Multi-parameter Mechanisms with Implicit Payment Computation*

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Abstract

In this paper we show that payment computation essentially does not present any obstacle in designing truthful mechanisms, even for multi-parameter domains, and even when we can only call the allocation rule once. We present a general reduction that takes any allocation rule which satisfies "cyclic monotonicity" (a known necessary and sufficient condition for truthfulness) and converts it to a truthful mechanism using a single call to the allocation rule, with arbitrarily small loss to the expected social welfare.

A prominent example for a multi-parameter setting in which an allocation rule can only be called once arises in sponsored search auctions. These are multi-parameter domains when each advertiser has multiple possible ads he may display, each with a different value per click. Moreover, the mechanism typically does not have complete knowledge of the click-realization or the click-through rates (CTRs); it can only call the allocation rule a single time and observe the click information for ads that were presented. On the negative side, we show that an allocation that is truthful for any realization essentially cannot depend on the bids, and hence cannot do better than random selection for one agent. We then consider a relaxed requirement of truthfulness, only in expectation over the CTRs. Even for that relaxed version, making any progress is challenging as standard techniques for construction of truthful mechanisms (as using VCG or an MIDR allocation rule) cannot be used in this setting. We design an allocation rule with non-trivial performance and directly prove it is cyclic-monotone, and thus it can be used to create a truthful mechanism using our general reduction.

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The multi-parameter transformation from Section 3 has appeared in [Babaioff et al., 2012], the full version of [Babaioff et al., 2010], but it is not a part of the conference version of [Babaioff et al., 2010]. This result is an integral part of this manuscript and the corresponding conference submission.

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1 Introduction

In this paper we show that payment computation essentially does not present any obstacle in designing truthful mechanisms, even for multi-parameter domains, and even when we can only call the allocation rule once. This extends the result of [Babaioff et al., 2010] for single parameter domains to multi-parameter domains. We present a general reduction that takes any allocation rule which satisfies "cyclic monotonicity" (a known necessary and sufficient condition for truthfulness) and convert it to a truthful mechanism using a single call to the allocation rule, with arbitrarily small loss to the expected social welfare. The mechanism does not compute the payments explicitly but rather charges random payments having the right expectation.

Such a reduction is particularly attractive as it can handle multi-parameter settings where it is impossible to decouple the computation of the allocation from the actual execution of the allocation. In such situations, the entire mechanism — including the payment computation — can only execute a single call to the allocation rule. We call this the "no-simulation" constraint; it can arise when a mechanism interacts with the environment, and the information revealed by the environment depends on the choices made by the allocation rule. The no-simulation constraint is a significant hurdle because the existing approaches to payment computation require multiple calls to the allocation rule, with different vectors of bids.

Sponsored search auctions supply a prominent example of a multi-parameter setting with the no-simulation constraint. In this setting each advertiser has multiple possible ads he is interested in displaying, each with a different value per click, and the mechanism does not have complete knowledge of the click-realization or the click-through rates (CTRs). Instead, it can only allocate ad impressions and observe the click information for ads that were presented. The no-simulation constraint also arises in other contexts, such as packet routing [Shnayder et al., 2012].

We note that our reduction — the multi-parameter transformation — has other uses beyond settings with the no-simulation constraint. For example, it can also be used to speed up the computation of payments in most multi-parameter mechanisms. Indeed, it has already been used for this purpose by two recent papers. Jain et al. [2011] used it to speed up the payment computation for a mechanism that allocates batch jobs in a cloud system. Huang and Kannan [2012] used it to compute payments for their privacy-preserving procurement auction for spanning trees, which is based on the well-known "exponential privacy mechanism" from prior work [McSherry and Talwar, 2007].

Sponsored search mechanisms with unknown CTRs. In the remainder of the paper we focus on the problem of designing truthful mechanisms for an archetypical multi-parameter setting with the no-simulation constraint: sponsored search auctions with unknown click-through rates (CTRs). The difficulty in designing such allocation rules stems from the fact that the welfare of a given allocation depends on clicks of the allocated ads, which are unknown to the bidders and to the mechanism. This prevents us from using the VCG mechanism since it depend on choosing a welfare-maximizing allocation. Yet, it is possible that welfare can at least be approximated.

We focus on a simple single-shot ad auction in which the allocation rule unfolds over time (and the CTRs are not known). As such, we contribute to a growing literature on ad auctions that unfold over time, as they do in practice. The non-strategic version of our model is a well-understood variant of the *multi-armed bandit* problem.

Mechanisms that are truthful for every realization of the clicks would be most attractive, as the strategic behavior in such mechanisms would not depend on the agents' beliefs about the process generating the clicks — for example, the belief that clicks for each ad are i.i.d. from a fixed distribution. Such mechanisms were constructed in Babaioff et al. [2009, 2010] for the single-parameter version of the problem. Unfortunately, the multi-parameter setting is much harder. In the setting of sponsored search with multiple ads per bidder and unknown CTRs, we show that if the mechanism is required to be truthful for every realization of the clicks, then it must be a trivial mechanism that outputs a fixed allocation (or distribution over allocations) with no dependence on the bids.

In light of this negative result we consider a weaker notion of truthfulness. Assume that clicks are stochastic (meaning that each ad has a CTR, and clicks are independent Bernoulli trials with the specified click probabilities) but the CTRs are not known. The mechanism is required to be truthful for every vector of CTRs; we call mechanisms with this property *stochastically truthful*. The VCG mechanism still cannot be used as we cannot maximize the expected welfare without knowing the CTRs. An alternative is to use a maximal-in-distributional-range (MIDR) allocation rule combined with VCG-based payment rule, but we show that for a natural family of MIDR allocation rules (in which the set of distributions the rule optimizes over is independent of the CTRs) the performance of such rules is no better than randomly selecting an ad to present.

There are a few examples in the literature of non-VCG-based truthful multi-parameter mechanisms in which bidders freely choose an option from a hand-crafted menu of allocations and prices, e.g. [Bartal et al., 2003, Dobzinski et al., 2006, Dobzinski and Nisan, 2011], but this technique similarly fails in our setting because the bidders do not have a dominant strategy for choosing from such a menu when they do not know their own CTRs.

Given all these negative results we turn to our multi-parameter transformation which reduces the problem of designing truthful randomized mechanisms to the (seemingly simpler) problem of designing cyclically monotone (CMON) allocation rules. In contrast to the negative result for truthfulness for every realization, we directly craft an allocation rule that satisfies stochastic CMON; to our knowledge, the only previous paper to successful apply this approach is [Lavi and Swamy, 2007]. Using the transformation we construct a stochastically truthful mechanism that outperforms the naïve random allocation for a single agent, when the difference in value-per-impression of his ads is sufficiently large. While this is clearly just a small step, it proves to be rather challenging, and relies heavily on the multi-parameter transformation described above.

Related work. Our earlier paper [Babaioff et al., 2010] considers the limited case of single parameter domains. It introduced the technique of designing black-box transformations that perform implicit payment computation while evaluating a given monotone allocation function only once. The same paper introduced monotone allocation rules with strong welfare guarantees for sponsored search auctions with unknown CTRs, by modifying multi-armed bandit algorithms to achieve the requisite monotonicity property. As all the results in our earlier paper are limited to single-parameter settings, they only apply to sponsored search when each advertiser has only *one* ad to display. In the present paper, we show that the black-box transformation extends readily from single-parameter to multi-parameter settings, whereas extending the results on sponsored search to multi-parameter settings is much more delicate, and in some cases (i.e. for the strongest notion of truthfulness) outright impossible.

Wilkens and Sivan [2012] extended the results of [Babaioff et al., 2010] to multi-parameter domains under some limitations. Their work provides a black-box transformation that allows implicit payment computation when the allocation function is maximal-in-distributional-range (MIDR). While the MIDR property is the most widely used method for achieving truthfulness in multi-parameter settings, it is not a necessary condition for truthfulness. In fact several papers (including this one) depend on multi-parameter mechanisms that are not MIDR. By presenting an implicit payment computation procedure that works *whenever* there exists a truthful mechanism utilizing the given allocation function, we believe that we have posed the multi-parameter transformation at the appropriate level of generality for future applications.

The literature contains surprisingly few examples of truthful multi-parameter mechanisms that are not based on MIDR allocation rules. Mechanisms designed by Bartal et al. [2003], Dobzinski et al. [2006], Dobzinski and Nisan [2011] for various combinatorial auction domains make use of what might be termed the *pricing technique*: each agent is allowed to choose freely from a menu of alternatives, each specifying an allocation and price. The menu presented to a given agent may depend on the others' bids, but it must be carefully constructed so that self-interested agents each choosing from their own menu will never jointly select an infeasible allocation. The taxation principle [Guesnerie, 1981, Hammond, 1979] implies that *every* dominant-strategy truthful mechanism can actually be represented this way, provided that agents are able to evaluate their own utilities for different allocations before the allocation is actually executed. In settings with the no-simulation constraint, the taxation principle does not apply because agents can only evaluate their utility *ex post*. In the sponsored search setting, for example, agents have no dominant strategy for choosing from a menu listing bundles of ad impressions, because without knowing CTRs they can't precisely determine the value of an impression; on the other hand, the mechanism is powerless to offer a menu listing bundles of clicks, because there is no way to guarantee that a bidder who chooses a certain bundle will receive the specified number of clicks.

Apart from mechanisms with `MIDR` allocation rules and those based on the pricing technique, we are aware of only one other mechanism in the literature that is dominant-strategy truthful in a multi-parameter setting: the scheduling mechanism of Lavi and Swamy [2007] for unrelated machines that have only two possible processing times. Their mechanism, like ours, is designed by directly constructing an allocation function that satisfies the cyclic monotonicity constraints.

2 Preliminaries

We study reductions from allocations to truthful mechanisms for multi-parameter domains. A CS-oriented background on multi-parameter mechanisms can be found in Archer and Kleinberg [2008b,a], while an Economics-oriented background can be found in Ashlagi et al. [2010]. Our main result holds for a very general framework for multi-parameter mechanisms, described below, where agents' types are defined as mappings from outcomes to valuations. Our reduction invokes the allocation rule only once, which make it particularly useful in domains in which the allocation rule cannot be invoked (or simulated) more than once due to informational constraints.

Types, outcomes, and mechanisms. Multi-parameter mechanisms are defined as follows. There are n agents and a set $\mathcal{O}$ of outcomes. Each agent i is characterized by his *type* $\mathbf{x}_i : \mathcal{O} \to \Re$, where $\mathbf{x}_i(o)$ is interpreted as the agent's valuation for the outcome $o \in \mathcal{O}$. For each agent i there is a set of feasible types, denoted $\mathcal{T}_i$. Denote $\mathcal{T} = \mathcal{T}_1 \times \ldots \times \mathcal{T}_n$ and call it the *type space*; call $\mathcal{T}_i$ the type space of agent i. The mechanism knows $(n, \mathcal{O}, \mathcal{T})$, but not the actual types $\mathbf{x}_i$; each type $\mathbf{x}_i$ is known only to the corresponding agent i. Formally, a problem instance, also called a *multi-parameter domain*, is a tuple $(n, \mathcal{O}, \mathcal{T})$.

