Total colorings of F_5-free planar graphs with maximum degree 8

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Abstract

The total chromatic number of a graph G, denoted by $\chi''(G)$, is the minimum number of colors needed to color the vertices and edges of G such that no two adjacent or incident elements get the same color. It is known that if a planar graph G has maximum degree $\Delta \geqslant 9$, then $\chi''(G) = \Delta + 1$. The join $K_1 \vee P_n$ of K_1 and P_n is called a fan graph F_n. In this paper, we prove that if G is an F_5-free planar graph with maximum degree 8, then $\chi''(G) = 9$.

Keywords: planar graph; total coloring; cycle

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1 Introduction

All graphs considered in this paper are simple, finite, and undirected. We follow [2] for the terminology and notation not defined here. For a graph G, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$ (or simply V, E and Δ), respectively. For a face f of G, the *degree* $d(f)$ is the number of edges incident with it, where each cut-edge is counted twice. The join $K_1 \vee P_n$ of K_1 and P_n is called a *fan graph* F_n. We say that a graph G is F_n*-free* if G contains no F_n as a subgraph. A *k-cycle* is a cycle of length k. We say that two cycles are *adjacent* if they share at least one edge.

A *total k-coloring* of G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ is the smallest integer k such that G has a total k-coloring. Clearly, $\chi''(G) \geqslant \Delta + 1$. Behzad [1] and Vizing [16] independently posed the following famous conjecture, which is known as the total coloring conjecture (**TCC**).

Conjecture A. *For any graph G, $\chi''(G) \leqslant \Delta + 2$.*

This conjecture was confirmed for general graphs with $\Delta \leqslant 5$. In recent years, the study of total colorings for the class of planar graphs has attracted considerable attention. For planar graphs the only open case is $\Delta = 6$ ([8, 13]), and for planar graphs with large maximum degree, there is a stronger result. It is shown that $\chi''(G) = \Delta + 1$ if G is a planar graph with $\Delta \geqslant 9$ ([9]). This stronger result does not hold for planar graphs of maximum degree at most 3. For $4 \leqslant \Delta \leqslant 8$, it is unknown that $\chi''(G) = \Delta + 1$ if G is a planar graph with maximum degree Δ. For $\Delta = 8$, the following four results have been recently proved.

Theorem A. ([7]) *Let G be a planar graph with $\Delta = 8$. If G contains no adjacent 3-cycles, then $\chi''(G) = \Delta + 1$.*

Theorem B. ([15]) *Let G be a planar graph with $\Delta \geqslant 8$. If G contains no adjacent 4-cycles, then $\chi''(G) = \Delta + 1$.*

Theorem C. ([14]) *Let G be a planar graph with $\Delta \geqslant 8$. If G contains no 5- or 6-cycles with chords, then $\chi''(G) = \Delta + 1$.*

Theorem D. ([5]) *Let G be a planar graph with $\Delta \geqslant 8$. If G contain no 5-cycles with two chords, then $\chi''(G) = \Delta + 1$.*

Here, we generalize these results and get the following result.

Theorem 1. *If G be an F_5-free planar graph with $\Delta \geqslant 8$, then $\chi''(G) = \Delta + 1$.*

Recently, neighbor sum distinguishing total colorings have received much attention ([10]). In [11, 12] neighbor sum distinguishing total colorings of planar graphs have been studied.

Now, we introduce some more notations and definitions. Let G be a planar graph with a plane drawing, denote by F the face set of G. For a vertex v of G, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v. A *k-vertex*, a *k^--vertex* or a *k^+-vertex* is a vertex of degree k, at most k or at least k, respectively. Similarly, we can define a *k-face*, a *k^--face* and a *k^+-face*. We use $(v_1, v_2, \cdots, v_k)$ to denote a cycle (or a face) whose boundary vertices are $v_1, v_2, \cdots, v_k$ in the clockwise order in G. Denote by $n_d(v)$ the number of d-vertices adjacent to v, by $f_d(v)$ the number of d-faces incident with v.

2 Proof of Theorem 1

According to [9], planar graphs with $\Delta \geqslant 9$ have a total $(\Delta + 1)$-coloring, so to prove Theorem 1, in the following we assume that $\Delta = 8$. Let $G = (V, E, F)$ be a minimal counterexample to Theorem 1, such that $|V| + |E|$ is minimum. Then every proper subgraph of G has a total 9-coloring. Let L be the color set $\{1, 2, \cdots, 9\}$ for simplicity. It is easy to prove that G is 2-connected and hence the boundary of each face f is exactly a cycle. We first show some known properties on G.
(a) G contains no edge uv with $\min\{d(u), d(v)\} \leqslant 4$ and $d(u) + d(v) \leqslant 9$ (see [3]).
(b) G contains no even cycle $(v_1, v_2, \cdots, v_{2t})$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = 2$ (see [3]).

It follows from (a) that, the two neighbors of a 2-vertex are all 8-vertices, and any two 4^--vertices are not adjacent. Note that in all figures of the paper, vertices marked • have no edges of G incident with them other than those shown.

Lemma 2. ([5], [6]) *G has no configurations depicted in Figure 1, (1)–(6).*

Lemma 3. ([4]) *Suppose that v is an 8-vertex and $v_1, v_2, \cdots, v_k$ are consecutive neighbors of v with $d(v_1) = d(v_k) = 2$ and $d(v_i) \geqslant 3$ for $2 \leqslant i \leqslant k - 1$, where $k \in \{3, 4, 5, 6, 7\}$. If the face incident with v, v_i, v_{i+1} is a 4-face for all $1 \leqslant i \leqslant k - 1$, then at least one vertex in $\{v_2, v_3, \cdots, v_{k-1}\}$ is a 4^+-vertex.*

Lemma 4. ([17]) *Suppose that v is an 8-vertex and $u, v_1, v_2, \cdots, v_k$ are consecutive neighbors of v with $d(u) = d(v_1) = 2$ and $d(v_i) \geqslant 3$ for $2 \leqslant i \leqslant k$, where $k \in \{3, 4, 5, 6, 7\}$. If the face incident with v, v_i, v_{i+1} is a 4-face for all $1 \leqslant i \leqslant k - 2$, and the face incident with v, v_{k-1}, v_k is a 3-face, then at least one vertex in $\{v_2, v_3, \cdots, v_{k-1}\}$ is a 4^+-vertex.*

Lemma 5. ([5]) *Suppose that v is an 8-vertex and $u, v_1, v_2, \cdots, v_k$ are consecutive neighbors of v with $d(u) = 2$ and $d(v_i) \geqslant 3$ for $1 \leqslant i \leqslant k$, where $k \in \{4, 5, 6, 7\}$. If the face incident with v, v_i, v_{i+1} is a 4-face for all $2 \leqslant i \leqslant k - 2$, and the face incident with v, v_j, v_{j+1} is a 3-face for all $j \in \{1, k - 1\}$, then at least one vertex in $\{v_2, v_3, \cdots, v_{k-1}\}$ is a 4^+-vertex.*

Let φ be a (partial) total 9-coloring of G. For a vertex v of G, we denote by $C(v)$ the set of colors of edges incident with v. Call φ is *nice* if only some 4^--vertices are not colored. Note that every nice coloring can be greedily extended to a total 9-coloring of G, since each 4^--vertex is adjacent to at most four vertices and incident with at most four

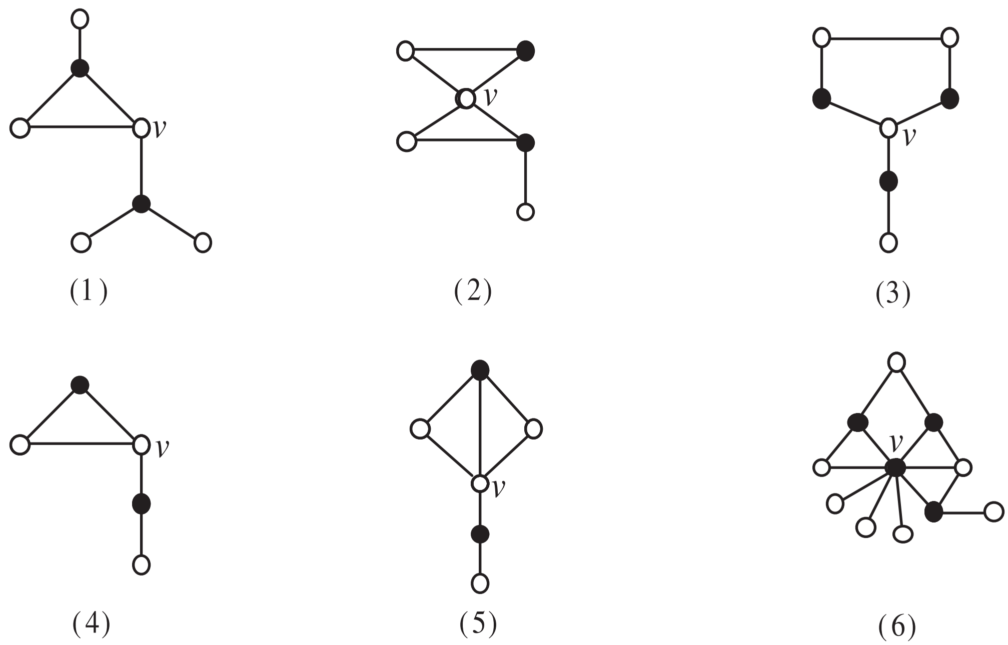

Figure 1: Reducible Configurations in G: $d(v) = 7$ in (1)

edges. Therefore, in the rest of this paper, we shall always suppose that such vertices are colored at the very end.

