On a new SDP-SOCP method for acoustic source localization problem

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Abstract

Acoustic source localization has many important applications. Convex relaxation provides a viable approach of obtaining good estimates very efficiently. There are two popular convex relaxation methods using either semi-definite programming or second order cone programming. However, the performances of the methods have not been studied properly in the literature and there is no comparison in terms of accuracy and performance. The aims of this paper are twofold. First of all, we study and compare several convex relaxation methods. We demonstrate by numerical examples that most of the convex relaxation methods cannot localize the source exactly even in the performance limit when the time difference of arrival (TDOA) information is exact. In addressing this problem, we propose a novel mixed SDP-SOCP relaxation model and study the characteristics of the optimal solutions and its localizable region. Furthermore, an error correction scheme for the proposed SDP-SOCP model is developed so that exact localization can be achieved in the performance limit. Experimental data have been collected in a room with two different array configurations to demonstrate our proposed approach.

I. Introduction

Acoustic source localization remains to be an important problem in the signal processing literature owing to their importance in many applications including radar, sonar, teleconferencing, wireless communications and voice control. Typically, the source location is estimated in two stages. In the first stage, the TDOA between each pair of microphones is estimated and transformed into distance measurements between sensors, resulting in a set of nonlinear hyperbolic equations. There are various techniques that can be used to compute the TDOA, such as cross-correlation (CC) method [1], phase transform (PHAT) and the maximum likelihood estimator (ML) [2]. In the second stage, efficient algorithms are needed to find the intersection of these nonlinear hyperbolic equations and obtain an estimate to the location of the source. However, finding the intersection of hyperbolic equations is a highly nonlinear problem.

One approach is to employ convex relaxation to convert the localization problem into a convex optimization problem, such as a semi-definite program or a second order cone program. In this way, very efficient algorithms can be derived. Semi-definite programming (SDP) is a generalization of linear programming (LP) where the decision variables are arranged in a symmetric matrix instead of a vector, and the non-negative orthant is replaced by the cone of positive semi-definite matrices [3], [4], [5]. Since interior point algorithms can be employed, semi-definite programming has polynomial time computational complexity. It has been successfully applied to a number of signal processing problems, such as filter design [6], antenna design [7], beamformer design [8], window design [9], filter bank design [10], sensor network localization [11] and configuration design [12]. The SDP and SOCP relaxation models have also been considered for solving optimal power flow problem in [13]. Sufficient conditions under which the convex relaxation is exact was presented in [14]. One formulation based on semi-definite programming for solving localization problems has been studied in [15]. Besides, another formulation using second order cone programming (SOCP) has been applied in [16]. For the SOCP relaxation, it has been proven in [16] that the optimized source location always lies in the convex hull of the microphone array. It is unfavourable for acoustic localization since the speaker is usually standing in front of the microphone array instead of being surrounded by the microphones.

Motivated by the above works, in this paper, we study the convex relaxation method for solving the source localization problem extensively and propose a novel mixed relaxation model. We first study and compare several convex relaxation methods. When the sound source is placed in front of the sensor array, we find that most existing convex relaxation methods in the literature cannot localize the source exactly even in the performance limit when the time difference of arrival (TDOA) information is exact. Furthermore, we find that the geometry of the array and the speaker location relative to the array are important factors affecting the accuracy of the solutions. In order to address these problems, we propose a novel mixed SDP-SOCP relaxation model and derive the characteristics of the optimal solution. We study the speaker localizable region, in the sense that the exact source location can be sought directly in the performance limit. We derive the exact geometry of the localizable region for the proposed mixed model. Furthermore, an error correction scheme is developed to recover the exact source location in the performance limit when the speaker is outside the localizable region. To demonstrate the method for practical applications, we also employ real recordings with two different array configurations collected in a meeting room.

The rest of the paper is organized as follows. In Section 2, we present different convex relaxation models for solving the source localization problem; then we formulate the mixed SDP-SOCP relaxation model. In Section 3, the characteristics of the optimal solution to the mixed SDP-SOCP model are derived and the geometry of the exact localizable region is derived. We present an error correction scheme in section 4 to recover the exact source location for the mixed SDP-SOCP model in the performance limit. Numerical results are shown in Section 5; comparison results with other convex relaxation models are also given. Several examples with real recordings are presented to illustrate the effectiveness of the proposed method.

II. Formulation and convex relaxation models

Assume we have microphones with locations $\boldsymbol{a}_i = (a_{i1}, a_{i2})^T$, $i = 1, \cdots, m$. Given a source location $\mathbf{s} = (s_1, s_2)^T$, the time difference of arrivals (TDOA) are given by

$$\tau_{ij} \equiv T(\{\boldsymbol{a}_i, \boldsymbol{a}_j\}, \mathbf{s}) = \frac{\|\mathbf{s} - \boldsymbol{a}_i\| - \|\mathbf{s} - \boldsymbol{a}_j\|}{c_0}, \qquad i, j = 1, \cdots, m, \tag{1}$$

where c_0 is the speed of sound in the air. The estimate is denoted by $\hat{\tau}_{ij}$ using the signals received at the ith and jth microphone. The source localization problem is to estimate the source location $\hat{\mathbf{s}}$ from the set of nonlinear hyperbolic equations (1). In the following, we introduce three relaxation models for solving this source localization problem. The first two models are using semi-definite programming and second order cone programming, respectively. Then the mixed SDP-SOCP relaxation model is presented.

A. SDP Relaxation Model

With a set of delay estimates, the problem is to find $\hat{\mathbf{s}}$ such that

$$\|\mathbf{s} - \boldsymbol{a}_i\| - \|\mathbf{s} - \boldsymbol{a}_j\| = c_0 \hat{\tau}_{ij}, \qquad i, j = 1, \cdots, m. \tag{2}$$

These equations can also be expressed as [1]

$$\begin{cases} \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, \qquad i, j = 1, \cdots, m, \\ \|\mathbf{s} - \boldsymbol{a}_i\|^2 = \alpha_i, \qquad i = 1, \cdots, m, \\ \alpha_i = \beta_i^2, \qquad i = 1, \cdots, m, \\ \beta_i \geq 0, \qquad i = 1, \cdots, m. \end{cases} \tag{3}$$

We relax the equality constraints $\alpha_i = \beta_i^2$ to $\begin{pmatrix} 1 & \beta_i \\ \beta_i & \alpha_i \end{pmatrix} \succeq 0$, which ensures that $\alpha_i \geq \beta_i^2$. Without the relaxation, α_i should be equal to the β_i^2; therefore, α_i should be as small as possible after the relaxation. Thus we set the objective function be $\min \sum_{i=1}^m \alpha_i$ and transform the source localization problem into

$$\begin{cases} \min \quad \displaystyle\sum_{i=1}^m \alpha_i \\ \text{ s.t. } \quad \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, \quad i, j = 1, \cdots, m, \\ \qquad \|\mathbf{s} - \boldsymbol{a}_i\|^2 = \alpha_i, \quad i = 1, \cdots, m, \\ \qquad \begin{pmatrix} 1 & \beta_i \\ \beta_i & \alpha_i \end{pmatrix} \succeq 0, \qquad i = 1, \cdots, m, \\ \quad \beta_i \geq 0, \qquad i = 1, \cdots, m. \end{cases} \tag{4}$$

Noting that

$$\|\boldsymbol{a}_i - \mathbf{s}\|^2 = \begin{pmatrix} \boldsymbol{a}_i^T & -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} \\ \mathbf{s}^T \end{pmatrix} [\boldsymbol{I} \ \mathbf{s}] \begin{pmatrix} \boldsymbol{a}_i \\ -1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_i^T & -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \mathbf{s} \\ \mathbf{s}^T & \mathbf{s}^T\mathbf{s} \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_i \\ -1 \end{pmatrix}. \tag{5}$$

Let $y = \mathbf{s}^T\mathbf{s}$. Then the above problem is equivalent to

[1]A different way of transforming (2) has been proposed in [15], which ends up with a different optimization model.

$$
\begin{cases}
\min & \sum_{i=1}^{m} \alpha_i \\
\text{s.t.} & \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, \quad i,j = 1, \cdots, m, \\
& \begin{pmatrix} \boldsymbol{a}_i^T & -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \mathbf{s} \\ \mathbf{s}^T & y \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_i \\ -1 \end{pmatrix} = \alpha_i, \quad i = 1, \cdots, m, \\
& \begin{pmatrix} 1 & \beta_i \\ \beta_i & \alpha_i \end{pmatrix} \succeq 0, \quad i = 1, \cdots, m, \\
& \beta_i \geq 0, \qquad i = 1, \cdots, m, \\
& y = \mathbf{s}^T \mathbf{s}.
\end{cases} \tag{6}
$$

An effective method for solving this problem is to relax $y = \mathbf{s}^T\mathbf{s}$ to

$$
\boldsymbol{Z} := \begin{pmatrix} \boldsymbol{I} & \mathbf{s} \\ \mathbf{s}^T & y \end{pmatrix} \succeq 0.
$$

Then, the relaxed version of the problem (6) can be represented by the following mixed SDP relaxation model

$$
\begin{cases}
\min & \sum_{i=1}^{m} \alpha_i \\
\text{s.t.} & \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, \quad i,j = 1, \cdots, m, \\
& \begin{pmatrix} \boldsymbol{a}_i^T & -1 \end{pmatrix} \boldsymbol{Z} \begin{pmatrix} \boldsymbol{a}_i \\ -1 \end{pmatrix} = \alpha_i, \quad i = 1, \cdots, m, \\
& \begin{pmatrix} 1 & \beta_i \\ \beta_i & \alpha_i \end{pmatrix} \succeq 0, \quad i = 1, \cdots, m, \\
& \beta_i \geq 0, \qquad i = 1, \cdots, m, \\
& \boldsymbol{Z}_{1:d,1:d} = I_d, \boldsymbol{Z} \succeq 0, \mathbf{s} = \boldsymbol{Z}_{1:d,d+1}.
\end{cases} \tag{7}
$$

B. SOCP Relaxation Model

For equation (2), instead of (3), it can also be expressed as

$$
\begin{cases}
\beta_i - \beta_j = c_0 \hat{\tau}_{ij}, & i,j = 1, \cdots, m, \\
\|\mathbf{s} - \boldsymbol{a}_i\| = \beta_i, & i = 1, \cdots, m.
\end{cases} \tag{8}
$$

Similar to the method in [16], we relax the equality constraints $\|\mathbf{s} - \boldsymbol{a}_i\| = \beta_i$ to " $\leq$ " inequality constraints, which yields a second order cone problem. Since without the relaxation, $\|\mathbf{s} - \boldsymbol{a}_i\|$ should be equal to the β_i; therefore β_i should be as small as possible after relaxation. Thus we set the objective function be $\min \sum_{i=1}^{m} \beta_i$ and transform the source localization problem into the following SOCP relaxation model

$$
\begin{cases}
\min & \sum_{i=1}^{m} \beta_i \\
\text{s.t.} & \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, \quad i,j = 1, \cdots, m, \\
& \|\mathbf{s} - \boldsymbol{a}_i\| \leq \beta_i, \quad i = 1, \cdots, m.
\end{cases} \tag{9}
$$

C. Mixed SDP-SOCP Relaxation Model

For equation (2), we can expressed it as

$$
\begin{cases}
\beta_i - \beta_j = c_0 \hat{\tau}_{ij}, & i,j = 1, \cdots, m, \\
\|\mathbf{s} - \boldsymbol{a}_i\| = \beta_i, & i = 1, \cdots, m.
\end{cases} \tag{10}
$$

Adding some redundant constraints

$$
\begin{cases}
\beta_i - \beta_j = c_0 \hat{\tau}_{ij}, & i,j = 1, \cdots, m, \\
\|\mathbf{s} - \boldsymbol{a}_i\| = \beta_i, & i = 1, \cdots, m, \\
\|\mathbf{s} - \boldsymbol{a}_i\|^2 = \alpha_i, & i = 1, \cdots, m, \\
\alpha_i = \beta_i^2, & i = 1, \cdots, m,
\end{cases} \tag{11}
$$

it is equivalent to the following problem

$$\begin{cases} \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, & i,j = 1, \cdots, m, \\ \|\mathbf{s} - \boldsymbol{a}_i\| = \beta_i, & i = 1, \cdots, m, \\ \|\mathbf{s} - \boldsymbol{a}_i\|^2 = \alpha_i, & i = 1, \cdots, m, \\ \alpha_i \geq \beta_i^2, & i = 1, \cdots, m, \\ \alpha_i \leq \beta_i^2, & i = 1, \cdots, m. \end{cases} \tag{12}$$

