions:

(C1) F has at most two $\diamond$-columns.

(C2) If F has at least three columns, then at most one $\diamond$-column of F has nonzero height.

(C3) Let $i < i'$ be a pair of rows, let $j < j'$ be a pair of columns. If the row i' intersects column j' inside F, and if the 2×2 submatrix of F induced by rows i, i' and columns j, j' is equal to the matrix $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, then either the two columns j, j' both belong to the left part, or they both belong to the right part (in other words, the configuration depicted in Figure 5 is forbidden).

(C4) The subfilling induced by the left part of F avoids 312.

(C5) The subfilling induced by the right part of F avoids 12.

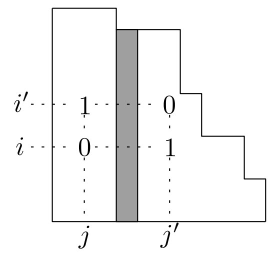

Figure 5: The configuration forbidden by condition (C3) of Observation 5.2. The column j is in the left part of the diagram, while j' is in the right part.

(C6) The subfilling induced by bottom-left part of F avoids 21.

Proof. It is easy to see that if any of the six conditions fails, then F contains the pattern 312.

To prove the converse, assume that F has an occurrence of the pattern 312 that intersects three columns $j < j' < j''$. Choose the occurrence of 312 in such a way that among the three columns j, j' and j'', there are as many $\diamond$-columns as possible.

If all the three columns j, j', j'' are $\diamond$-columns, (C1) fails. If two of the three columns are $\diamond$-columns, (C2) fails. If j is a $\diamond$-column and j' and j'' are standard, (C5) fails.

Assume j and j'' are standard columns and j' is a $\diamond$-column. If j' is the leftmost $\diamond$-column, (C3) fails, otherwise (C2) fails. Assume j'' is a $\diamond$-column and j and j' are standard. If j'' is the leftmost $\diamond$-column, (C6) fails, otherwise (C2) fails.

Assume all the three columns are standard. Let $i < i' < i''$ be the three rows that are intersected by the chosen occurrence of 312. If there is a $\diamond$-column that intersects all the three rows i, i', i'', we may find an occurrence of 312 that uses this $\diamond$-column, contradicting our choice of j, j' and j''. On the other hand, if no $\diamond$-column intersects the three rows, then the whole submatrix inducing 312 is in the left part and (C4) fails. $\square$

Next, we state a similar description of 231-avoiding partial transversals.

Observation 5.3. A partial transversal F of a Ferrers diagram avoids the pattern 231 if and only if it satisfies the following conditions (the first three conditions are the same as the corresponding three conditions of Observation 5.2):

(C1') F has at most two $\diamond$-columns.

(C2') If F has at least three columns, then at most one $\diamond$-column of F has nonzero height.

(C3') Let $i < i'$ be a pair of rows, let $j < j'$ be a pair of columns. If the row i' intersects column j' inside F, and if the 2×2 submatrix of F induced by rows i, i' and columns j, j' is equal to the matrix $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, then either the two columns j, j' both belong to the left part, or they both belong to the right part.

(C4') The subfilling induced by the left part of F avoids 231.

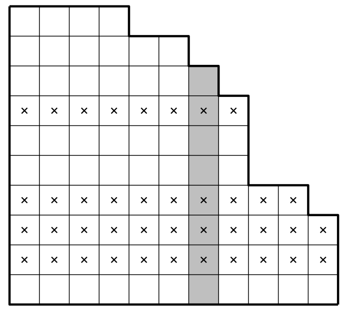

Figure 6: A Ferrers diagram with one $\diamond$-column, indicated in gray. The rows with crosses are the rightist rows of D.

(C5') The subfilling induced by the right part of F avoids 21.

(C6') The subfilling induced by bottom-left part of F avoids 12.

The proof of Observation 5.3 is analogous to the proof of Observation 5.2, and we omit it.

In the next part of our argument, we will look in more detail at fillings satisfying some of the Conditions (C1) to (C6), or some of the Conditions (C1') to (C6').

For later reference, we state explicitly the following easy facts about transversal fillings of Ferrers diagrams that avoid permutation matrices of size 2 (see, e.g., [3]).

Fact 5.4. *Assume that D is a Ferrers diagram that has at least one (non-partial) transversal. The following holds.*

- *The diagram D has exactly one 12-avoiding transversal. To construct this transversal, take the rows of D in top-to-bottom order, and in each row i, insert a 1-cell into the leftmost column that has no 1-cell in any of the rows above row i.*
- *The diagram D has exactly one 21-avoiding transversal. To construct this transversal, take the rows of D in top-to-bottom order, and in each row i, insert a 1-cell into the rightmost column that has no 1-cell in any of the rows above row i.*

Our next goal is to give a more convenient description of the partial fillings that satisfy Conditions (C1), (C2) and (C3) (which are equal to (C1'), (C2') and (C3'), respectively). Let D be a Ferrers diagram with a prescribed set of $\diamond$-columns, and with k rows in its bottom part. We will distinguish two types of rows of D, which we refer to as *rightist rows* and *leftist rows* (see Figure 6). The rightist rows are defined inductively as follows. None of the rows in the top part is rightist. The k-th row (i.e., the highest row in the bottom part) is rightist if and only if it has at least one cell in the right part of D. For any $i < k$, the i-th row is rightist if and only if the number of cells in the i-th row belonging to the right part of D is greater than the number of rightist rows that are above row i. A row is leftist if it is not rightist.

The distinction between leftist and rightist rows is motivated by the following lemma.

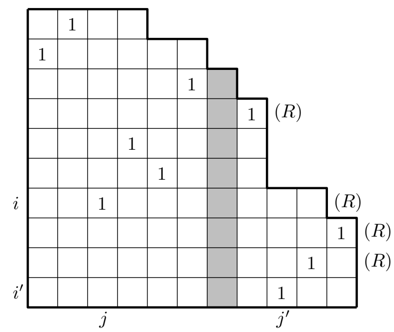

Figure 7: An example of a partial transversal violating condition (c) of Lemma 5.5. Rightist rows are marked by (R).

Lemma 5.5. *Let D be a Ferrers diagram, and let F be a partial transversal of D that satisfies (C1) and (C2). The following statements are equivalent.*

(a) F satisfies (C3).

(b) All the 1-cells in the leftist rows appear in the left part of F.

(c) All the 1-cells in the rightist rows appear in the right part of F.

Proof. Let us first argue that the statements (b) and (c) are equivalent. To see this, notice first that in all the partial transversals of D, the number of 1-cells in the right part is the same, since each non-degenerate column in the right part has exactly one 1-cell. Consequently, all the partial transversals of D also have the same number of 1-cells in the bottom-left part, because the number of 1-cells in the bottom-left part is equal to the number of bottom rows minus the number of non-degenerate right columns.

We claim that the number of rightist rows is equal to the number of non-degenerate columns in the right part. To see this, consider the (unique) partial transversal F_{21} of D in which no two standard columns contain the pattern 21. The characterization of Fact 5.4 easily implies that in F_{21}, a row has a 1-cell in the right part, if and only if it is a rightist row. Thus, in the partial filling F_{21}, and hence in any other partial transversal of D, the number of rightist rows is equal to the number of 1-cells in the right part of D, which is equal to the number of non-degenerate right columns.

Thus, if in a partial transversal F there is a leftist row that has a 1-cell in the right part of D, there must also be a rightist row with a 1-cell in the left part of D, and vice versa. In other words, conditions (b) and (c) are indeed equivalent for any partial transversal F.

Assume now that F is a partial transversal that satisfies (a). We claim that F satisfies (c) as well. For contradiction, assume that there is a rightist row i that contains a 1-cell in the left part of F. Choose i as large as possible. Let j be the column containing the 1-cell in row i. See Figure 7.

Since i is a rightist row, it follows that the number of cells in the right part of i is greater than the number of rightist rows above i. We may thus find a column j' in the right part of D that intersects row i and whose 1-cell does not belong to any of the rightist rows above row i. Let i' be the row that contains the 1-cell in column j'. If $i' < i$, then the two rows i, i' and the two columns j, j' induce the pattern that was forbidden by (C3), which contradicts statement (a).

Thus, we see that $i' > i$. By the choice of j', this implies that i' is a leftist row. Furthermore, by the choice of i, we know that all the rightist rows above i, and hence all the rightist rows above i', have a 1-cell in the right part. Since row i' has a 1-cell in the right part as well, it means that the number of cells in the right part of row i' is greater than the number of rightist rows above row i'. This contradicts the fact that i' is a leftist row. This contradiction proves that (a) implies (c).

It remains to show that statement (c) implies statement (a). Assume F is a partial transversal that satisfies (c), and hence also (b). For contradiction, assume that F contains the pattern forbidden by statement (a). Assume that the forbidden pattern is induced by a pair of rows $i < i'$ and a pair of columns $j < j'$, where the column j' is in the right part and the column j in the left part, and the two cells (i', j) and (i, j') are 1-cells, as in Figure 5.

By statement (c), the row i' must be leftist, since it has a 1-cell in the left part. However, the number of cells in the right part of row i' must be greater than the number of rightist rows above row i', because all the rightist rows above row i' have 1-cells in distinct right columns intersecting row i', and all these columns must be different from column j', whose 1-cell is in row i below row i'. This contradicts the fact that i' is a leftist row, and completes the proof of the lemma. □

Lemma 5.5, together with Observations 5.2 and 5.3, shows that in any partial transversal avoiding 312 or 231, each 1-cell is either in the intersection of a rightist row with a right column, or the intersection of a leftist row and a left column.

The next lemma provides the main ingredient of our proof of Theorem 5.1.

Lemma 5.6 (Key Lemma)**.** *Let $k \geq 1$ be an integer, and let D be a proper Ferrers diagram with the property that the bottom k rows of D all have the same length. Let $\mathcal{F}^{(k)}(D, 312, 21)$ be the set of all (non-partial) transversals of D that avoid 312 and have the additional property that their bottom k rows avoid 21. Let $\mathcal{F}^{(k)}(D, 231, 12)$ be the set of all (non-partial) transversals of D that avoid 231 and have the additional property that their bottom k rows avoid 12. Then $|\mathcal{F}^{(k)}(D, 312, 21)| = |\mathcal{F}^{(k)}(D, 231, 12)|$.*

Before we prove the Key Lemma, let us explain how it implies Theorem 5.1.

Proof of Theorem 5.1 from Lemma 5.6. Let D be a Ferrers diagram with a prescribed set of $\diamond$-columns. Assume that D has at least one partial transversal. Our goal is to show that the number of 312-avoiding partial transversals of D is equal to the number of its 231-avoiding partial transversals.

