

Evaluating regions of attraction of LTI systems with saturation in IQS framework*

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Abstract

The integral quadratic separation (IQS) framework is a specialization of topological separation theory to the case where operators are described by (or embedded in) integral quadratic constraints. IQS applies both to regular and descriptor system representations. Thanks to this feature, IQS provides a methodology for deriving sequences of LMI conditions with reducing conservatism. It has been applied for robust performance analysis of systems with uncertainties and delays. In this paper the methodology is adapted for the first time for region of attraction evaluation in the case of systems with saturations. Results are illustrated on a launcher attitude control example with saturated inputs.

Keywords: Stability analysis, Saturation, Convex optimization, Launcher control .

1 Introduction

The work exposed in the paper is part of a project dedicated to robust analysis of a launcher control system. The global goal is to provide efficient tools for assessing stability and performance of non-linear, time-varying, models with uncertainties, delays, saturations, dead-zones and hybrid features. Among all these features, the paper focuses on saturations and considers the launcher in ballistic phase. The issue is formulated in terms of regions of stability estimation. It corresponds to the situation where, because of stage separation related disturbances or temporary loss of actuators/sensors, the attitude control of the launcher starts with initial conditions far from the required equilibrium. Guaranteeing, robustly, large regions of attraction is crucial in the validation of control laws at this phase of the launcher trajectory.

Region of attraction estimation is a long standing problem that has been considered by many authors. Actually, the exact geometric characterization of the region of attraction of the origin for a system subject to input saturation is not an easy task [SLT96], [Rom96], [Kha92]. The region of attraction can be non-convex, open and unlimited. Thus, in general, this region can be approximately determined only by intensive simulations, but its visualization is practically impossible for systems with more than 3 states. Hence, it is important to characterize subsets of the region of attraction with a well-defined analytical representation, such as for example ellipsoidal or polyhedral sets. It is the reason for which region of attraction estimation, and therefore the determination of region of stability, has been a long standing problem considered by several authors, most of the approaches being developed based on the use of Lyapunov functions (quadratic, polyhedral, piecewise quadratic, ...) and an adequate model for the saturated terms [BM95], [Joh99], [HL01], [KE02], [AaDLC06], [Fia10], [TGGQ11].

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A necessary tradeoff has to be done between the available performance and/or robustness properties and the size of the region of the guaranteed stability (see, for example, [KE02], [TGGQ11] and references therein). Yet, these robust results are not as rich as what would permit merging Lyapunov-based approaches with Integral Quadratic Constraints (IQC) results. Unfortunately, most IQC-type results for systems with saturations deal with global stability: [MR97, Meg01, VKS09]. Some papers such as [FLR06, MS09] deal with region of attraction estimation by means of IQCs but room is clearly left for better insight in that direction.

The contribution of this paper is to consider region of attraction estimation in a framework closely related to IQCs. This framework named Integral Quadratic Separation (IQS) is a specialization of topological separation proposed by [Saf80] for linear systems in feedback-loop with non-linear, time-varying uncertain operators. It generalizes quadratic separation results developed by [IH98, IS01]. In [PAHG07] it is shown how descriptor modeling permitted in IQS framework contributes to building LMI conditions with reduced conservatism by means of including information on higher derivatives of the state of the system. In [PBG09] a technique is proposed to deal with information on initial conditions. The contribution of the present paper is to combine these two features. It provides results for attraction estimation which are for the first time formulated in a non Lyapunov framework.

The paper is organized as follows. First a section is devoted to some mathematical preliminaries and recalls the main features of IQS. Second, the launcher region of attraction estimation problem is formulated and divided in two steps. The first step is tackled in section 4. It provides two LMI conditions for estimating sets of initial conditions for which the control input is bounded. The second step guarantees stability and is described in section 5. Concluding remarks are finally drawn about conservatism of the results and the multiple perspectives for further conservatism reduction and for robust region of stability evaluation.

2 Preliminaries

2.1 Notation

$\mathbb{R}^{m \times p}$ and $\mathbb{C}^{m \times p}$ are the sets of m -by- p real and complex matrices respectively. A^T is the transpose of the matrix A and A^* is the transpose conjugate. $\mathbf{1}$ and $\mathbf{0}$ are respectively the identity and the zero matrices of appropriate dimensions. $A > (\geq) B$ means that $A - B$ is Hermitian positive (semi) definite. $A^\perp$ is a full rank matrix whose columns span the null-space of A : $AA^\perp = \mathbf{0}$. Let $L_2^{m \times p}[0, +\infty[$ be the set of all measurable functions $f : [0, +\infty[\rightarrow \mathbb{C}^m$ bounded with respect to the following norm $\|f\| = \left(\int_0^\infty f^*(t)f(t)dt \right)^{1/2}$. Associated to that norm, is the inner product $\langle f|g \rangle = \left(\int_0^\infty f^*(t)g(t)dt \right)$. To alleviate notation, in the following $L_2^m[0, +\infty[$ is denoted for simplicity L_2 , whatever the dimensions. For the ease of notation define as well the truncated inner product and norm

$$\langle f|g \rangle_{\bar{t}} = \left(\int_0^{\bar{t}} f^*(t)g(t)dt \right), \quad \|f\|_{\bar{t}} = \langle f|f \rangle_{\bar{t}}^{1/2}$$