A (direct revelation) mechanism $\mathcal{M}$ consists of the pair $(\mathcal{A}, \mathcal{P})$, where $\mathcal{A} : \mathcal{T} \to \mathcal{O}$ is the *allocation rule* and $\mathcal{P} : \mathcal{T} \to \Re^n$ is the *payment rule*. Both $\mathcal{A}$ and $\mathcal{P}$ can be randomized, possibly with a common random seed. Each agent i reports a type $\mathbf{b}_i \in \mathcal{T}_i$ to the mechanism, which is called the *bid* of this agent. We denote the vector of bids by $\mathbf{b} = (\mathbf{b}_1 , \ldots , \mathbf{b}_n) \in \mathcal{T}$. The mechanism receives the bid vector $\mathbf{b} \in \mathcal{T}$, selects an outcome $\mathcal{A}(\mathbf{b})$, and charges each agent i a payment of $\mathcal{P}_i(\mathbf{b})$. The utilities are quasi-linear and agents are risk-neutral: if agent i has type $\mathbf{x}_i \in \mathcal{T}_i$ and the bid vector is $\mathbf{b} \in \mathcal{T}$, then this agent's utility is

$$u_i(\mathbf{x}_i; \mathbf{b}) = \mathbb{E}_{\mathcal{M}} \left[\mathbf{x}_i(\mathcal{A}(\mathbf{b})) - \mathcal{P}_i(\mathbf{b}) \right]. \tag{1}$$

For each type $\mathbf{x}_i \in \mathcal{T}_i$ of agent i we use a standard notation $(\mathbf{b}_{-i}, \mathbf{x}_i)$ to denote the bid vector $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}_i = \mathbf{x}_i$ and $\hat{\mathbf{b}}_j = \mathbf{b}_j$ for every agent $j \neq i$.

Game-theoretic properties. A mechanism is *truthful* if for every agent i truthful bidding is a *dominant strategy*:

$$u_i(\mathbf{x}_i; (\mathbf{b}_{-i}, \mathbf{x}_i)) \geq u_i(\mathbf{x}_i; \mathbf{b}) \quad \forall \mathbf{x}_i \in \mathcal{T}_i,\ \mathbf{b} \in \mathcal{T}. \tag{2}$$

An allocation rule is called *truthfully implementable* if it is the allocation rule in some truthful mechanism.

A mechanism is *individually rational (IR)* if each agent i never receives negative utility by participating in the mechanism and bidding truthfully:

$$u_i(\mathbf{x}_i; (\mathbf{b}_{-i}, \mathbf{x}_i)) \geq 0 \quad \forall \mathbf{x}_i \in \mathcal{T}_i,\ \mathbf{b}_{-i} \in \mathcal{T}_{-i}. \tag{3}$$

The right-hand side in Equation (3) represents the maximal guaranteed utility of an "outside option" (i.e., from not participating in the mechanism). For example, our definition of IR is meaningful whenever this utility is 0, which is a typical assumption for most multi-parameter domains studied in the literature.

Note that if the mechanism is randomized, the above properties are defined in expectation over the internal random seed. We can also define utility (and, accordingly, truthfulness and IR) for a given realization of the random seed. We say a mechanism is *universally truthful* if it is truthful for all realizations of the random seed; similarly for IR and other properties.

Our assumptions. We make two assumptions on the type space $\mathcal{T}$:

- *non-negative types*: $\mathbf{x}_i(o) \geq 0$ for each agent i, type $\mathbf{x}_i \in \mathcal{T}_i$, each outcome $o \in \mathcal{O}$.
- *rescalable types*: $\lambda \mathbf{x}_i \in \mathcal{T}_i$ for each agent i, type $\mathbf{x}_i \in \mathcal{T}_i$, and any parameter $\lambda \in [0, 1]$. ($\lambda \mathbf{x}_i$ denotes the type $\mathbf{x}'_i$ whose valuation for every outcome o satisfies $\mathbf{x}'_i(o) = \lambda \mathbf{x}_i(o)$.)

In particular, for each agent i there exists a *zero type*: a type $\mathbf{x}_i \in \mathcal{T}_i$ such that $\mathbf{x}_i(\cdot) \equiv 0$. Let us say that a mechanism is *normalized* if for each agent i, the expected payment of this agent is 0 whenever she submits the zero type. For domains with non-negative types, it is desirable that all agents are charged a non-negative amount; this is known as the *no-positive-transfers* property.

Dot-product valuations. An important special case is *dot-product valuations*, where the type $\mathbf{x} \in \mathcal{T}_i$ of each agent i can be decomposed as a dot product $\mathbf{x}(o) = \beta_{\mathbf{x}} \cdot a_i(o)$, for each outcome $o \in \mathcal{O}$, where $\beta_{\mathbf{x}}, a_i(o) \in \Re^d$ are some finite-dimensional vectors. Here the term $a_i(o)$ is the same for all types $\mathbf{x} \in \mathcal{T}_i$ (and known to the mechanism), whereas $\beta_{\mathbf{x}}$ is the same for all outcomes $o \in \mathcal{O}$ and is known only to agent i. The term $a_i(o)$ is usually called an "allocation" of agent i for outcome o, and $\beta_{\mathbf{x}}$ is called the "private value". Single-parameter domains correspond to the case $d = 1$.

Note that the type $\mathbf{x}$ of each agent i is determined by the corresponding private value $\beta_{\mathbf{x}}$, and his type space $\mathcal{T}_i$ is determined by $D_i = \{\beta_{\mathbf{x}} :\ \mathbf{x} \in \mathcal{T}_i\} \subset \Re^d$. Because of this, in the literature on dot-product valuations the term "type" often refers to $\beta_{\mathbf{x}}$. To avoid ambiguity, in this section we will refer to $\beta_{\mathbf{x}}$ as "private value" rather than "type", and call $D_1 \times \ldots \times D_n$ the *private value space*.

In a domain with dot-product valuations, types are rescalable if and only if for each $\beta_{\mathbf{x}} \in D_i$ and each $\lambda \in [0, 1]$ it holds that $\lambda \beta_{\mathbf{x}} \in D_i$. In other words, if and only if the set D_i is star-convex at 0. To ensure non-negative types, it suffices to assume that $D_i \subset \Re^d_+$ for each agent i, and all allocations are non-negative: $a_i(o) \in \Re^d_+$ for all $o \in \mathcal{O}$.

Truthfulness characterization. We will use a characterization of truthful mechanisms via a property called "cycle-monotonicity" (henceforth abbreviated as `CMON`). A (randomized) allocation rule $\mathcal{A}$ satisfies `CMON` if the following holds: for each bid vector $\mathbf{b} \in \mathcal{T}$, each agent i, each $k \geq 2$, and each k-tuple $\mathbf{x}_{i,0},\ \mathbf{x}_{i,1}\ ,\ \ldots\ , \mathbf{x}_{i,k} \in \mathcal{T}_i$ of this agent's types, we have

$$\mathbb{E}_{\mathcal{A}} \left[\textstyle\sum_{j=0}^{k}\ \mathbf{x}_{i,j}\,(o_{i,j}) - \mathbf{x}_{i,\,(j-1) \bmod k}\,(o_{i,j}) \right] \geq 0, \qquad \text{where } o_{i,j} = \mathcal{A}\,(\mathbf{b}_{-i},\, \mathbf{x}_{i,j}) \in \mathcal{O}. \tag{4}$$

Recall that we are using a general notion of agents' types (and bids), which are defined as functions from outcomes to real-valued valuations.

It is known that $\mathcal{A}$ is truthfully implementable if and only if it is cycle-monotone, in which case the corresponding payment rule is essentially fixed.

Theorem 2.1 (Rochet [1987])**.** *Consider an arbitrary multi-parameter domain $(n, \mathcal{O}, \mathcal{T})$. A (randomized) allocation rule $\mathcal{A}$ is truthfully implementable if and only if it is cycle-monotone. Assuming rescalable types, for any cycle-monotone allocation rule $\mathcal{A}$, a mechanism $(\mathcal{A}, \mathcal{P})$ is truthful and normalized if and only if*

$$\mathbb{E}_{\mathcal{A}}\left[\mathcal{P}_i(\mathbf{b})\right] = \mathbb{E}_{\mathcal{A}}\left[\mathbf{b}_i(\mathcal{A}(\mathbf{b})) - \int_{t=0}^{1} \mathbf{b}_i(\mathcal{A}(\mathbf{b}_{-i},\, t\,\mathbf{b}_i))\, dt\right]. \tag{5}$$

This characterization generalizes a well-known truthfulness characterization of single-parameter mechanisms in terms of monotonicity, due to [Myerson, 1981, Archer and Tardos, 2001]. Recall that for single-parameter domains, the type of each agent i is captured by a single number (the private value v_i), and the outcome pertinent to this agent is also captured by a single number (this agent's allocation $a_i(o)$). The bid of agent i is represented by $b_i \in \Re$. Cycle-monotonicity is then equivalent to a much simpler property called *monotonicity*: for each agent, fixing the bids of other agents, increasing this agent's bid cannot decrease this agent's allocation. The payment formula (5) can also be simplified, e.g. for non-negative valuations it is

$$\mathcal{P}_i(\mathbf{b}) = b_i\, \mathcal{A}_i(b_{-i}, b_i) - \int_0^{b_i} \mathcal{A}_i(b_{-i}, u)\, du. \tag{6}$$

External seed. We allow allocation rules to receive input from the environment; a canonical example is pay-per-click auctions where such input consists of user clicks. Formally, the allocation rule and the payment rule depend on the additional argument ω which captures all relevant input from the environment. (To simplify the notation, we keep the dependence on ω implicit.) We call ω the *external seed*, to distinguish from the internal random seed of the mechanism. We assume that ω is an independent sample from some fixed distribution $\mathcal{D}_{\texttt{ext}}$; this distribution may be unknown to the mechanism.

All game-theoretic properties defined above carry over to mechanisms with external seed if all expectations are over both internal and external seed. In particular, Theorem 2.1 carries over with no other modification.

We are primarily interested in properties that hold in expectation over the external seed, for all possible distributions $\mathcal{D}_{\texttt{ext}}$ over the external seed. The corresponding version of a given property P is called *stochastically P*. For example, we are interested in mechanisms that are stochastically truthful, and this requires the allocation rules to be stochastically CMON.

We also define a stronger version of truthfulness: one that holds for each realization of the external seed. For each game-theoretic property P described above, such as truthfulness, IR and CMON, a version that holds for each realization of the external seed will be called *ex-post P*. Theorem 2.1 holds for every given realization of the external seed (but requires the allocation rule to satisfy ex-post CMON).

A crucial way in which the external seed is different from the internal randomness is that a given run of the allocation rule might not observe the entire external seed. More precisely, runs of the allocation rule on different bid vectors might observe different portions of the external seed. For example, if an ad is not displayed to a given user, the mechanism does not observe whether this user would have clicked on this ad if it were displayed. It follows that the mechanism might not be able to simulate the allocation rule on different bid vectors – this is precisely the "no-simulation" constraint discussed in the Introduction. Moreover, this issue can affect payment computation: the payment prescribed by Equation (5), although well-defined as a

Mechanism 1: The single-parameter mechanism $\mathcal{M}_\delta$ from [Babaioff et al., 2010]

1. Collect bid vector b.
2. Independently for each agent $i \in [n]$, randomly sample $\chi_i = 1$ with probability $1 - \delta$ and otherwise $\chi_i = \gamma_i^{1/(1-\delta)}$, where $\gamma_i \in [0, 1]$ is sampled uniformly at random.
3. Construct the vector of modified bids, $x = (\chi_1 b_1, \ldots, \chi_n b_n)$.
4. Allocate according to $\widetilde{\mathcal{A}}(b) = \mathcal{A}(x)$.
5. Compute payments using the formula $\widetilde{\mathcal{P}}_i(b) = b_i \cdot \mathcal{A}_i(x) \cdot \begin{cases} 1 & \text{if } \chi_i = 1 \\ 1 - \frac{1}{\delta} & \text{if } \chi_i < 1 \end{cases}$.

mathematical expression, might not be computable given the information available to the mechanism.[1]

To address this issue formally, we say that the mechanism is *information-feasible* if for each run of of the mechanism (i.e., for each bid vector $\mathbf{b}$, each realization of the mechanism's internal randomness, and every possible value of the external seed) the payments are uniquely determined given the information available to the mechanism.