Assume that D satisfies conditions (C1) and (C2), otherwise it has no 312-avoiding or 231-avoiding partial transversal. Let k be the number of leftist rows in the bottom part of D. Let D_L be the subdiagram of D formed by the cells that are intersections of leftist rows and left columns of D, and let D_R be the subdiagram formed by the intersections of rightist rows and right columns. Notice that neither D_L nor D_R have any $\diamond$-columns, and the k bottom rows of D_L have the same length.

By Lemma 5.5, in any partial transversal F of D that satisfies (C3), each 1-cell of F is either in D_L or in D_R. Thus, F can be decomposed uniquely into two transversals F_L and F_R, induced by D_L and D_R, respectively. Conversely, if F_L and F_R are any transversals of D_L and D_R, then the two fillings give rise to a unique partial transversal F of D satisfying (C3).

Let F be a partial transversal of D that satisfies condition (C3). Note that F satisfies condition (C4) of Observation 5.2 if and only if F_L avoids 312, and F satisfies (C6) if and only if F_L avoids 21 in its bottom k rows. Thus, F satisfies (C4) and (C6) if and only if $F_L \in \mathcal{F}^{(k)}(D_L, 312, 21)$. Observe also that F satisfies (C5) if and only if F_R avoids 12. By Fact 5.4, this determines F_R uniquely.

By combining the above remarks, we conclude that a partial transversal F of the diagram D avoids 312 if and only if F_L belongs to the set $\mathcal{F}^{(k)}(D_L, 312, 21)$ and F_R is the unique transversal filling of D_R that avoids 12. By analogous reasoning, a partial transversal F' of D avoids 231, if and only if its subfilling F'_L induced by D_L belongs to $\mathcal{F}^{(k)}(D_L, 231, 12)$ and the subfilling F'_R induced by D_R is the unique transversal of D_R avoiding 21.

The Key Lemma asserts that $\mathcal{F}^{(k)}(D_L, 312, 21)$ and $\mathcal{F}^{(k)}(D_L, 231, 12)$ have the same cardinality, which implies that the number of 312-avoiding partial transversals of D is equal to the number of its 231-avoiding partial transversals. $\square$

The rest of this section is devoted to the proof of the Key Lemma.

Although the proof of the Key Lemma could in principle be presented in the language of fillings and diagrams, it is more convenient and intuitive to state the proof in the (equivalent) language of matchings. This will allow us to apply previously known results on pattern-avoiding matchings in our proof.

Let us now introduce the relevant terminology. A *matching of order* n is a graph $M = (V, E)$ on the vertex set $V = \{1, 2, \ldots, 2n\}$, with the property that every vertex is incident to exactly one edge. We will assume that the vertices of matchings are represented as points on a horizontal line, ordered from left to right in increasing order, and that edges are represented as circular arcs connecting the two corresponding endpoints and drawn above the line containing the vertices. If e is an edge connecting vertices i and j, with $i < j$, we say that i is the *left-vertex* and j is the *right-vertex* of e. Clearly, a matching of order n has n left-vertices and n right-vertices. Let $\mathrm{L}(M)$ denote the set of left-vertices of a matching M.

If M is a matching of order n, we define the *reversal* of M, denoted by $\overline{M}$, to be the matching on the same vertex set as M, such that $\{i, j\}$ is an edge of $\overline{M}$ if and only if $\{2n - j + 1, 2n - i + 1\}$ is an edge of M. Intuitively, reversing corresponds to flipping the matching along a vertical axis.

Let $e = ij$ and $e' = i'j'$ be two edges of a matching M, with $i < j$ and $i' < j'$. If $i < i' < j < j'$ we say that e *crosses* e' *from the left* and e' *crosses* e *from the right*. If $i < i' < j' < j$, we say that e' *is nested below* e. Moreover, if k is a vertex such that

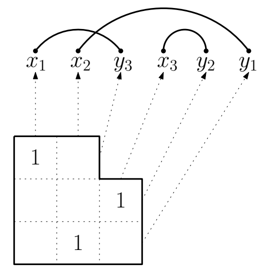

Figure 8: The bijection μ between transversals of Ferrers diagrams and matchings. The dotted arrows show the correspondence between rows and columns of the diagram and vertices of the matching.

$i < k < j$, we say that k is *nested below* the edge $e = ij$, or that $e = ij$ *covers* the vertex k.

A set of k edges of a matching is said to form a *k-crossing* if each two edges in the set cross each other, and it is said to form a *k-nesting* if each two of its edges are nested.

If $M = (V, E)$ is a matching of order n and $M' = (V', E')$ a matching of order n', we say that M *contains* M' if there is an edge-preserving increasing injection from V' to V. In other words, M contains M' if there is a function $f\colon V' \to V$ such that for each $u, v \in V'$, if $u < v$ then $f(u) < f(v)$ and if uv is an edge of M' then $f(u)f(v)$ is an edge of M. If M does not contain M', we say that M *avoids* M'. More generally, if $\mathcal{F}$ is a set of matchings, we say that M avoids $\mathcal{F}$ if M avoids all the matchings in $\mathcal{F}$.

Let $\mathcal{M}_n$ denote the set of all matchings of order n. For a set of matchings $\mathcal{F}$ and for a set of integers $X \subseteq [2n]$, define the following sets of matchings:

$$\mathcal{M}_n(X) = \{M \in \mathcal{M}_n;\ \mathrm{L}(M) = X\}$$
$$\mathcal{M}_n(X, \mathcal{F}) = \{M \in \mathcal{M}_n(X);\ M \text{ avoids } \mathcal{F}\}$$

If the set $\mathcal{F}$ contains a single matching F, we will write $\mathcal{M}_n(X, F)$ instead of $\mathcal{M}_n(X, \{F\})$.

De Mier [16] has pointed out a one-to-one correspondence between transversals of (proper) Ferrers diagrams with n rows and n columns and matchings of order n. This correspondence allows to translate results on pattern-avoiding transversals of Ferrers diagrams to equivalent results on pattern-avoiding matchings. We describe the correspondence here, and state its main properties.

Let F be a transversal of a proper Ferrers diagram D. Let n be the number of rows (and hence also the number of columns) of D. We encode F into a matching $\mu(F) \in \mathcal{M}_n$ defined as follows. First, we partition the vertex set $[2n]$ into two disjoint sets $X(D) = \{x_1 < x_2 < \cdots < x_n\}$ and $Y(D) = \{y_1 > y_2 > \cdots > y_n\}$, with the property that $x_j < y_i$ if and only if the j-th column of D intersects the i-th row of D (note that the elements of Y are indexed in decreasing order). The diagram D determines $X(D)$ and $Y(D)$ uniquely. Let $\mu(F)$ be the matching whose edge-set is the set

$$E = \{x_j y_i;\ F \text{ has a 1-cell in column } j \text{ and row } i\}.$$

Figure 9: The matching M_{312}, corresponding to the permutation pattern 312 (left), and the matching M_{231}, corresponding to the permutation pattern 231 (right).

Figure 8 shows an example of this correspondence.

We state, without proof, several basic properties of μ (see [16]).

Fact 5.7. *The mapping μ has the following properties.*

- *The mapping μ is a bijection between transversals of Ferrers diagrams and matchings, with fillings of the same diagram corresponding to matchings with the same left-vertices. If F is a transversal of a proper Ferrers diagram D, then $\mu(F)$ is a matching whose left-vertices are precisely the vertices from the set $X(D)$. Conversely, for any matching M there is a unique proper Ferrers diagram D such that $X(D)$ is the set of left-vertices of M, and a unique transversal F of D satisfying $\mu(F) = M$.*
- *F is a permutation matrix of order n (i.e., a filling of an $n \times n$ square diagram) if and only if $\mu(F)$ is a matching with $L(M) = \{1, 2, \ldots, n\}$.*
- *Assume that F' is a permutation matrix. A filling F avoids the pattern F' if and only if the matching $\mu(F)$ avoids the matching $\mu(F')$.*
- *D is a proper Ferrers diagram whose k bottom rows have the same length, if and only if $Y(D)$ contains the k numbers $\{2n, 2n-1, \ldots, 2n-k+1\}$. In such case, in any matching representing a transversal of D, all the k rightmost vertices are right-vertices.*

In the rest of this section, we will say that a matching M *corresponds to* a filling F, if $M = \mu(F)$. We will also say that M corresponds to a permutation p if it corresponds to the permutation matrix of p. Specifically, we let M_{312} be the matching corresponding to the permutation 312, and let M_{231} be the matching corresponding to the permutation 231 (see Fig. 9).

Let D be a proper Ferrers diagram with n rows and n columns, whose bottom k rows have the same length. To prove the Key Lemma, we need a bijection between the sets of fillings $\mathcal{F}^{(k)}(D, 312, 21)$ and $\mathcal{F}^{(k)}(D, 231, 12)$. Let $\mathcal{M}^{(k)}(D, 312, 21)$ be the set of matchings that correspond to the fillings from the set $\mathcal{F}^{(k)}(D, 312, 21)$, and similarly let $\mathcal{M}^{(k)}(D, 231, 12)$ be the set of matchings corresponding to the fillings from $\mathcal{F}^{(k)}(D, 231, 12)$.

By definition, a matching M belongs to $\mathcal{M}^{(k)}(D, 312, 21)$ if and only if $\mathrm{L}(M) = X(D)$, M avoids M_{312}, and the k edges incident to the rightmost k vertices of M form a k-nesting.

(Notice that all the rightmost k vertices of M are right-vertices, since the bottom k rows of D are assumed to have the same length.) Similarly, a matching M belongs to $\mathcal{M}^{(k)}(D, 231, 12)$ if and only if $\mathrm{L}(M) = X(D)$, M avoids M_{231}, and the edges incident to the rightmost k vertices form a k-crossing.

Let M be a matching. A sequence of edges $(e_1, e_2, \ldots, e_p)$ is called *a chain of order p from e_1 to e_p*, if for each $i < p$, the edge e_i crosses the edge e_{i+1} from the left. A chain is *proper* if each of its edges only crosses its neighbors in the chain. It is not difficult to see that every chain from e_1 to e_p contains, as a subsequence, a proper chain from e_1 to e_p.

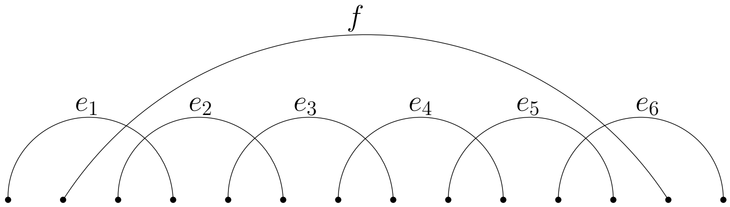

Figure 10: The cyclic chain of order 7.

A *cyclic chain of order $p+1$* is a $(p+1)$-tuple of edges $(f, e_1, \ldots, e_p)$, with the following properties.