Let the following truncation operator $x \mapsto \mathcal{T}_\theta x$:

$$[\mathcal{T}_\theta x](t) = \begin{cases} x(t) & \text{if } t \in [0, \theta] \\ 0 & \text{if } t > \theta \end{cases}$$

and the square-root delta operator (see [PBG09]) $x \mapsto \varphi_\theta x$:

$$\langle \varphi_{\theta_1} y | \varphi_{\theta_2} x \rangle_{\bar{t}} = \begin{cases} y^*(\theta_1)x(\theta_1) & \text{if } \theta_1 = \theta_2 \leq \bar{t}, \\ 0 & \text{if } \theta_1 \neq \theta_2 \text{ or if } \theta_1 > \bar{t}. \end{cases}$$

Both of these operators are linear and map signals of L_2 to L_2 . In the paper we assume the signals admit a derivative, almost everywhere, defined by the following integral relation $\int_0^{\bar{t}} \dot{x}(t)dt + x(0) = x(\bar{t})$. Based on this notation, we define the integrator operator on truncated signals as:

$$\mathcal{I}_F \begin{pmatrix} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \end{pmatrix} = \begin{pmatrix} \mathcal{T}_\theta x \\ \varphi_\theta x \end{pmatrix}. \quad (1)$$

This formula states that for all time instants $\bar{t} \leq \theta$ the state results from a mapping from the initial conditions and its derivative

$$\int_0^{\bar{t}} [\mathcal{T}_\theta \dot{x}](t) dt + x(0) = \int_0^{\bar{t}} \dot{x}(t) dt + x(0) = x(\bar{t}) = [\mathcal{T}_\theta x](\bar{t})$$

and for any $\bar{t} \geq \theta$ the integral of the truncated derivative maps to the final value $\int_0^{\bar{t}} [\mathcal{T}_\theta \dot{x}](t) dt + x(0) = x(\theta)$.

To make some formulas more concise ‘diag’ defines block diagonal matrices. It is used as follows on a three-block example

$$\text{diag} \begin{bmatrix} A & B & C \end{bmatrix} = \text{diag} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C \end{bmatrix}.$$

2.2 Well-posedness

Let the following interconnected system

$$\mathcal{E}z(t) = \mathcal{A}w(t), \quad w(t) = [\nabla z](t) \quad (2)$$

where w and z are signals describing the system; $\mathcal{E}$ and $\mathcal{A}$ are constant matrices, possibly not square, and $\mathcal{E}$ is assumed to be full column rank; ∇ is an operator, that may be non-causal and may be uncertain, it is defined as belonging to a closed set $\nabla \in \mathbf{\nabla}$.

Definition 1 ([Saf80]) *The interconnected systems (2) is said to be well-posed if whatever bounded signals $(\tilde{z}, \tilde{w})$ perturbing the system equations*

$$\mathcal{E}z(t) = \mathcal{A}(t)w(t) + \tilde{z}(t), \quad w(t) = [\nabla z](t) + \tilde{w}(t) \quad (3)$$

and whatever uncertain operator $\nabla \in \mathbf{\nabla}$, the internal signals (z, w) are unique and bounded in L_2 :

$$\exists \bar{\gamma} > 0 : \quad \left\| \begin{bmatrix} z \\ w \end{bmatrix} \right\| \leq \bar{\gamma} \left\| \begin{bmatrix} \tilde{z} \\ \tilde{w} \end{bmatrix} \right\|, \quad \forall \begin{pmatrix} \tilde{z} \\ \tilde{w} \end{pmatrix} \in L_2, \quad \forall \nabla \in \mathbf{\nabla} \quad (4)$$

To illustrate the definition consider the following interconnected system

$$E\dot{x} = Ax, \quad x = \nabla \dot{x}$$

where $\nabla = \mathcal{I}$ is the integration operator with zero initial conditions: $\mathcal{I}\dot{x} = x$. The exogenous signal $\tilde{w}$ plays in that case the role of initial conditions (when non zero) and $\tilde{z}$ is a perturbation on the dynamics. Well-posedness implies that for zero initial conditions $\tilde{w} = 0$ and no perturbations $\tilde{z} = 0$, the only trajectory of the system is $w = 0$, $z = 0$ (i.e. the origin is an equilibrium point). It also implies that for bounded initial conditions and bounded perturbations, the state remains bounded. In this case well-posedness of the interconnected system is equivalent to the stability of $E\dot{x} = Ax$.