The inequality constraints $\alpha_i \geq \beta_i^2$ is equivalent to $\begin{pmatrix} 1 & \beta_i \\ \beta_i & \alpha_i \end{pmatrix} \succeq 0$, and $\alpha_i \leq \beta_i^2$ is equivalent to $\|\mathbf{s} - \boldsymbol{a}_i\| \leq \beta_i$. Thus we relax the equality constraints $\|\mathbf{s} - \boldsymbol{a}_i\| = \beta_i$ to " $\leq$ " inequality constraints, which yields a second order cone problem. The source localization problem can be transformed into

$$\begin{cases} \min & \sum_{i=1}^{m} \alpha_i \\ \text{s.t.} & \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, \quad i,j = 1, \cdots, m, \\ & \|\mathbf{s} - \boldsymbol{a}_i\| \leq \beta_i, \quad i = 1, \cdots, m, \\ & \|\mathbf{s} - \boldsymbol{a}_i\|^2 = \alpha_i, \quad i = 1, \cdots, m, \\ & \begin{pmatrix} 1 & \beta_i \\ \beta_i & \alpha_i \end{pmatrix} \succeq 0, \qquad i = 1, \cdots, m. \end{cases} \tag{13}$$

The above problem is equivalent to

$$\begin{cases} \min & \sum_{i=1}^{m} \alpha_i \\ \text{s.t.} & \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, \quad i,j = 1, \cdots, m, \\ & \|\mathbf{s} - \boldsymbol{a}_i\| \leq \beta_i, \quad i = 1, \cdots, m, \\ & \begin{pmatrix} \boldsymbol{a}_i^T & -1 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \mathbf{s} \\ \mathbf{s}^T & y \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_i \\ -1 \end{pmatrix} = \alpha_i, \quad i = 1, \cdots, m, \\ & \begin{pmatrix} 1 & \beta_i \\ \beta_i & \alpha_i \end{pmatrix} \succeq 0, \quad i = 1, \cdots, m, \\ & y = \mathbf{s}^T \mathbf{s}. \end{cases} \tag{14}$$

After relaxation of problem (14), the mixed SDP-SOCP relaxation model becomes

$$\begin{cases} \min & \sum_{i=1}^{m} \alpha_i \\ \text{s.t.} & \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, \quad i,j = 1, \cdots, m, \\ & \|\mathbf{s} - \boldsymbol{a}_i\| \leq \beta_i, \quad i = 1, \cdots, m, \\ & \begin{pmatrix} \boldsymbol{a}_i^T & -1 \end{pmatrix} \boldsymbol{Z} \begin{pmatrix} \boldsymbol{a}_i \\ -1 \end{pmatrix} = \alpha_i, \quad i = 1, \cdots, m, \\ & \begin{pmatrix} 1 & \beta_i \\ \beta_i & \alpha_i \end{pmatrix} \succeq 0, \quad i = 1, \cdots, m, \\ & \boldsymbol{Z}_{1:d,1:d} = I_d, \boldsymbol{Z} \succeq 0, \mathbf{s} = \boldsymbol{Z}_{1:d,d+1}. \end{cases} \tag{15}$$

In the following section, we investigate thoroughly the optimality conditions for this problem. Indeed, it is obvious that the solution of (2) is a feasible solution for (15). On the contrary, it is important to learn the conditions for which the solution of (15) satisfies the original problem (2) in the performance limit when all the TDOA's are correct. In fact, it boils down to the relative position of the speaker to the array and the array geometry. These relative positions form a region in which accurate speaker localization is possible. We will establish this region in the following section.

III. Speaker localizable region

A. A Representation Theorem for The Mixed SDP-SOCP

For the source localization problem with time difference information of m microphones, we denote

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m)^T, \quad \boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_m)^T, \quad \mathbf{s} = (s_1, s_2)^T.$$

Let

$$\boldsymbol{a}_i = (a_{i1}, a_{i2})^T, \quad i = 1, 2, \cdots, m$$

be m points in $\mathbb{R}^2$ with $m \geq 3$, and assume

$$\text{rank}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m\} = 2.$$

For ease of analysis, the mixed SDP-SOCP (15) written in the following algebraic form will be used:

$$\begin{cases} \min & \alpha_1 + \alpha_2 + \cdots + \alpha_m \\ \text{s.t.} & \beta_i - \beta_j = c_0 \hat{\tau}_{ij}, \quad i, j = 1, 2, \cdots, m, \\ & \|\mathbf{s} - \boldsymbol{a}_i\|^2 + (y - \|\mathbf{s}\|^2) = \alpha_i, \quad i = 1, 2, \cdots, m, \\ & y \geq \|\mathbf{s}\|^2, \quad \alpha_i \geq \beta_i^2, \quad \beta_i \geq \|\mathbf{s} - \boldsymbol{a}_i\|, \quad i = 1, 2, \cdots, m. \end{cases} \tag{16}$$

Also, we use the notation $(\mathbf{s}, y, \boldsymbol{\beta}, \boldsymbol{\alpha})$ to denote a feasible solution for (16). Let $\mathbf{s}^* = (s_1^*, s_2^*)^T$ be the true source location, i.e., it satisfies

$$\|\mathbf{s}^* - \boldsymbol{a}_i\| - |\mathbf{s}^* - \boldsymbol{a}_j\| = c_0 \hat{\tau}_{ij}, i, j = 1, 2, \cdots, m.$$

Denote by $\beta_i^* = \|\mathbf{s}^* - \boldsymbol{a}_i\|$, $i = 1, 2, \cdots, m$. In general, we can first establish a simple relationship between the optimal solution of (16) and the true solution $(\mathbf{s}^*, y^*, \boldsymbol{\beta}^*, \boldsymbol{\alpha}^*)$ by the following lemma.

Lemma 1. *Let $(\hat{\mathbf{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ be an optimal solution of (16). Then there exists a $\lambda \geq 0$ such that*

$$\hat{\beta}_i = \beta_i^* - \lambda, \quad i = 1, 2, \cdots, m.$$

Proof. Set $\boldsymbol{\beta^*} = (\beta_1^*, \beta_2^*, \cdots, \beta_m^*)^T$, $\boldsymbol{\alpha^*} = ((\beta_1^*)^2, (\beta_2^*)^2, \cdots, (\beta_m^*)^2)^T$, and $y^* = \|\mathbf{s}^*\|^2$. Then $(\mathbf{s}^*, y^*, \boldsymbol{\beta^*}, \boldsymbol{\alpha^*})$ is a feasible solution of the mixed SDP-SOCP (16). Since

$$\alpha_i^* = (\beta_i^*)^2 = \|\mathbf{s}^* - \boldsymbol{a}_i\|^2, \; i = 1, 2, \cdots, m \tag{17}$$

and $(\hat{\mathbf{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ is an optimal solution of (16), then we have

$$\sum_{i=1}^m \hat{\beta}_i^2 \leq \sum_{i=1}^m \hat{\alpha}_i \leq \sum_{i=1}^m \alpha_i^* = \sum_{i=1}^m (\beta_i^*)^2.$$

Notice that $\hat{\beta}_i, i = 1, \cdots, m$ and $\beta_i^*, i = 1, \cdots, m$ are the solution of the linear equations with m variables and 2 equations:

$$x_m - x_i = c_0 \hat{\tau}_{mi}, \quad i = 1, 2, \cdots, m - 1,$$

and the rank of the coefficient matrix of the above linear equations is 2, then there exists $\lambda \in \mathbb{R}$ such that

$$\hat{\beta}_i = \beta_i^* - \lambda, \quad i = 1, \cdots, m. \tag{18}$$

From

$$\sum_{i=1}^m (\beta_i^* - \lambda)^2 \leq \sum_{i=1}^m (\beta_i^*)^2,$$

we have $\lambda > 0$. $\square$

As indicated by the above lemma, it is clear that if $\lambda > 0$, then the solution of (16) will not be the solution of the original problem (2) and hence is not the true speaker location. In order to study the property of this λ, we further assume the array form a triangle. In fact, since most array configurations can be represented by a union of triangular array, it will be sufficient to derive the conditions for a triangular array and then generalize to other configurations.

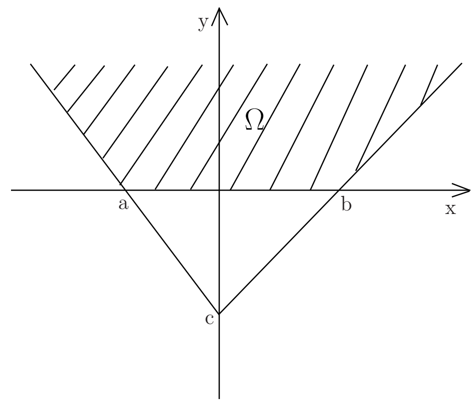

Fig. 1.

When $m=3$, we have $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)^T$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)^T$, $\mathbf{s}=(s_1,s_2)^T$, and $\boldsymbol{a}_i=(a_{i1},a_{i2})^T$, $i=1,2,3$, be three microphone locations in $\mathbb{R}^2$. The mixed SDP-SOCP (15) model becomes

$$\begin{cases} \min & \alpha_1+\alpha_2+\alpha_3 \\ \text{s.t.} & \beta_1-\beta_2=c_0\hat{\tau}_{12}, \\ & \beta_1-\beta_3=c_0\hat{\tau}_{13}, \\ & \|\mathbf{s}-\boldsymbol{a}_i\|^2+(y-\|\mathbf{s}\|^2)=\alpha_i, \quad i=1,2,3, \\ & y\geq\|\mathbf{s}\|^2, \quad \alpha_i\geq\beta_i^2, \quad \beta_i\geq\|\mathbf{s}-\boldsymbol{a}_i\|, \quad i=1,2,3. \end{cases} \tag{19}$$

Let $\mathbf{s}^*=(s_1^*,s_2^*)^T$ be the exact source location satisfying

$$\|\mathbf{s}^*-\boldsymbol{a}_1\|-\|\mathbf{s}^*-\boldsymbol{a}_2\|=c_0\tau_{12}, \ \|\mathbf{s}^*-\boldsymbol{a}_1\|-\|\mathbf{s}^*-\boldsymbol{a}_3\|=c_0\tau_{13}.$$

Define $\beta_i^*=\|\mathbf{s}^*-\boldsymbol{a}_i\|$, $i=1,2,3$, $\boldsymbol{\beta}^*=(\beta_1^*,\beta_2^*,\beta_3^*)^T$, $\boldsymbol{\alpha}^*=(\beta_1^{*2},\beta_2^{*2},\beta_3^{*2})^T$, and $y^*=\|\mathbf{s}^*\|^2$. Clearly $(\mathbf{s}^*,y^*,\boldsymbol{\beta}^*,\boldsymbol{\alpha}^*)$ is a feasible solution of (19). However, the opposite is not straightforward. As a start, we first show that, if the speaker is surrounded by the microphones, then the optimality of this solution is proven in the following theorem.

Theorem 1. *If $\mathbf{s}^*$ is in the triangle region $\Delta_{a_1a_2a_3}$ determined by the three points $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, then $(\mathbf{s}^*,y^*,\boldsymbol{\beta}^*,\boldsymbol{\alpha}^*)$ is an optimal solution of (19).*

Proof. The proof will be given in the appendix. □

For the more general situation when the speaker is standing in front of the microphone array, the optimal solution of (19) is not necessary the optimal solution to the original problem. In order to understand the conditions for this to be true, we need the following lemmas, which summarize some properties of the optimal solution of (19).

Lemma 2. *Let $(\hat{\mathbf{s}},\hat{y},\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\alpha}})$ be an optimal solution of (19). Then $\|\hat{\mathbf{s}}-\boldsymbol{a}_i\|<\hat{\beta}_i, \ i=1,2,3$ cannot be satisfied at the same time.*

Proof. The proof will be given in the appendix. □

Lemma 3. *Let $(\hat{\mathbf{s}},\hat{y},\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\alpha}})$ be an optimal solution of (19). Then $\hat{\beta}_i^2<\hat{\alpha}_i, \ i=1,2,3$ cannot be satisfied at the same time.*

Proof. The proof will be given in the appendix. □

In the following, for simplicity of notation, we locate the microphones as: $\boldsymbol{a}_1=(a,0)^T$, $\boldsymbol{a}_2=(b,0)^T$ and $\boldsymbol{a}_3=(0,c)^T$ where $a\leq 0$, $b>0$, $c<0$. As illustrated in Figure 1, let Ω be the region enclosed by lines $u_2=0$, $u_1-a=-\frac{a}{c}u_2$ and $u_1-b=-\frac{b}{c}u_2$, i.e.

$$\Omega=\left\{\boldsymbol{u}=(u_1,u_2)^T \,|\ u_2\geq 0, u_1-a>-\tfrac{a}{c}u_2 \text{ and } u_1-b<-\tfrac{b}{c}u_2 \ \right\}.$$

Since $(\mathbf{s}^*,y^*,\boldsymbol{\beta}^*,\boldsymbol{\alpha}^*)$ is an optimal solution of (19) if $\mathbf{s}^*$ is in the triangle region $\Delta_{a_1a_2a_3}$ determined by the three points $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$, we only consider the case when the speaker is in front of the array, which means $\mathbf{s}^*\in\Omega$ with $s_2^*>0$.