By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V}(2d(v) - 6) + \sum_{f \in F}(d(f) - 6) = -12 < 0.$$

We define ch to be the initial charge. Let $ch(v) = 2d(v) - 6$ for each $v \in V$ and $ch(f) = d(f) - 6$ for each $f \in F$. So $\sum_{x \in V \cup F} ch(x) = -12 < 0$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V \cup F$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have $\sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -12$. If we can show that $ch'(x) \geqslant 0$ for each $x \in V \cup F$, then we get an obvious contradiction to

$$0 \leqslant \sum_{x \in V \cup F} ch'(x) = \sum_{x \in V \cup F} ch(x) = -12,$$

which completes our proof.

For $f = (v_1, v_2, \cdots, v_k) \in F$, we use $(d(v_1), d(v_2), \cdots, d(v_k)) \to (c_1, c_2, \cdots, c_k)$ to denote that the vertex v_i sends f the amount of charge c_i for $i = 1, 2, \cdots, k$. Now we define the discharging rules as follows.

R1. Each 2-vertex receives 1 from each of its neighbors.

R2. For a 3-face (v_1, v_2, v_3), let

$(3^-, 7^+, 7^+) \to \big(0, \frac{3}{2}, \frac{3}{2}\big)$,

$(4, 6^+, 6^+) \to \big(\frac{1}{2}, \frac{5}{4}, \frac{5}{4}\big)$,

$(5^+, 5^+, 5^+) \to \big(1, 1, 1\big)$.

R3. For a 4-face (v_1, v_2, v_3, v_4), let

$(3^-, 7^+, 3^-, 7^+) \to \big(0, 1, 0, 1\big)$,

$(3^-, 7^+, 4^+, 7^+) \to \big(0, \frac{3}{4}, \frac{1}{2}, \frac{3}{4}\big)$,

$(4^+, 4^+, 4^+, 4^+) \to \big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big)$.

R4. For a 5-face $(v_1, v_2, v_3, v_4, v_5)$, let

$(3^-, 7^+, 3^-, 7^+, 7^+) \to \big(0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\big)$,

$(3^-, 7^+, 4^+, 4^+, 7^+) \to \big(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\big)$,

$(4^+, 4^+, 4^+, 4^+, 4^+) \to \big(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\big)$.

Next we show that $ch'(x) \geqslant 0$ for each $x \in V \cup F$. Since our discharging rules are designed such that $ch'(f) \geqslant 0$ for all $f \in F$ and $ch'(v) \geqslant 0$ for all 2-vertices $v \in V$, it suffices to check that $ch'(v) \geqslant 0$ for all 3^+-vertices in G. Let $v \in V$. Suppose $d(v) = 3$. Then $ch'(v) = ch(v) = 0$. Suppose $d(v) = 4$. Then v sends at most $\frac{1}{2}$ to each of its incident faces and $ch'(v) \geqslant ch(v) - \frac{1}{2} \times 4 = 0$. Suppose $d(v) = 5$. Then $f_3(v) \leqslant 3$, and v sends at most 1 to each of its incident 3-faces by R2, at most $\frac{1}{2}$ to each of its incident 4^+-faces by R3 and R4. So $ch'(v) \geqslant ch(v) - f_3(v) \times 1 - (5 - f_3(v)) \times \frac{1}{2} = \frac{3}{2} - \frac{1}{2} f_3(v) \geqslant 0$. Suppose $d(v) = 6$. Then $f_3(v) \leqslant 4$, and v sends at most $\frac{5}{4}$ to each of its incident 3-faces, at most $\frac{1}{2}$ to each of its incident 4^+-faces. So $ch'(v) \geqslant ch(v) - f_3(v) \times \frac{5}{4} - (6 - f_3(v)) \times \frac{1}{2} = 3 - \frac{3}{4} f_3(v) \geqslant 0$.

Call a 3-face is *bad* if it has a 3^--vertex, a 4-face is *bad* if it has two 3^--vertices, *good* otherwise.

Suppose $d(v) = 7$. Note that $f_3(v) \leqslant 5$. If $f_3(v) \leqslant 2$, then $ch'(v) \geqslant ch(v) - f_3(v) \times \frac{3}{2} - (7 - f_3(v)) \times 1 = 1 - \frac{1}{2} f_3(v) \geqslant 0$. Suppose $3 \leqslant f_3(v) \leqslant 5$, then v is incident with at most two bad 3-faces by Figure 1(1). If $3 \leqslant f_3(v) \leqslant 4$, then $ch'(v) \geqslant ch(v) - \max\{2 \times \frac{3}{2} + (f_3(v) - 2) \times \frac{5}{4} + (7 - f_3(v)) \times \frac{1}{2}, \frac{3}{2} + (f_3(v) - 1) \times \frac{5}{4} + \frac{3}{4} + (7 - f_3(v) - 1) \times \frac{1}{2}, f_3(v) \times \frac{5}{4} + 2 \times 1 + (7 - f_3(v) - 2) \times \frac{3}{4}\} = \frac{9}{4} - \frac{1}{2} f_3(v) \geqslant \frac{1}{4} > 0$. If $f_3(v) = 5$, then $ch'(v) \geqslant ch(v) - \max\{2 \times \frac{3}{2} + 3 \times \frac{5}{4} + 2 \times \frac{1}{2}, \frac{3}{2} + 4 \times \frac{5}{4} + \frac{3}{4} + \frac{1}{2}\} = \frac{1}{4} > 0$.

Suppose $d(v) = 8$. Let $v_1, v_2, \cdots, v_8$ be neighbors of v and $f_1, f_2, \cdots, f_8$ be faces incident with v in an clockwise order, where f_i is incident with v_i, v_{i+1}, and $i \in \{1, 2, \cdots, 8\}$. Note that all the subscripts in the paper are taken modulo 8. First, we prove some lemmas.

Lemma 6. *Suppose that v is an 8-vertex and $v_1, v_2, \cdots, v_k, v_{k+1}, v_s, v_{s+1}$ are consecutive neighbors of v with $d(v_1) = 2$ and $d(v_i) = 3$ for $2 \leqslant i \leqslant k$, where $3 \leqslant k + 1 \leqslant s$ and $s \in \{3, 5, \cdots, 7\}$. If v is incident with 3-faces (v, v_k, v_{k+1}) and (v, v_s, v_{s+1}), and incident with 4-faces (v, v_j, x_j, v_{j+1}) for all $1 \leqslant j \leqslant k - 1$, then* $\min\{d(v_s), d(v_{s+1})\} \geqslant 4$.

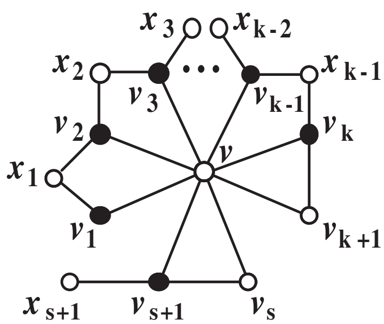

Figure 2: Reducible Configuration in G

Proof. By Figure 1(2), we have $\min\{d(v_s), d(v_{s+1})\} \geqslant 3$. Assume to be contradictory that $d(v_s) = 3$ or $d(v_{s+1}) = 3$. Without loss of generality, suppose that $d(v_{s+1}) = 3$, and $N(v_{s+1}) = \{v, v_s, x_{s+1}\}$ (see Figure 2). Consider a nice coloring φ of $G' = G - vv_1$. If $\varphi(v_1x_1) \in C(v)$, then the forbidden colors for vv_1 number at most 8, so vv_1 can be properly colored. Then we can suppose $\varphi(v_1x_1) \notin C(v)$. Without loss of generality, suppose that $\varphi(v) = 9$, $\varphi(v_1x_1) = 1$, and $\varphi(vv_j) = j$ for $j \in \{2, \cdots, k, k+1, s, s+1\}$. It is easy to see that $1 \in C(v_j)$ for $j \in \{2, \cdots, k, s+1\}$, since otherwise, we can recolor vv_j with 1, color vv_1 with j, a contradiction. So $\varphi(v_2x_2) = \cdots = \varphi(v_{k-1}x_{k-1}) = \varphi(v_kv_{k+1}) = 1$ and $1 \in \{\varphi(v_sv_{s+1}), \varphi(v_{s+1}x_{s+1})\}$. Note that $\varphi(v_kx_{k-1}) = k+1$, since otherwise, we may get a contradiction by exchange the colors on vv_{k+1} and v_kv_{k+1}, color vv_1 with $k+1$. Thus $\varphi(v_{k-1}x_{k-2}) = k+1$, since otherwise, we exchange the colors on vv_{k+1} and v_kv_{k+1}, v_kx_{k-1} and $v_{k-1}x_{k-1}$, color vv_1 with $k+1$, also a contradiction. Similarly, $\varphi(v_{k-2}x_{k-3}) = \cdots = \varphi(v_2x_1) = k+1$.