2.3 Integral quadratic separation

Theorem 1 *The interconnected system (2) is well-posed if there exists a Hermitian matrix Θ satisfying simultaneously the LMI:*

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp*} \Theta \begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^{\perp} > \mathbf{0} \quad (5)$$

and the IQC:

$$\left\langle \begin{pmatrix} z \\ \nabla z \end{pmatrix} \middle| \Theta \begin{pmatrix} z \\ \nabla z \end{pmatrix} \right\rangle \leq 0, \quad \forall z \in L_2, \quad \forall \nabla \in \mathbf{\nabla}. \quad (6)$$

The result is proved in [PAHG07]. It is directly related to topological separation of [Saf80] where the topological separator is an integral quadratic form defined by the matrix Θ :

$$\theta(z, w) = \left\langle \begin{pmatrix} z \\ w \end{pmatrix} \middle| \Theta \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle$$

Topological separation is proved by the fact that the separator is positive on the graph of $\mathcal{E}z = \mathcal{A}w$ and negative on the inverse graph of $w = \nabla z$. If Θ satisfies both conditions (5) and (6) it is said that it defines an integral quadratic separator (IQS). Moreover, any Θ that satisfies the IQC (6) is called an IQS-candidate.

Applied to the loop $E\dot{x} = Ax$, $x = \mathcal{I}\dot{x}$ the theorem applies with the following choice of IQS-candidate

$$\Theta = \begin{bmatrix} \mathbf{0} & -P \\ -P & \mathbf{0} \end{bmatrix}, \quad P > \mathbf{0}$$

which indeed satisfies for zero initial conditions:

$$\left\langle \begin{pmatrix} \dot{x} \\ \mathcal{I}\dot{x} \end{pmatrix} \middle| \Theta \begin{pmatrix} \dot{x} \\ \mathcal{I}\dot{x} \end{pmatrix} \right\rangle_{\bar{t}} = -x(\bar{t})^T P x(\bar{t}) \leq 0, \quad \forall \bar{t} > 0.$$

The results are coherent with Lyapunov theory and the matrix P used in the separator happens to define a quadratic Lyapunov function $V(x) = x^T P x$ for the system.

3 Launcher stability problem

Our aim is to investigate the usage of the IQS framework for stability analysis of a launcher attitude control in ballistic phase. As a first approximation (no atmospheric friction, neglected sloshing modes, no external perturbation, no coupling between axes) the launcher dynamics along one axis are those of a double integrator: $I\ddot{\theta} = T$. Keeping the problem as simple as possible, the actuator dynamics are neglected, as well as any thresholds on the control around zero inputs. The only complication considered is that the torque T is saturated at some value $\bar{T}$. This saturation is modeled as the difference between a unitary gain and a dead-zone: $T = u - \bar{T} \text{dz}(\frac{1}{\bar{T}}u)$, where the dead-zone is defined by:

$$\text{dz}(\hat{z}) = \begin{cases} \hat{z} + 1 & \text{if } \hat{z} \leq -1 \\ 0 & \text{if } |\hat{z}| \leq 1 \\ \hat{z} - 1 & \text{if } \hat{z} \geq 1 \end{cases} \quad (7)$$

Sensor dynamics and delays are neglected and the control is assumed to be a state-feedback (proportional-derivative type): $u = -K_P \theta - K_D \dot{\theta}$.

Numerical values for this problem are provided by the partners from ASTRIUM-ST: $I = 20\,000$, $K_P = 1\,800$, $K_D = 14\,400$ et $\bar{T} = 240$. The resulting closed-loop system reads:

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{z}} \end{pmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -0.09 & -0.72 & 0.012 \\ \hline 7.5 & 60 & 0 \end{array} \right] \begin{pmatrix} x \\ \hat{w} \end{pmatrix} = \begin{bmatrix} A & B \\ C & \mathbf{0} \end{bmatrix} \begin{pmatrix} x \\ \hat{w} \end{pmatrix}$$

where $\hat{w} = \text{dz}(\hat{z})$. In the zone where the input does not saturate (when $\text{dz}(\hat{z}) = 0$) the associated linear system is trivially stable. Unfortunately, as $|\hat{z}|$ goes to infinity the dynamics of the system become asymptotically identical to the open-loop unstable double integrator. It is well known that under some open-loop assumptions [SSY94], the global stabilization of linear systems subject to input saturation can be carried out. The double integrator enters in the kind of systems, which can be globally stabilized by a linear saturated control law. Nevertheless, since, in the current case, the double integrator system results from some approximation of the real system, there is no real sense to study the global stability/stabilization of the approximated system. It is furthermore important to note that it is impossible to assess global stability by simply embedding the non-linearity in the sector $\hat{w}/\hat{z} \in [0, 1]$.