For $\lambda\in[0,\min_{1\leq i\leq 3}\beta_i^*]$, as illustrated in Figure 2, we have

$$B(\boldsymbol{a}_1,\beta_1^*-\lambda)\cap B(\boldsymbol{a}_2,\beta_2^*-\lambda)\subset B(\boldsymbol{a}_3,\beta_3^*-\lambda),$$

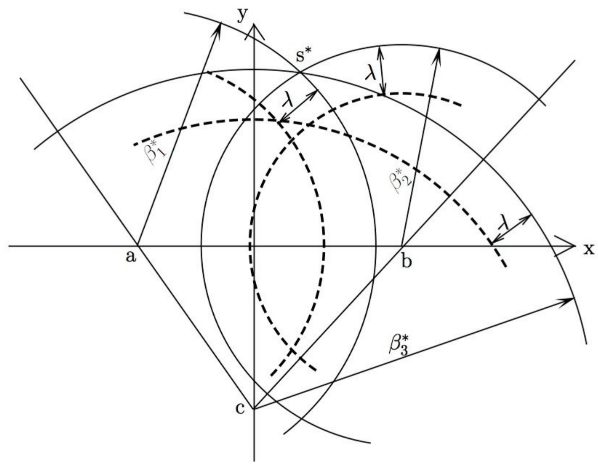

Fig. 2.

where $B(\boldsymbol{a}, r) = \{\boldsymbol{u}|\ |\boldsymbol{u} - \boldsymbol{a}| \leq r\}$. In particular, $\beta_3^* - \lambda > -c$.

Let $(\hat{\mathbf{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ be an optimal solution of (19). By Lemma 1, there exists $\lambda \in [0, \min_{1\leq i\leq 3} \beta_i^*]$ such that

$$\hat{\beta}_i = \beta_i^* - \lambda, \quad i = 1, 2, 3.$$

Then

$$\hat{\mathbf{s}} \in \cap_{i=1}^3 B(\boldsymbol{a}_i, \hat{\beta}_i) = \cap_{i=1}^3 B(\boldsymbol{a}_i, \beta_i^* - \lambda) = B(\boldsymbol{a}_1, \beta_1^* - \lambda) \cap B(\boldsymbol{a}_2, \beta_2^* - \lambda). \tag{20}$$

By Lemma 2 and Lemma 3, we have one of the following seven cases :

(C1):

$$\begin{cases} \|\hat{\mathbf{s}} - \boldsymbol{a}_1\|^2 = (\beta_1^* - \lambda)^2, \\ \|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 = (\beta_2^* - \lambda)^2, \\ \|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2 \leq (\beta_3^* - \lambda)^2 = \hat{\alpha}_3; \end{cases} \tag{21}$$

(C2):

$$\begin{cases} \|\hat{\mathbf{s}} - \boldsymbol{a}_1\|^2 = (\beta_1^* - \lambda)^2, \\ (\beta_2^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 = (\beta_3^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2 > 0; \end{cases} \tag{22}$$

(C3):

$$\begin{cases} \|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 = (\beta_2^* - \lambda)^2, \\ (\beta_1^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_1\|^2 = (\beta_3^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2 > 0; \end{cases} \tag{23}$$

(C4): $\|\hat{\mathbf{s}} - \boldsymbol{a}_1\|^2 = (\beta_1^* - \lambda)^2$, $\|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 < (\beta_2^* - \lambda)^2 = \hat{\alpha}_2$, $\|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2 \leq (\beta_3^* - \lambda)^2$, and $(\beta_2^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 > (\beta_3^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2$;

(C5): $\|\hat{\mathbf{s}} - \boldsymbol{a}_1\|^2 = (\beta_1^* - \lambda)^2$, $\|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 < (\beta_2^* - \lambda)^2$, $\|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2 \leq (\beta_3^* - \lambda)^2 = \hat{\alpha}_3$, and $(\beta_2^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 < (\beta_3^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2$;

(C6): $\|\hat{\mathbf{s}} - \boldsymbol{a}_1\|^2 < (\beta_1^* - \lambda)^2 = \hat{\alpha}_1$, $\|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 = (\beta_2^* - \lambda)^2$, $\|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2 \leq (\beta_3^* - \lambda)^2$, and $(\beta_1^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_1\|^2 > (\beta_3^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2$;

(C7): $\|\hat{\mathbf{s}} - \boldsymbol{a}_1\|^2 < (\beta_1^* - \lambda)^2$, $\|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 = (\beta_2^* - \lambda)^2$, $\|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2 \leq (\beta_3^* - \lambda)^2 = \hat{\alpha}_3$, and $(\beta_1^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_1\|^2 < (\beta_3^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2$.

We will prove that cases *(C4)*, *(C5)*, *(C6)* and *(C7)* can never happen by the following theorem. In this way, the optimal solution of (19) must satisfy only (C1), (C2) or (C3).

Theorem 2. *(Representation theorem). Assume that $-c \geq |a+b|$. Let $(\hat{\mathbf{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ be an optimal solution of (19). Then $\hat{s}_2 \geq 0$ and there exists $\lambda \geq 0$ such that $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}^* - \boldsymbol{\lambda}$, and one of (C1), (C2) and (C3) holds. Furthermore, if $\hat{s}_1 \in [a, b]$, then (C1) must be true.*

Proof. The proof will be completed by lemma 4-8 given in the appendix. □

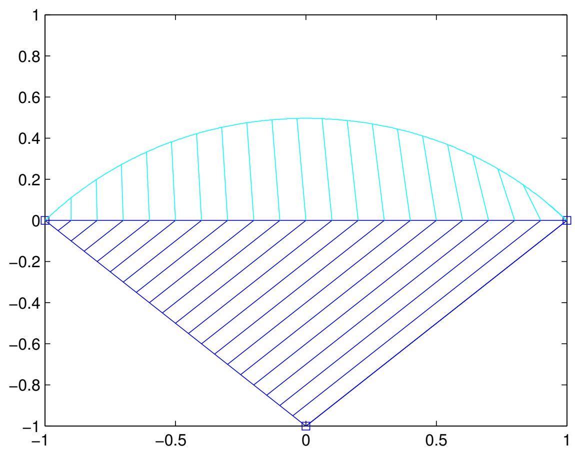

Fig. 3. Exact localizable region

B. A Characterization Theorem

In this section, we study the conditions such that $(s^*, y^*, \boldsymbol{\beta}^*, \boldsymbol{\alpha}^*)$ is also the optimal solution of (19) when $s_1^* \in [a, b]$. Let $(\hat{\mathbf{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ be the optimal solution of (19) and assume that $\hat{s}_1 \in [a, b]$. Then by Theorem 2, $\hat{s}_2 > 0$ and $\hat{\mathbf{s}}$ is a solution of equation (21). Define

$$D = \left\{ \lambda | \lambda \in (0, \min\{\beta_1^*, \beta_2^*, \beta_3^*\}) \right\}. \tag{24}$$

Assume $\hat{\mathbf{s}}$ to be a solution of equation (21), define

$$M(\lambda) := (\beta_1^* - \lambda)^2 + (\beta_3^* - \lambda)^2 + \|\hat{\mathbf{s}} - \boldsymbol{a}_2\|^2 + 2((\beta_3^* - \lambda)^2 - \|\hat{\mathbf{s}} - \boldsymbol{a}_3\|^2), \quad \lambda \in D.$$

It follows from Theorem 2 that $(\mathbf{s}^*, y^*, \boldsymbol{\beta}^*, \boldsymbol{\alpha}^*)$ is an optimal solution of (19) if and only if

$$g(\lambda) := M(\lambda) - ((\beta_1^*)^2 + (\beta_2^*)^2 + (\beta_3^*)^2) > 0 \quad \forall \lambda \in D. \tag{25}$$

The following theorem gives a characterization such that the mixed SDP-SOCP (19) model can give the true solution under the condition $-c \geq |a + b|$ in Theorem 2.

Theorem 3. *(Characterization theorem). Assume that $-c \geq |a+b|$. Let $(\hat{\mathbf{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ be an optimal solution of (19) and assume that $\hat{s}_1 \in [a, b]$. Set*

$$\begin{aligned} \Psi(s^*) := & -\left((\beta_1^*)^2 - (\beta_3^*)^2 + \frac{a((\beta_1^*)^2 - (\beta_2^*)^2)}{b-a} + ab + c^2 \right) \\ & \times \left(\beta_1^* - \beta_2^* - 3\beta_3^* + \frac{2a(\beta_1^* - \beta_2^*)}{b-a} \right) \\ & - 2c^2 \left(-2\,\beta_1^* + \frac{((\beta_1^*)^2 - (\beta_2^*)^2 + (b-a)^2)(\beta_1^* - \beta_2^*)}{(b-a)^2} \right). \end{aligned} \tag{26}$$

If $\Psi(s^*) > 0$, then $\hat{\mathbf{s}} = s^*$, i.e., the mixed SDP-SOCP (19) model gives the true solution. Inversely, if the mixed SDP-SOCP (19) model gives the true solution, then $\Psi(s^*) \geq 0$.

Proof. The proof will be given in the appendix. □

To illustrate the characterization theorem, we give a simple example here. Assume $a = -1$, $b = 1$, $c = -1$, by Theorem 3, we find the localizable region of the mixed SDP-SOCP model (Figure 3). For the SOCP relaxation model, the localizable region is the convex hull of the microphone array [16]. It is unfavorable for acoustic localization since the speaker is usually standing in front of the microphone array instead of surrounded by the microphones. From Figure 3, for the mixed SDP-SOCP relaxation model, we can find that there is an extended region to the convex hull of the triangular array which has the exact localizable property. That means when the sound source is placed in front of the microphone array, we can still localize the source exactly when the TDOA information is exact. This result is better than the SOCP relaxation model.

From this illustrative example for a triangular array, we know that the localizable region is shaped like a fan Figure 4. We further consider and simulate a more complex situation. Assume there are four microphones located at the coordinates

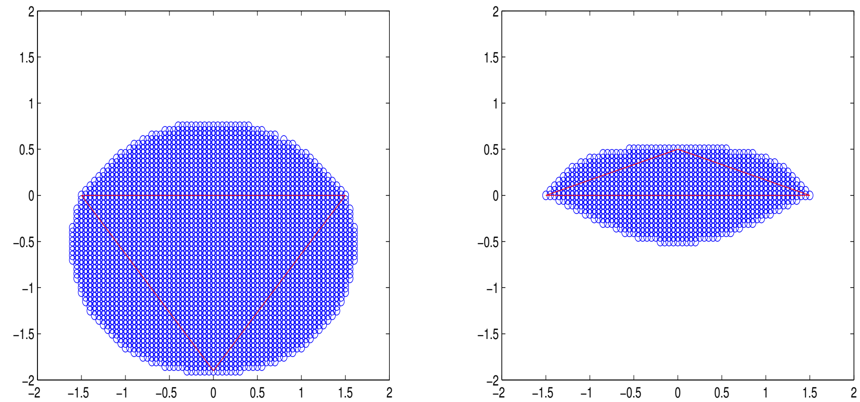

Fig. 4.

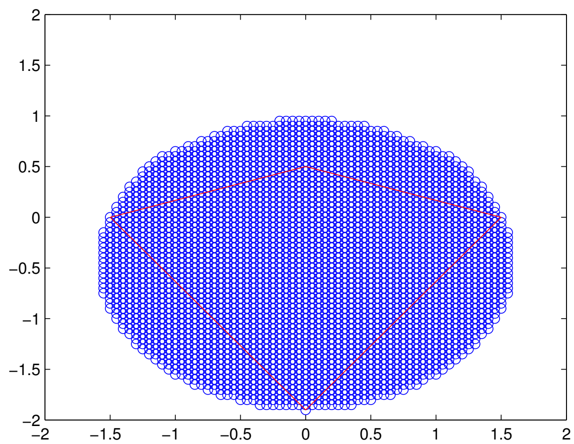

Fig. 5. Localizable region with four microphones

$\{(-1.5,0),(0,0.5),(1.5,0),(0,-1.9)\}$, its localizable region is shown in Figure 5. We find that the localizable region is like a combination of several fan-shaped regions. Since the distance between two points is invariant under translation and rotation, we can state a general form of Theorem 3 in the following theorem.

Theorem 4. *Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ be three points in $\mathbb{R}^2$ and $\boldsymbol{r}$ be the foot of the perpendicular from point $\boldsymbol{a}_3$ to the line segment between $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Set*

$$a = -\|\boldsymbol{a}_1 - \boldsymbol{r}\|, b = \|\boldsymbol{a}_2 - \boldsymbol{r}\|, c = -\|\boldsymbol{a}_3 - \boldsymbol{r}\|,$$

and

$$\beta_i^* = \|\mathbf{s}^* - \boldsymbol{a}_i\|, \quad i = 1, 2, 3.$$

Assume that $\boldsymbol{r}$ is between $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, and $-c \geq |a+b|$. Let $(\hat{\mathbf{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ be an optimal solution of (19) and assume that $\hat{s}_1 \in [a,b]$. If $\Psi(\mathbf{s}^*) > 0$, then $\hat{\mathbf{s}} = \mathbf{s}^*$. Conversely, if the mixed SDP-SOCP (19) model gives the true solution, then $\Psi(\mathbf{s}^*) \geq 0$.