Implicit payment computation for single-parameter domains. Babaioff et al. [2010] provide an implicit payment computation result for single-parameter domains. They prove that any monotone allocation rule for any single-parameter domain can be transformed into a truthful, information-feasible mechanism with an arbitrarily small loss in expected welfare. The allocation rule is only invoked once. Below we quote a special case of this result that is most relevant to the present paper.[2]

Theorem 2.2 (Babaioff et al. [2010])**.** *Consider an arbitrary single-parameter domain where the private values of each agent lie in the interval $[0, 1]$. Let $\mathcal{A}$ be a stochastically monotone allocation rule for this domain. Then for each $\delta \in (0, 1)$, mechanism $\mathcal{M}_\delta = (\widetilde{\mathcal{A}}, \widetilde{\mathcal{P}})$ (described in Mechanism 1) is information-feasible and has the following properties.*

(a) [Incentives] $\mathcal{M}_\delta$ is stochastically truthful, universally ex-post individually rational. If $\mathcal{A}$ is ex-post monotone, then $\mathcal{M}_\delta$ is ex-post truthful.

(b) [Performance] For n agents and any bid vector b (and any fixed external seed) allocations $\widetilde{\mathcal{A}}(b)$ and $\mathcal{A}(b)$ are identical with probability at least $1 - n\delta$. Moreover, if $\mathcal{A}$ is α-approximate (for social welfare), then mechanism $\mathcal{M}_\delta$ is $\alpha/(1 - \frac{\delta}{2-\delta})$-approximate.

(c) [Payments] $\mathcal{M}_\delta$ is ex-post no-positive-transfers; and although it is not universally so, for all realizations of the internal seed it never pays any agent i more than $b_i \cdot \mathcal{A}_i(x) \cdot (\frac{1}{\delta} - 1)$. $\mathcal{M}_\delta$ is universally ex-post normalized.

[1] This has been proved in [Babaioff et al., 2009, Devanur and Kakade, 2009] in the context of multi-armed bandit mechanisms, see Section 4 for more details.

[2] We restate the result slightly, to make it consistent with our notation. [Babaioff et al., 2010] states the mechanism more abstractly, in terms of a general *self-resampling procedure*. The simple description of $\mathcal{M}_\delta$ that we present here was first published in [Shnayder et al., 2012].

3 The multi-parameter transformation

In this section we present our first main contribution: the implicit payment computation result for multi-parameter domains. For a given multi-parameter domain and a given `CMON` allocation rule for this domain,[3] our goal is to design a truthful, information-feasible mechanism with outcome that is almost always identical to that of the original allocation rule, and this, in particular, ensures a small loss in expected welfare. We achieve this goal for *every* `CMON` allocation rule and *every* multi-parameter domain (under a mild assumption of rescalable, non-negative types). More precisely, we give a general "multi-parameter transformation" which takes an arbitrary `CMON` allocation rule $\mathcal{A}$ and transforms it into a truthful, information-feasible mechanism which implements the same outcome as $\mathcal{A}$ with probability arbitrarily close to 1. This mechanism requires evaluating $\mathcal{A}$ only once; its allocation rule randomly modifies the submitted bids, and then calls $\mathcal{A}$ on the modified bids.[4] The technical contribution here is showing that the natural generalization of the reduction for the single-parameter setting, to the multi-parameter setting, preserves all desired properties. The non-trivial part of the proof is showing that although the single-parameter transformation only ensures that each agent does not have an incentive to deviate by scaling all his bids by the same scalar in $[0, 1]$, he also does not have an incentive to deviate to any other arbitrary bids.

The transformation. Our multi-parameter transformation is a remarkably straightforward generalization of the single-parameter transformation specified in Mechanism 1. In fact, there is no need to rewrite the five steps; the only thing that changes is the interpretation of the notation. Specifically, the bids $\mathbf{b}_1, \ldots, \mathbf{b}_n$ should now be interpreted as elements of the type spaces $\mathcal{T}_1, \ldots, \mathcal{T}_n$ rather than as scalars, and for each i the modified bid $\mathbf{x}_i = \chi_i \, \mathbf{b}_i$ is obtained by multiplying the abstract type $\mathbf{b}_i$ (a function from outcomes to reals) by the random scalar χ_i. (Note that $\chi_i \, \mathbf{b}_i$ is well-defined because we are assuming the rescalable types property.) The notation $b_i \cdot \mathcal{A}_i(x)$ from the single-parameter case is now interpreted as $\mathbf{b}_i(\mathcal{A}_i(\mathbf{x}))$, where $\mathbf{x} = (\mathbf{x}_1 \, , \, \ldots \, , \mathbf{x}_n)$ is the vector of re-sampled bids. With this interpretation, the payment rule is as follows:

$$\widetilde{\mathcal{P}}_i(\mathbf{b}) = \mathbf{b}_i(\mathcal{A}_i(\mathbf{x})) \cdot \begin{cases} 1 & \text{if } \chi_i = 1 \\ 1 - \frac{1}{\delta} & \text{if } \chi_i < 1 \end{cases} .$$

In the remainder of this section we analyze the properties of the multi-parameter transformation, proving an analogue of Theorem 2.2. The subtlest step, which occupies most of the analysis, is to prove that the modified allocation rule $\widetilde{\mathcal{A}}$ satisfies `CMON`.

Induced single-parameter domains. To aid in the analysis, it will be helpful to introduce the following notation. Consider a bid vector $\mathbf{b} \in \mathcal{T}$ and a vector of "rescaling coefficients" $\lambda \in [0, 1]^n$. Denote

$$\lambda \otimes b = (\lambda_1 \mathbf{b}_1 \, , \, \ldots \, , \lambda_n \mathbf{b}_n) \in \mathcal{T}.$$

In other words, $\lambda \otimes b$ is the rescaled bid vector where the bid of each agent i is $\lambda_i \mathbf{b}_i$. Note that for each $\mathbf{b}$ the subset

$$\mathcal{T}_\mathbf{b} = \{\lambda \otimes b : \ \lambda \in [0, 1]^n\} \subset \mathcal{T}$$

forms a single-parameter type space where each agent i has private value $\lambda_i \in [0, 1]$ and allocation $b_i(o)$ for every outcome o. By abuse of notation, let us treat the allocation and payment rules for $\mathcal{T}_\mathbf{b}$ as functions from the private value space $[0, 1]^n$ rather than the type space $\mathcal{T}_\mathbf{b}$.

[3] Recall that `CMON` is a necessary and sufficient condition for truthfulness.

[4] The transformation presented here is certainly not the only reduction that transforms multi-parameter allocation rules satisfying `CMON` into truthful, information-feasible mechanisms. One appealing feature of our transformation, in comparison to alternatives, is its simplicity. It also optimizes the trade-off between the worst-case bid-to-payment ratio and the probability of adopting the original allocation, as was shown by Wilkens and Sivan (2012) in the single-parameter context.

We want to prove that the mechanism $\mathcal{M}_\delta = (\widetilde{\mathcal{A}}, \widetilde{\mathcal{P}})$ defined by our transformation is truthful. As a starting observation, note that when one applies the single-parameter transformation given in Section 2 to the allocation rule defined by $\mathcal{A}_{\mathbf{b}}(\lambda) = \mathcal{A}(\lambda \otimes \mathbf{b})$, one obtains a mechanism that coincides with the restriction of $\mathcal{M}_\delta$ to $\mathcal{T}_{\mathbf{b}}$. By Theorem 2.2, we may conclude that the restriction of $\mathcal{M}_\delta$ to the single-parameter type space $\mathcal{T}_{\mathbf{b}}$ is truthful. Yet this conclusion is not sufficient, since this truthfulness condition is actually weaker than what we are aiming for: it ensures that a deviation inside the single-parameter type space $\mathcal{T}_{\mathbf{b}}$ is not beneficial, but says nothing about deviation to other types in $\mathcal{T} \setminus \mathcal{T}_{\mathbf{b}}$. Nevertheless, our proof will show that if the original allocation rule was `CMON`, the transformed allocation rule is also `CMON` for the domain $\mathcal{T}$, and thus is truthful as needed.

Theorem 3.1. *Consider an arbitrary multi-parameter domain $(n, \mathcal{O}, \mathcal{T})$ with rescalable, non-negative types. Let $\mathcal{A}$ be a stochastically* `CMON` *allocation rule for this domain. Let $\mathcal{M}_\delta = (\widetilde{\mathcal{A}}, \widetilde{\mathcal{P}})$ be the transformed mechanism for some parameter $\delta \in (0,1)$. Then $\mathcal{M}_\delta$ has the following properties:*

(a) [Structure] $\mathcal{M}_\delta$ is information-feasible.

(b) [Incentives] $\mathcal{M}_\delta$ is stochastically truthful and universally ex-post individually rational. If $\mathcal{A}$ is ex-post `CMON`*, then $\mathcal{M}$ is ex-post truthful.*

(c) [Performance] For n agents and any bid vector b (and any fixed external seed) allocations $\widetilde{\mathcal{A}}(b)$ and $\mathcal{A}(b)$ are identical with probability at least $1 - n\delta$. Moreover, if $\mathcal{A}$ is α-approximation to the maximal social welfare then $\widetilde{\mathcal{A}}$ is $\alpha / \left(1 - \frac{2}{1-\delta}\right)$-approximation to the maximal social welfare.

(d) [Payments] $\mathcal{M}$ is ex-post no-positive-transfers; and although it is not universally so, for all realizations of the internal seed it never pays any agent i more than $\mathbf{b}_i(o)(\frac{1}{\delta} - 1)$, where $o = \mathcal{A}(\mathbf{b}) \in \mathcal{O}$. Additionally, $\mathcal{M}$ is universally ex-post normalized.

Proof. $\mathcal{M}_\delta$ is information-feasible by construction, since so are the single-parameter mechanisms obtained from Theorem 2.2. All claimed properties except truthfulness follow immediately from Theorem 2.2. Below we prove truthfulness.