Our goal is to build LMI conditions for proving local stability and this will be done in two steps:

- The first step is to build sets X_0 of initial conditions $x(0)$ such that $|\hat{z}(t)| \leq \bar{z}$ is guaranteed along the trajectories,

where $\bar{z}$ is some given upper bound.

- The second one is to prove stability assuming that $|\hat{z}(t)| \leq \bar{z}$ holds along the trajectories.

The combination of the two steps implies that for all initial conditions $x(0) \in X_0$ the origin is stable (local stability). The strategy is intrinsically conservative since it relies on an intermediate constraint on the trajectories.

4 Building sets of initial conditions

Following the reasoning developed in [PBG09] for studying impulse-to-peak performance (which has interpretations in terms of peak response for given initial conditions), let us introduce the following model involving initial and final conditions over some time interval $[0, \theta]$:

$$\underbrace{\begin{pmatrix} \varphi_0 x \\ \mathcal{T}_\theta \dot{x} \\ \mathcal{T}_\theta \hat{z} \\ \varphi_\theta \hat{z} \end{pmatrix}}_z = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ A & \mathbf{0} & B & \mathbf{0} \\ C & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C & \mathbf{0} & \mathbf{0} \end{bmatrix}}_A \underbrace{\begin{pmatrix} \mathcal{T}_\theta x \\ \varphi_\theta x \\ \mathcal{T}_\theta \hat{w} \\ \varphi_0 x \end{pmatrix}}_w \quad (8)$$

In this model, the initial conditions are described by the signal $\varphi_0 x$. Ellipsoidal sets of initial conditions are described by a symmetric positive definite matrix $Q > \mathbf{0}$:

$$\langle \varphi_0 x | Q \varphi_0 x \rangle = x^T(0) Q x(0) \leq 1.$$

It is assumed that the initial conditions enforce the output z to be bounded over the time interval $[0, \theta]$:

$$|\hat{z}(t)| = |[\mathcal{T}_\theta \hat{z}](t)| \leq \bar{z} \quad \forall t \in [0, \theta].$$

Since $\mathcal{T}_\theta \hat{w} = dz(\mathcal{T}_\theta \hat{z})$ the boundedness of $\hat{z}$ implies that the pair $([\mathcal{T}_\theta \hat{z}](t), [\mathcal{T}_\theta \hat{w}](t))$ is inside the sector $[0, \frac{\bar{z}-1}{\bar{z}}]$ (see Figure 1). Mathematically this property reads as the following quadratic constraint: for all $t \in [0, \theta]$

$$\begin{pmatrix} [\mathcal{T}_\theta \hat{z}](t) \\ [\mathcal{T}_\theta \hat{w}](t) \end{pmatrix}^T \begin{bmatrix} 0 & \frac{1-\bar{z}}{\bar{z}} \\ \frac{1-\bar{z}}{\bar{z}} & 2 \end{bmatrix} \begin{pmatrix} [\mathcal{T}_\theta \hat{z}](t) \\ [\mathcal{T}_\theta \hat{w}](t) \end{pmatrix} \leq 0. \quad (9)$$

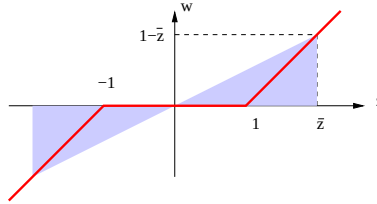


Figure 1: Sector constraint for bounded $|\hat{z}| \leq \bar{z}$

This assumption being made, our aim is at proving that the final condition is strictly bounded by that same bound:

$$|\hat{z}(\theta)| = \|\varphi_\theta \hat{z}\| < \bar{z}.$$

If this property holds whatever $\theta \geq 0$ it proves that the initial conditions enforce $|\hat{z}(t)| \leq \bar{z}$ for all t .

Our aim is therefore formulated in terms of assessing that

$$\langle \varphi_0 x | Q \varphi_0 x \rangle = x^T(0) Q x(0) \leq 1$$

implies $\|\varphi_\theta \hat{z}\| = |\hat{z}(\theta)| < \bar{z}$. Equivalently (see [PBG09]), it corresponds to assessing that the only admissible solution to $\bar{z}^2 \langle \varphi_0 x | Q \varphi_0 x \rangle \leq \|\varphi_\theta \hat{z}\|^2$ is the trivial solution $(\varphi_\theta \hat{z}, \varphi_0 x) = 0$. It is a well-posedness issue where $\varphi_0 x$ is expressed as dependent on $\varphi_\theta \hat{z}$ and the signals are constrained by the IQC:

$$\left\langle \begin{pmatrix} \varphi_\theta \hat{z} \\ \varphi_0 x \end{pmatrix} \middle| \begin{bmatrix} -1 & \mathbf{0} \\ \mathbf{0} & \bar{z}^2 Q \end{bmatrix} \begin{pmatrix} \varphi_\theta \hat{z} \\ \varphi_0 x \end{pmatrix} \right\rangle \leq 0. \quad (10)$$