Corollary 1. *For any three point $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ with $\|\boldsymbol{a}_1 - \boldsymbol{a}_3\| = \|\boldsymbol{a}_2 - \boldsymbol{a}_3\|$, set*

$$b = \|\boldsymbol{a}_1 - \boldsymbol{a}_2\|/2, \quad c = -\sqrt{\|\boldsymbol{a}_2 - \boldsymbol{a}_3\|^2 - b^2}.$$

Let $(\hat{\mathbf{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ be an optimal solution of (19) and assume that $\hat{s}_1 \in [a,b]$. If

$$3\left((\beta_1^*)^2 + (\beta_2^*)^2 - 2(\beta_3^*)^2 - 2b^2 + 2c^2\right)\beta_3^* - \frac{c^2}{b^2}\left(((\beta_1^*)^2 - (\beta_2^*)^2) - 4b^2(\beta_1^* + \beta_2^*)\right) > 0,$$

then the mixed SDP-SOCP (19) model gives the true solution.

Conversely, if the mixed SDP-SOCP (19) model gives the true solution, then

$$3\left((\beta_1^*)^2+(\beta_2^*)^2-2(\beta_3^*)^2-2b^2+2c^2\right)\beta_3^*-\frac{c^2}{b^2}\left(((\beta_1^*)^2-(\beta_2^*)^2)-4b^2(\beta_1^*+\beta_2^*)\right)\geq 0.$$

Proof. Since under the condition $a=-b$, a_3 can be simplified to

$$\begin{aligned} a_3=8\Psi(\mathbf{s}^*)=&\ 12\left((\beta_1^*)^2+(\beta_2^*)^2-2(\beta_3^*)^2-2b^2+2c^2\right)\beta_3^* \\ &-\frac{4c^2}{b^2}\left(((\beta_1^*)^2-(\beta_2^*)^2)-4b^2(\beta_1^*+\beta_2^*)\right). \end{aligned}$$

Thus, Corollary 1 is a consequence of Theorem 4. □

Corollary 2. *Let $\boldsymbol{a}_1,\boldsymbol{a}_2,\cdots,\boldsymbol{a}_n$ be n points in $\mathbb{R}^2$, then the interior of the union of all the localizable regions determined by any three points in the convex hull formed by these n points is a localizable region for the n points.*

Proof. If $\mathbf{s}^*$ is in this interior, then there exists a triangular convex hull containing $\mathbf{s}^*$. Thus we have $\Psi(s^*)>0$. Therefore the mixed SDP-SOCP (19) model gives the true solution.

□

IV. An error correction algorithm

From the previous analysis, we know that if the source location is outside the exact localizable region, then the proposed mixed SDP-SOCP model cannot find the source location exactly. In this section, a method is developed to recover the exact solution when the source location is outside the exact localizable region. The method is based on the idea of correcting the solution to the mixed SDP-SOCP relaxation model by a second order polynomial equation. We consider source localization with time-difference information of m points. By Lemma 1, from (17) and (18), we know there exists $\lambda\geq 0$ such that $\beta^*=\hat{\beta}+\lambda$, i.e, $\mathbf{s}^*$ is a solution of the equations

$$\|\mathbf{s}-\boldsymbol{a}_i\|=\hat{\beta}_i+\lambda, i=1,2,\cdots,m. \tag{27}$$

This implies

$$\|\mathbf{s}-\boldsymbol{a}_m\|^2-\|\mathbf{s}-\boldsymbol{a}_i\|^2=(\hat{\beta}_m+\lambda)^2-(\hat{\beta}_i+\lambda)^2, i=1,2,\cdots,m-1.$$

That is

$$2\sum_{j=1}^d(a_{ij}-a_{mj})s_j=\sum_{j=1}^d(a_{ij}^2-a_{mj}^2)+(\hat{\beta}_m^2-\hat{\beta}_i^2)+2\lambda(\hat{\beta}_m-\hat{\beta}_i),\ i=1,\cdots,m-1. \tag{28}$$

Define

$$\boldsymbol{B}=(b_{ij})_{(m-1)\times d},\ \boldsymbol{C}=(c_1,\cdots,c_{m-1})^T,\ \boldsymbol{D}=(d_1,\cdots,d_{m-1})^T,$$

where

$$b_{ij}=2(a_{ij}-a_{mj}),\quad i=1,2,\cdots,m-1,\ j=1,\cdots,d,$$

and

$$c_i=\sum_{j=1}^d(a_{ij}^2-a_{mj}^2)+(\hat{\beta}_m^2-\hat{\beta}_i^2),\quad d_i=2(\hat{\beta}_m-\hat{\beta}_i),\qquad i=1,2,\cdots,m-1.$$

Then the equations (28) can be re-written as

$$\boldsymbol{B}\mathbf{s}=\boldsymbol{C}+\lambda\boldsymbol{D}, \tag{29}$$

which implies

$$\boldsymbol{B}^T\boldsymbol{B}\mathbf{s}=\boldsymbol{B}^T\boldsymbol{C}+\lambda\boldsymbol{B}^T\boldsymbol{D}.$$

Therefore,

$$\mathbf{s}=(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{C}+\lambda(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{D}. \tag{30}$$

Substituting $(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{C}+\lambda(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{D}$ for $\mathbf{s}$ in $\|\mathbf{s}-\boldsymbol{a}_m\|^2=(\hat{\beta}_m+\lambda)^2$, we obtain

$$\|(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{C}+\lambda(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{D}-\boldsymbol{a}_m\|^2=(\hat{\beta}_m+\lambda)^2. \tag{31}$$

It can be re-arranged into the following form

$$\begin{aligned} &\lambda^2 \left(1 - ((\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{D})^T(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{D}\right) \\ &+ \lambda \left(2\hat{\beta}_m - 2\left((\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{C} - \boldsymbol{a}_m\right)^T (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{D}\right) \\ &+ \hat{\beta}_m^2 - ((\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{C} - \boldsymbol{a}_m)^T((\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{C} - \boldsymbol{a}_m) = 0. \end{aligned} \tag{32}$$

By solving this second order polynomial equation, we can obtain the exact solution for the source location from the solution of the mixed SDP-SOCP relaxation. We summarize the result in the following theorem.

Theorem 5. *Let $\boldsymbol{a}_i = (a_{i1}, a_{i2}, \cdots, a_{id})^T$, $i = 1, 2, \cdots, m$, be m points in $\mathbb{R}^d$ satisfying*

$$\operatorname{rank}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_m\} = d.$$

Let $(\hat{\mathrm{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ be an optimal solution of (16) and let $\hat{\lambda}$ (error corrector) be a positive solution of the equation (32). Define

$$\mathrm{s}^* = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{C} + \hat{\lambda}(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{D}, \tag{33}$$

then s^* is the true source location, i.e.,

$$\|\mathrm{s}^* - \boldsymbol{a}_i\| - |\mathrm{s}^* - \boldsymbol{a}_j\| = c_0\hat{\tau}_{ij}, \quad i, j = 1, 2, \cdots, m. \tag{34}$$

Remark 1. *If $m = d + 1$, then*

$$(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T = \boldsymbol{B}^{-1}.$$

Therefore, equation (30) and (32) can be simplified to

$$\mathrm{s}^* = \boldsymbol{B}^{-1}\boldsymbol{C} + \hat{\lambda}\boldsymbol{B}^{-1}\boldsymbol{D}, \tag{35}$$

$$\begin{aligned} &\lambda^2\left(1 - \boldsymbol{D}^T(\boldsymbol{B}^{-1})^T\boldsymbol{B}^{-1}\boldsymbol{D}\right) + \lambda\left(2\hat{\beta}_{d+1} - 2\left(\boldsymbol{B}^{-1}\boldsymbol{C} - \boldsymbol{a}_{d+1}\right)^T\boldsymbol{B}^{-1}\boldsymbol{D}\right) \\ &+\hat{\beta}_{d+1}^2 - (\boldsymbol{B}^{-1}\boldsymbol{C} - \boldsymbol{a}_{d+1})^T(\boldsymbol{B}^{-1}\boldsymbol{C} - \boldsymbol{a}_{d+1}) = 0. \end{aligned} \tag{36}$$

Remark 2. *If $d = 2$ and $m = 3$, then*

$$B = \begin{pmatrix} 2(a_{11} - a_{31}) & 2(a_{12} - a_{32}) \\ 2(a_{21} - a_{31}) & 2(a_{22} - a_{32}) \end{pmatrix},$$

$$C = \begin{pmatrix} a_{11}^2 + a_{12}^2 - a_{31}^2 - a_{32}^2 + \hat{\beta}_3^2 - \hat{\beta}_1^2 \\ a_{21}^2 + a_{22}^2 - a_{31}^2 - a_{32}^2 + \hat{\beta}_3^2 - \hat{\beta}_2^2 \end{pmatrix},$$

$$D = \begin{pmatrix} 2(\hat{\beta}_3 - \hat{\beta}_1) \\ 2(\hat{\beta}_3 - \hat{\beta}_2) \end{pmatrix}.$$

V. Examples and applications

In this section, we first present some numerical results for comparison between the three convex relaxation models described in Section II and also YWL's model ([15]), under the condition that there are no TODA errors. The SDPT3 ([17]) will be employed as the solver. Assume there are three microphones located at the coordinates $\{(-1, 0), (0, -1), (1, 0)\}$ and the source location is $(0.5, 0.35)$. The results of using SDP relaxation model, SOCP relaxation model, YWL's model and mixed SDP-SOCP relaxation model are shown in Figure 6. From Table I, we can see that the proposed mixed SDP-SOCP can indeed find the exact location subject to some numerical errors, and is therefore more accurate than the other convex relaxation models.

	SDP	SOCP	YWL	SDP-SOCP
ESL	$(0, -0.1707)$	$(0.4650, -0.0023)$	$(0.7033, 0.5092)$	$(0.5000, 0.3498)$
Error	0.7219	0.3540	0.2582	$1.6939e-04$

TABLE I

If the source location is outside the exact localizable region, we can use the developed error correction method to recover the solution. Assume there are three microphones located at the coordinates $\{(-1, 0), (0, -1), (1, 0)\}$ and the source location is $(0.8, 2.8)$. The results of using SDP relaxation model, SOCP relaxation model, YWL's model, mixed SDP-SOCP relaxation

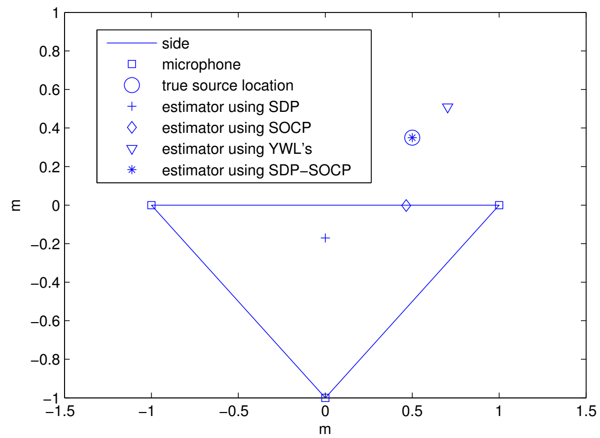

Fig. 6. Comparion between various relaxation models

	SDP	SOCP	YWL	SDP-SOCP	SDP-SOCP-EC
ESL	$(-0.0004,$ $0.0762)$	$(0.2608,$ $-0.0004)$	$(-0.0240,$ $0.1153)$	$(0.2775,$ $0.3518)$	$(0.8000,$ $2.8000)$
Error	2.8389	2.8519	2.8083	2.5033	$6.0688e-13$

TABLE II

model together with the error correction method are shown in Figure 7. From Table II, we can see that the proposed error correction method can indeed find the exact location if the source location is outside the exact localizable region.