If $k+1 = s$, then $\varphi(v_{s+1}x_{s+1}) = 1$. We exchange the colors on vv_{k+1} and v_kv_{k+1}, v_kx_{k-1} and $v_{k-1}x_{k-1}$, $\cdots$, v_1x_1 and v_2x_1, recolor vv_{s+1} with $k+1$, color vv_1 with $s+1$, a contradiction. So we can suppose $k+1 < s$. Then $k+1 \in \{\varphi(v_sv_{s+1}), \varphi(v_{s+1}x_{s+1})\}$, since otherwise, we can exchange the colors on vv_{k+1} and v_kv_{k+1}, v_kx_{k-1} and $v_{k-1}x_{k-1}$, $\cdots$, v_1x_1 and v_2x_1, recolor vv_{s+1} with $k+1$, color vv_1 with $s+1$, a contradiction. We first exchange the colors on vv_s and v_sv_{s+1}. If $\varphi(v_sv_{s+1}) = k+1$, we additionally exchange the colors on vv_{k+1} and v_kv_{k+1}, v_kx_{k-1} and $v_{k-1}x_{k-1}$, $\cdots$, v_1x_1 and v_2x_1. Then we color vv_1 with s, also a contradiction. □

Lemma 7. *Suppose that v is an 8-vertex and $N(v) = \{v_i | i = 1, 2, \cdots, 8\}$ with $d(v_2) = 3$. If vv_2 is incident with two 3-faces (v, v_1, v_2) and (v, v_2, v_3), then there exists at most one 3-vertex $v_j (j \neq 2)$ such that vv_j is incident with a 3-face.*

Proof. By Property (a), we have $\min\{d(v_1), d(v_3)\} \geqslant 7$. Suppose, to be contradictory, that there are two 3-vertices v_j and v_k $(4 \leqslant j < k \leqslant 8)$, such that vv_j is incident with a 3-face and vv_k is incident with another 3-face. Consider a nice coloring φ of $G' = G - vv_2$. Without loss of generality, suppose that $\varphi(v) = 2$ and $\varphi(vv_i) = i$ for $i \in \{1, 3, 4, 5, 6, 7, 8\}$. If $9 \notin C(v_2)$, then we can obtain a nice coloring of G by coloring vv_2 with 9, a contradiction.

So $9 \in C(v_2)$, that is, $\varphi(v_1v_2) = 9$ or $\varphi(v_2v_3) = 9$. Without loss of generality, suppose that $\varphi(v_1v_2) = 9$. At the same time, we have the following results:

(1) For some $i \in \{j, k\}$, if $\varphi(v_2v_3) \neq i$ then $9 \in C(v_i)$;

(2) For some $i \in \{j, k\}$, if $\varphi(v_2v_3) \notin \{1, i\}$, then $C(v_i) = \{1, i, 9\}$;

(3) For some $i \in \{j, k\}$, if $\varphi(v_2v_3) = 1$, then $C(v_i) = \{3, i, 9\}$.

For (1), if $9 \notin C(v_i)$, then we can recolor vv_i with 9, and color vv_2 with i to obtain a nice coloring of G, a contradiction. For (2), if $\{1, i, 9\} \subset C(v_i)$, then we exchange the colors on vv_1 and v_1v_2, recolor vv_i with 1, and color vv_2 with i, a contradiction again. For (3), if $\{3, i, 9\} \subset C(v_i)$, then we exchange the colors on vv_1 and v_1v_2, vv_3 and v_2v_3, recolor vv_i with 3, and color vv_2 with i, a contradiction.

Case 1. $v_1v_k \notin E(G)$ and $v_3v_j \notin E(G)$.

Without loss of generality, suppose that $N(v_j) = \{v, v_{j+1}, x_j\}$ and $N(v_k) = \{v, v_{k-1}, x_k\}$ (see Fig. 3(1)). It is obvious that $v_{j+1} \neq v_k$. Suppose $\varphi(v_2v_3) = 1$. Then $C(v_j) = \{3, j, 9\}$ by (3). We exchange the colors on vv_{j+1} and v_jv_{j+1}, color vv_2 with $j+1$. If $\varphi(v_jv_{j+1}) = 3$, then we additionally exchange the colors on vv_1 and v_1v_2, vv_3 and v_2v_3. Thus we obtain a nice coloring of G, a contradiction.

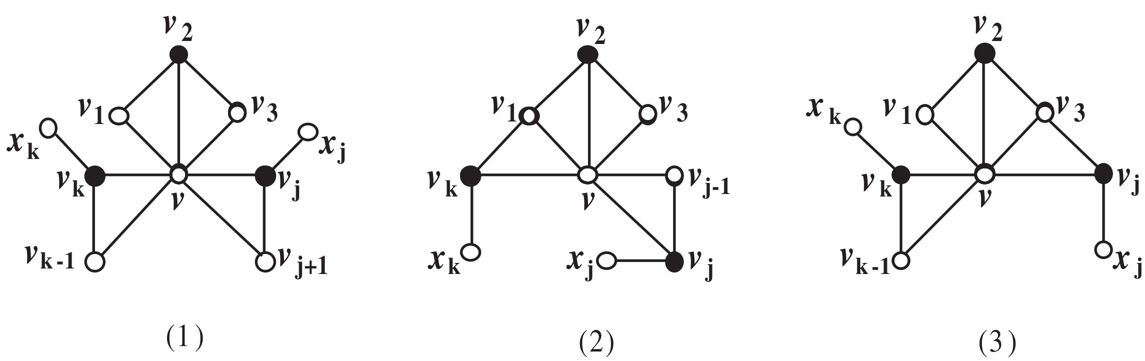

Figure 3: Reducible Configurations in G

Suppose $\varphi(v_2v_3) = j + 1$. Then $C(v_j) = \{1, j, 9\}$ and $C(v_k) = \{1, k, 9\}$ by (2). We exchange the colors on vv_{j+1} and v_jv_{j+1}, recolor vv_k with $j + 1$, and color vv_2 with k. If $\varphi(v_jv_{j+1}) = 1$, then we additionally exchange the colors on vv_1 and v_1v_2. Thus we also obtain a nice coloring of G, a contradiction, too. So we have $\varphi(v_2v_3) \notin \{1, j + 1\}$. Since $\varphi(v_2v_3)$ is different from either j or k, we may assume that $\varphi(v_2v_3) \neq j$. Then $C(v_j) = \{1, j, 9\}$ by (2). We exchange the colors on vv_{j+1} and v_jv_{j+1}, color vv_2 with $j+1$. If $\varphi(v_jv_{j+1}) = 1$, then we additionally exchange the colors on vv_1 and v_1v_2. Thus we obtain a nice coloring of G, a contradiction.

Case 2. $v_1v_k \in E(G)$.

Without loss of generality, suppose that $N(v_j) = \{v, v_{j-1}, x_j\}$ and $N(v_k) = \{v, v_1, x_k\}$ (see Figure 3(2)). If $\varphi(v_2v_3) \notin \{1, k\}$, then $C(v_k) = \{1, k, 9\}$ by (2), so $\varphi(v_1v_k) = 1$ or $\varphi(v_1v_k) = 9$, a contradiction. Suppose $\varphi(v_2v_3) = 1$. Then $C(v_j) = \{3, j, 9\}$ and

$C(v_k) = \{3, k, 9\}$ by (3). If $v_3 = v_{j-1}$, then $\varphi(v_3v_j) = 9$ and $\varphi(v_1v_k) = 3$. We exchange the colors on vv_1 and v_1v_2, v_2v_3 and v_3v_j, recolor vv_k with 1, and color vv_2 with k, a contradiction. So we can suppose $v_3 \neq v_{j-1}$. We exchange the colors on vv_{j-1} and $v_{j-1}v_j$, and color vv_2 with $j-1$. If $\varphi(v_{j-1}v_j) = 3$, then we additionally exchange the colors on vv_1 and v_1v_2, vv_3 and v_2v_3. Thus we obtain a nice coloring of G, a contradiction. Suppose $\varphi(v_2v_3) = k$. Then $C(v_j) = \{1, j, 9\}$ by (2). We exchange the colors on vv_{j-1} and $v_{j-1}v_j$, color vv_2 with $j-1$. If $\varphi(v_{j-1}v_j) = 1$, then we additionally exchange the colors on vv_1 and v_1v_2. Thus we also obtain a nice coloring of G, a contradiction, too.

Case 3. $v_3v_j \in E(G)$, but $v_1v_k \notin E(G)$.