The upper reasoning concludes in formulating a well-posedness problem defined by $\mathcal{E} = \mathbf{1}$, $\mathcal{A}$ given in (8) and the block diagonal operator $\nabla = \text{diag} \begin{bmatrix} \mathcal{I}_F & \nabla_1 \nabla_2 \end{bmatrix}$ where the notation $\mathcal{T}_\theta \hat{w} = \nabla_1 \mathcal{T}_\theta \hat{z}$ means that the constraint (9) holds and where $\varphi_0 x = \nabla_2 \varphi_\theta \hat{z}$ means that the constraint (10) holds. Well-posedness of this feedback system implies that for all initial conditions satisfying $x^T(0)Qx(0) \leq 1$ the signal $\hat{z}$ is bounded by $|\hat{z}(t)| \leq \bar{z}$ along the trajectories.

Corollary 1 Assume $\mathcal{A}$ given in (8), if there exist two symmetric matrices P and $Q > \mathbf{0}$ and a scalar $\tau > 0$ such that the following LMI holds

$$\begin{bmatrix} \mathcal{A} \\ \mathbf{1} \end{bmatrix}^T \underbrace{\begin{bmatrix} \text{diag} \begin{bmatrix} -P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \text{diag} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -P & \mathbf{0} \end{bmatrix} \\ \text{diag} \begin{bmatrix} \mathbf{0} & -P \\ \mathbf{0} & \mathbf{0} \end{bmatrix} & \text{diag} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \end{bmatrix}}_{\Theta} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -P & \mathbf{0} \\ \tau \frac{1-\bar{z}}{\bar{z}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P \\ 2\tau & \bar{z}^2 Q \end{bmatrix} \begin{bmatrix} \mathcal{A} \\ \mathbf{1} \end{bmatrix} > \mathbf{0} \quad (11)$$

then for any initial condition in

$$X_0 = \{x(0) : x^T(0)Qx(0) \leq 1\}$$

the trajectories of $\dot{x} = Ax + Bdz(Cx)$ are such that $|Cx| \leq \bar{z}$.

Proof The result is a direct corollary of Theorem 1 applied to the well-posedness problem defined by $\mathcal{E} = \mathbf{1}$, $\mathcal{A}$ given in (8) and the block diagonal operator $\nabla = \text{diag} \begin{bmatrix} \mathcal{I}_F & \nabla_1 \nabla_2 \end{bmatrix}$. The inequality (11) is exactly (5) where

$$\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^\perp = \begin{bmatrix} \mathbf{1} & -\mathcal{A} \end{bmatrix}^\perp = \begin{bmatrix} \mathcal{A} \\ \mathbf{1} \end{bmatrix}.$$

The structure given to Θ makes it an IQS-candidate with respect to the operator ∇ . Indeed for the signal z and w defined in (8), the IQC (6) holds: all the terms involving P cancel since $2 \langle \mathcal{T}_\theta \dot{x} | P \mathcal{T}_\theta x \rangle = \langle \varphi_\theta x | P \varphi_\theta x \rangle - \langle \varphi_0 x | P \varphi_0 x \rangle$; terms involving $\tau > 0$ give a negative contribution due to (9); and the remaining term is exactly (10). ■

When developing (11), the LMI also reads as:

$$\begin{bmatrix} A^T P + P A & P B + \tau \frac{\bar{z}-1}{\bar{z}} C^T \\ B^T P + \tau \frac{\bar{z}-1}{\bar{z}} C & -2\tau \end{bmatrix} < \mathbf{0}, \quad \begin{matrix} C^T C < P \\ P < \bar{z}^2 Q \end{matrix} \quad (12)$$

which are exactly the classical conditions for proving invariance of the set $x^T P x \leq \bar{z}^2$ for initial conditions $x^T(0)Qx(0) \leq 1$ as proposed in [PTB97], [HB98], [PTGC02] using the Lyapunov framework and classical sector conditions. The link between IQS results and Lyapunov-type results is that the matrix P defining the separator with respect to the integrator operator $\mathcal{I}_F$ defines a Lyapunov function.

Conditions (12) are known to be conservative. One way to understand this conservatism is that the structure given to Θ to make it an IQS-candidate is sufficient but may not be necessary. In particular, it performs the separation with respect to all operators ∇_1 with bounded gain in the sector $[0, \frac{\bar{z}-1}{\bar{z}}]$ which includes the dead-zone as well as many other non-linear operators.