In the next example, we study the effect of reverberation via the image-source method (ISM) described in [18]. The ISM is a well-known technique that can be used in order to generate a synthetic room impulse response (RIR) in a given environment. Once such a RIR is available, a sample of audio data can be obtained by convolving the RIR with a given source signal [19]. This provides a simulated sample of the sound signal received at the sensor in the considered environment. Here we assume the room is 5 x 5 x 2.5m (L x W x H). There are nine microphones located at the coordinates $\{(0.9, 1.4, 1)$,$(1.3, 1.8, 1)$,$(1.7, 2.2, 1)$,$(2.1, 2.6, 1)$,$(2.5, 3, 1)$,$(2.9, 2.6, 1)$,$(3.3, 2.2, 1)$,$(3.7, 1.8, 1)$, $(4.1, 1.4, 1)\}$ and the true source location is $(2.5, 1, 1)$. The reverberation time T_{60} is 0.2s. First we use the ISM technique to generate a synthetic room impulse response. Then we estimate the $\hat{\tau}_{ij}$ using the signals received at the ith and jth microphone by the cross-correlation method [1]. After that we use the mixed SDP-SOCP relaxation model with error correction to estimate the source location. We can see from Figure 8 that the estimated source location denoted by star point is almost at the same position as

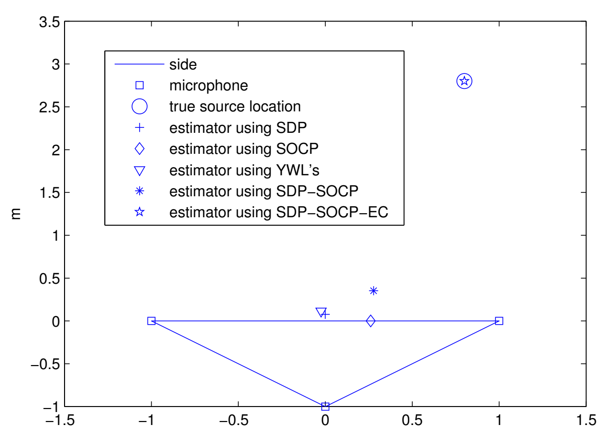

Fig. 7. Comparion between various relaxation models

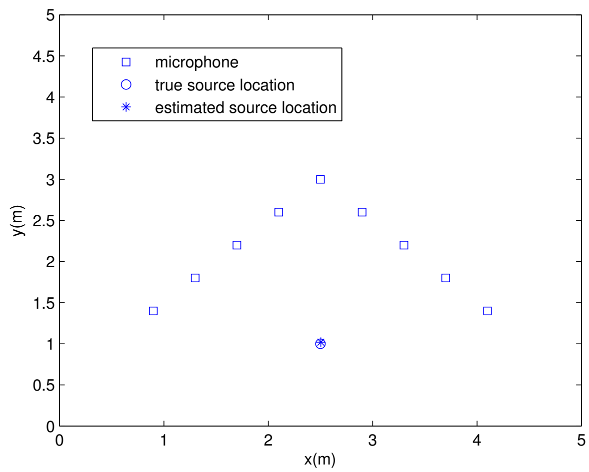

Fig. 8. SDP-SOCP-EC method

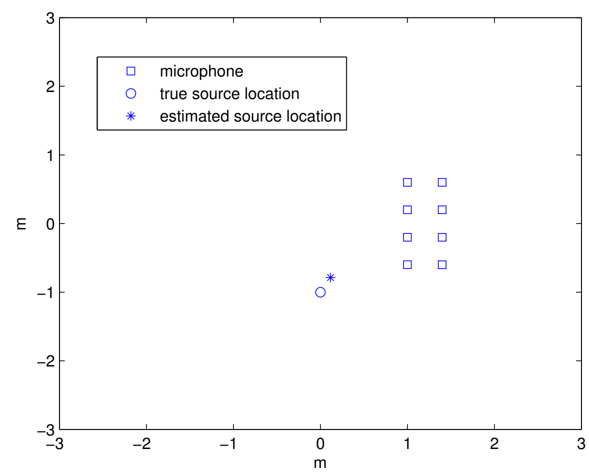

Fig. 9. SDP-SOCP-EC method with real data

the circle point which is the true source location. The corresponding error is 0.0209.

We continue to study the use of the mixed SDP-SOCP model with error correction method for localization using real data. In collecting the experimental data, the microphone arrays used for the recording consist of 8 microphones, rectangular shape and L- shape in a room with 4 x 4.5 x 2.4m (L x W x H). The microphones are custom built and are connected to OctaMic amplifier, which is then connected to ADI-648 by RME to Hammerfall DSP interface card in computer. All these equipments are from RME. The sound driver used is ASIO Hammerfall DSP. All the microphone has been calibrated before used. The speakers are connected using Delta 1010LT sound-card by M-Audio. Note that separate sound devices are used for recording and playback but their sampling frequency are set to be the same, which is 48000Hz. The speech signals used for the recordings are from NOIZEUS database.

First we use this real data to estimate the $\hat{\tau}_{ij}$, and then solve the formulated mixed SDP-SOCP model (15) to get $\hat{\mathbf{s}}$. A rectangular array is employed in Figure 9 while L-shape array is applied in Figures 10 and 11, where the circle point is the true source location and the star point is the estimated source location computed from using error correction method with $\hat{\mathbf{s}}$ as a starting point. Clearly, the results look promising and the proposed method is very efficient computationally.

VI. Conclusion

In this paper, we have proposed a novel mixed SDP-SCOP relaxation model for solving the source localization problem with TDOA information. The characteristics of the solution to the proposed model have been studied rigorously. Furthermore, based on the proven representation theorem, we have derived the geometry of the localizable region. When the source point is not

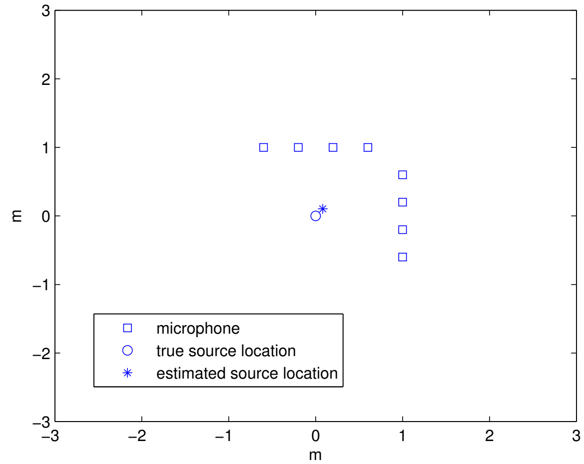

Fig. 10. SDP-SOCP-EC method with real data

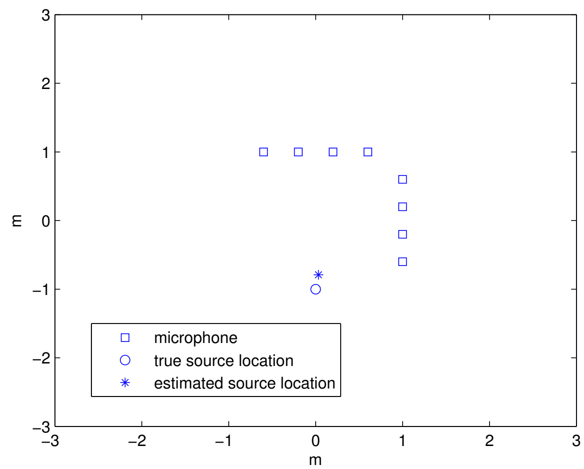

Fig. 11. SDP-SOCP-EC method with real data

within the localizable region, an error correction algorithm has been developed to recover the exact solution under exact TDOA information. Numerical results have shown that the proposed model is more accurate than other convex relaxation methods in the literature. By applying the proposed model on real data, we have demonstrated that the method is an accurate and effective way of solving the problem. As a future extension, the method can be generalized easily to problems with estimated microphone locations and this work is well-underway. Moreover, it is of interest to extend the method for localization of multiple sources.

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APPENDIX

Appendix

Proof of Theorem 1

Proof. Let $(\hat{\mathrm{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ be an optimal solution of (19), then there exists a $\lambda \geq 0$ such that

$$\hat{\beta}_i = \beta_i^* - \lambda, \quad i = 1, 2, 3.$$

Thus

$$\hat{\mathbf{s}} \in \cap_{i=1}^3 B(\boldsymbol{a}_i, \beta_i^* - \lambda) \subset \cap_{i=1}^3 B(\boldsymbol{a}_i, \beta_i^*).$$

But if $\mathbf{s}^* \in \Delta_{a_1 a_2 a_3}$, then

$$\cap_{i=1}^3 B(\boldsymbol{a}_i, \beta_i^*) = \{\mathbf{s} | \ \|\mathbf{s} - \boldsymbol{a}_i\| \leq \beta_i^* = \|\mathbf{s}^* - \boldsymbol{a}_i\|, i = 1, 2, 3\} = \{\mathbf{s}^*\}.$$

Therefore $\hat{\mathbf{s}}$=$\mathbf{s}^*$. $\square$

Proof of Lemma 2

Proof. If for all $i = 1, 2, 3$, $\|\hat{\mathrm{s}} - \boldsymbol{a}_i\| < \hat{\beta}_i$. Then it is clear that $\hat{y} > \|\hat{\mathrm{s}}\|^2$. Let $\delta = \min_{i=1,2,3}\{\hat{\beta}_i - \|\hat{\mathrm{s}} - \boldsymbol{a}_i\|\}$, $\tilde{\beta}_i = \hat{\beta}_i - \delta/2$. Given $\varepsilon > 0$ such that $\hat{y} - \|\hat{\mathrm{s}}\|^2 \geq \varepsilon$ and $(\hat{\beta}_i - \delta/2)^2 \leq \hat{\alpha}_i - \varepsilon, \ i = 1, 2, 3$, let $\tilde{\mathrm{s}} = \hat{\mathrm{s}}$, $\tilde{y} = \hat{y} - \varepsilon$, $\tilde{\alpha}_i = \hat{\alpha}_i - \varepsilon$, then $(\tilde{\mathrm{s}}, \tilde{y}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\alpha}})$ is also a feasible solution of (19) and $\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 < \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3$. Thus, $(\hat{\mathrm{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ is not optimal. $\square$

Proof of Lemma 3

Proof. If for all $i = 1, 2, 3$, $\hat{\beta}_i^2 < \hat{\alpha}_i$. Then it is clear that $\hat{y} > \|\hat{\mathrm{s}}\|^2$. Choose $\delta > 0$, such that $\hat{\alpha}_i - \delta \geq \hat{\beta}_i^2, \ i = 1, 2, 3$ and $\hat{y} - \|\hat{\mathrm{s}}\|^2 \geq \delta$. Let $\tilde{\mathrm{s}} = \hat{\mathrm{s}}$, $\tilde{y} = \hat{y} - \delta$, $\tilde{\beta}_i = \hat{\beta}_i$, $\tilde{\alpha}_i = \hat{\alpha}_i - \delta$, $i = 1, 2, 3$. Then $(\tilde{\mathrm{s}}, \tilde{y}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\alpha}})$ is also a feasible solution of (19) and $\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 < \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3$. Therefore, $(\hat{\mathrm{s}}, \hat{y}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}})$ is not optimal. $\square$

Lemma 4. *Let $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ be a feasible solution of (19). If $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ satisfies (C4) or (C6), then $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ is not an optimal solution of (19).*

Proof. We only consider case *(C4)* since case *(C6)* is similar. Assume that $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ satisfies *(C4)*. If $\hat{\mathrm{s}}$ is below of x-axis, let $\hat{\mathrm{s}}^*$ denote the reflection of $\hat{\mathrm{s}}$ with respect to x-axis, then $\|\hat{\mathrm{s}} - \boldsymbol{a}_2\| = \|\hat{\mathrm{s}}^* - \boldsymbol{a}_2\|$, $\|\hat{\mathrm{s}} - \boldsymbol{a}_1\| = \|\hat{\mathrm{s}}^* - \boldsymbol{a}_1\|$, and

$$\begin{aligned}(\beta_3^* - \lambda)^2 - \|\hat{\mathrm{s}}^* - \boldsymbol{a}_3\|^2 &\leq (\beta_3^* - \lambda)^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_3\|^2 \\ &< (\beta_2^* - \lambda)^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_2\|^2 = (\beta_2^* - \lambda)^2 - \|\hat{\mathrm{s}}^* - \boldsymbol{a}_2\|^2.\end{aligned}$$

Set

$$g(\lambda, \boldsymbol{u}) = (\beta_3^* - \lambda)^2 - \|\boldsymbol{u} - \boldsymbol{a}_3\|^2 - \left((\beta_2^* - \lambda)^2 - \|\boldsymbol{u} - \boldsymbol{a}_2\|^2\right),$$

$$\boldsymbol{u} \in \cap_{i=1}^3 B(\boldsymbol{a}_i, \beta_i^* - \lambda).$$

Then $g(\lambda, \mathrm{s}^\lambda) > 0$, $g(\lambda, \hat{\mathrm{s}}) < 0$ and $g(\lambda, \hat{\mathrm{s}}^*) < 0$, thus there exists $\tilde{\mathrm{s}}$ with $s_1^\lambda < \tilde{s}_1 < \hat{s_1}^*$, $\tilde{s}_2 > 0$, and $\|\tilde{\mathrm{s}} - \boldsymbol{a}_1\|^2 = (\beta_1^* - \lambda)^2$ and $\|\tilde{\mathrm{s}} - \boldsymbol{a}_2\|^2 < (\beta_2^* - \lambda)^2 = \hat{\alpha}_2$, such that

$$g(\lambda, \tilde{\mathrm{s}}) = 0.$$

Set $\tilde{y} = \|\tilde{\mathrm{s}}\|^2 + (\beta_2^* - \lambda)^2 - \|\tilde{\mathrm{s}} - \boldsymbol{a}_2\|^2$, $\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}^* - \boldsymbol{\lambda}$ and $\tilde{\alpha}_1 = (\beta_1^* - \lambda)^2 + (\beta_2^* - \lambda)^2 - \|\tilde{\mathrm{s}} - \boldsymbol{a}_2\|^2$, $\tilde{\alpha}_2 = \tilde{\beta}_2^2$, $\tilde{\alpha}_3 = \tilde{\beta}_3^2$. Then $(\tilde{\mathrm{s}}, \tilde{y}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\alpha}})$ is a feasible solution of (19), and $\tilde{\alpha}_1 < (\beta_1^* - \lambda)^2 + (\beta_2^* - \lambda)^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_2\|^2$, and so

$$\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 < \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3.$$

Thus, $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ is not optimal. $\square$