- The sequence $(e_1, \ldots, e_p)$ is a proper chain.
- The edge f crosses e_1 from the right and e_p from the left. Furthermore, for each $i \in \{2, 3, \ldots, p-1\}$, the edge e_i is nested below f.

Figure 10 shows an example of a cyclic chain of order 7. The matching of order $p+1$ whose edges form a cyclic chain will be denoted by C_{p+1}. The smallest cyclic chain is C_3, whose three edges form a 3-crossing. Let $\mathcal{C}$ denote the infinite set $\{C_q \colon q \geq 3\}$.

As shown in [22], there is a bijection ψ which maps the set of M_{312}-avoiding matchings to the set of $\mathcal{C}$-avoiding matchings, with the additional property that each M_{312}-avoiding matching M is mapped to a $\mathcal{C}$-avoiding matching $\psi(M)$ with the same order and the same set of left-vertices. Since the reversal of a M_{312}-avoiding matching is an M_{231}-avoiding matching, while the reversal of a $\mathcal{C}$-avoiding matching is again $\mathcal{C}$-avoiding, it is easy to see that the mapping $M \mapsto \overline{\psi(\overline{M})}$ is a bijection that maps an M_{231}-avoiding matching M to a $\mathcal{C}$-avoiding matching with the same set of left-vertices.

We will use the bijection ψ as a building block of our bijection between the sets $\mathcal{M}^{(k)}(D, 312, 21)$ and $\mathcal{M}^{(k)}(D, 231, 12)$. However, before we do so, we need to describe the bijection ψ, which requires more terminology.

Let M be a matching on the vertex set $[2n]$. For an integer $r \in [2n]$, we let $M[r]$ denote the subgraph of M induced by the leftmost r vertices of M. We will call $M[r]$ *the r-th prefix of M*. The graph $M[r]$ is a union of disjoint edges and isolated vertices. The isolated vertices of $M[r]$ will be called *the stubs* of $M[r]$.

If x and x' are two stubs of $M[r]$, with $x < x'$, we say that x and x' are *equivalent in $M[r]$*, if $M[r]$ contains a chain $(e_1, \ldots, e_p)$ (possibly containing a single edge) such that

x is nested below e_1 and x' is nested below e_p. We will also assume that each stub is equivalent to itself. As shown in [22], this relation is indeed an equivalence relation on the set of stubs. The blocks of this equivalence relation will be simply called *the blocks of* $M[r]$. It is easy to see that if x and x' are stubs belonging to the same block, and x'' is a stub satisfying $x < x'' < x'$, then x'' belongs to the same block as x and x'. Figure 11 shows an example.

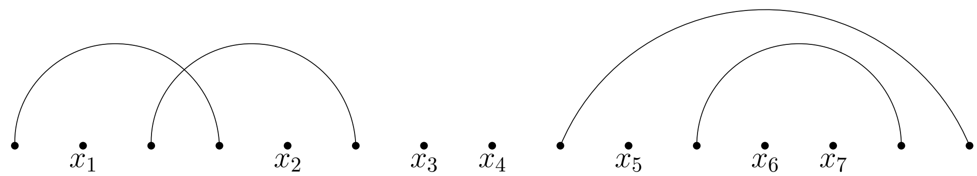

Figure 11: A prefix of a matching with seven stubs, forming four equivalence classes $\{x_1, x_2\}$, $\{x_3\}$, $\{x_4\}$, and $\{x_5, x_6, x_7\}$.

For a matching $M \in \mathcal{M}_n$, the sequence of prefixes $M[1], M[2], \dots, M[2n] = M$ will be called *the generating sequence of* M. We will interpret this sequence as a sequence of steps of an algorithm that generates the matching M by adding vertices one-by-one, from left to right, starting with the graph $M[1]$ that consists of a single isolated vertex.

Each prefix in the generating sequence defines an equivalence on the set of its stubs. To describe the bijection ψ, we first need to point how the blocks of these equivalences change when we pass from one prefix in the sequence to the next one.

Clearly, $M[1]$ consists of a single stub, so its equivalence has a single block $\{1\}$. Let us now show how the equivalence defined by $M[r]$ differs from the equivalence defined by $M[r-1]$. If a vertex $r > 1$ is a left-vertex of M, then the graph $M[r]$ is obtained from $M[r-1]$ by adding a new stub r. In such case, we say that $M[r]$ is obtained from $M[r-1]$ by *an L-step*. It is obvious that each block of $M[r-1]$ is also a block of $M[r]$, and apart from that $M[r]$ also has the singleton block $\{r\}$.

Assume now that $r > 1$ is a right-vertex of M. In this situation, we say that $M[r]$ is obtained from $M[r-1]$ by *an R-step*. Clearly, $M[r]$ is obtained from $M[r-1]$ by adding the vertex r and connecting it by an edge to a stub s of $M[r-1]$. We say that the stub s is *selected in step* r. In such case, s is no longer a stub in $M[r]$. Let $B_1, B_2, \dots, B_b$ be the blocks of $M[r-1]$ ordered left to right, and assume that s belongs to a block B_j. Then $B_1, B_2, \dots, B_{j-1}$ are also blocks in $M[r]$. The set $(B_j \setminus \{s\}) \cup B_{j+1} \cup \dots \cup B_b$ is either empty or forms a block of $M[r]$. Notice that the sizes of the blocks of $M[r]$ only depend on the value of j and on the sizes of the blocks of $M[r-1]$. We define two special types of R-steps: a *maximalist* R-step is an R-step in which the selected stub is the rightmost stub of its block (i.e., $s = \max B_j$), while a *minimalist* R-step is an R-step in which the selected stub is the leftmost stub in its block.

To connect our terminology with the results from [22], we need a simple lemma.

Lemma 5.8. *Let $M \in \mathcal{M}_n$ be an M_{312}-avoiding matching, let $r \in [2n]$ be an integer. Let s and s' be two distinct stubs of $M[r]$. The two stubs s and s' belong to the same block, if and only if $M[r]$ has an edge e that covers both s and s'.*

Proof. By definition, if two stubs are covered by a single edge of $M[r]$, they are equivalent and hence belong to the same block. To prove the converse, assume that $s < s'$ are stubs of $M[r]$ that belong to the same block. Let $C = (e_1, \ldots, e_p)$ be a chain in $M[r]$, such that e_1 covers s and e_p covers s'. Choose C to be as short as possible. If C consists of a single edge, then s and s' are both covered by this edge and we are done. For contradiction, assume that C has at least two edges. The edge e_2 does not cover s, because if it did, the chain $(e_2, \ldots, e_p)$ would contradict the minimality of C. Let f be the edge of M incident to the vertex s. Necessarily, the right endpoint of f is greater than r, otherwise s would not be a stub in $M[r]$. In particular, in the matching M, f intersects e_1 from the right, and e_2 is nested below f. Thus, the three edges e_1, e_2 and f form in M a copy of M_{312}, contradicting the assumption that M is M_{312}-avoiding. □

Combining Lemma 5.8 with [22, Lemma 3], we get the following result that gives characterizations of M_{312}-avoiding and $\mathcal{C}$-avoiding matchings.

Fact 5.9. *A matching $M \in \mathcal{M}_n$ avoids the pattern M_{312} if and only if, for every right-vertex $r > 1$ of M, $M[r]$ is obtained from $M[r-1]$ by a minimalist R-step. A matching $M \in \mathcal{M}_n$ avoids the set of patterns $\mathcal{C}$ if and only if, for every right-vertex $r > 1$ of M, $M[r]$ is obtained from $M[r-1]$ by a maximalist R-step.*

We are now ready to state the following key result from [22], which describes the properties of the bijection ψ.

Fact 5.10. *There is a bijection ψ between M_{312}-avoiding and $\mathcal{C}$-avoiding matchings. If M is an M_{312}-avoiding matching of order n, and $N = \psi(M)$ its corresponding $\mathcal{C}$-avoiding matching, then the following holds.*

- *M and N have the same set of left-vertices (and hence the same size).*
- *For any vertex $r \in [2n]$, the prefix $M[r]$ has the same number of blocks as the prefix $N[r]$. Moreover, if $B_1, \ldots, B_b$ are the blocks of $M[r]$ in left-to-right order, and $B'_1, \ldots, B'_b$ are the blocks $N[r]$ in left-to-right order, then $|B_i| = |B'_i|$ for each $i \leq b$.*
- *Assume that $r+1$ is a right-vertex of M (and hence also of N), and that $B_1, \ldots, B_b$ and $B'_1, \ldots, B'_b$ are blocks of $M[r]$ and $N[r]$, as above. If $M[r+1]$ is obtained from $M[r]$ by selecting a stub s from a block B_j, then $N[r+1]$ is obtained from $N[r]$ by selecting a stub s' from the corresponding block B'_j. In view of Fact 5.9, we must then have $s = \min B_j$ and $s' = \max B'_j$.*

The properties of ψ listed above in fact determine ψ uniquely.

Finally, we are ready to present the bijection between $\mathcal{M}^{(k)}(D, 312, 21)$ and $\mathcal{M}^{(k)}(D, 231, 12)$. Recall that the matchings from these two sets have the same set of left-vertices $X(D)$ and the same set of right-vertices $Y(D) = [2n] \setminus X(D)$. Let us write $X(D) = \{x_1 < x_2 < \ldots < x_n\}$ and $Y(D) = \{y_1 > y_2 > \ldots > y_n\}$. Recall also that by assumption, the rightmost k right-vertices $y_1, \ldots, y_k$ are to the right of any left-vertex.

The bijection we present is a composition of several steps, with the correctness of each step proved separately. An example is shown in Figure 12.

Step 1: apply ψ. Use the bijection ψ to map the set $\mathcal{M}^{(k)}(D, 312, 21)$ bijectively to the set $S_1 = \{\psi(M);\ M \in \mathcal{M}^{(k)}(D, 312, 21)\}$. As shown in Lemma 5.11 below, S_1 is precisely the set of all the matchings N satisfying the following properties:

(P1) N avoids $\mathcal{C}$.

(P2) $\mathrm{L}(N) = X(D)$.

(P3) In the prefix $N[2n-k]$ of N, each block has a single stub.

(P4) The edges of N incident to $y_1, \dots, y_k$ form a k-nesting.

Step 2: add edge. For a matching $M \in S_1$, let M^+ be the matching obtained from M by adding two new vertices x_{new} and y_{new} and a new edge $e_{\mathrm{new}} = x_{\mathrm{new}}y_{\mathrm{new}}$, such that the edge e_{new} covers precisely the vertices $y_1, \dots, y_k$. We relabel the $2n+2$ vertices of M^+, without altering their left-to-right order, so that their labels correspond to the integers $1, \dots, 2n+2$ in their usual order. With this labeling, we have $x_{\mathrm{new}} = 2n-k+1$ and $y_{\mathrm{new}} = 2n+2$.