To reduce the conservatism we suggest to include informations on the derivatives of $\hat{w}$. Indeed, since the considered non-linearity is a dead-zone, one gets that $\hat{w}$ is equal either to 0 or to $\hat{z}$ depending whether if $|\hat{z}| \leq 1$ holds or not. This added information reduces the class of operators from the original set of sector bounded operators. It is thus expected that results build with this added information may be less conservative.

The way to add the information on derivatives in the IQS framework is as simple as augmenting the modeling (8) with the information on derivatives. The result is given in equation (13).

The problem of finding initial conditions such that $|\hat{z}|$ is bounded is included in the well-posedness problem defined with these matrices $\mathcal{E}$ and $\mathcal{A}$ together with the block diagonal operator $\nabla = \text{diag} \begin{bmatrix} \mathcal{I}_F & \nabla_1 & \nabla_1 & \nabla_3 & \nabla_2 \end{bmatrix}$ where the notation $\mathcal{T}_\theta \hat{w} = \nabla_3 \mathcal{T}_\theta \hat{z}$ means that either $\mathcal{T}_\theta \hat{w} = 0$ or $\mathcal{T}_\theta \hat{w} = \mathcal{T}_\theta \hat{z}$.

$$\begin{aligned}
& \overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -C & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}^{\mathcal{E}} \overbrace{\begin{pmatrix} \varphi_0 x \\ \varphi_0 \hat{w} \\ \mathcal{T}_\theta \dot{x} \\ \mathcal{T}_\theta \dot{\hat{w}} \\ \mathcal{T}_\theta \dot{\hat{z}} \\ \varphi_\theta \dot{\hat{z}} \\ \mathcal{T}_\theta \dot{\hat{z}} \\ \varphi_\theta \dot{\hat{z}} \end{pmatrix}}^z \\
& = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} \mathcal{T}_\theta x \\ \mathcal{T}_\theta \hat{w} \\ \varphi_\theta x \\ \varphi_\theta \hat{w} \\ \mathcal{T}_\theta \hat{w} \\ \varphi_\theta \hat{w} \\ \mathcal{T}_\theta \hat{w} \\ \varphi_0 x \\ \varphi_0 \hat{w} \end{pmatrix}}_w
\end{aligned} \tag{13}$$

Corollary 2 Assume $\mathcal{E}$ and $\mathcal{A}$ are given in (13), if there exist two symmetric matrices P_1 and $Q_1 > 0$, five scalars $\tau_1 > 0$, $\tau_2 > 0$, $\mu_1 \leq 0$, $\mu_1 + 2\mu_2 + \mu_3 \leq 0$, such that the LMI (5) holds with

$$\Theta = \begin{bmatrix} \text{diag} \left[\begin{bmatrix} -P_1 & 0 \\ 0 & 0 \end{bmatrix} \right. \\ 0 \\ 0 \\ \mu_1 \\ -1 \\ \left. \begin{bmatrix} 0 & -P_1 \\ 0 & 0 \end{bmatrix} \right] \\ \text{diag} \left[\begin{bmatrix} 0 & -P_1 \\ 0 & 0 \end{bmatrix} \right. \\ \tau_1 \frac{1-\bar{z}}{\bar{z}} \\ \tau_2 \frac{1-\bar{z}}{\bar{z}} \\ \mu_2 \\ 0 \\ \left. \begin{bmatrix} 0 & 0 \\ 0 & P_1 \end{bmatrix} \right] \\ 2\tau_1 \\ 2\tau_2 \\ \mu_3 \\ \bar{z}^2 Q_1 \end{bmatrix} \tag{14}$$

then for any initial condition in

$$X_1 = \left\{ x(0) : \begin{pmatrix} x(0) \\ dz(Cx(0)) \end{pmatrix}^T Q_1 \begin{pmatrix} x(0) \\ dz(Cx(0)) \end{pmatrix} \leq 1 \right\}$$

the trajectories of $\dot{x} = Ax + Bdz(Cx)$ are such that $|Cx| \leq \bar{z}$.

Proof The result is obtained by assessing that the matrices structured as in (14) are IQS-candidates for $\nabla = \text{diag} \left[\mathcal{I}_F \quad \nabla_1 \quad \nabla_1 \quad \nabla_3 \quad \nabla_2 \right]$. Indeed, for the signal z and w defined in (13), the IQC (6) holds: all the terms involving P cancel as for the case of Corollary 1; terms involving τ_1 and τ_2 give a negative contributions due to (9); terms involving $\mu_{1,2,3}$ are negative because $\mu_1 \leq 0$ which is needed when $\mathcal{T}_\theta \dot{\hat{w}} = 0$ and because $\mu_1 + 2\mu_2 + \mu_3 \leq 0$ which is needed when $\mathcal{T}_\theta \dot{\hat{w}} = \mathcal{T}_\theta \dot{\hat{z}}$; and the remaining term is exactly (10). ■

The initial conditions of Corollary 2 are described in terms of the state and the signal exiting the dead-zone. The border of the set of initial conditions is piecewise quadratic and continuous. Similar types of initial condition sets can be found in [Joh99]. The difference is that here the sets are computed using one single LMI condition and not separately in each region with continuity constraints on the border.