Without loss of generality, suppose that $N(v_j) = \{v, v_3, x_j\}$ and $N(v_k) = \{v, v_{k-1}, x_k\}$ (see Figure 3(3)). It is obvious that $v_j \neq v_{k-1}$. Suppose $\varphi(v_2v_3) = 1$. Then $C(v_j) = \{3, j, 9\}$ and $C(v_k) = \{3, k, 9\}$ by (3). We exchange the colors on vv_1 and v_1v_2, v_2v_3 and v_3v_j, recolor vv_k with 1, and color vv_2 with k. a contradiction. Suppose $\varphi(v_2v_3) = j$. Then $C(v_k) = \{1, k, 9\}$ by (2). We exchange the colors on vv_{k-1} and $v_{k-1}v_k$, color vv_2 with $k-1$. If $\varphi(v_{k-1}v_k) = 1$, then we additionally exchange the colors on vv_1 and v_1v_2. Thus we also obtain a nice coloring of G, a contradiction. So we have $\varphi(v_2v_3) \notin \{1, j\}$. Then $C(v_j) = \{1, j, 9\}$ by (2). We exchange the colors on vv_3 and v_3v_j, color vv_2 with 3. If $\varphi(v_3v_j) = 1$, then we additionally exchange the colors on vv_1 and v_1v_2. Thus we also obtain a nice coloring of G, a contradiction, too. □

Lemma 8. *Suppose that $d(v_i) = d(v_k) = 2$ and $d(v_j) \geqslant 3$ for all $j = i+1, \cdots, k-1$, where $k \geqslant i+2$. If $\min\{d(f_i), d(f_{i+1}), \cdots, d(f_{k-1})\} \geqslant 4$, then v sends at most $\frac{3}{2} + (k-i-2)$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$.*

Proof. By Lemma 3, $\max\{d(v_{i+1}), \cdots, d(v_{k-1})\} \geqslant 4$ or $\max\{d(f_i), \cdots, d(f_{k-1})\} \geqslant 5$. If $\max\{d(v_{i+1}), \cdots, d(v_{k-1})\} \geqslant 4$, then v sends at most $2 \times \frac{3}{4} + (k-i-2)$ (in total) to $f_i, \cdots, f_{k-1}$ by R3. If $\max\{d(f_i), \cdots, d(f_{k-1})\} \geqslant 5$, then or v sends at most $\frac{1}{3} + (k-i-1)$ (in total) to $f_i, \cdots, f_{k-1}$ by R3 and R4. Since $2 \times \frac{3}{4} > 1 + \frac{1}{3}$, v sends at most $\frac{3}{2} + (k-i-2)$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. □

Lemma 9. *Suppose that $d(v_i) = d(v_{i+4}) = 2$ and $d(v_j) \geqslant 3$ for all $j = i+1, i+2, i+3$. If $\min\{d(f_i), d(f_{i+2}), d(f_{i+3})\} \geqslant 4$ and $d(f_{i+1}) = 3$, then v sends at most $\frac{15}{4}$ (in total) to f_i, f_{i+1}, f_{i+2} and f_{i+3}.*

Proof. If $d(v_{i+1}) = 3$, then $d(v_{i+2}) \geqslant 7$, and $d(f_i) \geqslant 5$ by Lemma 4, so v sends at most $\frac{1}{3} + \frac{3}{2} + \frac{3}{4} + 1 = \frac{43}{12}$ to f_i, f_{i+1}, f_{i+2} and f_{i+3}. If $d(v_{i+2}) = 3$, then $d(v_{i+1}) \geqslant 7$, and $d(v_{i+3}) \geqslant 4$ or there is at least one 5^+-face in $\{f_{i+2}, f_{i+3}\}$ by Lemma 4, so v sends at most $\frac{3}{4} + \frac{3}{2} + \max\{2 \times \frac{3}{4}, 1 + \frac{1}{3}\} = \frac{15}{4}$ to f_i, f_{i+1}, f_{i+2} and f_{i+3}. If $\min\{d(v_{i+1}), d(v_{i+2})\} \geqslant 4$, then v sends at most $\frac{3}{4} + \frac{5}{4} + \frac{3}{4} + 1 = \frac{15}{4}$ to f_i, f_{i+1}, f_{i+2} and f_{i+3}. Since $\frac{43}{12} < \frac{15}{4}$, v sends at most $\frac{15}{4}$ (in total) to f_i, f_{i+1}, f_{i+2} and f_{i+3}. □

Lemma 10. *Suppose that $d(v_i) = d(v_k) = 2$ and $d(v_j) \geqslant 3$ for all $j = i+1, \cdots, k-1$, where $k \geqslant i+3$. If $\min\{d(f_i), d(f_{k-1})\} \geqslant 4$ and $d(f_{i+1}) = \cdots = d(f_{k-2}) = 3$, then v sends at most $\frac{11}{4} + (k-i-3) \times \frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$.*

Proof. We note that if $k \geqslant i+4$, then $\min\{d(v_{i+2}), \cdots, d(v_{k-2})\} \geqslant 4$ by Figure 1(5). If $d(f_i) = d(f_{k-1}) = 4$, then $\min\{d(v_{i+1}), d(v_{k-1})\} \geqslant 4$ by Lemma 4, so v sends at most $2 \times \frac{3}{4} + (k-i-2) \times \frac{5}{4} = \frac{11}{4} + (k-i-3) \times \frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. If one of f_i and f_{k-1} is 4−face, then v sends at most $\frac{3}{4} + \frac{1}{3} + \frac{3}{2} + (k-i-3) \times \frac{5}{4} = \frac{31}{12} + (k-i-3) \times \frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. If $\min\{d(f_i), d(f_{k-1})\} \geqslant 5$, then v sends at most $2 \times \frac{1}{3} + 2 \times \frac{3}{2} + (k-i-4) \times \frac{5}{4} = \frac{29}{12} + (k-i-3) \times \frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. Since $\max\{\frac{11}{4}, \frac{31}{12}, \frac{29}{12}\} = \frac{11}{4}$, v sends at most $\frac{11}{4} + (k-i-3) \times \frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. □

Now, we come back to check the new charge of 8-vertex v and consider nine cases in the following.

Case 1. $n_2(v) = 8$. Note that $f_{6^+}(v) = 8$ by Figure 1(3) and (4). Then, no charge is discharged from v to its incident faces. So $ch'(v) = ch(v) - 8 \times 1 = 10 - 8 = 2 > 0$ by R1.

Case 2. $n_2(v) = 7$. Then $f_{6^+}(v) \geqslant 6$ and $f_3(v) = 0$ by Figure 1(4). So $ch'(v) \geqslant ch(v) - 7 \times 1 - 2 \times 1 = 10 - 9 = 1 > 0$.

Case 3. $n_2(v) = 6$. Then there are four possibilities in which 2-vertices are located. They are shown as configurations in Figure 4. For Figure 4(1), $f_{6^+}(v) \geqslant 5$ and $f_3(v) \leqslant 1$. So $ch'(v) \geqslant ch(v) - 6 \times 1 - \frac{3}{2} - 2 \times 1 = \frac{1}{2} > 0$. For Figure 4(2)–(4), $f_{6^+}(v) \geqslant 4$ and $f_3(v) = 0$. So $ch'(v) \geqslant ch(v) - 6 \times 1 - 4 \times 1 = 0$.

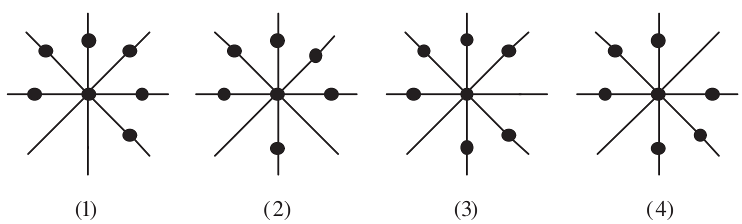

Figure 4: Fig. 4. $n_2(v) = 6$

Case 4. $n_2(v) = 5$. Then there are five possibilities in which 2-vertices are located. They are shown as configurations in Figure 5.

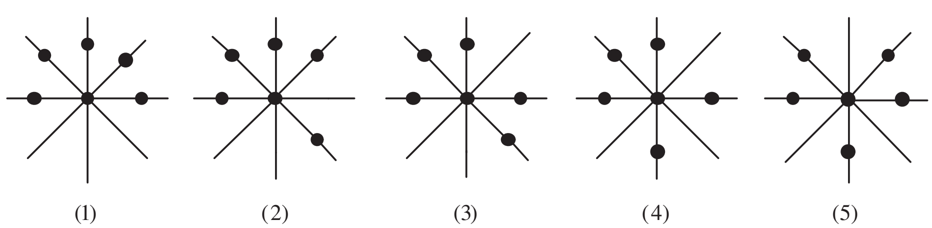

Figure 5: $n_2(v) = 5$

For Figure 5(1), $f_{6^+}(v) \geqslant 4$ and $f_3(v) \leqslant 2$. So $ch^{'}(v) \geqslant ch(v) - 5\times 1 - 2\times \frac{3}{2} - 2\times 1 = 0$. For Figure 5(2) and (3), $f_{6^+}(v) \geqslant 3$ and $f_3(v) \leqslant 1$. So $ch^{'}(v) \geqslant ch(v) - 5 \times 1 - \frac{3}{2} - \max\{\frac{11}{4}, \frac{3}{2}+1\} = \frac{3}{4} > 0$ by Lemma 8 and Lemma 10. For Figure 5(4) and (5), $f_{6^+}(v) \geqslant 2$ and $f_3(v) = 0$. So $ch^{'}(v) \geqslant ch(v) - 5 \times 1 - 3 \times \frac{3}{2} = \frac{1}{2} > 0$.