Let S_2 be the set $\{M^+;\ M \in S_1\}$. Clearly, all the matchings in S_2 share the same set of left-vertices and the same set of right-vertices. We call these sets X^+ and Y^+, respectively. It is also clear that the mapping $M \mapsto M^+$ is a bijection between S_1 and S_2. In Lemma 5.12, we will show that S_2 is precisely the set of all the matchings N that satisfy the following conditions:

(R1) N avoids $\mathcal{C}$.

(R2) $\mathrm{L}(N) = X^+$.

(R3) N contains the edge $e_{\mathrm{new}} = \{2n-k+1, 2n+2\}$.

Step 3: reverse. Recall that $\overline{M}$ denotes the reversal of a matching M. Let S_3 be the set $\{\overline{M};\ M \in S_2\}$. All the matchings in S_3 have the same set of left-vertices and right-vertices, denoted by $\overline{X^+}$ and $\overline{Y^+}$, respectively. From the previously stated properties of S_2 it follows that S_3 contains precisely the matchings N satisfying these conditions:

($\overline{\mathrm{R1}}$) N avoids $\mathcal{C}$.

($\overline{\mathrm{R2}}$) $\mathrm{L}(N) = \overline{X^+}$.

($\overline{\mathrm{R3}}$) N contains the edge $\{1, k+2\}$.

Step 4: apply ψ^{-1}. Let S_4 be the set $\{\psi^{-1}(M);\ M \in S_3\}$. In Lemma 5.13, we show that S_4 contains precisely the matchings N satisfying these three conditions:

(S1) N avoids M_{312}.

(S2) $\mathrm{L}(N) = \overline{X^+}$.

(S3) N contains the edge $\{1, k+2\}$.

Step 5: remove edge. For a matching $M \in S_4$, let M^- denote the matching obtained from M by removing the edge $\{1, k+2\}$ together with its endpoints. Relabel the vertices of M^- by integers $1, 2, \dots, 2n$, in their usual order. Let S_5 be the set $\{M^-;\ M \in S_4\}$. All the matchings in S_5 have the same set of left-vertices, denoted by $\overline{X}$. We show in Lemma 5.14 that S_5 contains precisely the following matchings N:

(S1$^-$) N avoids M_{312}.

(S2$^-$) $\mathrm{L}(N) = \overline{X}$.

(S3$^-$) The edges incident to the leftmost k vertices of N form a k-crossing.

Step 6: reverse back. The properties of S_5 stated above imply that the matchings in S_5 are exactly the reversals of the matchings in $\mathcal{M}^{(k)}(D, 231, 12)$. Thus, applying reversal to the elements of S_5 we complete the bijection from $\mathcal{M}^{(k)}(D, 312, 21)$ to $\mathcal{M}^{(k)}(D, 231, 12)$.

Next, we will prove the correctness of the individual steps. The proofs are mostly routine.

Lemma 5.11. *The set S_1 contains precisely those matchings that satisfy the four properties (P1)–(P4).*

Proof. Let N be a matching from the set S_1. Let $M \in \mathcal{M}^{(k)}(D, 312, 21)$ be the preimage of N under ψ. The properties of ψ stated in Fact 5.10 directly show that N satisfies (P1) and (P2).

We now show that N satisfies (P3). Fact 5.10 shows that N satisfies (P3) if and only if M satisfies (P3). It is thus enough to prove (P3) for M.

In the matching M, the k edges incident to $y_1, \dots, y_k$ form a k-nesting, by the definition of $\mathcal{M}^{(k)}(D, 312, 21)$. Assume that M does not satisfy (P3), i.e., in the prefix $M[2n-k]$, there are two stubs $s < s'$ belonging to the same block. By Lemma 5.8, this means that s and s' are both covered by a single edge $e \in M[2n-k]$. Let f and f' be the edges of M incident to s and s', respectively. The right endpoints of f and f' must belong to $\{y_1, \dots, y_k\}$, which means that f and f' are nested. This means that e, f, and f' form a copy of M_{312} in M, which is impossible.

Next, we show that N satisfies (P4), i.e., that the edges of N incident to $y_1, \dots, y_k$ form a k-nesting. This is equivalent to saying that for every $i \in [k]$, the prefix $N[2n-i+1]$ is obtained from $N[2n-i]$ by adding the right-vertex y_i and connecting it to the rightmost stub of $N[2n-i]$. We know that the edges of M incident to $\{y_1, \dots, y_k\}$ form a k-nesting. Hence, for each $i \in [k]$, $M[2n-i+1]$ is obtained from $M[2n-i]$ by an R-step in which the rightmost stub in $M[2n-i]$ is selected. By the properties of ψ, we also select the rightmost stub whenever we create $N[2n-i+1]$ from $N[2n-i]$. This shows that N has property (P4).

We now show that every matching satisfying the four properties (P1)–(P4) belongs to S_1. Let N be a matching satisfying (P1)–(P4). Since N is $\mathcal{C}$-avoiding, we may define $M = \psi^{-1}(N)$. To show that N belongs to S_1, we need to prove that M belongs to $\mathcal{M}^{(k)}(D, 312, 21)$. The properties of ψ guarantee that M is M_{312}-avoiding and that $\mathrm{L}(M) = X(D)$. It remains to show that rightmost k vertices of M are incident to a k-nesting.

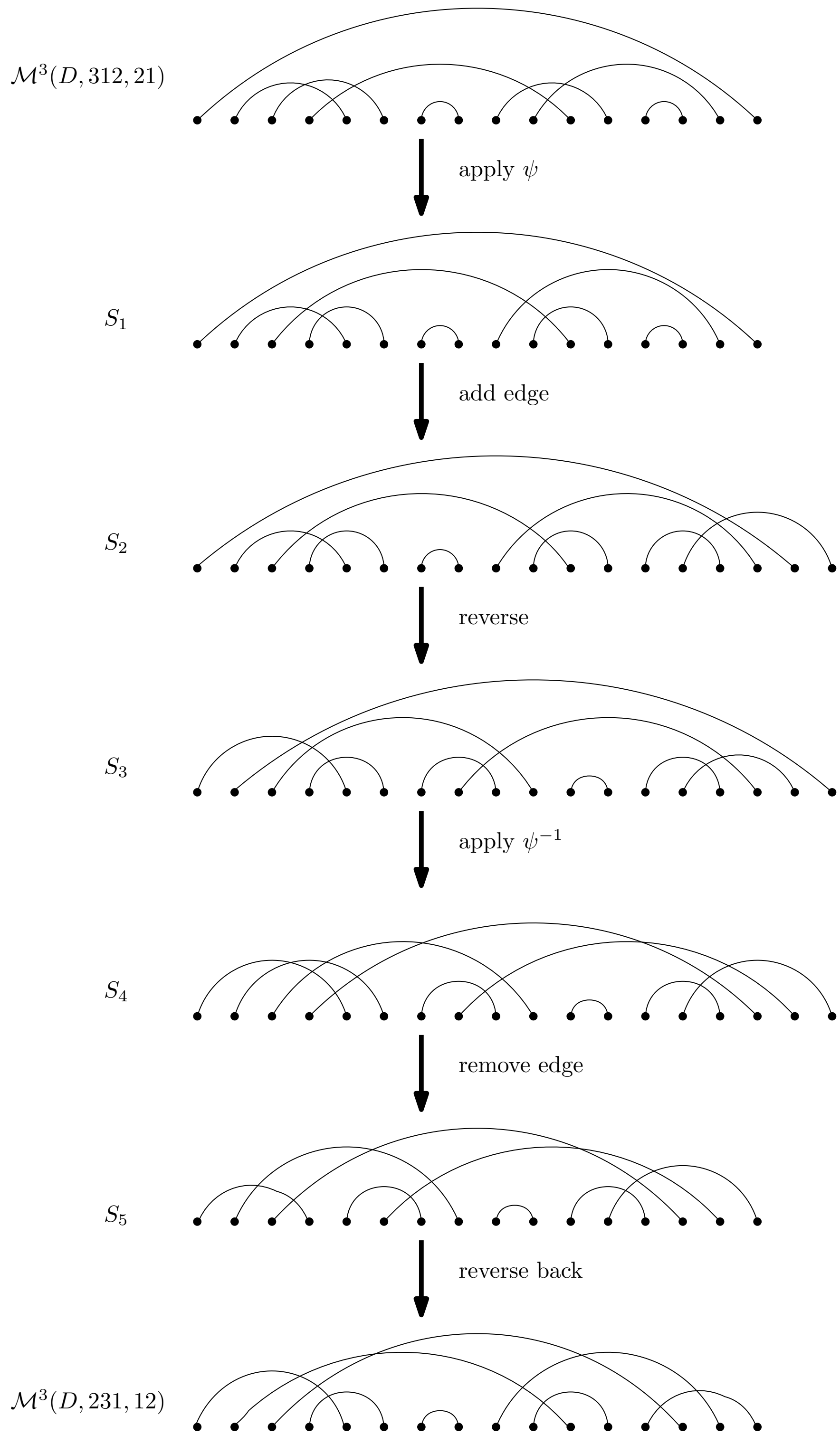

Figure 12: The six steps of a bijection from $\mathcal{M}^{(k)}(D, 312, 21)$ to $\mathcal{M}^{(k)}(D, 231, 12)$ (with $k = 3$).

Since N satisfies (P3), so does M. Moreover, since N satisfies (P4), we know that for each $i \leq k$, the prefix $N[2n-i+1]$ is obtained from $N[2n-i]$ by adding the vertex y_i and connecting it to the rightmost stub of $N[2n-i]$. From this description, we easily notice that for each $i \leq k$, each block in the matching $N[2n-i]$ is a singleton. By the properties of ψ, each block of $M[2n-i]$ is also a singleton, and $M[2n-i+1]$ is created form $M[2n-i]$ by connecting y_i to the rightmost stub of $M[2n-i]$. This shows that in M, the vertices $y_1, \ldots, y_k$ are indeed incident to a k-nesting, and hence $M \in \mathcal{M}^{(k)}(D, 312, 21)$. ☐

Lemma 5.12. *A matching N belongs to S_2 if and only if it satisfies (R1)–(R3).*

Proof. Suppose $N \in S_2$. Let M be a matching from S_1 such that $N = M^+$. By construction, N satisfies (R2) and (R3). We need to show that N also satisfies (R1), i.e., that it is $\mathcal{C}$-avoiding. For contradiction, assume that N contains a copy C of a cyclic chain formed by $p+1$ edges $(f, e_1, \ldots, e_p)$. Recall that in a cyclic chain, the sequence $(e_1, \ldots, e_p)$ is a proper chain, and f crosses e_1 from the right and e_p from the left, while the edges $e_2, \ldots, e_{p-1}$ are nested below f.