We claim that $\widetilde{\mathcal{A}}$ satisfies `CMON`. Indeed, fix bid vector $\mathbf{b} \in \mathcal{T}$, agent i, some $k \geq 2$, and a k-tuple $\mathbf{x}_{i,0},\ \mathbf{x}_{i,1}\,,\ \ldots\ , \mathbf{x}_{i,k} \in \mathcal{T}_i$ of this agent's types. Let us consider a fixed realization of the random vector $\chi \in [0,1]^n$ in step (2) of mechanism $\mathcal{M}_\delta$. For each type $\mathbf{x}_{i,j}$, note that we have

$$\widetilde{\mathcal{A}}(\mathbf{x}_{i,j}, \mathbf{b}_{-i}) = \mathcal{A}\left(\chi \otimes (\mathbf{x}_{i,j},\, \mathbf{b}_{-i})\right) \in \mathcal{O}.$$

Denote this outcome by $o_{i,j}(\chi)$. Let us apply the cycle-monotonicity of $\mathcal{A}$ for bid vector $\chi \otimes (\mathbf{x}_{i,j}, \mathbf{b}_{-i})$:

$$\mathbb{E}_{\mathcal{A}}\left[\textstyle\sum_{j=0}^{k}\ \mathbf{x}_{i,j}(o_{i,j}(\chi)) - \mathbf{x}_{i,\,(j-1) \bmod k}(o_{i,j}(\chi))\right] \geq 0. \tag{7}$$

Recalling that $o_{i,j}(\chi) = \widetilde{\mathcal{A}}(\mathbf{x}_{i,j}, \mathbf{b}_{-i})$, we observe that for this fixed realization of χ, Equation (7) is exactly the inequality in the definition of cycle-monotonicity for $\widetilde{\mathcal{A}}$. Therefore taking expectation over χ, we obtain the desired inequality Equation (4) for $\widetilde{\mathcal{A}}$. Claim proved.[5]

It remains to prove that in the transformed mechanism $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{P}})$, the payment rule satisfies Equation (5). Fix bid vector $\mathbf{b}$ and consider the transformed single-parameter mechanism $(\widetilde{\mathcal{A}}_{\mathbf{b}}, \widetilde{\mathcal{P}}_{\mathbf{b}})$ for the single-parameter

[5] Note that the proof of cycle-monotonicity of $\widetilde{\mathcal{A}}$ did not use any other property of M_δ other than that the re-scaling factors χ_i are chosen independently from a distribution that does not depend on $\mathcal{A}$. The truthfulness of the single-parameter mechanisms $(\widetilde{\mathcal{A}}_{\mathbf{b}}, \widetilde{\mathcal{P}}_{\mathbf{b}})$ is used in the forthcoming argument about payments.

type space $\mathcal{T}_{\mathbf{b}}$. In the terminology of single-parameter domains, each agent i receives an allocation $\widetilde{\mathcal{A}}_{\mathbf{b},\,i}(\lambda) = b_i(\widetilde{\mathcal{A}}_{\mathbf{b}}(\lambda))$ whenever the bid vector is $\lambda \in [0,1]^n$. Since this is a truthful and normalized single-parameter mechanism, it follows that

$$\mathbb{E}\left[\widetilde{\mathcal{P}}_{\mathbf{b}}(\lambda)\right] = \mathbb{E}\left[\lambda_i\,\widetilde{\mathcal{A}}_{\mathbf{b},\,i}(\lambda) - \int_0^{\lambda_i} \widetilde{\mathcal{A}}_{\mathbf{b},\,i}(\lambda_{-i}, t)\,dt\right], \quad \forall \lambda \in [0,1]^n.$$

Plugging in $\lambda = \vec{1}$ and using the definitions of $\widetilde{\mathcal{A}}_{\mathbf{b}}, \widetilde{\mathcal{P}}_{\mathbf{b}}$, we obtain the desired Equation (5). □

4 Multi-parameter MAB mechanisms

Let us define a natural multi-parameter extension to the MAB mechanism design problem studied in [Babaioff et al., 2009, Devanur and Kakade, 2009, Babaioff et al., 2010].[6]

Problem formulation. There are n agents. For each agent there is a known and fixed set of ads he is interested in; we assume that these sets are disjoint. The total number of ads is denoted by m.

As is common in the literature on sponsored search we assume that agents only value clicks; they have no value for an impression when the ad is not clicked. For every ad j there is a *value-per-click* v_j such that the unique agent that is interested in that ad receives utility v_j whenever this ad is clicked; this value is the agent's private information.

A mechanism for this domain proceeds as follows. There are T rounds, where the time horizon T is fixed and known to everyone. In each round the mechanism either decides to *skip* this round or chooses one ad to display. Then the ad is either clicked or not clicked. All agents bid once, before the first round. The bid of a given agent consists of a tuple of reported values for his ads. The bid reported for ad j is denoted b_j; the entire bid vector of all agents for the m ads is denoted $b = (b_1\,,\ \ldots\ ,b_m)$. Payments are assigned after the last round.

For each ad j, the click probability is fixed over time and denoted μ_j. In each round when this ad is displayed, it is clicked independently with probability μ_j. Click probabilities are called *click-though rates* (*CTRs*) in the industry. We assume that the CTRs are not known neither to the mechanism nor to the agents. For brevity, let $\mu = (\mu_1\,,\ \ldots\ ,\mu_m)$ be the vector of all CTRs.

Interpretation as a multi-parameter domain. For our setting, stochastic truthfulness (and similarly stochastic CMON, etc.) is a property that holds in expectation over clicks, for all possible CTR vectors μ.

Following the prior work, the external seed is defined as *click realization* ρ, in the following sense. For every ad j and every round t, realization $\rho(t,j) \in \{0,1\}$ says whether this ad would be clicked if it is shown in this round. In particular, ex-post truthfulness corresponds to truthfulness for every click realization. Note that a given run of a mechanism does not observe the entire click realization: it only observes clicks for ads that are displayed in a given round.

For every bid vector b and each click realization ρ, let $C_j(b,\rho)$ be the expected total number of clicks received by ad j, where the expectation is over the internal randomness in the mechanism. Denote $C(b,\rho) = (C_1(b,\rho)\,,\ \ldots\ ,C_m(b,\rho))$ and call it the *click vector*. We interpret the click vectors as the "outcomes" in the multi-parameter domain. Note that a given click vector $C(b,\rho)$ corresponds to expected welfare $\sum_j v_j C_j(b,\rho)$.

[6]Here and elsewhere, *MAB* stands for *multi-armed bandits*.

Note that with this interpretation of the "outcomes", the allocation rule is not free to choose any well-defined outcome. Instead, the collection of outcomes that can be implemented on a given run of the mechanism is constrained by the click realization.[7]

For a given CTR vector μ, let $C(b, \mu) = \mathbb{E}_{\rho\sim\mu} C(b, \rho)$, where the expectation is taken over click realizations ρ according to the corresponding CTRs. Call it the μ-click vector. In expectation over the clicks, the welfare is $\sum_j v_j C_j(b, \mu)$. When considering stochastic truthfulness, it will be more convenient to re-define outcomes as μ-click vectors.

Discussion and background. If not for the issue of incentives and the requirement of truthfulness, the welfare-maximization problem for the allocation rule is precisely the *multi-armed bandit* problem (henceforth, *MAB*), a well-studied problem in Machine Learning and Operations Research. MAB *mechanisms* can be seen as a version of the MAB problem that incorporates incentives. MAB mechanisms (in the limited single-parameter case, with one ad per agent), were introduced and studied in [Babaioff et al., 2009, Devanur and Kakade, 2009] for the deterministic case. Subsequently, Babaioff et al. [2010] studied randomized MAB mechanisms. Below we recap some of the contributions made in [Babaioff et al., 2009, Devanur and Kakade, 2009].

MAB mechanisms were suggested as a simple model in which one can study the interplay between incentives and learning, two major issues that arise in pay-per-click auctions. Pay-per-click is (along with pay-per-impression) one of the two prevalent business models in the advertising on the Internet, and *the* prevalent pricing model in sponsored search. Compared to pay-per-impression, pay-per-click reduces the risk that advertisers take, as they only pay when the ad is clicked. The seller, who has some control over clicks, bears the risk instead. Moreover, advertisers typically have very little or no information about their CTRs, and should not be required to learn more. The pay-per-click model essentially shields the advertisers from this uncertainty.

The crucial assumption in our model of MAB mechanisms is that the CTRs are initially not known to the mechanism. This assumption reflects the fact that the CTRs are learned over time, while the ads are being allocated, and so the process of learning should be treated as a part of the game.[8]

The focus of the investigation in [Babaioff et al., 2009, Devanur and Kakade, 2009] was whether and how the requirement of truthfulness restricts the performance of MAB algorithms when types are single-parameter. They found a very severe restriction for deterministic, ex-post truthful mechanisms: the allocation rule can only have a very simple, "naïve" structure (separating exploration and exploitation), which severely impacts performance compared to the best MAB algorithms. They capitalize on the "no-simulation" constraint to prove that if an allocation rule does not conform to this simple structure, then a truthful mechanism with this allocation rule cannot be information-feasible.

The obstacle of information-feasibility for the single parameter case is circumvented in Babaioff et al. [2010] by moving from deterministic to randomized MAB mechanisms. The single-parameter transformation (Theorem 2.2) reduces the design of truthful, information-feasible MAB mechanisms to the design of monotone allocation rules for this domain. Further, the authors provide monotone allocation rules whose performance matches that of optimal MAB algorithms. Specifically, they show that (a minor modification of) a standard MAB algorithm `UCB1` [Auer et al., 2002] is stochastically monotone, and they design a new MAB algorithm which is ex-post monotone and has essentially the same performance.

[7] Alternatively, we could have defined "outcomes" via impressions rather than clicks. But then an agent would not have a full knowledge of his value for each outcome (his type) as the CTRs are not known to him. Such a definition necessitates some cumbersome modifications to the framework in Section 2. Both versions lead to the same results.

[8] If some information on CTRs is known before the allocation starts, this can be modeled via Bayesian priors on CTRs. Following [Babaioff et al., 2009, Devanur and Kakade, 2009, Babaioff et al., 2010], we focus on the non-Bayesian version.

5 Multi-parameter MAB mechanisms: Impossibility result for ex-post truthful mechanisms

In this section we present our second main contribution: a very strong impossibility result for ex-post truthful multi-parameter MAB mechanisms. Consider one of the agents and fix the bids of the others. Essentially, we show that an allocation rule which satisfies ex-post `CMON` for that agent, cannot depend on the bid of that agent. More precisely, this holds for a deterministic allocation rule if the bids are large enough, as well as for any allocation rule (deterministic or randomized) that never skips a round. For randomized allocation rules that may skip a round, we show that if the allocation rule satisfies ex-post `CMON` then it cannot achieve a nontrivial worst-case approximation ratio.

Theorem 5.1. *Let $\mathcal{A}$ be an allocation rule for multi-parameter MAB which satisfies ex-post* `CMON`*. Fix any agent i, and fix bids submitted by all other agents.*

(a) If $\mathcal{A}$ is any allocation rule (deterministic or randomized) that never skips a round, and if agent i is the only agent, then the allocation has no dependence on his bids.

(b) If $\mathcal{A}$ is deterministic, then there exists a finite B such that the allocation for agent i does not depend on his bids, as long as all his bids are larger than B.

(c) If $\mathcal{A}$ is randomized, then its worst-case approximation ratio (over all bid vectors of agent i) is no better than that of the trivial randomized allocation rule that ignores agent i's bid, samples one of his ads uniformly at random, and allocates all impressions to that ad.

The first conclusion presumes there is only a single agent, and to prove the remaining two conclusions it suffices to consider the case of a single agent, because from the perspective of any given agent the ads allocated to other agents can be represented as skips. (In particular, allowing skips in single-agent allocation rules is essential for the generalization to multiple agents.) In the rest of this section we assume a single agent with m ads.

To prove our result we need to set up some notation. Recall that the bids of the agent are represented by a vector $b = (b_1, \ldots, b_m) \in \mathbb{R}_+^m$. For a given allocation rule $\mathcal{A}$ and a given click-realization ρ, the *impression allocation* $\mathcal{A}(b,t,\rho) \in \mathbb{R}_+^m$ is a vector of probabilities, in expectation over the random seed of the algorithm, so that $\mathcal{A}_i(b,t,\rho)$ is the probability that ad i is chosen in round t given bid vector b and realization ρ.