Lemma 5. *Assume that $-c \geq |a + b|$. Let $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ be a feasible solution of (19). If $\hat{s}_2 < 0$ or $\hat{s}_1 \in [a, b]$, and $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ satisfies either (C2) or (C3), then $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ is not an optimal solution of (19).*

Proof. We only consider condition ($C2$). If $\hat{\mathrm{s}}$ is below x-axis, denote by $\hat{\mathrm{s}}^*$ the reflection of $\hat{\mathrm{s}}$ with respect to x-axis. Then

$$(\beta_3^* - \lambda)^2 - \|\hat{\mathrm{s}}^* - \boldsymbol{a}_3\|^2 < (\beta_3^* - \lambda)^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_3\|^2.$$

If $(\beta_3^*-\lambda)^2-\|\hat{\mathbf{s}}^*-\boldsymbol{a}_3\|^2\geq(\beta_2^*-\lambda)^2-\|\hat{\mathbf{s}}^*-\boldsymbol{a}_2\|^2$, set $\tilde{y}=\|\hat{\mathbf{s}}^*\|^2+(\beta_3^*-\lambda)^2-\|\hat{\mathbf{s}}^*-\boldsymbol{a}_3\|^2$, $\tilde{\boldsymbol{\beta}}=\boldsymbol{\beta}^*-\boldsymbol{\lambda}$ and $\tilde{\alpha}_1=(\beta_1^*-\lambda)^2+(\beta_3^*-\lambda)^2-\|\hat{\mathbf{s}}^*-\boldsymbol{a}_3\|^2$, $\tilde{\alpha}_2=\|\hat{\mathbf{s}}^*-\boldsymbol{a}_2\|^2+(\beta_3^*-\lambda)^2-\|\hat{\mathbf{s}}^*-\boldsymbol{a}_3\|^2$, $\tilde{\alpha}_3=\tilde{\beta}_3^2$. Then $(\hat{\mathbf{s}}^*,\tilde{y},\tilde{\boldsymbol{\beta}},\tilde{\boldsymbol{\alpha}})$ is a feasible solution of (19), and $\tilde{\alpha}_i<\hat{\alpha}_i$ for $i=1,2$, and so

$$\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3<\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3.$$

Thus, $(\hat{\mathbf{s}},\hat{y},\boldsymbol{\beta}^*-\boldsymbol{\lambda},\hat{\boldsymbol{\alpha}})$ is not optimal. If $(\beta_3^*-\lambda)^2-\|\hat{\mathbf{s}}^*-\boldsymbol{a}_3\|^2<(\beta_2^*-\lambda)^2-\|\hat{\mathbf{s}}^*-\boldsymbol{a}_2\|^2$, then there exists $\tilde{\mathbf{s}}$ with $\hat{s}_2<\tilde{s}_2<\hat{s}_2^*$, $\|\tilde{\mathbf{s}}-\boldsymbol{a}_1\|^2=(\beta_1^*-\lambda)^2$ and $\|\tilde{\mathbf{s}}-\boldsymbol{a}_2\|^2<(\beta_2^*-\lambda)^2$, such that $g(\lambda,\tilde{\mathbf{s}})=0$. Set $\tilde{y}=\|\tilde{\mathbf{s}}\|^2+(\beta_2^*-\lambda)^2-\|\tilde{\mathbf{s}}-\boldsymbol{a}_2\|^2$, $\tilde{\boldsymbol{\beta}}=\boldsymbol{\beta}^*-\boldsymbol{\lambda}$ and $\tilde{\alpha}_1=(\beta_1^*-\lambda)^2+(\beta_2^*-\lambda)^2-\|\tilde{\mathbf{s}}-\boldsymbol{a}_2\|^2$, $\tilde{\alpha}_2=\tilde{\beta}_2^2$, $\tilde{\alpha}_3=\tilde{\beta}_3^2$. Then $(\tilde{\mathbf{s}},\tilde{y},\tilde{\beta},\tilde{\alpha})$ is a feasible solution of (19), and $\tilde{\alpha}_1<(\beta_1^*-\lambda)^2+(\hat{\beta}_2-\lambda)^2-\|\hat{\mathbf{s}}-\boldsymbol{a}_2\|^2$, and so

$$\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3<\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3,$$

which implies that $(\hat{\mathbf{s}},\hat{y},\boldsymbol{\beta}^*-\boldsymbol{\lambda},\hat{\boldsymbol{\alpha}})$ is also not optimal. Therefore, $\hat{\mathbf{s}}$ must be above x-axis, i.e., $\hat{s}_2\geq 0$, and $s_1^\lambda<\hat{s}_1$.

The equation $(\beta_3^*-\lambda)^2-\|\boldsymbol{u}-\boldsymbol{a}_3\|^2=(\beta_2^*-\lambda)^2-\|\boldsymbol{u}-\boldsymbol{a}_2\|^2$ can be written as

$$2cu_2=2bu_1+(\beta_2^*-\lambda)^2-(\beta_3^*-\lambda)^2-b^2+c^2,$$

which is a straight line with a negative slope. Assume that the straight line intersects the circle $\{\boldsymbol{u}|\ \|\boldsymbol{u}-\boldsymbol{a}_1\|=\beta_1^*-\lambda\}$ at $\hat{\mathbf{s}}^{(1)}=(\hat{s}_1^{(1)},\hat{s}_2^{(1)})^T$ and $\hat{\mathbf{s}}^{(2)}=(\hat{s}_1^{(2)},\hat{s}_2^{(2)})^T$, where $\hat{s}_2^{(1)}\leq\hat{s}_2^{(2)}$. Without loss of generality, assume that $\hat{s}_2^{(1)}<\hat{s}_2^{(2)}$, then $\hat{s}_2^{(2)}>0$, and $\hat{\mathbf{s}}=\hat{\mathbf{s}}^{(1)}$ or $\hat{\mathbf{s}}=\hat{\mathbf{s}}^{(2)}$. If $\hat{\mathbf{s}}=\hat{\mathbf{s}}^{(1)}$, for $\varepsilon>0$, set

$$\begin{aligned}u_1^\epsilon&=\hat{s}_1-\varepsilon,\\ u_2^\varepsilon&=\frac{b}{c}u_1^\varepsilon+\frac{(\beta_2^*-\lambda)^2-(\beta_3^*-\lambda)^2-b^2+c^2}{2c}=\hat{s}_2-\frac{b}{c}\varepsilon.\end{aligned}$$

Then when ε is small enough, $\boldsymbol{u}^\varepsilon=(u_1^\varepsilon,u_2^\varepsilon)^T\in B(\boldsymbol{a}_1,\beta_1^*-\lambda)\cap B(\boldsymbol{a}_2,\beta_2^*-\lambda)$,

$$\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_1\|^2+(\beta_2^*-\lambda)^2-\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_2\|^2>(\beta_1^*-\lambda)^2,$$

and

$$(\beta_3^*-\lambda)^2-\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_3\|^2=(\beta_2^*-\lambda)^2-\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_2\|^2<(\beta_2^*-\lambda)^2-\|\hat{\mathbf{s}}-\boldsymbol{a}_2\|^2.$$

Set $\tilde{y}=\|\boldsymbol{u}^\varepsilon\|^2+(\beta_2^*-\lambda)^2-\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_2\|^2$, $\tilde{\boldsymbol{\beta}}=\boldsymbol{\beta}^*-\boldsymbol{\lambda}$ and $\tilde{\alpha}_1=\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_1\|^2+(\beta_2^*-\lambda)^2-\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_2\|^2$, $\tilde{\alpha}_2=\tilde{\beta}_2^2$, $\tilde{\alpha}_3=\tilde{\beta}_3^2$. Then $(\boldsymbol{u}^\varepsilon,\tilde{y},\tilde{\boldsymbol{\beta}},\tilde{\boldsymbol{\alpha}})$ is a feasible solution of (19), and $\tilde{\alpha}_1<(\beta_1^*-\lambda)^2+(\beta_2^*-\lambda)^2-\|\hat{\mathbf{s}}-\boldsymbol{a}_2\|^2$, and so

$$\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3<\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3.$$

Therefore, $(\hat{\mathbf{s}},\hat{y},\hat{\boldsymbol{\beta}},\hat{\boldsymbol{\alpha}})$ is not optimal.

If $\hat{\mathbf{s}}=\hat{\mathbf{s}}^{(2)}$ and $\hat{s}_1\geq a$, for $\varepsilon>0$, set

$$u_1^\varepsilon=\hat{s}_1-\varepsilon,\quad u_2^\varepsilon=\hat{s}_2-\varepsilon.$$

Then

$$\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_1\|^2=(\beta_1^*-\lambda)^2-2\varepsilon(|\hat{s}_1-a|+\hat{s}_2)+O(\varepsilon^2),$$

and so, for ε small enough, $\boldsymbol{u}^\varepsilon\in B(\boldsymbol{a}_1,\beta_1^*-\lambda)$. Set

$$\lambda^\varepsilon=\lambda+(\beta_1^*-\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_1\|)\,\frac{\beta_1^*-\lambda}{|\hat{s}_1-a|+\hat{s}_2}.$$

Then

$$\lambda^\varepsilon=\lambda+\varepsilon+O(\varepsilon^2),$$

and

$$\frac{b}{c}u_1^\varepsilon+\frac{(\beta_2^*-\lambda^\varepsilon)^2-(\beta_3^*-\lambda^\varepsilon)^2-b^2+c^2}{2c}=\hat{s}_2-\frac{\beta_2^*-\beta_3^*+b}{c}\varepsilon+O(\varepsilon^2).$$

Since $|\beta_2^*-\beta_3^*|<\sqrt{b^2+c^2}<b-c$, we have $\beta_2^*-\beta_3^*+b>c$, i.e., $\frac{\beta_2^*-\beta_3^*+b}{-c}>-1$ which yields

$$u_2^\varepsilon<\frac{b}{c}u_1^\varepsilon+\frac{(\beta_2^*-\lambda^\varepsilon)^2-(\beta_3^*-\lambda^\varepsilon)^2-b^2+c^2}{2c}.$$

Noting that $\hat{s}_2>0$ and $s_1^\lambda<\hat{s}_1$, we see that $(\beta_2^*-\lambda)^2>\|\hat{\mathbf{s}}-\boldsymbol{a}_2\|^2$. Then we can choose $\varepsilon_0>0$ small enough such that for all $\varepsilon\in(0,\varepsilon_0)$, $\boldsymbol{u}^\varepsilon\in B(\boldsymbol{a}_1,\beta_1^*-\lambda^\varepsilon)\cap B(\boldsymbol{a}_2,\beta_2^*-\lambda^\varepsilon)$,

$$\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_1\|^2+(\beta_3^*-\lambda^\varepsilon)^2-\|\boldsymbol{u}^\varepsilon-\boldsymbol{a}_3\|^2>(\beta_1^*-\lambda^\varepsilon)^2,$$

and

$$u_2^\varepsilon < \frac{b}{c}u_1^\varepsilon + \frac{(\beta_2^* - \lambda^\varepsilon)^2 - (\beta_3^* - \lambda^\varepsilon)^2 - b^2 + c^2}{2c}.$$

Noting that

$$(\beta_3^* - \lambda^\varepsilon)^2 - \|\boldsymbol{u} - \boldsymbol{a}_3\|^2 > (\beta_2^* - \lambda^\varepsilon)^2 - \|\boldsymbol{u} - \boldsymbol{a}_2\|^2$$

if and only if

$$u_2 < \frac{b}{c}u_1 + \frac{(\beta_2^* - \lambda^\varepsilon)^2 - (\beta_3^* - \lambda^\varepsilon)^2 - b^2 + c^2}{2c},$$

thus we have

$$(\beta_3^* - \lambda^\varepsilon)^2 - \|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_3\|^2 > (\beta_2^* - \lambda^\varepsilon)^2 - \|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_2\|^2.$$

Set $\tilde{y} = \|\boldsymbol{u}^\varepsilon\|^2 + (\beta_3^* - \lambda^\varepsilon)^2 - \|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_3\|^2$, $\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}^* - \boldsymbol{\lambda}^{\boldsymbol{\varepsilon}}$ and $\tilde{\alpha}_1 = \|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_1\|^2 + (\beta_3^* - \lambda^\varepsilon)^2 - \|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_3\|^2$, $\tilde{\alpha}_2 = \|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_2\|^2 + (\beta_3^* - \lambda^\varepsilon)^2 - \|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_3\|^2$, $\tilde{\alpha}_3 = \tilde{\beta}_3^2$. Then $(\boldsymbol{u}^\varepsilon, \tilde{y}, \tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\alpha}})$ is a feasible solution of (19). Since

$$\begin{aligned}
\|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_1\|^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_1\|^2 &= -2\varepsilon\left((\hat{s}_1 - a) + \hat{s}_2\right) + O(\varepsilon^2),\\
\|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_2\|^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_2\|^2 &= -2\varepsilon\left((\hat{s}_1 - b) + \hat{s}_2\right) + O(\varepsilon^2),\\
\|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_3\|^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_3\|^2 &= -2\varepsilon\left(\hat{s}_1 + (\hat{s}_2 - c)\right) + O(\varepsilon^2),
\end{aligned}$$

and

$$(\beta_3^* - \lambda^\varepsilon)^2 - (\beta_3^* - \lambda)^2 = -2\varepsilon(\beta_3^* - \lambda) + O(\varepsilon^2),$$

we have

$$\begin{aligned}
&\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 - (\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3) = \|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_1\|^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_1\|^2 + \|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_2\|^2\\
&- \|\hat{\mathrm{s}} - \boldsymbol{a}_2\|^2 - 2\left(\|\boldsymbol{u}^\varepsilon - \boldsymbol{a}_3\|^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_3\|^2\right) + 3\left((\beta_3^* - \lambda^\varepsilon)^2 - (\beta_3^* - \lambda)^2\right)\\
&= -4c\varepsilon + 2\varepsilon(a+b) - 6\varepsilon(\beta_3^* - \lambda) + O(\varepsilon^2).
\end{aligned}$$