Case 5. $n_2(v) = 4$. Then there are eight possibilities in which 2-vertices are located. They are shown as configurations in Figure 6.

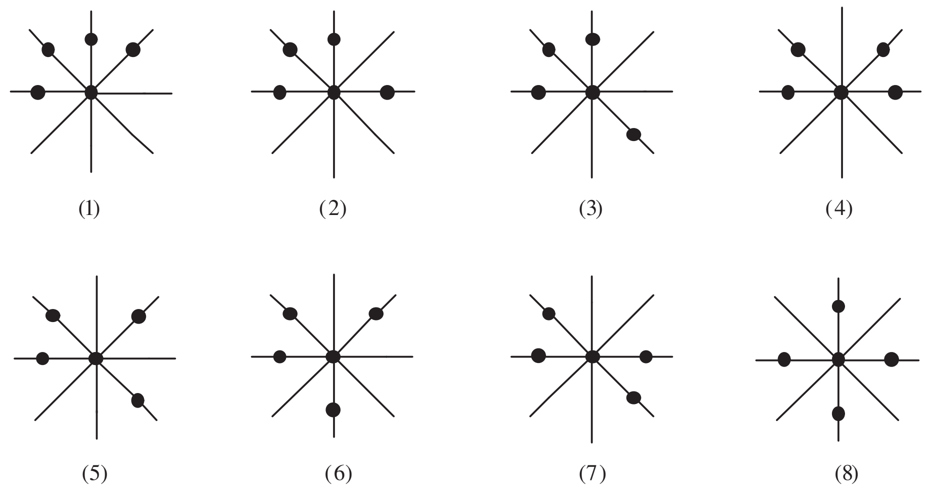

Figure 6: $n_2(v) = 4$

For Figure 6(1), $f_{6^+}(v) \geqslant 3$ and $f_3(v) \leqslant 3$. If $f_3(v) = 3$, then $ch^{'}(v) \geqslant ch(v) - 4 \times 1 - (\frac{11}{4} + 2 \times \frac{5}{4}) = \frac{3}{4} > 0$. Otherwise, $ch^{'}(v) \geqslant ch(v) - 4 \times 1 - f_3(v) \times \frac{3}{2} - (5 - f_3(v)) \times 1 = 1 - \frac{1}{2}f_3(v) \geqslant 0$. For Figure 6(2) and (4), $f_{6^+}(v) \geqslant 2$ and $f_3(v) \leqslant 2$. If $f_3(v) = 2$, then $ch^{'}(v) \geqslant ch(v) - 4 \times 1 - \frac{3}{2} - (\frac{11}{4} + \frac{5}{4}) = \frac{1}{2} > 0$ by Lemma 8 and Lemma 10. Otherwise, $ch^{'}(v) \geqslant ch(v) - 4\times 1 - \frac{3}{2} - f_3(v)\times\frac{3}{2} - (4 - f_3(v))\times 1 = \frac{1}{2} - \frac{1}{2}f_3(v) \geqslant 0$. For Figure 6(3) and (7), $f_{6^+}(v) \geqslant 2$ and $f_3(v) \leqslant 2$. So $ch^{'}(v) \geqslant ch(v) - 4\times 1 - f_3(v)\times\frac{11}{4} - (2 - f_3(v))\times(\frac{3}{2}+1) = 1 - \frac{1}{4}f_3(v) > 0$ by Lemma 8 and Lemma 10. For Figure 6(5) and (6), $f_{6^+}(v) \geqslant 1$ and $f_3(v) \leqslant 1$. So $ch^{'}(v) \geqslant ch(v) - 4\times 1 - 2\times\frac{3}{2} - f_3(v)\times\frac{11}{4} - (1 - f_3(v))\times(\frac{3}{2}+1) = \frac{1}{2} - \frac{1}{4}f_3(v) > 0$. For Figure 6(8), $f_3(v) = 0$. So $ch^{'}(v) \geqslant ch(v) - 4 \times 1 - 4 \times \frac{3}{2} = 0$.

Case 6. $n_2(v) = 3$. Then there are five possibilities in which 2-vertices are located. They are shown as configurations in Figure 7.

For Figure 7(1), note that $\min\{d(f_1), d(f_2)\} \geqslant 6$, $\min\{d(f_3), d(f_8)\} \geqslant 4$, and $f_3(v) \leqslant 3$. If $f_3(v) \leqslant 2$, then $ch^{'}(v) \geqslant ch(v) - 3 \times 1 - f_3(v) \times \frac{3}{2} - (6 - f_3(v)) \times 1 = 1 - \frac{1}{2}f_3(v) \geqslant 0$. Suppose $f_3(v) = 3$, Then $\min\{d(f_4), d(f_7)\} = 3$. Without loss of generality, suppose that $d(f_4) = 3$, then v sends at most $\frac{3}{4}$ to f_3 by Lemma 4. If $d(f_7) = 3$, then $ch^{'}(v) \geqslant ch(v) - 3 \times 1 - 1 - 2 \times \frac{3}{4} - 3 \times \frac{3}{2} = 0$. Otherwise, $d(f_4) = d(f_5) = d(f_6) = 3$, then f_5 is good by Figure 1(5). So $ch^{'}(v) \geqslant ch(v) - 3 \times 1 - 2 \times 1 - \frac{3}{4} - \frac{5}{4} - 2 \times \frac{3}{2} = 0$.

For Figure 7(2), $d(f_1) \geqslant 6$, $\min\{d(f_2), d(f_3), d(f_4), d(f_8)\} \geqslant 4$, and $f_3(v) \leqslant 3$. If

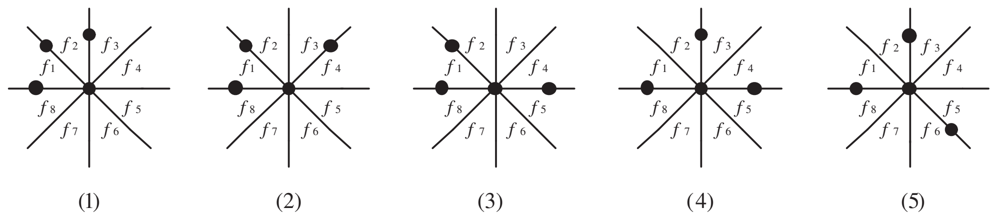

Figure 7: $n_2(v) = 3$

$f_3(v) \leqslant 1$, then $ch'(v) \geqslant ch(v) - 3\times 1 - \frac{3}{2} - f_3(v)\times\frac{3}{2} - (5 - f_3(v))\times 1 = \frac{1}{2} - \frac{1}{2}f_3(v) \geqslant 0$ by Lemma 8. If $f_3(v) = 3$, then $d(f_5) = d(f_6) = d(f_7) = 3$, so $ch'(v) \geqslant ch(v) - 3\times 1 - \frac{3}{2} - (\frac{11}{4} + 2\times\frac{5}{4}) = \frac{1}{4} > 0$ by Lemma 8 and Lemma 10. Suppose $f_3(v) = 2$. If $\max\{d(f_4), d(f_8)\} \geqslant 5$, then $ch'(v) \geqslant ch(v) - 3\times 1 - \frac{3}{2} - 2\times\frac{3}{2} - \frac{1}{3} - 2\times 1 = \frac{1}{6} > 0$. Otherwise, without loss of generality, suppose that $d(f_5) = 3$. If $d(f_6) = 3$, then f_4 and f_5 are good by Figure 1(5) and Lemma 4. So $ch'(v) \geqslant ch(v) - 3\times 1 - \frac{3}{2} - \frac{3}{4} - \frac{5}{4} - \frac{3}{2} - 2\times 1 = 0$. If $d(f_7) = 3$, then f_4 and f_8 are good. So $ch'(v) \geqslant ch(v) - 3\times 1 - \frac{3}{2} - 2\times\frac{3}{4} - 2\times\frac{3}{2} - 1 = 0$.

For Figure 7(3), $d(f_1) \geqslant 6$, $\min\{d(f_2), d(f_4), d(f_5), d(f_8)\} \geqslant 4$, and $f_3(v) \leqslant 3$. If $f_3(v) = 3$, then $ch'(v) \geqslant ch(v) - 3\times 1 - (\frac{11}{4} + \frac{5}{4}) - \frac{11}{4} = \frac{1}{4} > 0$ by Lemma 10. Otherwise, $f_3(v) \leqslant 2$. If $d(f_3) = 3$, then $ch'(v) \geqslant ch(v) - 3\times 1 - \frac{11}{4} - \max\{\frac{15}{4}, 4\times 1\} = \frac{1}{4} > 0$. If $d(f_3) \geqslant 4$, then $ch'(v) \geqslant ch(v) - 3\times 1 - (\frac{3}{2} + 1) - \max\{\frac{11}{4} + \frac{5}{4}, \frac{15}{4}, 4\times 1\} = \frac{1}{2} > 0$.