Weak monotonicity. We use `CMON` through a special case where $k = 2$ in Equation (4); this special case is known in the literature as *weak monotonicity*, henceforth abbreviated `WMON`. `WMON` is equivalent to `CMON` if there are finitely many outcomes and the type space is convex [Saks and Yu, 2005]. It follows that in our setting, ex-post `WMON` is equivalent to ex-post `CMON` for deterministic allocation rules. For more background on `WMON`, see [Archer and Kleinberg, 2008a].

Let us restate `WMON` in the notation of multi-parameter MAB mechanisms. Recall that the click vector $C(b,\rho)$ is a vector such that $C_j(b,\rho)$ is the total expected number of clicks for ad j, given bid vector b and realization ρ. Then

$$C_j(b,\rho) = \sum_{t=1}^{T} \rho(t,j)\, \mathcal{A}_j(b,t,\rho) = \sum_{t=1}^{T} \Delta_t(\rho)\, \mathcal{A}(b,t,\rho),$$

where $\Delta_t(\rho)$ is the $m \times m$ diagonal matrix with diagonal entries $(\rho(t,1)\,,\;\ldots\;,\rho(t,m))$. Ex-post WMON states the following: for any realization ρ and any bid vectors $b, \tilde{b} \in \mathbb{R}_+^m$,

$$(\tilde{b} - b) \cdot (\; C(\tilde{b},\rho) - C(\tilde{b},\rho)\;) \geq 0$$

Re-writing this in terms of the impression allocation, we obtain:

$$(\tilde{b} - b)^\dagger \;\sum_{t=1}^T \;\Delta_t(\rho)(\;\mathcal{A}(\tilde{b},t,\rho) - \mathcal{A}(b,t,\rho)\;) \geq 0. \tag{8}$$

Here and elsewhere, $M^\dagger$ denotes a transpose of a matrix M.

Analysis for allocation rules with no skips (Theorem 5.1(a)). For the sake of contradiction, assume that $\mathcal{A}(b,t,\rho) \neq \mathcal{A}(b',t,\rho)$ for some round t, click-realization ρ, and bid vectors $b, b' \in \mathbb{R}_+^m$. Pick the smallest t for which such counterexample exists. Assume w.l.o.g. that $\rho \equiv 0$ for all rounds after t. For each ad i, let ρ_i be a realization that coincides with ρ on all rounds but t, and in round t ad i is clicked and all other ads are not clicked.

Let $\tilde{b} = \vec{1} + \max(b,b') \in \mathbb{R}_+^m$, where $\max(b,b')$ is the coordinate-wise maximum of b and b'. Since $\mathcal{A}(b,t,\rho) \neq \mathcal{A}(b',t,\rho)$, we can w.l.o.g. assume that $\mathcal{A}(\tilde{b},t,\rho) \neq \mathcal{A}(b,t,\rho)$. Since $\mathcal{A}$ never skips a round,

$$\textstyle\sum_{i=1}^m \mathcal{A}_i(\tilde{b},t,\rho) = 1 = \sum_{i=1}^m \mathcal{A}_i(b,t,\rho). \tag{9}$$

Combining $\mathcal{A}(\tilde{b},t,\rho) \neq \mathcal{A}(b,t,\rho)$ with Equation (9) we deduce that for some ad i, $\mathcal{A}_i(\tilde{b},t,\rho) < \mathcal{A}_i(b,t,\rho)$. We claim that WMON is violated for bids $b, \tilde{b}$ and realization ρ_i. Indeed, consider Equation (8) for realization ρ_i. The sum in Equation (8) is 0 for all rounds other than t because $\mathcal{A}(\tilde{b},s,\rho) = \mathcal{A}(b,s,\rho)$ for all rounds $s < t$ (by minimality of t), and $\rho_i \equiv 0$ for all rounds $s > t$. For round t, the sum in Equation (8) is 0 for all ads other than i, by definition of ρ_i. Thus, the sum is simply equal to $(b_i - \tilde{b}_i) \cdot [\mathcal{A}_i(b,t,\rho) - \mathcal{A}_i(\tilde{b},t,\rho)]$, which is negative, contradicting Equation (8).

Analysis for the deterministic case (Theorem 5.1(b)). We now address deterministic allocation rules that may skip rounds. The analysis of this case captures the main ideas of the randomized case while being significantly easier to present.

Fix click-realization ρ and round t. Let $\mathcal{A}$ be the deterministic allocation rule for agent i that is induced by fixing the bids of all other agents. If $\mathcal{A}$ skips round t, write $\mathcal{A}(b,t,\rho) = \texttt{skip}$. For a vector $b = (b_1\,,\;\ldots\;,b_m) \in \mathbb{R}_+^m$, denote $\max(b\,) = \max_{1 \leq i \leq m} b_i$. Define $\min(b)$ similarly.

One technicality in the analysis is handling skips; we deal with it using the following notions:[9]

$$\begin{aligned} b_{\texttt{min}}(t,\rho) &= \sup\{\max(b) :\; b \in \mathbb{R}_+^m \text{ and } \mathcal{A}(b,t,\rho) = \texttt{skip}\}. \\ B &= \max\left(\quad \{0\} \cup \{b_{\texttt{min}}(t,\rho) :\; \exists\, t,\rho \text{ such that } b_{\texttt{min}}(t,\rho) < \infty\} \quad\right). \end{aligned} \tag{10}$$

Note that $B = 0$ if $b_{\texttt{min}}(t,\rho) = \infty$ for all t and ρ. For a given round t and realization ρ, $b_{\texttt{min}}(t,\rho)$ is defined such that if all m bids are larger than $b_{\texttt{min}}(t,\rho)$ then the allocation does not skip at round t on realization ρ. B is defined such that for every realization and every round, if all bids are larger than B then the allocation rule never skips.

Claim 5.2. *Let $\mathcal{A}$ be a deterministic single-agent allocation rule which satisfies ex-post* WMON. *Then for each click-realization ρ and each round t, $\mathcal{A}$ does not depend on the bid vector b for all bid vectors $b \in (B,\infty)^m$, where B is defined in Equation (10).*

[9] We use a standard convention that $\sup(\emptyset) = -\infty$.

Proof. For the sake of contradiction, assume that $\mathcal{A}(b,t,\rho) \neq \mathcal{A}(b',t,\rho)$ for some round t, click-realization ρ, and bid vectors $b, b' \in (B,\infty)^m$. Pick the smallest t for which such counterexample exists. Assume w.l.o.g. that $\rho \equiv 0$ for all rounds after t. For each ad i, let ρ_i be a realization such that it coincides with ρ on all rounds but t, and in round t ad i is clicked and all other ads are not clicked.

Let us consider two cases, depending on whether $b_{\texttt{min}}(t,\rho)$ is finite.

Case 1: $b_{\texttt{min}}(t,\rho) = \infty$**.** At least one of $\mathcal{A}(b,t,\rho)$, $\mathcal{A}(b',t,\rho)$ is not equal to `skip`. Since $\mathcal{A}(b,t,\rho) \neq \mathcal{A}(b',t,\rho)$, we can w.l.o.g. assume that $\mathcal{A}(b,t,\rho) \neq \texttt{skip}$. Hence, $\mathcal{A}_i(b,t,\rho) = 1$ for some ad i. Since $b_{\texttt{min}}(t,\rho) = \infty$, there exists $\tilde{b} \in (\max(b),\ \infty)^m$ such that $\mathcal{A}(\tilde{b},t,\rho) = \texttt{skip}$.

We claim `WMON` is violated for bids $b, \tilde{b}$ and realization ρ_i. As in the first case, we see that the sum in Equation (8) is 0 for all rounds other than t, and for round t the sum is 0 for all ads other than i. Again, it follows that the sum is simply equal to $b_i - \tilde{b}_i$, which is negative, contradicting Equation (8). Claim proved.

Case 2: $b_{\texttt{min}}(t,\rho) < \infty$**.** The proof of this case is very similar to the proof of Theorem 5.1(b).

Recall that in case 1 it holds that $b_{\texttt{min}}(t,\rho) < \infty$. Let $\tilde{b} = \vec{1} + \max(b,b') \in \mathbb{R}_+^m$, where $\max(b,b')$ is the coordinate-wise maximum of b and b'. Since $b_{\texttt{min}}(t,\rho) < \infty$, it follows that $B \geq b_{\texttt{min}}(t,\rho)$, so neither $\mathcal{A}(b,t,\rho)$ nor $\mathcal{A}(b',t,\rho)$ nor $\mathcal{A}(\tilde{b},t,\rho)$ is equal to `skip`. Since $\mathcal{A}(b,t,\rho) \neq \mathcal{A}(b',t,\rho)$, we can w.l.o.g. assume that $\mathcal{A}(\tilde{b},t,\rho) \neq \mathcal{A}(b,t,\rho)$. In particular, $\mathcal{A}_i(\tilde{b},t,\rho) = 0$ and $\mathcal{A}_i(b,t,\rho) = 1$ for some ad i.

We claim that `WMON` is violated for bids $b, \tilde{b}$ and realization ρ_i. Indeed, consider Equation (8) for realization ρ_i. The sum in Equation (8) is 0 for all rounds other than t because $\mathcal{A}(\tilde{b},s,\rho) = \mathcal{A}(b,s,\rho)$ for all rounds $s < t$ (by minimality of t), and $\rho_i \equiv 0$ for all rounds $s > t$. For round t, the sum in Equation (8) is 0 for all ads other than i, by definition of ρ_i. Thus, the sum is simply equal to $b_i - \tilde{b}_i$, which is negative, contradicting Equation (8). Claim proved. □

5.1 Analysis of the randomized case: proof of Theorem 5.1(c)

The proof of the randomized case of Theorem 5.1 is technically more involved than the proof of Theorem 5.1(b)). In particular, even stating the analog of Claim 5.2 requires a considerable amount of setup.

Define functions $f, g, G : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ by the following recurrence: $f(0,y) = g(0,y) = G(0,y) = 0$ for all y; while for $t > 0$:

$$\begin{aligned} f(t,y) &= 3ym\, G(t-1,y) + 1 \\ g(t,y) &= 2\, f(t,y) + 2 + 3ym\, G(t-1,y) \\ G(t,y) &= g(t,y) + G(t-1,y) = \textstyle\sum_{s=1}^{t} g(s,y). \end{aligned}$$

For real numbers $B \geq 0, y \geq 1$, let $\mathcal{D}(B,y)$ denote the set

$$\mathcal{D}(B,y) = \{b\ :\ \min(b) \geq B, \max(b)/\min(b) \leq y\}.$$

We will refer to a bid vector as "y-balanced" if it satisfies $\max(b)/\min(b) \leq y$.