Therefore, under the condition $-c \geq |a+b|$, when ε is small enough, $\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 < \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3$, and so, $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ is also not optimal. This is a contradiction. □

Lemma 6. *Assume that $-3c \geq |a+b|$. Let $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ be a feasible solution of (19). If $\hat{s}_2 = 0$ and either (C5) or (C7) is true, then $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ is not an optimal solution of (19).*

Proof. We only consider condition (*C5*). Suppose that $\hat{s}_2 = 0$, then $\hat{s}_1 = \beta_1^* - \lambda + a$. If $(\beta_3^* - \lambda)^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_3\|^2 > (\beta_2^* - \lambda)^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_2\|^2$, then

$$\begin{aligned}
&\hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 = \|\hat{\mathrm{s}} - \boldsymbol{a}_1\|^2 + \|\hat{\mathrm{s}} - \boldsymbol{a}_2\|^2 + 2((\beta_3^* - \lambda)^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_3\|^2) + (\beta_3^* - \lambda)^2\\
&= (\beta_1^* - \lambda)^2 + (\beta_1^* - \lambda + a - b)^2 + 2((\beta_3^* - \lambda)^2 - (\beta_1^* - \lambda + a)^2 - c^2) + (\beta_3^* - \lambda)^2.
\end{aligned}$$

Choose $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0)$,

$$(\beta_3^* - \lambda^\varepsilon)^2 - \|\hat{\mathrm{s}}^\varepsilon - \boldsymbol{a}_3\|^2 > (\beta_2^* - \lambda^\varepsilon)^2 - \|\hat{\mathrm{s}}^\varepsilon - \boldsymbol{a}_2\|^2,$$

where $\lambda^\varepsilon = \lambda + \varepsilon$ and $\hat{\mathrm{s}}^\varepsilon = (\hat{s}_1 - \varepsilon, 0)^T$. Set $\tilde{y} = \|\hat{\mathrm{s}}^\varepsilon\|^2 + (\beta_3^* - \lambda^\varepsilon)^2 - \|\hat{\mathrm{s}}^\varepsilon - \boldsymbol{a}_3\|^2$, $\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta}^* - \boldsymbol{\lambda}^{\boldsymbol{\varepsilon}}$ and $\tilde{\alpha}_1 = \|\hat{\mathrm{s}}^\varepsilon - \boldsymbol{a}_1\|^2 + (\beta_3^* - \lambda^\varepsilon)^2 - \|\hat{\mathrm{s}}^\varepsilon - \boldsymbol{a}_3\|^2$, $\tilde{\alpha}_2 = \|\hat{\mathrm{s}}^\varepsilon - \boldsymbol{a}_2\|^2 + (\beta_3^* - \lambda^\varepsilon)^2 - \|\hat{\mathrm{s}}^\varepsilon - \boldsymbol{a}_3\|^2$, $\tilde{\alpha}_3 = \tilde{\beta}_3^2$. Then

$$\begin{aligned}
&\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 = \|\hat{\mathrm{s}} - \boldsymbol{a}_1\|^2 + \|\hat{\mathrm{s}} - \boldsymbol{a}_2\|^2 + 2((\beta_3^* - \lambda)^2 - \|\hat{\mathrm{s}} - \boldsymbol{a}_3\|^2) + (\beta_3^* - \lambda)^2 - 2\varepsilon(\beta_1^* - \lambda)\\
&+ \varepsilon^2 - 2\varepsilon(\beta_1^* - \lambda + a - b) + \varepsilon^2 - 6\varepsilon(\beta_3^* - \lambda) + 3\epsilon^2 + 4\varepsilon(\beta_1^* - \lambda + a) - 2\varepsilon^2\\
&= \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 + 2\varepsilon\left(a + b - 3(\beta_3^* - \lambda)\right) + 3\varepsilon^2.
\end{aligned}$$

Therefore, noting that $\beta_3^* - \lambda > -c$, under the condition $-3c \geq |a+b|$, when ε is small enough, $\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 < \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3$. Therefore, $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ is not optimal. □

Lemma 7. *Assume that $-c \geq |a+b|$. Let $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ be a feasible solution of (19). If one of (C5) and (C7), then $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ is not an optimal solution of (19).*

Proof. We only consider the condition (*C5*). Suppose that $(\hat{\mathrm{s}}, \hat{y}, \boldsymbol{\beta}^* - \boldsymbol{\lambda}, \hat{\boldsymbol{\alpha}})$ is an optimal solution of (19). Then by Lemma 6, $\hat{s}_2 > 0$, and $\hat{\mathrm{s}}$ is above x-axis by a similar argument in the proof of Lemma 5. Thus, $\hat{\mathrm{s}}$ is above x-axis with $\tilde{s}_2 > 0$ and $s_1^\lambda < \hat{s}_1$.

For $\varepsilon > 0$, set

$$u_1^\varepsilon = \hat{s}_1 - \frac{\varepsilon(\hat{s}_1 - a)}{|\hat{s}_1 - a|}, \quad u_2^\varepsilon = \hat{s}_2 - \varepsilon.$$

Then

$$\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_1\|^2=(\beta_1^*-\lambda)^2-2\varepsilon\left(|\hat{s}_1-a|+\hat{s}_2\right)+2\varepsilon^2+O(\varepsilon^2),$$

and so, for ε small enough, $\boldsymbol{u}^{\varepsilon}\in B(\boldsymbol{a}_1,\beta_1^*-\lambda)$. Set

$$\lambda^{\varepsilon}=\lambda+(\beta_1^*-\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_1\|)\,\frac{\beta_1^*-\lambda}{|\hat{s}_1-a|+\hat{s}_2}.$$

Then

$$\lambda^{\varepsilon}=\lambda+\varepsilon+O(\varepsilon^2),$$

Noting that for $\tilde{s}_2>0$ and $s_1^{\lambda}<\hat{s}_1$, we can see that $(\beta_2^*-\lambda)^2>\|\hat{\mathrm{s}}-\boldsymbol{a}_2\|^2$. Then we can choose $\varepsilon_0>0$ small enough such that for all $\varepsilon\in(0,\varepsilon_0)$, $\boldsymbol{u}^{\varepsilon}\in B(\boldsymbol{a}_2,\beta_2^*-\lambda^{\varepsilon})$.

Set $\tilde{y}=\|\boldsymbol{u}^{\varepsilon}\|^2+(\beta_3^*-\lambda^{\varepsilon})^2-\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_3\|^2$, $\tilde{\boldsymbol{\beta}}=\boldsymbol{\beta}^*-\boldsymbol{\lambda}^{\boldsymbol{\varepsilon}}$ and $\tilde{\alpha}_1=\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_1\|^2+(\beta_3^*-\lambda^{\varepsilon})^2-\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_3\|^2$, $\tilde{\alpha}_2=\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_2\|^2+(\beta_3^*-\lambda^{\varepsilon})^2-\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_3\|^2$, $\tilde{\alpha}_3=\tilde{\beta}_3^2$. Then $(\boldsymbol{u}^{\varepsilon},\tilde{y},\tilde{\boldsymbol{\beta}},\tilde{\boldsymbol{\alpha}})$ is a feasible solution of (19). Since

$$\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_1\|^2-\|\hat{\mathrm{s}}-\boldsymbol{a}_1\|^2=-2\varepsilon\left(\frac{(\hat{s}_1-a)}{|\hat{s}_1-a|}(\hat{s}_1-a)+\hat{s}_2\right)+O(\varepsilon^2),$$

$$\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_2\|^2-\|\hat{\mathrm{s}}-\boldsymbol{a}_2\|^2=-2\varepsilon\left(\frac{(\hat{s}_1-a)}{|\hat{s}_1-a|}(\hat{s}_1-b)+\hat{s}_2\right)+O(\varepsilon^2),$$

$$\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_3\|^2-\|\hat{\mathrm{s}}-\boldsymbol{a}_3\|^2=-2\varepsilon\left(\frac{(\hat{s}_1-a)}{|\hat{s}_1-a|}\hat{s}_1+(\hat{s}_2-c)\right)+O(\varepsilon^2),$$

and

$$(\beta_3^*-\lambda^{\varepsilon})^2-(\beta_3^*-\lambda)^2=-2\varepsilon(\beta_3^*-\lambda)+O(\varepsilon^2),$$

we have

$$\begin{aligned}&\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3-(\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3)=\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_1\|^2-\|\hat{\mathrm{s}}-\boldsymbol{a}_1\|^2+\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_2\|^2-\|\hat{\mathrm{s}}-\boldsymbol{a}_2\|^2\\&-2\left(\|\boldsymbol{u}^{\varepsilon}-\boldsymbol{a}_3\|^2-\|\hat{\mathrm{s}}-\boldsymbol{a}_3\|^2\right)+3\left((\beta_3^*-\lambda^{\varepsilon})^2-(\beta_3^*-\lambda)^2\right)\\&=-4c\varepsilon+2(a+b)\frac{\varepsilon(\hat{s}_1-a)}{|\hat{s}_1-a|}-6\varepsilon(\beta_3^*-\lambda)+O(\varepsilon^2).\end{aligned}$$

Therefore, under the condition $-c\geq|a+b|$, when ε is small enough, $\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3<\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3$, and so, $(\hat{\mathrm{s}},\hat{y},\boldsymbol{\beta}^*-\boldsymbol{\lambda},\hat{\boldsymbol{\alpha}})$ is also not optimal. This is a contradiction. □

Lemma 8. *Let $(\hat{\mathrm{s}},\hat{y},\boldsymbol{\beta}^*-\boldsymbol{\lambda},\hat{\boldsymbol{\alpha}})$ be an optimal solution of (19). If $(\hat{\mathrm{s}},\hat{y},\boldsymbol{\beta}^*-\boldsymbol{\lambda},\hat{\boldsymbol{\alpha}})$ satisfies (C1), then $\hat{s}_2>0$.*

Proof. If $\hat{s}_2<0$, denote by $\hat{\mathrm{s}}^*$ the reflection of $\hat{\mathrm{s}}$ with respect to x-axis. Then

$$\|\hat{\mathrm{s}}^*-\boldsymbol{a}_1\|^2=(\beta_1^*-\lambda)^2,\quad\|\hat{\mathrm{s}}^*-\boldsymbol{a}_2\|^2=(\beta_2^*-\lambda)^2,\quad\|\hat{\mathrm{s}}^*-\boldsymbol{a}_3\|^2\leq(\beta_3^*-\lambda)^2$$

and

$$(\beta_3^*-\lambda)^2-\|\hat{\mathrm{s}}^*-\boldsymbol{a}_3\|^2<(\beta_3^*-\lambda)^2-\|\hat{\mathrm{s}}-\boldsymbol{a}_3\|^2.$$

Set $\tilde{y}=\|\hat{\mathrm{s}}^*\|^2+(\beta_3^*-\lambda)^2-\|\hat{\mathrm{s}}^*-\boldsymbol{a}_3\|^2$, $\tilde{\boldsymbol{\beta}}=\boldsymbol{\beta}^*-\boldsymbol{\lambda}$ and $\tilde{\alpha}_1=(\beta_1^*-\lambda)^2+(\beta_3^*-\lambda)^2-\|\hat{\mathrm{s}}^*-\boldsymbol{a}_3\|^2$, $\tilde{\alpha}_2=\|\hat{\mathrm{s}}^*-\boldsymbol{a}_2\|^2+(\beta_3^*-\lambda)^2-\|\hat{\mathrm{s}}^*-\boldsymbol{a}_3\|^2$, $\tilde{\alpha}_3=\tilde{\beta}_3^2$. Then $(\hat{\mathrm{s}}^*,\tilde{y},\tilde{\boldsymbol{\beta}},\tilde{\boldsymbol{\alpha}})$ is a feasible solution of (19), and $\tilde{\alpha}_i<\hat{\alpha}_i$ for $i=1,2$, and so

$$\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3<\hat{\alpha}_1+\hat{\alpha}_2+\hat{\alpha}_3.$$

Thus, $(\hat{\mathrm{s}},\hat{y},\boldsymbol{\beta}^*-\boldsymbol{\lambda},\hat{\boldsymbol{\alpha}})$ is not optimal.