Since M is $\mathcal{C}$-avoiding, C must contain the new edge $\{2n-k+1, 2n+2\}$. Necessarily, the new edge is the edge e_p, which is incident to the rightmost vertex of C. In the matching M, the edges incident to $y_1, \ldots, y_k$ form a k-nesting. In particular, the edges f and e_{p-1} are nested, and hence C has at least four edges. Let s and s' be the left endpoints of the edges f and e_{p-1}, respectively. Consider now the prefix $M[2n-k]$. This prefix contains the nonempty proper chain $e_1, \ldots, e_{p-2}$ (possibly consisting of a single edge), and the two vertices s and s' are stubs of $M[2n-k]$. Since s is covered by e_1 and s' is covered by e_{p-2}, the two stubs belong to the same block of $M[2n-k]$, which is impossible, since the matching $M \in S_1$ must satisfy (P3).

We conclude that every matching from S_2 satisfies (R1)–(R3).

To prove the converse, assume that N is a matching satisfying (R1)–(R3). By (R3), there is a matching M such that $M^+ = N$. We need to show that M belongs to S_1, i.e., that it satisfies (P1)–(P4). It is clear that M satisfies (P1) and (P2). If M fails (P4), then N must contain a 3-crossing, which is impossible, since a 3-crossing is a special case of a cyclic chain. If M satisfies (P4) but fails (P3), then N contains a cyclic chain of length at least four, which is also impossible. Thus M belongs to S_1, and hence N belongs to S_2, as claimed. ☐

Lemma 5.13. *A matching N belongs to S_4 if and only if it satisfies (S1)–(S3).*

Proof. Choose $N \in S_4$, and set $M = \psi(N)$. By definition, M belongs to S_3, so it satisfies ($\overline{\text{R1}}$)–($\overline{\text{R3}}$). It follows directly that N satisfies (S1) and (S2).

We know that M satisfies ($\overline{\text{R3}}$). Consider the R-step from $M[k+1]$ to $M[k+2]$. Since $M[k+1]$ consists of $k+1$ stubs, all its blocks are singletons. Since M has the edge $\{1, k+2\}$, the prefix $M[k+2]$ has been constructed from $M[k+1]$ by selecting the leftmost stub of $M[k+1]$ and connecting it to the vertex $k+2$. By the properties of ψ, this means that $N[k+2]$ was obtained from $N[k+1]$ in the same way, and in particular, N contains the edge $\{1, k+2\}$. We conclude that N satisfies (S1)–(S3).

The same argument shows that every matching satisfying (S1)–(S3) belongs to S_4. ☐

Lemma 5.14. *A matching N belongs to S_5 if and only if it satisfies (S1$^-$)–(S3$^-$).*

Proof. Choose $N \in S_5$, and let M be the matching from S_4 such that $N = M^-$. Clearly, N satisfies (S1$^-$) and (S2$^-$). Assume for contradiction that N fails (S3$^-$). In such case, N has two nested edges e_1 and e_2, whose left endpoints are among the leftmost k vertices of N. Since the leftmost k vertices of N are all left-vertices, we know that in the matching M, the two edges e_1 and e_2 are both crossed from the left by the edge $\{1, k+2\}$, forming the pattern M_{312} forbidden by (S1).

It is easy to see that any matching satisfying (S1$^-$)–(S3$^-$) belongs to S_5. □

This completes the proof of Lemma 5.6, and hence also of Theorem 5.1.

6 The k-Wilf-equivalence of patterns of length $k+2$

We will now consider the structure of pattern-avoiding partial permutations in which the number of holes is close to the length of the forbidden pattern.

Let us begin by an easy observation.

Observation 6.1. Let p be a pattern of length ℓ. Obviously any partial permutation with at least ℓ holes contains p. Almost as obviously, a partial permutation with $\ell - 1$ holes and of length at least ℓ, contains p as well. In particular, $s_n^k(p) = 0$ for every $k \geq \ell - 1$ and $n \geq \ell$, and all patterns of length ℓ are k-Wilf-equivalent.

In the rest of this section, we will deal with k-Wilf-equivalence of patterns of length $\ell = k+2$. As we will see, an important part will be played by Baxter permutations, which we now define.

Definition 6.2. A permutation $p \in \mathcal{S}_\ell$ is called a *Baxter permutation*, if there is no four-tuple of indices $a < b < c < d \in [\ell]$ such that

- $c = b + 1$, and
- the subpermutation p_a, p_b, p_c, p_d is order-isomorphic to 2413 or to 3142.

In the terminology of Babson and Steingrímsson [2], Baxter permutations are exactly the permutations avoiding simultaneously the two patterns 2-41-3 and 3-14-2.

Baxter permutations were originally introduced by G. Baxter [5] in 1964, in the study of common fixed points of commuting continuous functions [5, 12]. Later, it has been discovered that Baxter permutations are also closely related to other combinatorial structures, such as plane bipolar orientations [11], noncrossing triples of lattice paths [19], and standard Young tableaux [17]. An explicit formula for the number of Baxter permutations has been found by Chung et al. [13], with several later refinements [27, 35, 18].

To deal with k-Wilf equivalence of patterns of length $k+2$, we first need to introduce more notation. Let $\pi \in \mathcal{S}_n^H$ be a partial permutation, with $|H| = k$. Let $h_1 < h_2 < \cdots < h_k$ be the elements of H. Let I denote the set $[n] \setminus H$, i.e., I is the set of indices of the non-holes of π. We may decompose the set I into $k+1$ (possibly empty) intervals $I_1, I_2, \ldots, I_{k+1}$, by defining $I_1 = \{i \in I;\ i < h_1\}$, $I_{k+1} = \{i \in I;\ i > h_k\}$, and for each $a \in \{2, \ldots, k\}$, $I_a = \{i \in I;\ h_{a-1} < i < h_a\}$.

Lemma 6.3. *Let ℓ and n be integers, let $k = \ell - 2$. Let $p = p_1 \cdots p_\ell$ be a permutation and let $\pi = \pi_1 \cdots \pi_n$ be a partial permutation with k holes. Assume that $H, I, I_1, \ldots, I_{k+1}$ are as above. The partial permutation π avoids the pattern p if and only if for each two distinct indices $i \in I_a$ and $j \in I_b$ such that $i < j$, the relative order of π_i and π_j is different from the relative order of p_a and p_{b+1} (i.e., $\pi_i < \pi_j \iff p_a > p_{b+1}$).*

Consequently, for each such p, n and H, we have $s_n^H(p) \leq 1$.

Proof. Assume that $i < j$ are distinct indices from the set I, with $i \in I_a$ and $j \in I_b$. Necessarily, $a \leq b$. Note that in π there are $a - 1$ holes to the left of π_i, there are $b - a$ holes between π_i and π_j, and there are $k - b + 1$ holes to the right of π_j.

Assume that for some $i \in I_a$ and $j \in I_b$, with $i < j$, the symbols π_i and π_j have the same relative order as p_a and p_{b+1}. Then π contains an occurrence of p, in which π_i corresponds to p_a, π_j corresponds to p_{b+1}, and the k holes correspond to the remaining k symbols of p.

Conversely, assume that π contains an occurrence of p. This means that there is an ℓ-tuple P of indices, such that the subsequence $(\pi_h;\ h \in P)$ is a copy of p. It is not hard to see that in such case we may always find a copy of p that uses all the k holes of π. In other words, we may assume that H is a subset of P. Let i and j be the two indices of P not belonging to H, with $i < j$. Fix a and b such that $i \in I_a$ and $j \in I_b$. In the ℓ-tuple $(\pi_h;\ h \in P)$, the element π_i is the a-th element, since it has $a - 1$ holes to the left of it, while j is $(b+1)$-th element, since it has $b - 1$ holes and the symbol π_i to the left of it. Since $(\pi_h;\ h \in P)$ is assumed to be a copy of p, we conclude that π_i and π_j have the same relative order as p_a and p_{b+1}.

This shows that π avoids p if and only if for each two distinct indices $i < j$ with $i \in I_a$ and $j \in I_b$, the relative order of π_i and π_j differs from the relative order of p_a and p_{b+1}.

For a fixed $p \in \mathcal{S}_\ell$, for each n and for each set $H \subseteq [n]$ of size k, if π is a partial permutation from $\mathcal{S}_n^H(p)$, the relative order of every two non-holes of π is uniquely determined by the relative order of the symbols of p. In particular, π is uniquely determined by p, n and H, implying that $s_n^H(p) \leq 1$. □

Motivated by Lemma 6.3, we introduce the following notation. Let $p \in \mathcal{S}_\ell$ be a pattern, let $k = \ell - 2$, let n be an integer, and let $H \subseteq [n]$ be a k-element set of integers. The *order graph* $G_n^H(p)$ is a directed graph on the vertex set $I = [n] \setminus H$, whose edge-set is defined by the following condition: for every $i < j$, such that $i \in I_a$ and $j \in I_b$, the graph $G_n^H(p)$ has an edge from i to j if $p_a > p_{b+1}$, and it has an edge from j to i if $p_a < p_{b+1}$.

Note that $G_n^H(p)$ is a tournament, i.e., for each pair of distinct vertices i and j, the graph $G_n^H(p)$ has an edge from i to j or an edge from j to i, but not both.

Let $\pi = \pi_1 \cdots \pi_n$ be a partial permutation from the set $\mathcal{S}_n^H$. Using the notion of order graphs, Lemma 6.3 can be restated in the following equivalent way: π avoids p if and only if, for each two distinct vertices i, j of $G_n^H(p)$, if the graph $G_n^H(p)$ has a directed edge from i to j then $\pi_i < \pi_j$. Notice that in this statement, we no longer need to assume that $i < j$.

Lemma 6.4. *Let $p \in \mathcal{S}_\ell$ be a pattern, let $k = \ell - 2$, let n be an integer, and let $H \subseteq [n]$ be a k-element set of integers. The following statements are equivalent:*

1. *$s_n^H(p) = 1$.*
2. *$G_n^H(p)$ has no directed cycle.*
3. *$G_n^H(p)$ has no directed cycle of length 3.*

Proof. Since $G_n^H(p)$ is a tournament, the statements 2 and 3 are easily seen to be equivalent.

Let us now show that (1) implies (2). Assume that $s_n^H(p) = 1$, and let $\pi = \pi_1 \cdots \pi_n$ be the partial permutation from $\mathcal{S}_n^H(p)$. As we have pointed out before, if $G_n^H(p)$ has an edge from i to j, then $\pi_i < \pi_j$. This clearly shows that $G_n^H(p)$ may have no directed cycle.

Conversely, if $G_n^H(p)$ has no directed cycle, we may topologically order its vertices, i.e., we can assign to every vertex i a value π_i in such a way that if the graph has an edge from i to j, then $\pi_i < \pi_j$. The values π_i then define a p-avoiding partial permutation $\pi \in \mathcal{S}_n^H(p)$, showing that $s_n^H(p) = 1$. □

We are now ready to demonstrate the significance of Baxter permutations. Note that for any pattern of length $\ell = k+2$, and for any n from the set $\{k, k+1, k+2\}$, we always have $s_n^k(p) = \binom{n}{k}$. Thus, for these small values of n, all patterns have the same behavior. However, for all larger values of n, the Baxter patterns are separated from the rest, as the next proposition shows.