Let $\mathcal{A}$ be a (potentially randomized) allocation rule. Fix realization ρ. For all times t and all $\epsilon > 0, y \geq 1$ let

$$A_{\texttt{max}}(t,\rho,y) = \limsup_{x\to\infty} \{\ \|\mathcal{A}(b,t,\rho)\|_1\ :\ b \in \mathcal{D}(x,y)\ \}$$

$$b_{\texttt{max}}(t,\rho,\epsilon,y) = \sup \{x:\ \exists b \in \mathcal{D}(x,y) \quad \|\mathcal{A}(b,t,\rho)\|_1 < A_{\texttt{max}}(t,\rho,y) - \epsilon\, f(t,y)\ \}$$

$$B_\epsilon(y) = \begin{cases} 0 & \text{if } b_{\texttt{max}}(t,\rho,\epsilon,y) = \infty \text{ for all } t,\rho,\epsilon \\ \sup\left(\mathbb{R} \cap \{b_{\texttt{max}}(t,\rho,\epsilon,y):\ t \in \mathbb{N}, \epsilon > 0\}\right) & \text{otherwise.} \end{cases}$$

Here $A_{\texttt{max}}(t, \rho, y)$ is the maximal expected number of impressions at time t such that the agent can obtain this number with arbitrarily large y-balanced bids. The meaning of $b_{\texttt{max}}$ is as follows: if every component of a y-balanced vector b is above $b_{\texttt{max}}$, the expected number of impressions for time t is guaranteed to be within $\epsilon\, f(t, y)$ of the best possible. Note that $B_\epsilon(y) = 0$ if $b_{\texttt{max}}(t, \rho, \epsilon, y)$ is infinite for all t and all ρ.

Claim 5.3. *Let $\mathcal{A}$ be a single-agent allocation rule which satisfies ex-post* `WMON`*. Then for any $y \geq 1$, any $\epsilon > 0$, any bid vectors $b, b' \in \mathcal{D}(B_\epsilon(y), y)$, any realization ρ, and any round t, we have*

$$\left\| \mathcal{A}(b, t, \rho) - \mathcal{A}(b', t, \rho) \right\|_1 \leq \epsilon\, g(t, y)$$

Proof. Fix $\epsilon > 0$ and realization ρ. Let us use induction on t. Case $t = 0$ is trivial, interpreting $\mathcal{A}(b, 0, \rho) = \vec{0}$ for all ρ, b. Now assume the claim is true for all times $s < t$. For the sake of contradiction, assume the claim does not hold for time t and some realization ρ.

By definition of $A_{\texttt{max}}$, there exists a number M^* such that

$$\sup_{b \in \mathcal{D}(M^*, y)} \|\mathcal{A}(b, t, \rho)\|_1 < A_{\texttt{max}}(t, \rho, y) + \epsilon.$$

For each ad i, define a new realization ρ_i as follows: it coincides with ρ before time t, only i gets clicked at time t, and there are no clicks after t.

Fix bid vectors $b, b' \in \mathcal{D}(B_\epsilon(y), y)$. Pick some bid vector $\tilde{b} \in \mathcal{D}(\widetilde{M}, y)$, where

$$\widetilde{M} = \max(M^*,\ 3y\ \left\| b + b' \right\|_\infty).$$

Let $M = \max(\tilde{b})$.

`WMON` for realization ρ_i, applied to bid vectors b and $\tilde{b}$, states the following:

$$(\tilde{b} - b\,)^\dagger\ \Delta_t(\rho_i)\ \Big(\ \mathcal{A}(\tilde{b}, t, \rho) - \mathcal{A}(b, t, \rho)\ \Big) \tag{11}$$

$$+\ (\tilde{b} - b\,)^\dagger\ \sum_{s=1}^{t-1} \Delta_s(\rho)\ \Big(\mathcal{A}(\tilde{b}, s, \rho) - \mathcal{A}(b, s, \rho)\Big) \geq 0. \tag{12}$$

The first summand in Equation (11) is simply $(\tilde{b}_i - b_i)\ \Big(\ \mathcal{A}_i(\tilde{b}, t, \rho) - \mathcal{A}_i(b, t, \rho)\ \Big)$.

By the induction hypothesis, for each time $s < t$ it holds that

$$(\tilde{b} - b\,)^\dagger\ \Delta_s(\rho)\ \Big(\mathcal{A}(\tilde{b}, s, \rho) - \mathcal{A}(b, s, \rho)\Big) \leq M\,\epsilon\, g(s, y)$$

It follows that

$$(\tilde{b} - b\,)^\dagger\ \sum_{s=1}^{t-1} \Delta_s(\rho)\ \Big(\mathcal{A}(\tilde{b}, s, \rho) - \mathcal{A}(b, s, \rho)\Big) \leq M\,\epsilon \sum_{s=1}^{t-1} g(s, y) = M\,\epsilon\, G(t-1, y)$$

Plugging this into Equation (11), we obtain

$$(\tilde{b}_i - b_i)\ \Big(\ \mathcal{A}_i(\tilde{b}, t, \rho) - \mathcal{A}_i(b, t, \rho)\ \Big) \geq -M\,\epsilon\, G(t-1, y)$$

$$\mathcal{A}_i(\tilde{b}, t, \rho) - \mathcal{A}_i(b, t, \rho) \geq -(\tilde{b}_i - b_i)^{-1} M\,\epsilon\, G(t-1, y).$$

We have $\tilde{b}_i \geq M/y$ since $\tilde{b}$ is y-balanced. Also $b_i \leq M/(3y)$ by our choice of M. Therefore $\tilde{b}_i - b_i \geq \frac{2M}{3y}$ and

$$\mathcal{A}_i(\tilde{b}, t, \rho) - \mathcal{A}_i(b, t, \rho) \geq -\tfrac{3y}{2}\, \epsilon\, G(t-1, y). \tag{13}$$

Case 1: $b_{\mathtt{max}}(t, \rho, \epsilon, y) < \infty$**.** Denote $X_i = \mathcal{A}_i(b, t, \rho)$ and $X_i' = \mathcal{A}_i(b', t, \rho)$. Let $\min(X_i, X_i')$ be the coordinate-wise minimum of X_i and X_i'; define $\max(X_i, X_i')$ similarly.

In this notation, our goal is to bound $\|X - X'\|_1$ above by $\epsilon\, g(t, y)$. Assume $\tilde{b} = M\, \vec{1}$ for some $M \geq \widetilde{M}$. By Equation (13), noting that this argument applies to both b and b', we have:

$$\mathcal{A}_i(\tilde{b}, t, \rho) \geq \max(X_i, X_i') - \tfrac{3y}{2}\, \epsilon\, G(t-1, y).$$

Summing this over all ads:

$$\left\|\mathcal{A}(\tilde{b}, t, \rho)\right\|_1 \geq \left\|\max(X, X')\right\|_1 - \tfrac{3y}{2}\, \epsilon\, m\, G(t-1, y).$$

Recall that $\left\|\mathcal{A}(\tilde{b}, t, \rho)\right\|_1 \leq A_{\mathtt{max}}(t, \rho, y) + \epsilon$ by our choice of M. Therefore:

$$\left\|\max(X, X')\right\|_1 \leq A_{\mathtt{max}}(t, \rho, y) + \epsilon\, (1 + \tfrac{3y}{2}\, m\, G(t-1, y)).$$

Note that

$$\begin{aligned}
\|X\|_1 + \left\|X'\right\|_1 &= \left\|\max(X, X')\right\|_1 + \left\|\min(X, X')\right\|_1 \\
\left\|X - X'\right\|_1 &= \left\|\max(X, X')\right\|_1 - \left\|\min(X, X')\right\|_1 \\
\|X\|_1 + \left\|X'\right\|_1 + \left\|X - X'\right\|_1 &= 2\left\|\max(X, X')\right\|_1
\end{aligned}$$

Because $b_{\mathtt{max}}(t, \rho, \epsilon, y) < \infty$ and $b, b' \in \mathcal{D}(B_\epsilon(y), y)$, both $\|X\|_1$ and $\|X'\|_1$ are at least $A_{\mathtt{max}}(t, \rho, y) - \epsilon\, f(t, y)$. Therefore:

$$\begin{aligned}
2A_{\mathtt{max}}(t, \rho) - 2\epsilon\, f(t, y) + \left\|X - X'\right\|_1 &\leq 2\ \left\|\max(X, X')\right\|_1 \\
&\leq 2A_{\mathtt{max}}(t, \rho, y) + 2\,\epsilon\, (1 + \tfrac{3y}{2}\, G(t-1, y)).
\end{aligned}$$

It follows that

$$\left\|X - X'\right\|_1 \leq 2\epsilon \left(\ 1 + f(t, y) + \tfrac{3y}{2}\, m\, G(t-1, y)\ \right) = \epsilon\, g(t, y).$$

Thus, we have proved the induction step assuming $b_{\mathtt{max}}(t, \rho, \epsilon, y)$ is finite.

Case 2: $b_{\mathtt{max}}(t, \rho, \epsilon, y) = \infty$**.** This case is impossible: we will arrive at a contradiction.

By definition of $A_{\mathtt{max}}$, there exists a bid vector $b \in \mathcal{D}(B_\epsilon(y), y)$ such that

$$\|\mathcal{A}(b, t, \rho)\|_1 > A_{\mathtt{max}}(t, \rho, y) - \tfrac{1}{2}\, \epsilon\, f(t, y).$$

Since $b_{\mathtt{max}}(t, \rho, \epsilon, y) = \infty$, we can pick $\tilde{b} \in \mathcal{D}(\widetilde{M}, y)$ such that

$$\left\|\mathcal{A}(\tilde{b}, t, \rho)\right\|_1 \leq A_{\mathtt{max}}(t, \rho, y) - \epsilon\, f(t, y) \leq \|\mathcal{A}(b, t, \rho)\|_1 - \tfrac{1}{2}\, \epsilon\, f(t, y).$$

It follows that

$$\begin{aligned}
\sum_{i=1}^{m} \left[\mathcal{A}_i(b, t, \rho) - \mathcal{A}_i(\tilde{b}, t, \rho)\right] &\geq \tfrac{1}{2}\, \epsilon\, f(t, y) \\
\exists i \quad \mathcal{A}_i(b, t, \rho) - \mathcal{A}_i(\tilde{b}, t, \rho) &\geq \tfrac{1}{2m}\, \epsilon\, f(t, y).
\end{aligned}$$

Using Equation (13), for this i we have:

$$\tfrac{1}{2m}\,\epsilon\, f(t,y) \le \mathcal{A}_i(b,t,\rho) - \mathcal{A}_i(\tilde{b},t,\rho) \le \tfrac{3y}{2}\,\epsilon\, G(t-1,y).$$

Thus, $f(t,y) \le 3ym\, G(t-1,y)$, contradicting the definition of f. □

Using Claim 5.3, it is now easy to prove Theorem 5.1(c).

of Theorem 5.1(c). For any $\delta > 0$, let

$$y = 2m/\delta$$
$$\epsilon = \tfrac{\delta}{2mg(T,y)}$$
$$B = B_\epsilon(y).$$

In our proof we will considering applying $\mathcal{A}$ to the bid vector $b^0 = B\vec{1}$ as well as the vectors b^j defined for $j = 1,\ldots,m$ by changing the j^{th} of b^0 from B to yB. The vectors $b^0,\ldots,b^m$ all belong to $\mathcal{D}(B,y)$.