Proposition 1 *If there exists a solution to Corollary 1 then there exists a positive scalar ϵ such that Corollary 2 holds as well with*

$$P_1 = \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & \epsilon \end{bmatrix}, \quad \begin{matrix} \tau_1 = \tau, & \mu_2 = 0, \\ \tau_2 = 0, & \mu_3 = -\mu_1 = \epsilon, \end{matrix} \quad Q_1 = \begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & \epsilon \frac{1+\epsilon}{\bar{z}^2} \end{bmatrix}.$$

Proof Omitted for space limitation reasons. It needs to build by hand the matrix $\begin{bmatrix} \mathcal{E} & -\mathcal{A} \end{bmatrix}^\perp$ for matrices of Corollary 2 and develop the LMI (5). It is then trivial to see that for a small enough $\epsilon > 0$ the LMIs (12) implies those issued from Corollary 2. ■

Proposition 1 demonstrates that including the information about derivatives in the well-posedness problem indeed contributes to build an LMI condition with reduced (at least not increased) conservatism. To test this conservatism reduction, both corollaries are applied to the launcher example.

In order to maximize the sets of initial conditions we minimize $\text{Tr}(Q)$ when solving the LMI of Corollary 1. In the case of Corollary 2 a tradeoff of the two pieces of the piecewise quadratic function is considered. The minimization criterion is:

$$\text{Tr} \left(\begin{bmatrix} \mathbf{1}_n \\ \mathbf{0} \end{bmatrix}^T Q_1 \begin{bmatrix} \mathbf{1}_n \\ \mathbf{0} \end{bmatrix} \right) + \frac{1}{20} \text{Tr} \left(\begin{bmatrix} \mathbf{1}_n \\ C \end{bmatrix}^T Q_1 \begin{bmatrix} \mathbf{1}_n \\ C \end{bmatrix} \right).$$

These objective functions have the advantage of being linear. They correspond to maximizing the sum of the lengths of the semi-axes of the ellipsoids. This is one way to maximize the sets. Another possibility is to maximize the volume (minimize the determinant of Q).

Up to $\bar{z} = 4$ the two corollaries give almost the same sets of initial conditions. For $\bar{z} = 4$ Figure 2 shows in blue the set found by Corollary 1 and in red the one found by Corollary 2. The solid straight lines are the limits of the region of linearity $|Cx| \leq 1$. The dotted lines are the limits of $|\hat{z}| \leq \bar{z} = 4$.

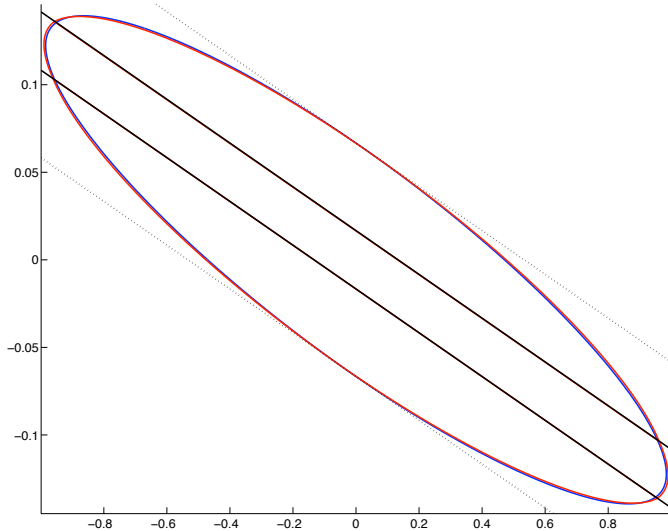


Figure 2: Sets of initial conditions for $\bar{z} = 4$ in the $x = (\theta, \dot{\theta})$ phase plane

For values of $\bar{z} \geq 4$ the two corollaries give noticeably different results. For $\bar{z} = 8$ the corresponding sets are plotted in Figure 3. It can be clearly seen that the red curve is continuous, piecewise quadratic.