If $\hat{s}_2=0$, then $\hat{s}_1=\beta_1^*-\lambda+a$ and $\hat{s}_1=\beta_2^*-\lambda-b$. Therefore, $b-a=\beta_2^*-\beta_1^*$. This is a contradiction of that $b-a,\beta_1^*,\beta_2^*$ are three edges of the triangle $\triangle_{\boldsymbol{a}_1\boldsymbol{a}_2\mathrm{s}^*}$. □

Proof of Theorem 3

Proof. By $\|\hat{\mathbf{s}}-\boldsymbol{a}_1\|^2=(\beta_1^*-\lambda)^2$ and $\|\hat{\mathbf{s}}-\boldsymbol{a}_2\|^2=(\beta_2^*-\lambda)^2$, we have

$$(b-a)\left(2\hat{s}_1-(a+b)\right)=(\beta_1^*-\lambda)^2-(\beta_2^*-\lambda)^2=(\beta_1^*-\beta_2^*)\left(\beta_1^*+\beta_2^*-2\lambda\right),$$

which yields

$$\hat{s}_1=\frac{(\beta_1^*-\beta_2^*)}{2(b-a)}\left(\beta_1^*+\beta_2^*-2\lambda\right)+\frac{a+b}{2}.$$

Therefore,

$$\hat{s}_2^2=(\beta_1^*-\lambda)^2-\left(\frac{(\beta_1^*-\beta_2^*)}{2(b-a)}\left(\beta_1^*+\beta_2^*-2\lambda\right)+\frac{b-a}{2}\right)^2.$$

Noting that $\hat{s}_2>0$, we obtain

$$\hat{s}_2=\sqrt{(\beta_1^*-\lambda)^2-\left(\frac{(\beta_1^*-\beta_2^*)}{2(b-a)}\left(\beta_1^*+\beta_2^*-2\lambda\right)+\frac{b-a}{2}\right)^2}.$$

By some algebraic manipulations, we have

$$\begin{aligned}&(\beta_1^*-\lambda)^2-\left(\frac{(\beta_1^*-\beta_2^*)}{2(b-a)}\left(\beta_1^*+\beta_2^*-2\lambda\right)+\frac{b-a}{2}\right)^2\\&=\frac{1}{(b-a)^2}\left((b-a)^2-(\beta_1^*-\beta_2^*)^2\right)\left(\lambda-\frac{\beta_1^*+\beta_2^*-(b-a)}{2}\right)\times\left(\lambda-\frac{\beta_1^*+\beta_2^*+(b-a)}{2}\right).\end{aligned}$$

Noting that the sum of the lengths of any two sides of a triangle is always greater than the length of the third one, we have

$$0<\frac{\beta_1^*+\beta_2^*-(b-a)}{2}<\min\{\beta_1^*,\beta_2^*\}.$$

Let $\hat{\mathbf{s}}_0$ denote the point of intersection of the lines $\boldsymbol{a}_3\mathbf{s}^*$ and $\boldsymbol{a}_1\boldsymbol{a}_2$. Then we also have $\beta_1^*-\|\boldsymbol{a}_1-\hat{\mathbf{s}}_0\|<\beta_3^*,\ \beta_2^*-\|\boldsymbol{a}_2-\hat{\mathbf{s}}_0\|<\beta_3^*,$ and $\|\boldsymbol{a}_1-\hat{\mathbf{s}}_0\|+\|\boldsymbol{a}_2-\hat{\mathbf{s}}_0\|=b-a$. Thus

$$\frac{\beta_1^*+\beta_2^*-(b-a)}{2}<\beta_3^*,$$

and so

$$0<\frac{\beta_1^*+\beta_2^*-(b-a)}{2}<\min\{\beta_1^*,\beta_2^*,\beta_3^*\}.$$

Thus, when $0<\lambda<\frac{\beta_1^*+\beta_2^*-(b-a)}{2}$,

$$(\beta_1^*-\lambda)^2-\left(\frac{(\beta_1^*-\beta_2^*)}{2(b-a)}\left(\beta_1^*+\beta_2^*-2\lambda\right)+\frac{b-a}{2}\right)^2>0;$$

and when $\frac{\beta_1^*+\beta_2^*-(b-a)}{2}<\lambda<\frac{\beta_1^*+\beta_2^*+(b-a)}{2}$,

$$(\beta_1^*-\lambda)^2-\left(\frac{(\beta_1^*-\beta_2^*)}{2(b-a)}\left(\beta_1^*+\beta_2^*-2\lambda\right)+\frac{b-a}{2}\right)^2<0.$$

Since $\min\{\beta_1^*,\beta_2^*,\beta_3^*\}<\frac{\beta_1^*+\beta_2^*+(b-a)}{2}$, from $0\le\lambda<\min\{\beta_1^*,\beta_2^*,\beta_3^*\}$ and

$$(\beta_1^*-\lambda)^2-\left(\frac{(\beta_1^*-\beta_2^*)}{2(b-a)}\left(\beta_1^*+\beta_2^*-2\lambda\right)+\frac{b-a}{2}\right)^2\ge 0,$$

we obtain $\lambda<\frac{\beta_1^*+\beta_2^*-(b-a)}{2}$, that is, $D=\left(0,\frac{\beta_1^*+\beta_2^*-(b-a)}{2}\right)$.

Now, since

$$
\begin{aligned}
&(\beta_3^* - \lambda)^2 - \|\hat{\mathbf{s}} - a_3\|^2 = (\beta_3^* - \lambda)^2 - \hat{s}_1^2 - \hat{s}_2^2 + 2c\hat{s}_2 - c^2 \\
&= (\beta_3^* - \lambda)^2 - \left(\frac{(\beta_1^* - \beta_2^*)}{2(b-a)} (\beta_1^* + \beta_2^* - 2\lambda) + \frac{a+b}{2} \right)^2 \\
&- (\beta_1^* - \lambda)^2 + \left(\frac{(\beta_1^* - \beta_2^*)}{2(b-a)} (\beta_1^* + \beta_2^* - 2\lambda) + \frac{b-a}{2} \right)^2 - c^2 \\
&+ 2c\sqrt{(\beta_1^* - \lambda)^2 - \left(\frac{(\beta_1^* - \beta_2^*)}{2(b-a)} (\beta_1^* + \beta_2^* - 2\lambda) + \frac{b-a}{2} \right)^2} \\
&= -(\beta_1^* - \beta_3^*)(\beta_1^* + \beta_3^* - 2\lambda) - a\left(\frac{(\beta_1^* - \beta_2^*)}{b-a} (\beta_1^* + \beta_2^* - 2\lambda) + b \right) - c^2 \\
&+ 2c\sqrt{(\beta_1^* - \lambda)^2 - \left(\frac{(\beta_1^* - \beta_2^*)}{2(b-a)} (\beta_1^* + \beta_2^* - 2\lambda) + \frac{b-a}{2} \right)^2},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&g(\lambda) = 3\lambda^2 - 2\lambda\left(\beta_1^* + \beta_2^* + \beta_3^*\right) - 2(\beta_1^* - \beta_3^*)(\beta_1^* + \beta_3^* - 2\lambda) \\
&- 2a\left(\frac{(\beta_1^* - \beta_2^*)}{b-a} (\beta_1^* + \beta_2^* - 2\lambda) + b \right) - 2c^2 \\
&+ 4c\sqrt{(\beta_1^* - \lambda)^2 - \left(\frac{(\beta_1^* - \beta_2^*)}{2(b-a)} (\beta_1^* + \beta_2^* - 2\lambda) + \frac{b-a}{2} \right)^2}.
\end{aligned}
$$

From Theorem 2, we know that $(\mathbf{s}^*, y^*, \boldsymbol{\beta}^*, \boldsymbol{\alpha}^*)$ is an optimal solution of (19) if and only if $g(\lambda) > 0$, which is equivalent to $\tilde{g}(\lambda) > 0$, where

$$
\begin{aligned}
&\tilde{g}(\lambda) := \Bigg(3\lambda^2 - 2\lambda\left(\beta_1^* + \beta_2^* + \beta_3^*\right) - 2(\beta_1^* - \beta_3^*)(\beta_1^* + \beta_3^* - 2\lambda) \\
&- 2a\left(\frac{\beta_1^* - \beta_2^*}{b-a} (\beta_1^* + \beta_2^* - 2\lambda) + b \right) - 2c^2 \Bigg)^2 \\
&- 16c^2\left((\beta_1^* - \lambda)^2 - \left(\frac{\beta_1^* - \beta_2^*}{2(b-a)} (\beta_1^* + \beta_2^* - 2\lambda) + \frac{b-a}{2} \right)^2 \right).
\end{aligned}
$$

Notice that

$$
\tilde{g}(0) = 0, \quad \tilde{g}\left(\frac{\beta_1^* + \beta_2^* - (b-a)}{2} \right) \geq 0,
$$

and for any $\frac{\beta_1^* + \beta_2^* - (b-a)}{2} \leq \lambda \leq \frac{\beta_1^* + \beta_2^* + (b-a)}{2}$,

$$
(\beta_1^* - \lambda)^2 - \left(\frac{\beta_1^* - \beta_2^*}{2(b-a)} (\beta_1^* + \beta_2^* - 2\lambda) + \frac{b-a}{2} \right)^2 \leq 0,
$$

which implies that for any $\frac{\beta_1^* + \beta_2^* - (b-a)}{2} \leq \lambda \leq \frac{\beta_1^* + \beta_2^* + (b-a)}{2}$,

$$
\tilde{g}(\lambda) \geq 0.
$$

Therefore, we can write

$$
\tilde{g}(\lambda) = \lambda\left(9\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3\right),
$$

where

$$\begin{aligned}
a_1 :&= 12\left(\beta_1^* - \beta_2^* - 3\beta_3^* + \frac{2a(\beta_1^* - \beta_2^*)}{b-a}\right) = 12\left(-3\beta_3^* + \frac{(a+b)(\beta_1^* - \beta_2^*)}{b-a}\right) \\
&\leq 12\left(-3\beta_3^* + |a+b|\right) \leq -24\beta_3^* < 0, \\
a_2 :&= 4\left(\beta_1^* - \beta_2^* - 3\beta_3^* + \frac{2a(\beta_1^* - \beta_2^*)}{b-a}\right)^2 - 12((\beta_1^*)^2 - (\beta_3^*)^2) - \frac{12a((\beta_1^*)^2 - (\beta_2^*)^2)}{b-a} \\
&+ \frac{16(\beta_1^* - \beta_2^*)^2}{(b-a)^2} - 12ab - 28c^2, \\
a_3 :&= 8\Psi(\mathbf{s}^*) \\
&= -8\left((\beta_1^*)^2 - (\beta_3^*)^2 + \frac{a((\beta_1^*)^2 - (\beta_2^*)^2)}{b-a} + ab + c^2\right) \\
&\times \left(\beta_1^* - \beta_2^* - 3\beta_3^* + \frac{2a(\beta_1^* - \beta_2^*)}{b-a}\right) - 16c^2\left(-2\,\beta_1^* + \frac{((\beta_1^*)^2 - (\beta_2^*)^2 + (b-a)^2)(\beta_1^* - \beta_2^*)}{(b-a)^2}\right).
\end{aligned}$$

This implies that $(s^*, y^*, \boldsymbol{\beta}^*, \boldsymbol{\alpha}^*)$ is an optimal solution of (19) if and only if for any $\lambda \in D$

$$\tilde{f}(\lambda) := 9\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 > 0.$$

In particular, $a_3 = \tilde{f}(0) \geq 0$, and so, $a_3 \geq 0$ is a necessary condition.

Conversely, if $a_3 > 0$, then by $\lim_{\lambda \to -\infty} \tilde{f}(\lambda) = -\infty$, we see that there exists $\lambda_0 < 0$ such that

$$\tilde{f}\left(\lambda_0\right) = 0.$$

Let λ_1 and λ_2 denote other two roots of the equation $\tilde{f}(\lambda) = 0$. Then

$$\lambda_0 + \lambda_1 + \lambda_2 = -\frac{1}{9}a_1.$$

Therefore, by $-\frac{1}{9}a_1 \geq \frac{8}{3}\beta_3^* > \beta_1^* + \beta_2^* - (b-a)$,

$$\lambda_1 + \lambda_2 > \beta_1^* + \beta_2^* - (b-a),$$

which implies that one of λ_1 and λ_2 is bigger than $(\beta_1^* + \beta_2^* - (b-a))/2$. On the other hand, since

$$\tilde{f}\left(0\right) > 0, \text{ and } \tilde{f}\left(\frac{\beta_1^* + \beta_2^* - (b-a)}{2}\right) \geq 0,$$

if $\tilde{f}\left(\frac{\beta_1^* + \beta_2^* - (b-a)}{2}\right) > 0$, then the number of roots in $((0, \beta_1^* + \beta_2^* + b - a)/2)$ of $\tilde{f}(\lambda) = 0$ must be even number. Therefore, either $\tilde{f}\left(\frac{\beta_1^* + \beta_2^* - (b-a)}{2}\right) = 0$, or both λ_1 and λ_2 are in $((\beta_1^* + \beta_2^* + b - a)/2, \infty)$; thus, the mixed SDP-SOCP (19) model gives the true solution. The proof is completed. $\square$