Let ρ be a realization such that $\rho(t,j) = 1$ for all t,j, i.e. every ad is always clicked. Since $\mathcal{A}$ can never allocate more than T impressions, we have $\sum_{t=1}^{T}\sum_{i=1}^{m}\mathcal{A}_i(b^0,t,\rho) \le T$. Hence, there is at least one $j \in [m]$ such that

$$\sum_{t=1}^{T}\mathcal{A}_j(b^0,t,\rho) \le T/m. \tag{14}$$

Now, for every round t, we have

$$\mathcal{A}_j(b^j,t,\rho) - \mathcal{A}_j(b^0,t,\rho) \le \left\|\mathcal{A}(b^j,t,\rho) - \mathcal{A}(b^0,t,\rho)\right\|_1 \le \epsilon\, g(t,y) = \tfrac{\delta}{2m}, \tag{15}$$

where the second inequality follows from Claim 5.3. Summing Equation (15) over $t = 1,\ldots,T$ and combining with Equation (14), we deduce that

$$\sum_{t=1}^{T}\mathcal{A}_j(b^j,t,\rho) \le \left(1\tfrac{\delta}{2}\right)\tfrac{T}{m}.$$

The optimal allocation for bid vector b^j assigns every impression to ad j, achieving a total value of yBT. Instead, the allocation computed by $\mathcal{A}$ achieves a total value bounded above by $\left(1+\tfrac{\delta}{2}\right)\tfrac{yBT}{m} + BT$, where the first term accounts for impressions allocated to ad j and the second term accounts for all other impressions. We have

$$\left(1+\tfrac{\delta}{2}\right)\tfrac{yBT}{m} + BT = \tfrac{yBT}{m}\cdot\left(1+\tfrac{\delta}{2}+\tfrac{m}{y}\right) = yBT\cdot\tfrac{1+\delta}{m}.$$

Since $\delta > 0$ was an arbitrarily small positive constant, we conclude that the worst-case approximation ratio of $\mathcal{A}$ is no better than $1/m$, which is trivially achieved by a random allocation. □

6 Multi-parameter MAB mechanisms: A stochastic CMON allocation rule

In this section we consider the problem of designing stochastically truthful multi-parameter MAB mechanisms. As discussed in the introduction, the VCG mechanism cannot be used as it is informationally-infeasible. Additionally, pricing based mechanisms do not seem to be feasible. The only other technique that is extensively exploited in the literature for multi-parameter domains is using *maximal in distributional range* (MIDR) allocation rules. We formalize the limitations of a natural family of MIDR allocation rules (in which the set of distributions the rule optimizes over is independent of the CTRs) in Section 6.3, showing that the performance of such rules is no better than randomly selecting an ad to present. We next discuss some simple approaches to create truthful mechanisms: the first disregards the bids, and the second uses randomization to reduce the problem to a single parameter problem.

The first approach is *bid-independent* allocation rules – ones that do not depend on the bids. Among those, we naturally focus on the allocation rule that achieves the best worst-case performance, that rule samples an ad independently and uniformly at random in each round; call it RND.

A slightly more sophisticated approach randomly reduces the problem to a single parameter problem as follows. One ad is selected independently for each agent, uniformly at random from this agent's ads. Then some truthful single-parameter mechanism $\mathcal{M}$ is run on the selected ads. Call this mechanism SubSample. This mechanism is truthful (ex-post or stochastically, same as $\mathcal{M}$) because for each realization of the selection described above, it is simply a truthful single-parameter mechanism. The performance of this mechanism is the same as the performance of the trivial RND mechanism when there is only one agent.

These two naïve approaches have poor performance. For example, for a single agent none performs better than uniformly randomizing over the ads. We call such a performance *trivial*. This gives rise to the following major open problem.

Open Problem: *Design a stochastically truthful mechanism for the multi-parameter MAB problem that achieves optimal approximation.*

A more modest goal is to design a stochastically truthful mechanism for the multi-parameter MAB problem that achieves *non-trivial* performance, even for some "well-behaved" subset of inputs. Unfortunately, it seems that all standard tools fail to achieve even this modest goal. Below we achieve this by designing a stochastically CMON allocation rule and then applying the multi-parameter transformation from Section 3. We interpret this result as an evidence that it is not completely hopeless to significantly improve over the trivial approaches.

6.1 The stochastically CMON allocation rule

We design a stochastically CMON allocation rule ALL whose expected welfare exceeds that of RND on all problem instances with at least two agents, and that of SubSample on an important family of problem instances which we characterize below. Structurally ALL depends on all submitted bids, is provably not MIDR, and, unlike SubSample, does not proceed through an explicit reduction to a single-parameter allocation rule. Implementing ALL as a truthful, information-feasible mechanism requires the full power of our multi-parameter transformation.

All results in this section require all private values to be bounded from above by 1. We will assume that without further notice.

Recap of notation. The term "expected welfare" refers to expectation over the randomness in the allocation rule and the clicks (for a given vector of CTRs). Let $W(\texttt{RND})$ denote the expected welfare of RND. Let $A_0 = \{1, \ldots, m\}$ be the set of m ads of all agents. Recall that v_j, b_j and μ_j be, resp., denote the private value, the submitted bid, and the CTR for ad j. Note that the expected value from each time a given ad j is displayed is $v_j\mu_j$.

Allocation rule ALL for ≥ 2 agents. Assume there are at least two agents. Define the following allocation rule, call it ALL. It consists of two phases: exploration and exploitation. Exploration lasts for T_0 rounds, where $T_0 \geq 1$ is fixed and chosen in advance. In each exploration round an ad is chosen uniformly at random among all ads. Let n_j be the number of clicks for ad j by the end of the exploration phase. In each round of exploitation ALL does the following:

(L1) pick each ad j with probability $b_j\, n_j/T_0$, where b_j is the bid for ad j.

(L2) with the remaining probability pick an ad uniformly at random.

This completes the specification of ALL. We note that even a single round of exploration suffices for our purposes. Using a small T_0 does not affect the expected performance, but results in a (very) high variance.

Discussion. We design ALL to ensure that the allocation probabilities depend on CTRs and bids in a simple, linear way. Below we explain why this "linear dependence" property is useful, and discuss some of the challenges in the analysis of ALL.

Let the *allocation-vector* be a vector $a \in \Re^m$ whose j-th component is the expected number of times ad j is allocated by ALL. For a given vector of CTRs, the *allocation-range* is the set of all allocation-vectors that can be realized by ALL. We conjecture that the allocation-range needs to depend on CTRs in order for an allocation rule to satisfy stochastic CMON and be, in some sense, non-trivial. (In Section 6.3, we prove a version of this conjecture that is restricted to stochastically MIDR allocation rules.) The "linear dependence" property of ALL ensures that the allocation-range does depend on CTRs.

For example, consider an allocation rule which has an exploration phase of fixed duration, picks the best (estimated) ad based on the clicks received so far, and sticks with this ad from then on. This allocation rule that is ex-post truthful in the single-parameter setting, and is perhaps the most natural candidate for a reasonable, easy-to-analyze allocation rule for our setting. However, the allocation-range of this allocation rule does not depend on CTRs (because the set of possible options for exploitation is fixed: any one ad can be chosen).

Further, the proof technique that we use in the analysis of ALL essentially requires us, for every given agent, to solve a system of equations where the unknowns are this agent's bids and the parameters are the CTRs and the components of the allocation vector. The allocation probabilities in ALL are explicitly defined in terms of bids in order to enable us to solve this system of equations in a desirable way; this is another place where the "linear dependence" property of ALL is helpful.

The subtle point in our analysis of ALL – or, it seems, in any analysis using the same proof technique – is that one needs to ensure that the allocation vector is a maximizer of a certain expression, which requires us to prove the positive-definiteness of the corresponding Hessian matrix. The "linear dependence" property of ALL enables us to argue about the Hessian matrix in a useful way.

As we discovered, the positive-definiteness of the Hessian should not be taken for granted: indeed, it fails for a number of otherwise promising allocation rules with better performance. We believe that further progress on stochastically CMON allocation rules would require a more systematic understanding of how changes in the allocation rule propagate through the analysis and affect the Hessian matrix.

Guarantees for ALL for ≥ 2 agents. A problem instance is called *uniform* if the product $v_j\mu_j$ is the same for all j, and *non-uniform* otherwise. Note that for uniform problem instances RND is optimal, and in fact all allocation rules without skips have the same expected welfare, and are all optimal. We will assume that all values-per-click are at most 1, and that all CTRs are strictly positive.

Note that instances on which RND performs very poorly are those where for one ad j the product $v_j\mu_j$ is large while for all other ads this product is very low. On the other hand, for such inputs ALL plays the best ad significantly more often.

We next present a parameter that aims to quantify the divergence of the instance from uniform and will be used to measure the performance of ALL. A problem instance is called σ-skewed, for some $\sigma \in [1, m]$, if it satisfies

$$(M_2)^2 \geq \sigma (M_1)^2, \quad \text{where } M_q = \left(\tfrac{1}{m} \textstyle\sum_{j=1}^{m} (v_j\, \mu_j)^q \right)^{1/q}. \tag{16}$$

Note that problem instances can be σ-skewed for any given $\sigma \in [1, m]$. It is 1-skewed for uniform problem instances, and m-skewed when only one ad is good while all other ads have value 0.

Let $W_0(\texttt{ALL})$ be the expected per-round welfare for the exploitation phase of ALL, and let $W_0(\texttt{RND})$ be the expected per-round welfare for RND. Note that $W_0(\texttt{RND}) = M_1$. The properties of ALL with at least two agents are captured by the next lemma (which is the main technical lemma in this section); its proof is deferred to Appendix 6.2.

Lemma 6.1. *With at least two agents, allocation rule* ALL *satisfies the following:*

(a) If the CTRs for all ads are strictly positive then ALL *satisfies stochastic* CMON*.*

(b) For $W_0(\texttt{ALL})$ and $W_0(\texttt{RND})$ as defined above it holds that

$$W_0(\texttt{ALL}) - W_0(\texttt{RND}) = M_2^2 - M_1^2, \quad \textit{where } M_q = \left(\tfrac{1}{m} \textstyle\sum_{j=1}^{m} (b_j\, \mu_j)^q \right)^{1/q}.$$

In particular, $W(\texttt{ALL}) > W(\texttt{RND})$ for all non-uniform problem instances.

The allocation rule ALL does not have the property that scaling all bids by a common factor scales the expected welfare by the same factor; therefore it is not MIDR (see Section 6.3 for the definition of MIDR, as it applies to our setting).

Reduction to the single-agent case. For a single agent, we define our allocation rule ALL as follows: we simulate a run of ALL with a single round of exploration and two agents, where the second agent is a dummy agent with a single ad. The dummy agent submits a bid of zero for his ad, and we fix its CTR to $\frac{1}{2}$ (any CTR works). This completes the specification of ALL.

Denote the resulting two-agent allocation rule by $\texttt{ALL}^*$. The single-agent allocation rule satisfies CMON because so does $\texttt{ALL}^*$. Since the dummy agent does not contribute welfare (because of the zero bid), we have $W_0(\texttt{ALL}) = W_0(\texttt{ALL}^*)$. Applying Lemma 6.1(a) to $\texttt{ALL}^*$, we see that

$$W_0(\texttt{ALL}) = M_1^* + (M_2^*)^2 - (M_1^*)^2, \text{ where } M_q^* = \left(\tfrac{1}{m+1} \textstyle\sum_{j=1}^{m} (b_j\, \mu_j)^q \right)^{1/q}. \tag{17}$$

We summarize the useful properties of ALL in the following lemma:

Lemma 6.2. *Consider the case of a single agent; assume $\mu_j > 0$ for all ads j. Then* ALL *satisfies stochastic* CMON*, and its welfare in exploitation rounds satisfies Equation (17). In the one exploration round,* ALL *obtains welfare $\frac{m}{m+1} W_0(\texttt{RND})$.*

Main provable guarantee. Let $\mathcal{M}_\delta$ be the mechanism obtained by applying Theorem 3.1 to ALL with parameter $\delta \in (0, 1)$. The main result of this section follows.