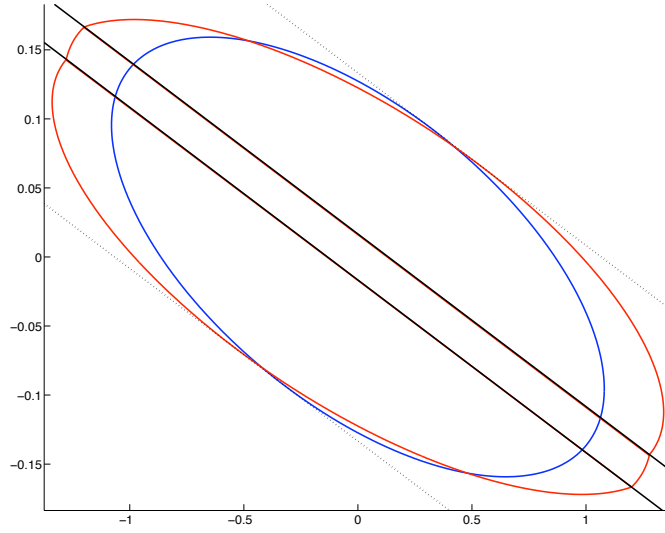


Figure 3: Sets of initial conditions for $\bar{z} = 8$ in the $x = (\theta, \dot{\theta})$ phase plane

Up to $\bar{z} = 11.5$ the LMIs of both corollaries are feasible. For $\bar{z} = 12$ (are greater values) both corollaries give infeasible LMIs. Since the system is known to be stable whatever bounded set of initial conditions, this result indicates that both corollaries are conservative. Following the proposed strategy conservatism may be further reduced by introducing information on the second derivative of the signal $\hat{w}$. The methodology is not quite identical to the one described for the introduction of the first derivative because $\dot{\hat{w}}$ is discontinuous. The authors are currently exploring ways to cope with this issue.

Reduction of conservatism from Corollary 1 to Corollary 2 goes along with increased numerical burden. For n the number of states of the system, the size of the LMI problems are given in Table 1 where *nb.var.* stand for the number of decision variables and *nb.rows.* the total number of rows of the constraints.

	<i>nb.var.</i>	<i>nb.rows.</i>
Corollary 1	$n(n+1) + 1$	$3n + 1$
Corollary 2	$(n+1)(n+2) + 5$	$3n + 4$

Table 1: Size of the LMI problems

All LMIs were coded in Matlab© using YALMIP [LÖ1] and solved using LMILab from the Robust Control toolbox.

5 Proving stability

The second step of the study is to prove that for all trajectories such that $|z(t)| \leq \bar{z}$, the system is stable. This issue amounts to the well-posedness problem of the following loop:

$$\underbrace{\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix}}_z = \underbrace{\begin{bmatrix} A & B \\ C & \mathbf{0} \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} x \\ \hat{w} \end{pmatrix}}_w, \quad \nabla = \begin{bmatrix} \mathcal{I} & \mathbf{0} \\ \mathbf{0} & \nabla_1 \end{bmatrix}. \quad (15)$$

Theorem 1 applies and gives the following result.

Corollary 3 Assume $\mathcal{A}$ is given in (15), if there exist a symmetric positive definite matrix $P > \mathbf{0}$ and a scalar $\tau > 0$ such that the following LMI holds

$$\begin{bmatrix} \mathcal{A} \\ \mathbf{1} \end{bmatrix}^T \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} & -P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tau \frac{1-\bar{z}}{\bar{z}} \\ -P & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tau \frac{1-\bar{z}}{\bar{z}} & \mathbf{0} & 2\tau \end{bmatrix}}_{\Theta} \begin{bmatrix} \mathcal{A} \\ \mathbf{1} \end{bmatrix} > \mathbf{0} \quad (16)$$

then for any initial condition guaranteeing $|Cx(t)| \leq \bar{z}$ along the trajectories of $\dot{x} = Ax + Bd_z(Cx)$, the state x converges asymptotically to the origin.

In the case of the launcher example, the LMI condition of the corollary holds whatever bounded $1 \leq \bar{z} < \infty$

Combination of this result with the one of the former section is as follows: for all initial conditions in X_0 or X_1 , the trajectories of the system are such that $|Cx(t)| \leq \bar{z}$ and are therefore stable thanks to Corollary 3. The result is intrinsically conservative since it relies on an intermediate constraint on the trajectories.

6 Concluding remarks

The study shows the applicability of the integral quadratic separation framework for estimating regions of attraction of saturated systems. The results presented in the paper are preliminary as they are based on very basic sector conditions. Improvements are however expected to be achievable in a near future by considering generalized sector conditions such as in [TGGQ11] or the advanced IQCs proposed in [Meg01].

In previous publications, integral quadratic separation was shown to be efficient to deal with robustness issues with respect to parametric uncertainties, for systems with delays and for input/output performance evaluation. Combining such features, and saturations, amounts to augmenting the operator ∇ in the feedback well-posedness problem. IQS can thus potentially provide new conditions for robust performance analysis of delayed and saturated systems. This will be tested on the launcher example which actually exhibits delays in measurements and uncertainties in parameters such as inertia.

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