For Figure 7(4), $f_3(v) \leqslant 2$. So $ch'(v) \geqslant ch(v) - 3\times 1 - 2\times\frac{3}{2} - \max\{\frac{11}{4} + \frac{5}{4}, \frac{15}{4}, 4\times 1\} = 0$. For Figure 7(5), $f_3(v) \leqslant 2$. So $ch'(v) \geqslant ch(v) - 3\times 1 - \frac{3}{2} - f_3(v)\times\frac{11}{4} - (2 - f_3(v))\times(\frac{3}{2} + 1) = \frac{1}{2} - \frac{1}{4}f_3(v) \geqslant 0$.

Case 7. $n_2(v) = 2$. Then there are four possibilities in which 2-vertices are located. They are shown as configurations in Figure 8.

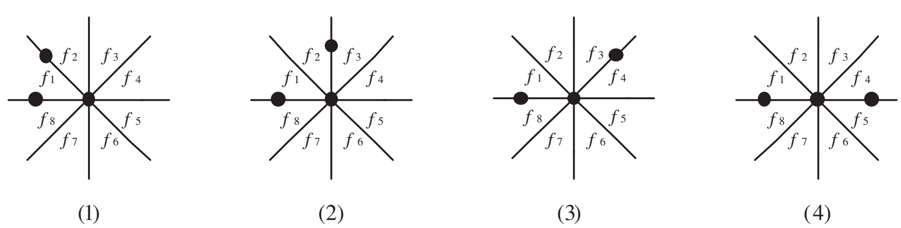

Figure 8: $n_2(v) = 2$

For Figure 8(1), note that $d(f_1) \geqslant 5$ and $f_3(v) \leqslant 4$. Suppose $f_3(v) = 4$. Then without loss of generality, let $d(f_3) = d(f_4) = d(f_7) = d(f_i) = 3$ ($i \in \{5, 6\}$). Then $d(v_4) \geqslant 4$ by Figure 1(5), and v sends at most $\max\{\frac{1}{3} + \frac{3}{2}, \frac{3}{4} + \frac{5}{4}\} = 2$ (in total) to f_2 and f_3. If $d(f_8) \geqslant 5$, then $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - 2 - 3\times\frac{3}{2} - \frac{3}{4} - \frac{1}{3} = \frac{1}{12} > 0$ by Lemma 5. Otherwise, $d(f_8) = 4$, then $d(v_8) \geqslant 4$ by Lemma 4, it follows that f_4 (if $i = 5$) or f_7 (if

$i = 6$) is good, and v sends at most $\max\{\frac{3}{2}+\frac{3}{2}+\frac{1}{3}, \frac{3}{2}+\frac{5}{4}+\frac{3}{4}\} = \frac{7}{2}$ (in total) to f_5, f_6 and f_7 (or f_4). So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - 2 - \frac{5}{4} - \frac{7}{2} - \frac{3}{4} = \frac{1}{6} > 0$.

Suppose $f_3(v) = 3$. If $f_{5^+}(v) \geqslant 3$, then $ch'(v) \geqslant ch(v) - 2\times 1 - f_{5^+}(v)\times\frac{1}{3} - 3\times\frac{3}{2} - (5 - f_{5^+}(v))\times 1 = \frac{2}{3}f_{5^+}(v) - \frac{3}{2} > 0$. If $f_{5^+}(v) = 2$, then except f_1, there is one 5^+-face incident with v, and there is at least one good 4-face which incident with v. So $ch'(v) \geqslant ch(v) - 2\times 1 - 2\times\frac{1}{3} - 3\times\frac{3}{2} - \frac{3}{4} - 2\times 1 = \frac{1}{12} > 0$. If $f_{5^+}(v) = 1$, then $d(f_i) \leqslant 4$ for all $2 \leqslant i \leqslant 8$. By symmetry, we need to consider the following cases in which 3-faces are located.

First, suppose $d(f_3) = d(f_4) = d(f_5) = 3$. Then $\min\{d(v_3), d(v_4), d(v_5)\} \geqslant 4$ and $\max\{d(v_6), d(v_7), d(v_8)\} \geqslant 4$ by Figure 1(5) and Lemma 4. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - 2\times\frac{5}{4} - \frac{3}{2} - 2\times\frac{3}{4} - 2\times 1 = \frac{1}{6} > 0$. Second, suppose $d(f_4) = d(f_5) = d(f_6) = 3$. Then $\min\{d(v_5), d(v_6)\} \geqslant 4$ by Figure 1(5), $\max\{d(v_3), d(v_4)\} \geqslant 4$ and $\max\{d(v_7), d(v_8)\} \geqslant 4$ by Lemma 4. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - \max\{\frac{5}{4}+2\times\frac{3}{2}+3\times\frac{3}{4}+1, 2\times\frac{5}{4}+\frac{3}{2}+2\times\frac{3}{4}+2\times 1\} = \frac{1}{6} > 0$. Third, suppose $d(f_3) = d(f_4) = d(f_6) = 3$. Then $d(v_4) \geqslant 4$ by Figure 1(5), $d(v_3) \geqslant 4$ and $\max\{d(v_7), d(v_8)\} \geqslant 4$ by Lemma 4, $\max\{d(v_5), d(v_6)\} \geqslant 4$ by Lemma 5. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - 2\times\frac{3}{2} - \frac{5}{4} - 3\times\frac{3}{4} - 1 = \frac{1}{6} > 0$. Fourth, suppose $d(f_3) = d(f_4) = d(f_7) = 3$. Then $\min\{d(v_3), d(v_4), d(v_8)\} \geqslant 4$ by Figure 1(5) and Lemma 4, $\max\{d(v_5), d(v_6), d(v_7)\} \geqslant 4$ by Lemma 5. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - 2\times\frac{3}{2} - \frac{5}{4} - 3\times\frac{3}{4} - 1 = \frac{1}{6} > 0$. Fifth, suppose $d(f_4) = d(f_5) = d(f_7) = 3$. Then $d(v_5) \geqslant 4$ by Figure 1(5), $d(v_8) \geqslant 4$ and $\max\{d(v_3), d(v_4)\} \geqslant 4$ by Lemma 4, $\max\{d(v_6), d(v_7)\} \geqslant 4$ by Lemma 5. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - 2\times\frac{3}{2} - \frac{5}{4} - 3\times\frac{3}{4} - 1 = \frac{1}{6} > 0$. Sixth, suppose $d(f_3) = d(f_5) = d(f_7) = 3$. Then f_2, f_4, f_6 and f_8 are good by Lemma 4 and Lemma 5, so $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - 3\times\frac{3}{2} - 4\times\frac{3}{4} = \frac{1}{6} > 0$.

Suppose $f_3(v) = 2$. Then without loss of generality, let $d(f_i) = d(f_j) = 3$ ($3 \leqslant i < j \leqslant 7$). If $f_{5^+}(v) \geqslant 2$, then $ch'(v) \geqslant ch(v) - 2\times 1 - f_{5^+}(v)\times\frac{1}{3} - 2\times\frac{3}{2} - (6 - f_{5^+}(v))\times 1 = \frac{2}{3}f_{5^+}(v) - 1 \geqslant 0$. Otherwise, $d(f_t) \leqslant 4$ for all $2 \leqslant t \leqslant 8$. If there is at least one good 3-face in $\{f_i, f_j\}$, then each face adjacent to good 3-face is good. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - \frac{5}{4} - \frac{3}{4} - \frac{3}{2} - 4\times 1 = \frac{1}{6} > 0$. Now we suppose both f_i and f_j are bad. If $j = i+1$, then $i \in \{4, 5\}$ by Figure 1(5) and Lemma 4, it follows that there are at least two good 4-faces in $\{f_2, f_3, f_4\}$, so $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - 2\times\frac{3}{2} - 3\times 1 - 2\times\frac{3}{4} = \frac{1}{6} > 0$. Otherwise, there are two 7^+-vertices in $\{v_i, v_{i+1}, v_j, v_{j+1}\}$. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - 2\times\frac{3}{2} - 2\times\frac{3}{4} - 3\times 1 = \frac{1}{6} > 0$.

Suppose $f_3(v) \leqslant 1$. Then $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{1}{3} - f_3(v)\times\frac{3}{2} - (7 - f_3(v))\times 1 = \frac{2}{3} - \frac{1}{2}f_3(v) \geqslant 0$.

For Figure 8(2), note that $f_3(v) \leqslant 3$, and v sends at most $\frac{3}{2}$ (in total) to f_1 and f_2 by Lemma 8. Suppose $f_3(v) = 3$, without loss of generality, let $d(f_4) = d(f_5) = d(f_i) = 3$ ($i \in \{6, 7\}$). Then v sends at most $\max\{\frac{3}{2}+\frac{1}{2}, \frac{5}{4}+\frac{3}{4}\} = 2$ (in total) to f_3 and f_4, and v sends at most $\max\{\frac{3}{2}+\frac{3}{2}+1+\frac{1}{3}, \frac{5}{4}+\frac{3}{2}+2\times\frac{3}{4}, \frac{5}{4}+\frac{5}{4}+\frac{3}{4}+1\} = \frac{13}{3}$ (in total) to f_5, f_6, f_7 and f_8 by Figure 1(5), Lemma 4 and Lemma 5. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{3}{2} - 2 - \frac{13}{3} = \frac{1}{6} > 0$.