Proposition 6.5. *Let p be a permutation pattern of size ℓ, and let $k = \ell - 2$. The following statements are equivalent.*

1. *The pattern p is a Baxter permutation.*
2. *For each $n \geq k$ and each k-element set $H \subseteq [n]$, $s_n^H(p) = 1$.*
3. *For $n = k + 3$ and each k-element set $H \subseteq [n]$, $s_n^H(p) = 1$.*
4. *There exists $n \geq k + 3$ such that for each k-element set $H \subseteq [n]$, $s_n^H(p) = 1$.*

Proof. Let us first prove that (1) implies (2). Assume that p is a Baxter permutation. Choose n and H as in (2). By Lemma 6.4, to show that $s_n^H(p) = 1$, it is enough to prove that the order graph $G_n^H(p)$ has no directed triangles. For contradiction, assume that the order graph contains a triangle induced by three vertices $h < i < j$.

Assume that $G_n^H(p)$ contains the edges from h to i, from i to j and from j to h (if the triangle is oriented in the other direction, the argument is analogous). Fix a, b and c, such that $h \in I_a$, $i \in I_b$ and $j \in I_c$. Necessarily, $1 \leq a \leq b \leq c \leq k+1 = \ell - 1$. Note that the three edges (hi), (ij), and (jh) imply, respectively, the three inequalities $p_a > p_{b+1}$,

$p_b > p_{c+1}$, and $p_a < p_{c+1}$. In other words, $p_{b+1} < p_a < p_{c+1} < p_b$. This shows that the four indices $a, b, b+1, c+1$ are all distinct, and they induce in p a pattern order-isomorphic to 2413, contradicting the assumption that p is a Baxter permutation. We conclude that for a Baxter permutation p, the graph $G_n^H(p)$ has no directed triangle, and hence $s_n^H(p) = 1$.

Clearly, (2) implies (3) and (3) implies (4). To complete the proof of the proposition, we will show that (4) implies (1). Assume that p is not a Baxter permutation, and that it contains a copy of 2413 induced by the indices $a < b < b+1 < c+1$ (the case when p contains 3142 is analogous). In other words, p satisfies $p_{b+1} < p_a < p_{c+1} < p_b$. Let $n \geq k+3$ be given. Select a k-element set $H \subseteq [n]$ in such a way that the three sets I_a, I_b and I_c are all nonempty. Choose $h \in I_a$, $i \in I_b$ and $j \in I_c$ arbitrarily. Necessarily, we have $h < i < j$, and the graph $G_n^H(p)$ contains the three directed edges hi, ij and jh. This means that $G_n^H(p)$ has a triangle, and hence $s_n^H(p) = 0$. □

The following result is a direct consequence of Proposition 6.5.

Theorem 6.6. *Let $p \in \mathcal{S}_\ell$ be a permutation pattern. Let $k = \ell - 2$. If p is a Baxter permutation then $s_n^k(p) = \binom{n}{k}$ for each $n \geq k$. If p is not a Baxter permutation, then $s_n^k(p) < \binom{n}{k}$ for each $n \geq k+3$. Moreover, all the Baxter permutations are strongly k-Wilf equivalent.*

We remark that by a slightly more careful analysis of the proof of Proposition 6.5, we could give a stronger upper bound for $s_n^k(p)$ when p is not a Baxter permutation. In particular, it is not hard to see that in such case, $s_n^k(p)$ is eventually a polynomial in n of degree at most $k-1$, with coefficients depending on k.

7 Short patterns

In the rest of this paper, we focus on explicit formulas for $s_n^k(p)$, where p is a pattern of length ℓ. We may assume that $k < \ell - 1$, and $\ell > 2$, since for any other values of (k, ℓ) the enumeration is trivial (see Observation 6.1). We also restrict ourselves to $k \geq 1$, since the case $k = 0$, which corresponds to classical pattern-avoidance in permutations, has already been extensively studied [10].

For a pattern p of length three, the situation is very simple. Theorem 6.6 implies that $s_n^1(p) = n$, since all permutations of length three are Baxter permutations.

Let us now deal with patterns of length four. In Figure 13, we depict the k-Wilf equivalence classes, where the four rows, top to bottom, correspond to the four values $k = 0, 1, 2, 3$. Since all the k-Wilf equivalences are closed under complements and reversals (but not inversions), we represent the 24 patterns of length four by eight representatives, one from each symmetry class. For instance, $\{1342, 1423\}$ in the second row represents the union of $\{1342, 2431, 3124, 4213\}$ and $\{1423, 2314, 3241, 4132\}$.

All patterns p of length four except 2413 and 3142 are Baxter permutations, and hence they satisfy $s_n^2(p) = \binom{n}{2}$ by Theorem 6.6.

Let us now compute $s_n^2(p)$ for a pattern $p \in \{2413, 3142\}$ and an integer n. Since 3142 is the complement of 2413, we know that $s_n^2(2413) = s_n^2(2413)$. Let i and j be two indices,

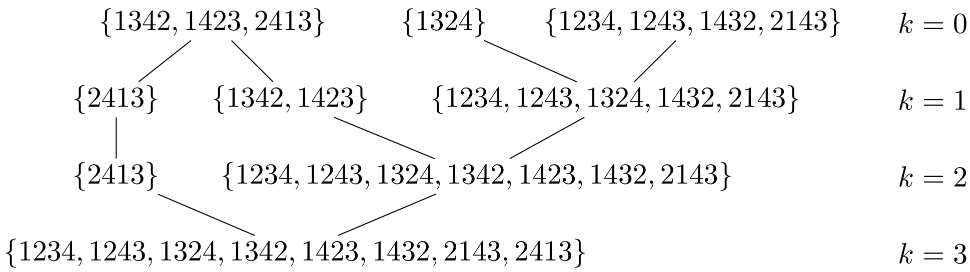

Figure 13: The k-Wilf-equivalence classes of permutations of size 4.

with $1 \le i < j \le n$, and let $H = \{i, j\}$. Let us determine the value of $s_n^H(2413)$. Define I_1, I_2 and I_3 as in the previous section, i.e., $I_1 = \{1, 2, \ldots, i-1\}$, $I_2 = \{i+1, \ldots, j-1\}$ and $I_3 = \{j+1, \ldots, n\}$.

Using the same argument as in the proof of Proposition 6.5, we deduce that if all the three sets I_1, I_2, and I_3 are nonempty, then $s_n^H(2413) = 0$. On the other hand, if at least one of the three sets is empty, then it is easy to see that the graph $G_n^H(2413)$ is acyclic, and hence $s_n^H(2413) = 1$ by Lemma 6.4.

For $n \ge 3$, there are $3n - 6$ possibilities to choose H in such a way that at least one of the sets I_1, I_2 and I_3 is empty. We conclude that $s_n^2(2413) = s_n^2(3142) = 3n - 6$.

In the rest of this section, we deal with 1-Wilf equivalence of patterns of length four, and with the enumeration of the corresponding avoidance classes. Theorem 4.4 and symmetry arguments imply that all the patterns 1234, 1243, 1432 and 2143 are strongly $\star$-Wilf-equivalent, and Theorem 5.1 with appropriate symmetry arguments shows that 1342 and 1423 are strongly $\star$-Wilf-equivalent as well. The only case not covered by these general theorems is the 1-Wilf equivalence of 1324 and 1234, which is handled separately by the next proposition.

Proposition 7.1. *The patterns 1234 and 1324 are strongly 1-Wilf-equivalent.*

Let $\pi = \pi_1\pi_2\cdots\pi_{j-1}\diamond\pi_{j+1}\cdots\pi_n$ be a partial permutation of length n with a single hole, appearing at position j. The sequence $\pi_1\pi_2\cdots\pi_{j-1}$ will be referred to as the *left part* of π and $\pi_{j+1}\cdots\pi_n$ will be the *right part* of π. The smallest element appearing in the left part of π will be called the *left minimum* of π. Left maximum, right minimum and right maximum are defined analogously.

The following two observations, which follow directly from the definitions, characterize the avoidance of 1234 and 1324 in partial permutations with a single hole.

Observation 7.2. A partial permutation $\pi = \pi_1\pi_2\cdots\pi_{j-1}\diamond\pi_{j+1}\cdots\pi_n$ avoids the pattern 1234 if and only if it satisfies the following conditions:

1. The left part of π avoids 123.

2. The elements of the left part that are smaller than the right maximum form a decreasing sequence.

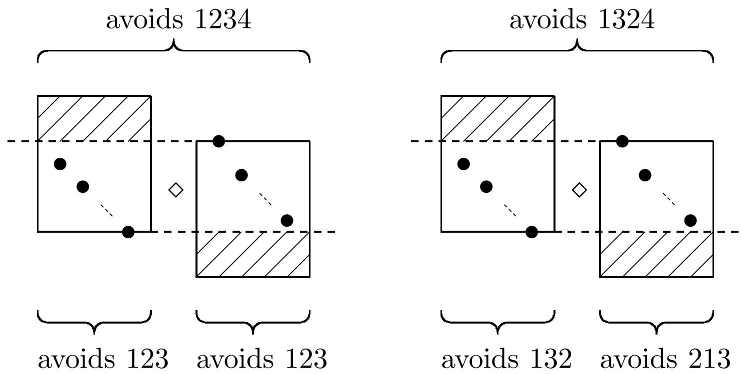

Figure 14: The structures of 1234- and 1324-avoiding partial permutations with one hole.

3. The right part of π avoids 123.
4. The elements of the right part that are larger than the left minimum form a decreasing sequence.

Observation 7.3. A partial permutation $\pi = \pi_1\pi_2 \cdots \pi_{j-1}\diamond\pi_{j+1} \cdots \pi_n$ avoids the pattern 1324 if and only if it satisfies the following conditions:

1. The left part of π avoids 132.
2. The elements of the left part that are smaller than the right maximum form a decreasing sequence.
3. The right part of π avoids 213.
4. The elements of the right part that are larger than the left minimum form a decreasing sequence.

Proof of Proposition 7.1. We describe a bijection between the sets $\mathcal{S}_n^1(1234)$ and $\mathcal{S}_n^1(1324)$. Choose an arbitrary $\pi \in \mathcal{S}_n^1(1234)$. In the first step, we permute the symbols of the left part of π, so that the left part is bijectively transformed from a sequence satisfying conditions 1 and 2 of Observation 7.2 to a sequence satisfying conditions 1 and 2 of Observation 7.3, while preserving the number of elements in the left part that are smaller than the right maximum. Actually, we can require a stronger statement, when under the transformation, in the left part of π the sequence of left-to-right minima will be preserved in value and in position, which can be done using the Simion-Schmidt bijection; see [31] and [15, Theorem 1].