Suppose $f_3(v) = 2$. Then without loss of generality, let $d(f_i) = d(f_j) = 3$ ($4 \leqslant i < j \leqslant 7$). If there is at least one 5^+-face in $\{f_t | 3 \leqslant t \leqslant 8\}$, then $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{3}{2} - 2\times\frac{3}{2} - \frac{1}{3} - 3\times 1 = \frac{1}{6} > 0$. Otherwise, $d(f_t) \leqslant 4$ for all $3 \leqslant t \leqslant 8$. If there is at least one good 3-face in $\{f_i, f_j\}$, then each 4-face adjacent to good 3-face is good. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{3}{2} - \frac{3}{2} - \frac{5}{4} - \frac{3}{4} - 3\times 1 = 0$. Now we suppose both f_i and f_j are bad. If $j=i+1$, then $i=5$, f_3, f_4, f_7, and f_8 are good by Figure 1(5) and Lemma 4. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{3}{2} - 4\times\frac{3}{4} - 2\times\frac{3}{2} = \frac{1}{2} > 0$. Otherwise, there are two 7^+-vertices in $\{v_i, v_{i+1}, v_j, v_{j+1}\}$. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{3}{2} - 2\times\frac{3}{2} - \frac{1}{2} - 3\times 1 = 0$.

Suppose $f_3(v) \leqslant 1$. Then $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{3}{2} - f_3(v)\times\frac{3}{2} - (6 - f_3(v))\times 1 = \frac{1}{2} - \frac{1}{2}f_3(v) \geqslant 0$.

For Figure 8(3), note that $f_3(v) \leqslant 4$. If $f_3(v) = 4$, then $d(f_2) = d(f_5) = d(f_6) = d(f_7) = 3$, so $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{11}{4} - (\frac{11}{4} + 2\times\frac{5}{4}) = 0$ by Lemma 10.

Suppose $f_3(v) = 3$. If $d(f_2) \geqslant 4$, then $d(f_5) = d(f_6) = d(f_7) = 3$, so $ch'(v) \geqslant ch(v) - 2\times 1 - (1 + \frac{3}{2}) - (\frac{11}{4} + 2\times\frac{5}{4}) = \frac{1}{4} > 0$. If $d(f_2) = 3$, then v sends at most $\frac{11}{4}$ (in total) to f_1, f_2 and f_3 by Lemma 10. Without loss of generality, let $d(f_5) = 3$. If $d(f_6) = 3$, then v sends at most 2 (in total) to f_4 and f_5, v sends at most $\frac{3}{4}$ to f_7. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{11}{4} - 2 - \frac{3}{2} - \frac{3}{4} - 1 = 0$. If $d(f_7) = 3$, then v sends at most $\frac{3}{4}$ to f_4, f_6 and f_8, respectively. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{11}{4} - 2\times\frac{3}{2} - 3\times\frac{3}{4} = 0$.

Suppose $f_3(v) = 2$. Then without loss of generality, let $d(f_i) = d(f_j) = 3$ $(i < j)$. If $i = 2$, then v sends at most $\frac{11}{4}$ (in total) to f_1, f_2 and f_3, v sends at most $\frac{3}{4}$ to f_{j-1} or f_{j+1}. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{11}{4} - \frac{3}{2} - \frac{3}{4} - 3\times 1 = 0$. Otherwise, v sends at most $\frac{5}{2}$ (in total) to f_1, f_2 and f_3 by Lemma 8, without loss of generality, let $i = 5$. If $j = 6$, then v sends at most 2 (in total) to f_4 and f_5. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{5}{2} - 2 - \frac{3}{2} - 2\times 1 = 0$. If $j = 7$, then v sends at most $\frac{3}{4}$ to f_4 and f_8, respectively. So $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{5}{2} - 2\times\frac{3}{4} - 2\times\frac{3}{2} - 1 = 0$.

Suppose $f_3(v) \leqslant 1$. If $d(f_2) = 3$, then $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{11}{4} - 5\times 1 = \frac{1}{4} > 0$. Otherwise, $ch'(v) \geqslant ch(v) - 2\times 1 - \frac{5}{2} - \frac{3}{2} - 4\times 1 = 0$.

For Figure 8(4), note that $f_3(v) \leqslant 4$. Suppose $f_3(v) = 4$. Then $d(f_2) = d(f_3) = d(f_6) = d(f_7) = 3$ and $ch'(v) \geqslant ch(v) - 2\times 1 - 2\times(\frac{11}{4} + \frac{5}{4}) = 0$ by Lemma 10. Suppose $f_3(v) = 3$. Then $ch'(v) \geqslant ch(v) - 2\times 1 - (\frac{11}{4} + \frac{5}{4}) - \frac{15}{4} = \frac{1}{4} > 0$ by Lemma 9. Suppose $f_3(v) = 2$. If two 3-faces incident with v are adjacent, then $ch'(v) \geqslant ch(v) - 2\times 1 - (\frac{11}{4} + \frac{5}{4}) - 4\times 1 = 0$. Otherwise, $ch'(v) \geqslant ch(v) - 2\times 1 - 2\times\frac{15}{4} = \frac{1}{2} > 0$. Suppose $f_3(v) \leqslant 1$. Then $ch'(v) \geqslant ch(v) - 2\times 1 - f_3(v)\times\frac{15}{4} - (2 - f_3(v))\times(4\times 1) = \frac{1}{4}f_3(v) \geqslant 0$.

Case 8. $n_2(v) = 1$. Without loss of generality, let v_1 be the unique 2-vertex adjacent to v. First, we consider the case that v_1 is not incident with any 3-face. Note that $f_3(v) \leqslant 5$.

Suppose $f_3(v) = 5$. Then $d(f_2) = d(f_3) = d(f_i) = d(f_6) = d(f_7) = 3$ $(i \in \{4, 5\})$, and at least two faces in $\{f_3, f_i, f_6\}$ are good by Figure 1(5) and Lemma 5. If $\min\{f_1, f_8\} \geqslant 5$, then $ch'(v) \geqslant ch(v) - 1 - 2\times\frac{1}{3} - 3\times\frac{3}{2} - 2\times\frac{5}{4} - 1 = \frac{1}{3} \geqslant 0$. Otherwise, $\min\{f_1, f_8\} \leqslant 4$, without loss of generality, let $d(f_1) = 4$. If $d(v_2) = 3$, then f_3, f_i, f_6 and f_7 are good by Lemma 6, so $ch'(v) \geqslant ch(v) - 1 - 1 - \frac{3}{2} - 4\times\frac{5}{4} - 2\times\frac{3}{4} = 0$. If $d(v_2) \geqslant 4$, we may assume that $d(f_8) \geqslant 5$ or $d(v_8) \geqslant 4$, then $ch'(v) \geqslant ch(v) - 1 - \frac{3}{4} - 3\times\frac{5}{4} - \frac{3}{2} - 1 - \max\{\frac{3}{2} + \frac{1}{3}, \frac{5}{4} + \frac{3}{4}\} = 0$.

Suppose $f_3(v) = 4$. Then there is at least one 3-face in $\{f_2, f_7\}$, without loss of generality, let $d(f_2) = d(f_i) = d(f_j) = d(f_t) = 3$, where $2 < i < j < t$ and $t \in \{6, 7\}$. If $f_{5^+}(v) \geqslant 2$, then $ch'(v) \geqslant ch(v) - 1 - f_{5^+}(v)\times\frac{1}{3} - 4\times\frac{3}{2} - (4 - f_{5^+}(v))\times 1 = \frac{2}{3}f_{5^+}(v) - 1 \geqslant 0$. Then $f_{5^+}(v) \leqslant 1$. We need to consider two cases. First, suppose there is one 5^+-face in $\{f_1\} \cup \{f_x | t+1 \leqslant x \leqslant 8\}$, then at least two faces in $\{f_3, f_4, f_5, f_6\}$ are good by Figure 1(5) and Lemma 5. So $ch'(v) \geqslant ch(v) - 1 - \frac{1}{3} - \max\{2\times\frac{3}{2} + 2\times\frac{5}{4} + 3\times 1, 3\times\frac{3}{2} + \frac{5}{4} + 2\times 1 +$

$\frac{3}{4}, 4 \times \frac{3}{2} + 1 + 2 \times \frac{3}{4}\} = \frac{1}{6} > 0$. Second, suppose $d(f_1) = d(f_x) = 4$ for all $t+1 \leqslant x \leqslant 8$. If $d(v_2) = 3$ or $d(v_y) = 3$ for all $t+1 \leqslant y \leqslant 8$, then v is incident with at least three good 3-faces and one good 4-face by Lemma 6. So $ch'(v) \geqslant ch(v) - 1 - \frac{3}{2} - 3 \times \frac{5}{4} - 3 \times 1 - \frac{3}{4} = 0$. Otherwise, $d(v_2) \geqslant 4$ and $\max\{d(v_y) | t+1 \leqslant y \leqslant 8\} \geqslant 4$, that is, there are at least two good 4-faces in $\{f_1\} \cup \{f_x | t+1 \leqslant x \leqslant 8\}$. Then $f_{5^+}(v) = 1$ or at least two faces in $\{f_3, f_4, f_5, f_6\}$ are good. So $ch'(v) \geqslant ch(v) - 1 - \max\{4 \times \frac{3}{2} + 1 + 2 \times \frac{3}{4} + \frac{1}{3}, 2 \times \frac{3}{2} + 2 \times \frac{5}{4} + 2 \times 1 + 2 \times \frac{3}{4}, 3 \times \frac{3}{2} + \frac{5}{4} + 1 + 3 \times \frac{3}{4}, 4 \times \frac{3}{2} + 4 \times \frac{3}{4}\} = 0$.