In the second step of the bijection, we perform an analogous transformation of the right part of the sequence, again using the Simion-Schmidt bijection. Indeed, we can achieve a bijective transformation between 123- and 213-avoiding permutations (corresponding to the right parts) preserving the sequence of right-to-left maxima in value and in place: applying reverse and complement, it is equivalent to preserving the sequence of left-to-right minima in value and in place in a bijection between 123- and 132-avoiding permutations, which we have by the Simion-Schmidt bijection. □

7.1 Enumeration

We now focus on explicit enumerations of $s_n^k(p)$ for $p \in \mathcal{S}_4$ and $k = 1, 2$. In what follows, for two sequences of numbers π_1 and π_2 we write $\pi_1 < \pi_2$ if each letter of π_1 is smaller than any letter of π_2. Let C_n denote the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$ and let $C(x)$ be the generating function

$$C(x) = \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

Theorem 7.4. *For $n \geq 1$, we have $s_n^1(1234) = \binom{2n-2}{n-1}$.*

Proof. From formula (1) in Subsection 1.2, we get $s_n^1(1234) = n s_{n-1}^0(123)$, and it is well-known (e.g., see [15]) that $s_{n-1}^0(123) = C_{n-1}$. This completes the proof. □

In Section 7.2, we also provide a bijective proof of Theorem 7.4, by mapping $\mathcal{S}_n^1(1234)$ to lattice paths from (0,0) to $(2n - 2, 0)$ with steps (1,1) and (1,-1).

Theorem 7.5. *For $n \geq 1$, we have $s_n^1(1342) = \binom{2n-2}{n-1} - \binom{2n-2}{n-5}$.*

Proof. The following observation, coming directly from the definitions, characterizes the avoidance of 1342 in partial permutations with a single hole.

Observation 7.6. A partial permutation $\pi = \pi_1\pi_2 \cdots \pi_{j-1}\diamond\pi_{j+1} \cdots \pi_n$ avoids the pattern 1342 if and only if it satisfies the following conditions (see also Figure 15):

1. The left part of π avoids 123.
2. The right part of π avoids 231.
3. The elements in the right part bigger than the left minimum form an increasing sequence.
4. If $a < b < c$ are three numbers such that b is in the right part of π while a and c are in the left part, then c must appear to the left of a in π.

Consequently, the structure of π is described by one of the following two cases.

(i) If the right part of π is an increasing (possibly empty) sequence, then the left part of π consists of a decreasing sequence of 123-avoiding possibly empty blocks as shown on the upper picture in Figure 15. Assuming that the right part is of size k, π can be decomposed as $B_1B_2 \cdots B_{k+1}\diamond a_k a_{k-1} \cdots a_1$ with $B_1 > a_1 > B_2 > a_2 > \cdots > a_k > B_{k+1}$, where B_i is a possibly empty 123-avoiding permutation, for $1 \leq i \leq k + 1$.

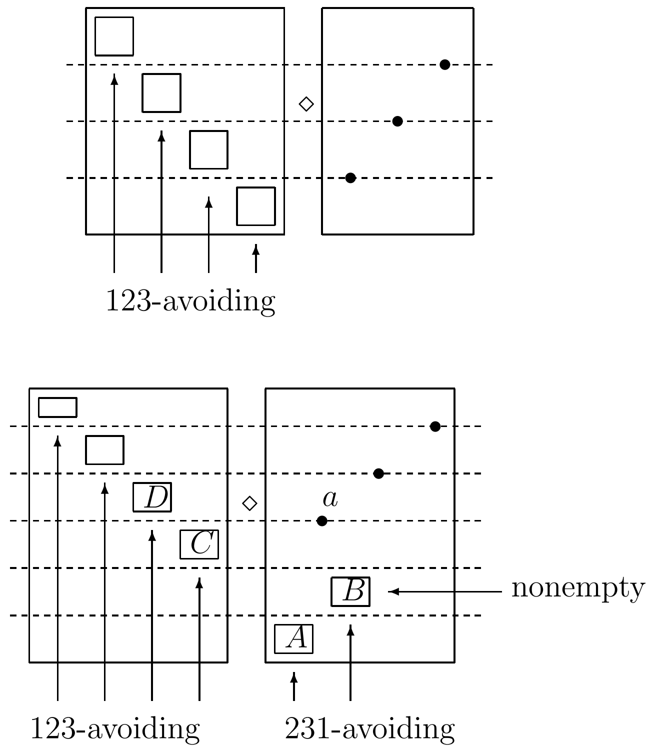

Figure 15: Two possible structures of partial permutations with one hole that avoid 1342.

(ii) Suppose the right part of π is not an increasing sequence. Let a be the smallest symbol in the right part of π such that all the symbols in the right part greater or equal to a form an increasing sequence (see the lower picture in Figure 15). Assuming there are k elements greater than a in the right part, π can be written as $B_1B_2\cdots B_kDC\diamond AaBa_ka_{k-1}\cdots a_1$, where D, C, and all the B_i are possibly empty 123-avoiding permutations (we distinguish D and C from the B_i's for enumeration arguments below), A and B are 231-avoiding permutations with B non-empty, such that

$$B_1 > a_1 > B_2 > a_2 > \cdots > B_k > a_k > D > a > C > B > A.$$

Using Observation 7.6, we will derive a closed-form formula for the generating function $\sum_{n\geq 1} s_n^1(1342)x^n$.

It is known that $C(x)$ is the generating function for 123-avoiding permutations, as well as for 231-avoiding permutations. The partial permutations considered in case (i) of Observation 7.6 then have the generating function $x\sum_{k\geq 0} x^kC^{k+1}(x) = xC(x)/(1-xC(x))$. Note that a factor x in the previous expression corresponds to the hole in the partial permutation.

On the other hand, the generating function corresponding to case (ii) in Observation 7.6 is

$$\frac{x^2C^3(x)(C(x)-1)}{1-xC(x)}$$

where in the numerator, one x corresponds to the hole, the other x corresponds to a; $C(x)-1$ corresponds to the nonempty B; $C^3(x)$ corresponds to A, C, and D.

We now sum the two functions and use the well-known relation $xC^2(x) = C(x) - 1$ to simplify the obtained expression:

$$\begin{aligned}\frac{x^2C^3(x)(C(x)-1)+xC(x)}{1-xC(x)} &= \frac{x^2C^4(x)(C(x)-1)+xC^2(x)}{C(x)-xC^2(x)} \\ &= x(C(x)-1)^2C^2(x)+C(x)-1 \\ &= (C(x)-1)(C^2(x)-2C(x)+2).\end{aligned}$$

From this, we get $s_n^1(1342) = \binom{2n-2}{n-1} - \binom{2n-2}{n-5}$, corresponding to sequence A026029 in [32] with indices shifted by one. □

Theorem 7.7. *For $n \geq 1$, we have $s_n^1(2413) = \frac{2}{n+1}\binom{2n}{n} - 2^{n-1}$.*

Proof. The following observation, coming directly from the definitions, characterizes the avoidance of 2413 in partial permutations with a single hole.

Observation 7.8. A partial permutation $\pi = \pi_1\pi_2\cdots\pi_{j-1}\diamond\pi_{j+1}\cdots\pi_n$ avoids the pattern 2413 if and only if it satisfies the following conditions (see also Figure 16):

1. The left part of π avoids 231.

2. The right part of π avoids 312.

3. If $a < b < c$ are three numbers such that a and c appear in the left part of π while b is in the right part, then c appears to the left of a.

4. If $a < b < c$ are three numbers such that a and c appear in the right part of π while b appears in the left part, then c appears to the left of a.

It follows that if both parts of π are nonempty, then π has one of the following two forms.

(i) The left minimum is larger than the right maximum, i.e., $\pi = A\diamond B$ with $A > B$ (see the right picture in Figure 16).

(ii) Otherwise, the left and the right parts of π must consist of decreasing sequences of blocks as shown in Figure 16 (the left picture) where A and B are nonempty; A (resp. B) is an arbitrary 231- (resp. 312-)avoiding permutation, and the remaining blocks are nonempty decreasing sequences. Moreover, in the places indicated by stars, we have possibly empty decreasing permutations. Formally speaking, in this case, π can be decomposed as $\pi = C_0C_1\ldots C_kA\diamond BD_1D_2\ldots D_{k+1}$ for some k so that

$$C_0 > B > C_1 > D_1 > C_2 > D_2 > \cdots > C_k > D_k > A > D_{k+1},$$

C_i's and D_i's are decreasing sequences, C_0 and D_{k+1} may be empty, and A and B are as described above.

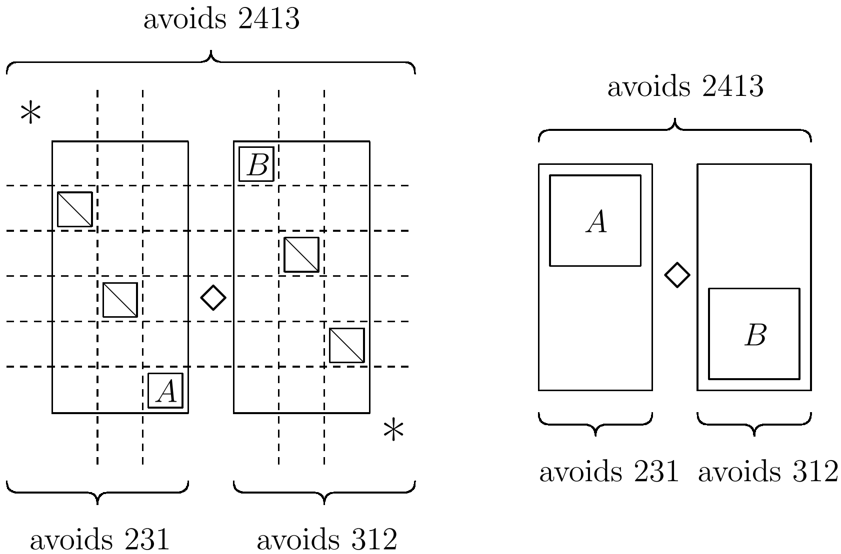

Figure 16: Two possible structures of partial permutations with one hole that avoid 2413.

Let us derive the generating function based on Observation 7.8.

If both the left and the right parts of π are empty, the corresponding generating function is x; if exactly one of the parts is empty, the generating function is $x(C(x)-1)$. Together, these cases have generating function $x(2C(x)-1)$. In what follows, we assume the parts are not empty.

The generating function for case (i) in Observation 7.8 is clearly $x(C(x)-1)^2$.

Consider case (ii) in Observation 7.8. The generating function for a nonempty decreasing block is $\frac{x}{1-x}$, whereas the generating function for a possibly empty such block is $\frac{1}{1-x}$. Thus, since the number of decreasing blocks in the left part is the same as that in the right part (not counting the places indicated by the stars), the number of partial permutations in this case has the following generating function (an extra x corresponds to the hole):

$$\frac{x}{(1-x)^2}\frac{(C(x)-1)^2}{1-\left(\frac{x}{1-x}\right)^2}=\frac{x(C(x)-1)^2}{1-2x}.$$

Summing the cases above, we see that the generating function for $s_n^1(2413)$ is

$$x(2C(x)-1)+x(C(x)-1)^2+\frac{x(C(x)-1)^2}{1-2x}=2C(x)-\frac{x}{1-2x}-2,$$

which gives $s_n^1(2413)=\frac{2}{n+1}\binom{2n}{n}-2^{n-1}$. □

7.2 Bijective proof of Theorem 7.4

Theorem 7.4 states that $s_n^1(1234)=\binom{2n-2}{n-1}$. We provide a bijective proof of this fact here.

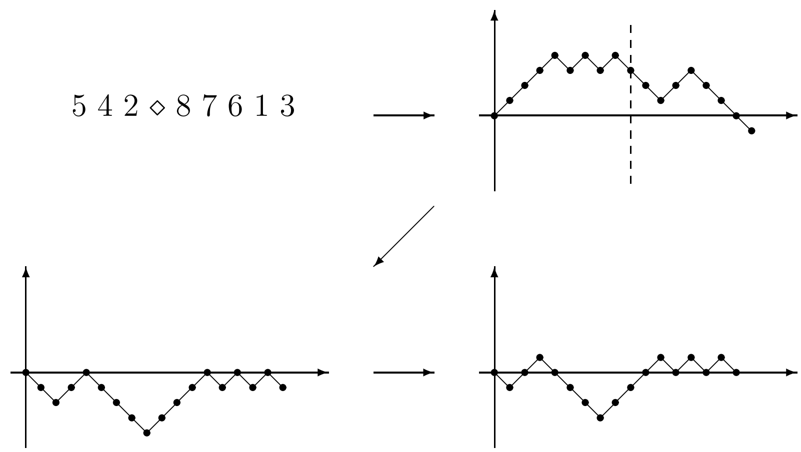

Figure 17: An example of a bijective map of $\pi \in \mathcal{S}_9^1(1234)$ to a lattice path from (0,0) to (16,0).

Theorem 7.9. *There is a bijection between partial permutations of length* n *with one hole that avoid 1234, and the set of all lattice paths from* $(0,0)$ *to* $(2n-2,0)$ *with steps* $(1,1)$ *and* $(1,-1)$.

Proof. Our proof is based on a known bijective proof of the fact that the number of Dyck paths of length $2n$ is given by the n-th Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Let $\pi \in \mathcal{S}_n^1(1234)$ and let the hole be in position i, $1 \le i \le n$. Remove the hole and map the obtained 123-avoiding permutation of length $n-1$, using any of your favorite bijections [15] to a Dyck path P from $(0,0)$ to $(2n-2,0)$. Now add a down-step at the end of P. Thus, P has n down-steps and $n-1$ up-steps. Cut P into two parts: P_1 is the (nonempty) part of all steps to the left, but *not* including the i-th down-step, and P_2 is the remaining, (nonempty) part. Move P_2 so that it starts from $(0,0)$ and append P_1 to it. We now have a path from $(0,0)$ to $(2n-1,-1)$ inducing, in an injective way, the desired path of length $2n-2$ from $(1,-1)$ to $(2n-1,-1)$.

The reverse to the procedure above is easy to see: append an extra down-step to the left of a given path from $(0,0)$ to $(2n-2,0)$ and shift the obtained path to start at $(0,0)$; find the leftmost minimum of the new path, cutting it into two parts and reassembling. Thus we get a bijection. In Figure 17 we provide an example using Krattenthaler's bijection from 123-avoiding permutations to Dyck paths (see [15]). □

8 Directions of further research

We have shown that classical Wilf equivalence may be regarded as a special case in a hierarchy of k-Wilf equivalence relations, and that many properties previously established in the context of Wilf equivalence can be generalized to all the k-Wilf equivalences. In many situations, understanding the k-Wilf equivalence class of a given pattern p becomes easier as k increases. Consider, for example, the identity permutation $\mathrm{id}_\ell = 123\cdots\ell$. We know that the $(\ell-1)$-Wilf equivalence class of id_ℓ contains every permutation of length ℓ, and we have shown that the $(\ell-2)$-Wilf equivalence class of id_ℓ contains exactly the Baxter permutations of length ℓ. What is the $(\ell-3)$-Wilf equivalence class of id_ℓ? For $\ell=3$ and $\ell=4$ that class contains exactly the layered permutations of length ℓ. Computer enumeration suggests that the same is true for $\ell=5$. We do not know whether this behavior generalizes to larger values of ℓ.

Another natural direction of further research is to extend known (general) results in permutation patterns theory to the setting of partial permutations. For example, it is natural to investigate the growth rate of $s_n^k(p)$ for a fixed k and a fixed pattern p, with $n\to\infty$. In the setting of non-partial permutations, Marcus and Tardos [28] have shown that for each pattern p, there is a constant K_p (known as the Stanley–Wilf limit of p), such that

$$\lim_{n\to\infty}\sqrt[n]{s_n^0(p)} = K_p \text{ or equivalently } s_n^0(p) = K_p^{n+o(n)}.$$

For a pattern p of length ℓ, a result of Valtr cited by Kaiser and Klazar [23] shows that $K_p \geq \Omega(\ell^2)$, while the best known general upper bound, due to Cibulka [14], is of order $2^{O(\ell\log\ell)}$. It is also easy to get a lower bound $K_p \geq \ell-1$ (see [10, Page 167, exc. 33]).

We do not know whether the limit $\lim_{n\to\infty}\sqrt[n]{s_n^k(p)}$ exists for every p and k. We can, however, bound the growth of $s_n^k(p)$ in terms of the Stanley–Wilf limits of certain subpermutations of p. To make this specific, let us introduce the following terminology: for two permutations $p\in\mathcal{S}_n$ and $q\in\mathcal{S}_m$, we say that q is a *consecutive subpattern of* p if for some $i\geq 0$ the consecutive subsequence $p_{i+1},p_{i+2},\dots,p_{i+m}$ of p is order-isomorphic to q.

Theorem 8.1. *Let $k\geq 0$ be an integer. Let p be a permutation pattern of length ℓ, with $\ell>k$. Let q be the consecutive subpattern of p of length $\ell-k$ chosen in such a way that its Stanley–Wilf limit K_q is as large as possible. We then have the bounds*

$$K_q^{n+o(n)} \leq s_n^k(p) \leq (k+1)^n K_q^{n+o(n)}.$$

Proof. Let us first prove the lower bound. Suppose that p has an occurrence of q at positions $i+1,i+2,\dots,i+\ell-k$, for some $i\geq 0$. Choose an $n\geq k$. Consider a partial permutation $\pi\in\mathcal{S}_n^k$ that begins with i holes, followed by a (non-partial) permutation π' of length $n-k$, followed by $k-i$ holes. It is easy to see that π avoids p if and only if π' avoids q, which means that $s_n^k(p)\geq s_{n-k}^0(q)$. This implies the desired lower bound.

To prove the upper bound, we fix an arbitrary $\varepsilon>0$, and we will show that $s_n^k(p)\leq C\binom{n}{k}(k+1)^{n-k}(K_q+\varepsilon)^n$ for some C depending on p, k and ε, but not on n. From this, the upper bound will easily follow.

Choose again an arbitrary $n\geq k$ and fix a set $H\subseteq[n]$ of size k. Let us estimate the size of $s_n^H(p)$. Let $i_1<i_2<\cdots<i_k$ be the elements of H. Let us also define $i_0=0$ and $i_{k+1}=n+1$. Each partial permutation $\pi\in s_n^H(p)$ can be written as

$$\pi = \pi^{(1)}\diamond\pi^{(2)}\diamond\cdots\diamond\pi^{(k)}\diamond\pi^{(k+1)},$$

where $\pi^{(j)}$ is a (possibly empty) subsequence of π of length $n_j = i_j - i_{j-1} - 1$ that does not contain any hole. Since π avoids p, it is clear that $\pi^{(j)}$ must avoid the consecutive subpattern $q^{(j)}$ of p that appears in p at positions $j, j+1, \ldots, j+\ell-k-1$. In other words, $\pi^{(j)}$ must be order-isomorphic to a $q^{(j)}$-avoiding permutation $\sigma^{(j)}$ of the set $[n_j]$.

Note that to describe a partial permutation $\pi \in s_n^H(p)$ uniquely, it is enough to specify for every $j \in [k+1]$ the $q^{(j)}$-avoiding permutation $\sigma^{(j)}$ of size n_j, and then, for each number $x \in [n-k]$, to specify which of the $k+1$ subsequences $\pi^{(j)}$ contains the value x.

Since each $q^{(j)}$ has Stanley–Wilf limit at most K_q, there is a constant Q (depending on p, k and ε) such that for every $m \in \mathbb{N}$ and every $j \in [k+1]$, there are at most $Q(K_q+\varepsilon)^m$ permutations of $[m]$ that avoid $q^{(j)}$. Thus, $s_n^H(p) \leq Q^{k+1}(k+1)^{n-k}(K_q+\varepsilon)^n$. Since there are $\binom{n}{k}$ possibilities for H, we get the desired bound for $s_n^k(p)$. □

We remark that for all the pattern-avoiding classes for which we can provide an enumeration, the limit $\lim_{n\to\infty} \sqrt[n]{s_n^k(p)}$ exists and is equal to the value K_q from Theorem 8.1. This means that the lower bound from Theorem 8.1 in general cannot be improved.

We close the section by summarizing the main open problems.

1. Find a combinatorial proof for the formulas for $s_n^1(1342)$ and $s_n^1(2413)$ derived in Theorems 7.5 and 7.7.

2. Which permutations are k-Wilf equivalent to id_{k+3}? Are they the layered permutations of length $k+3$?

3. We know that $s_n^H(\mathrm{id}_{k+3}) = C_{n-k}$ for any $n \geq k \geq 0$ and any set $H \subseteq [n]$ of size k, where C_m is the m-th Catalan number. Can we have $s_n^H(p) > C_{n-k}$ for some permutation p of length $k+3$, some set H of size k and some $n \geq k$? Can we even have $s_n^k(p) > s_n^k(\mathrm{id}_{k+3})$ for some $p \in \mathcal{S}_{k+3}$?

4. Does the limit $\lim_{n\to\infty} \sqrt[n]{s_n^k(p)}$ exist for each k and p? Is the limit equal to the value K_q defined in Theorem 8.1? Can the upper bound in Theorem 8.1 be improved?

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