Suppose $f_3(v) = 3$. If $f_{5^+}(v) \geqslant 1$, then $ch'(v) \geqslant ch(v) - 1 - f_{5^+}(v) \times \frac{1}{3} - 3 \times \frac{3}{2} - (5 - f_{5^+}(v)) \times 1 = \frac{2}{3} f_{5^+}(v) - \frac{1}{2} \geqslant 0$. Otherwise, at least two faces incident with v are good by Lemma 5 and Lemma 6. So $ch'(v) \geqslant ch(v) - 1 - \max\{2 \times \frac{3}{2} + \frac{5}{4} + \frac{3}{4} + 4 \times 1, 3 \times \frac{3}{2} + 2 \times \frac{3}{4} + 3 \times 1\} = 0$. Suppose $f_3(v) \leqslant 2$. Then $ch'(v) \geqslant ch(v) - 1 - f_3(v) \times \frac{3}{2} - (8 - f_3(v)) \times 1 = 1 - \frac{1}{2} f_3(v) \geqslant 0$.

Next, we consider the case that v_1 is incident with a 3-face. Then $f_3(v) \leqslant 6$, and the other 3-faces incident with v are good by Figure 1(2). If $f_3(v) = 6$, then $d(f_1) = d(f_2) = d(f_3) = d(f_5) = d(f_6) = d(f_7) = 3$, v sends at most $\frac{1}{2}$ to f_4, and v sends at most $\frac{3}{4}$ to f_8. So $ch'(v) \geqslant ch(v) - 1 - \frac{3}{2} - 5 \times \frac{5}{4} - \frac{1}{2} - \frac{3}{4} = 0$. Suppose $f_3(v) \leqslant 5$. If $f_{5^+}(v) \geqslant 1$, then $ch'(v) \geqslant ch(v) - 1 - \frac{3}{2} - 4 \times \frac{5}{4} - f_{5^+}(v) \times \frac{1}{3} - (3 - f_{5^+}(v)) \times 1 = \frac{2}{3} f_{5^+}(v) - \frac{1}{2} \geqslant 0$. Otherwise, $f_{5^+}(v) = 0$. If $f_3(v) = 5$, then at least two 4-faces incident with v are good. So $ch'(v) \geqslant ch(v) - 1 - \frac{3}{2} - 4 \times \frac{5}{4} - 1 - 2 \times \frac{3}{4} = 0$. If $f_3(v) \leqslant 4$, then at least one 4-face incident with v is good. So $ch'(v) \geqslant ch(v) - 1 - \frac{3}{2} - (f_3(v) - 1) \times \frac{5}{4} - (8 - f_3(v) - 1) \times 1 - \frac{3}{4} = 1 - \frac{1}{4} f_3(v) \geqslant 0$.

Case 9. $n_2(v) = 0$. Note that $f_3(v) \leqslant 6$. If $f_3(v) \leqslant 4$, then $ch'(v) \geqslant ch(v) - f_3(v) \times \frac{3}{2} - (8 - f_3(v)) \times 1 = 2 - \frac{1}{2} f_3(v) \geqslant 0$. Suppose $f_3(v) = 5$. Then there are two adjacent 3-cycles which incident with v, without loss of generality, let $d(f_i) = d(f_{i+1}) = 3$. If $f_{5^+}(v) \geqslant 1$, then $ch'(v) \geqslant ch(v) - 5 \times \frac{3}{2} - f_{5^+}(v) \times \frac{1}{3} - (3 - f_{5^+}(v)) \times 1 = \frac{2}{3} f_{5^+}(v) - \frac{1}{2} > 0$. Then $f_{5^+}(v) = 0$. If $d(v_{i+1}) = 3$, then v is incident with at most four bad 3-faces by Lemma 7, so $ch'(v) \geqslant ch(v) - 4 \times \frac{3}{2} - \frac{5}{4} - 2 \times 1 - \frac{3}{4} = 0$. Otherwise, no two 3-cycles have a common 3-vertex, then there are at least two good faces which incident with v by Figure 1(6), so $ch'(v) \geqslant ch(v) - 3 \times \frac{3}{2} - 1 - \max\{2 \times \frac{5}{4} + 2 \times 1, \frac{3}{2} + \frac{5}{4} + 1 + \frac{3}{4}, 2 \times \frac{3}{2} + 2 \times \frac{3}{4}\} = 0$. Suppose $f_3(v) = 6$. Then without loss of generality, let $d(f_1) = d(f_2) = d(f_3) = d(f_5) = d(f_6) = d(f_7) = 3$. If $\min\{d(v_2), d(v_3), d(v_6), d(v_7)\} = 3$, then v is incident with at most four bad 3-faces by Lemma 7. So $ch'(v) \geqslant ch(v) - 4 \times \frac{3}{2} - 2 \times \frac{5}{4} - 2 \times \frac{3}{4} = 0$. Otherwise, $\min\{d(v_2), d(v_3), d(v_6), d(v_7)\} \geqslant 4$. If $\max\{d(v_1), d(v_4), d(v_5), d(v_8)\} \geqslant 4$, then $ch'(v) \geqslant ch(v) - 3 \times \frac{3}{2} - 3 \times \frac{5}{4} - 1 - \frac{3}{4} = 0$. If $d(v_1) = d(v_4) = d(v_5) = d(v_8) = 3$, then $\min\{d(v_2), d(v_3), d(v_6), d(v_7)\} \geqslant 7$, so $ch'(v) \geqslant ch(v) - 4 \times \frac{3}{2} - 2 \times 1 - 2 \times 1 = 0$.

Hence we complete the proof of the theorem.

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References

[1] M. Behzad. Graphs and their chromatic numbers. Ph.D. Thesis, Michigan State University, 1965.

[2] J. A. Bondy and U. S. R. Murty. Graph Theory with Applications. MacMillan, London, 1976.

[3] O. V. Borodin, A. V. Kostochka, and D. R. Woodall. Total colorings of planar graphs with large maximum degree. *J. Graph Theory*, 26:53–59, 1997.

[4] G. J. Chang, J. F. Hou, and N. Roussel. Local condition for planar graphs of maximum degree 7 to be 8-totally colorable. *Discrete Appl. Math.*, 159:760–768, 2011.

[5] J. Chang, H. J. Wang, J. L. Wu, and Y. G. A. Total colorings of planar graphs with maximum degree 8 and without 5-cycles with two chords. *Theoretical Computer Sci.*, 476:16–23, 2013.

[6] J. S. Cai, G. H. Wang, and G. Y. Yan. Planar graphs with maximum degree 8 and without intersecting chordal 4-cycles are 9-totally colorable. *Sci. China Math.*, 55:2601–2612, 2012.

[7] D. Z. Du, L. Shen, and Y. Q. Wang. Planar graphs with maximum degree 8 and without adjacent triangles are 9-totally-colorable. *Discrete Appl. Math.*, 157:2778–2784, 2009.

[8] A. V. Kostochka. The total chromatic number of any multigraph with maximum degree five is at most seven. *Discrete Math.*, 162:199–214, 1996.

[9] L. Kowalik, J. S. Sereni, and R. Škrekovski. Total-Coloring of plane graphs with maximum degree nine. *SIAM J. Discrete Math.*, 22:1462–1479, 2008.

[10] H. L. Li, B. Q. Liu, and G. H. Wang. Neighbor sum distinguishing total colorings of K_4-minor free graphs. *Front. Math. China*, 8(6):1351–1366, 2013.

[11] H. L. Li, L. H. Ding, B. Q. Liu, and G. H. Wang. Neighbor sum distinguishing total colorings of planar graphs. *J. Comb. Optim.*, doi:10.1007/s10878-013-9660-6, 2013.

[12] M. Pilśniak and M. Woźniak. On the adjacent-vertex-distinguishing index by sums in total proper colorings. *Preprint MD 051*, available at http://www.ii.uj.edu.pl/preMD/index.php

[13] D. P. Sanders and Y. Zhao. On total 9-coloring planar graphs of maximum degree seven. *J. Graph Theory*, 31:67–73, 1999.

[14] L. Shen and Y. Q. Wang. Total colorings of planar graphs with maximum degree at least 8. *Sci. China*, 38:1356–1364, 2008 (in Chinese).

[15] X. Tan, H. Y. Chen, and J. L. Wu. Total colorings of planar graphs without adjacent 4-cycles. *in: ISORA'09*, pp. 167–173.

[16] V. G. Vizing. Some unsolved problems in graph theory. *Uspekhi Mat. Nauk*, 23:117–134, 1968 (in Russian).

[17] B. Wang and J. L. Wu. Total colorings of planar graphs with maximum degree seven and without intersecting 3-cycles. *Discrete Math.*, 311:2025–2030, 2011.