Theorem 6.3. *Consider a multi-parameter MAB domain with $v_j \leq 1$ and $\mu_j > 0$ for every ad j. Then mechanism $\mathcal{M}_\delta$ is stochastically truthful, for every $\delta \in (0,1)$.*

Consider σ-skewed problem instances, and assume $\max_{j \in A_0} v_j \mu_j > \epsilon > 0$. There exists $\delta \in (0,1)$ such that mechanism $\mathcal{M}_\delta$ satisfies the following:

(a) $W(\mathcal{M}) > W(\texttt{RND})$ *on all problem instances with at least two agents, as long as* $\sigma > 1$.

(b) $W(\mathcal{M}) > W(\texttt{RND}) = W(\texttt{SubSample})$ *on all problem instances with a single agent with m ads, as long as* $\sigma > 1 + \frac{m+1}{m\epsilon} + \frac{m+1}{\epsilon(T-1)}$.

(c) Suppose there exists an agent with $k > m/2$ ads; w.l.o.g. assume this is agent 1. Then $W(\mathcal{M}) > W(\texttt{SubSample})$ *on all problem instances such that* $\sigma > 1 + \frac{m(m-k)}{k\epsilon}$ *when for all agents $i > 1$ all private values are* 0.[10] [11]

The theorem follows from Lemma 6.1, Lemma 6.2 and Theorem 3.1 via straightforward computations, some of which we omit from this version. Recall that for each $\delta > 0$ we have $W(\mathcal{M}_\delta) > (1-\delta)\, W(\texttt{ALL})$.

Theorem 6.3(a). Assume $M_2 > (1+\epsilon) M_1$ and $\max_{j \in A_0} b_j \mu_j > \epsilon$ for some $\epsilon > 0$. Then, using the notation of Lemma 6.1(b), we have $M_1 \geq \epsilon/m$, and therefore

$$W_0(\texttt{ALL}) - W_0(\texttt{RND}) \geq M_1^2\left((1+\epsilon)^2 - 1\right) > M_1 \tfrac{2\epsilon^2}{m}.$$

Recall that T_0 is the duration of exploration in ALL, and T is the time horizon. Then:

$$\begin{aligned}
W(\texttt{RND}) &= T\, W_0(\texttt{RND}) = T\, M_1 \\
W(\texttt{ALL}) &= T_0\, W_0(\texttt{RND}) + (T - T_0)\, W_0(\texttt{ALL}) \\
&= W(\texttt{RND}) + (T - T_0)\,(W_0(\texttt{ALL}) - W_0(\texttt{RND})) \\
&> W(\texttt{RND}) + \gamma\, W(\texttt{RND}), \quad \text{where } \gamma = \tfrac{2\epsilon^2 (T - T_0)}{mT} \\
W(\mathcal{M}) &> (1-\eta)\, W(\texttt{ALL}) > (1-\eta)(1+\gamma)\, W(\texttt{RND}).
\end{aligned}$$

Thus, to ensure that $W(\mathcal{M}) > W(\texttt{RND})$, it suffices to take $\eta < 1 - \frac{1}{1+\gamma}$. □

Proof Sketch of Theorem 6.3(bc). For part (b), recall that $W_0(\texttt{RND}) = W_0(\texttt{SubSample}) = \frac{1}{m}\sum_{j=1}^{m} b_j\, \mu_j$. With a simple computation which we omit from this version, one derives that $W(\mathcal{A}) > W(\texttt{RND})$. We prove $W(\mathcal{M}) > W(\texttt{RND})$ using a computation similar to the one in the proof of part (a), we omit the details.

For part (c), note that $W_0(\texttt{RND}) = M_1$ and (under the assumptions in Theorem 6.3(c)), $W(\texttt{SubSample}) \leq \frac{1}{k} M_1$. Again, using a simple computation one can show that $W(\mathcal{A}) > W(\texttt{SubSample})$, and then pick a sufficiently small δ as in the proof of part (a). □

[10] One can also derive a version of this result where the private values for all agents $i > 1$ are smaller than δ, for some $\delta \ll \epsilon$. We omit the easy details.

[11] Note that the instances considered in this result are generalizing the instances we have discussed before. There are instances in which one agent have all but one ad, and only one of his ads has positive value, while all the rest of the ads (his and others) have value 0.

6.2 Proof of the main technical lemma (Lemma 6.1)

Let us set up some notation. Consider an exploitation round in the execution of ALL. For each ad j, let E_j be the event that ad j is chosen in line (L1) of the algorithm's specification. Let $E_{\mathtt{u}}$ be the remaining event in line (L2) when the ad is chosen uniformly at random. Denote $x_j = \Pr[E_j]$, and note that for each ad j,

$$x_j \triangleq \Pr[E_j] = b_j\, \mathbb{E}[n_j]/T_0 = \tfrac{1}{m}\, b_j \mu_j.$$

of Lemma 6.1(b). Consider a round in the exploitation phase of ALL. Partition this round into events $\mathcal{P} = \{E_1\,,\; \ldots\;, E_m; E_{\mathtt{u}}\}$. For each event $E \in \mathcal{P}$ in this partition, let $W_0(E)$ be the expected per-round welfare of ALL from this event, so that $W_0(\mathtt{ALL}) = \sum_{E\in\mathcal{P}} W_0(E)$. Note that $W_0(E_{\mathtt{u}}) = \Pr[E_{\mathtt{u}}]\, W_0(\mathtt{RND})$. Further, $W_0(E_j) = b_j \mu_j \Pr[E_j] = m x_j^2$ for each ad i.

It is easy to see that $W_0(\mathtt{RND}) = \frac{1}{m}\sum_j b_j \mu_j = \sum_j x_j$. It follows that

$$\begin{aligned} W_0(\mathtt{ALL}) - W_0(\mathtt{RND}) &= \textstyle\sum_{E\in\mathcal{P}} W_0(E) - \Pr[E]\, W_0(\mathtt{RND}) \\ &= \textstyle\sum_j W_0(E_j) - \sum_j \Pr[E_j]\, W_0(\mathtt{RND}) \\ &= \left(m \textstyle\sum_j x_j^2\right) - \left(\textstyle\sum_j x_j\right)^2 \\ &= M_2^2 - M_1^2. \end{aligned}$$

□

For Lemma 6.1(a), we rely on the following characterization of CMON from prior work:

Lemma 6.4. *Consider a function $f : S \to \Re^k$, where $S \subset \Re^k$. Let $f(S) \subset \Re^k$ be the image of f. Then f is* CMON *if and only if it is an affine maximizer, i.e.*

$$f(x) = \operatorname*{argmax}_{y\in f(S)} [x\cdot y - g(y)] \quad \textit{for some function } g : f(S) \to \Re.$$

Proof of Lemma 6.1(a). Assume that there are at least two agents, and all CTRs are strictly positive. Without loss of generality, let us focus on agent 1. We will use the following notation. Let $A = \{1\,,\; \ldots\;, k\}$ be the set of ads submitted by agent 1. Here k is the number of ads submitted by agent 1; note that $k < m$. Let $b = (b_1\,,\; \ldots\;, b_k)$ be the vector of bids for agent 1, where b_j is the bid on ad j. Let $B = [0,1]^k$ be the set of all possible bid vectors for agent 1. Let $\mu = (\mu_1\,,\; \ldots\;, \mu_k)$ be the vector of CTRs for agent 1. We will use both i and j to index ads.

Throughout the proof, let us keep the bids of all other agents fixed. Let $C_{i,t}(b)$ be the expected number of clicks that ad i receives in round t of ALL, given the bid vector b, where the expectation is taken over all realizations of the clicks and over the randomness in the algorithm.[12]

Let $\vec{C}_t(b) = (C_{1,t}(b)\,,\; \ldots\;, C_{k,t}(b))$ be the round-t vector over the ads of agent 1, and let $\vec{C}(b) = \sum_t \vec{C}_{i,t}(b)$ be the vector whose i-th component is the total expected number of clicks for ad i.

We need to prove that the function $\vec{C} : B \to \Re^k$ satisfies CMON. It suffices to prove that CMON is satisfied for each round t separately, i.e. that it is satisfied for each function $\vec{C}_t$. This is obvious if t is an exploration round. In the rest of the proof we fix t to be an exploitation round.

By Lemma 6.4, it suffices to prove that $\vec{C}_t(b)$ is an affine maximizer, i.e. that

$$\vec{C}_t(b) = \operatorname*{argmax}_{p\in\vec{C}_t(B)} \sum_{j\in A} b_i\, p_i - G(p,\mu) \tag{18}$$

[12] Here it is more convenient to use a slightly different notation for click-vectors, compared to Section 4.

for some function $G(p,\mu): \vec{C}_t(B) \times [0,1]^k \to \Re$, where $\vec{C}_t(B) \subset [0,1]^k$ is the image of $\vec{C}_t$. Crucially, the function G cannot depend on b. [13]

Denote $p^* = \vec{C}_t(b)$. If p^* is an interior point of $\vec{C}_t(B)$ and function G is differentiable, then Equation (18) implies the following:

$$\frac{\partial}{\partial p_i} G(p^*,\mu) = b_i \quad \text{for each ad } i \text{, bid vector } b \text{ and CTR vector } \mu. \tag{19}$$

We will construct a function $G(p,\mu)$ so that it satisfies Equation (19).

Here and on, $i \in A$ denotes an arbitrary ad of agent 1. Recall that $x_i = \Pr[E_i] = \frac{1}{m} b_i \mu_i$. Thus:

$$\begin{aligned} \Pr[E_{\mathtt{u}}] &= 1 - \textstyle\sum_{j\in A_0} \Pr[E_j] = 1 - \sum_{j\in A_0} x_j \\ C_{i,t}(b) &= \mu_i \left(\Pr[E_i] + \tfrac{1}{m}\Pr[E_{\mathtt{u}}]\right) = \mu_i \left(x_i + \tfrac{1}{m} - \tfrac{1}{m}\textstyle\sum_{j\in A_0} x_j\right). \end{aligned}$$

Recalling the notation $p^* = \vec{C}_t(b)$ and solving for x_i, we obtain

$$\begin{aligned} p_i^*/\mu_i &= x_i + \tfrac{1}{m} - \tfrac{1}{m}\textstyle\sum_{j\in A_0} x_j \\ \textstyle\sum_{j\in A} p_j^*/\mu_j &= \textstyle\sum_{j\in A} x_j + \tfrac{k}{m} - \tfrac{k}{m}\textstyle\sum_{j\in A_0} x_j \\ &= (\tfrac{k}{m} - Y) + (1 - \tfrac{k}{m})\textstyle\sum_{j\in A_0} x_j, \qquad \text{where } Y = \textstyle\sum_{j\in A_0\setminus A} x_j. \\ p_i^*/\mu_i &= x_i - \alpha \textstyle\sum_{j\in A} p_j^*/\mu_j + \beta. \end{aligned}$$

where $\alpha = \frac{1}{m-k}$ and $\beta = \frac{1}{m} - \alpha(Y - \frac{k}{m})$. It follows that

$$b_i = \frac{m}{\mu_i}\, x_i = p_i^* \frac{m}{\mu_i^2} + \sum_{j\in A} p_j^* \frac{\alpha m}{\mu_i \mu_j} - \frac{\beta m}{\mu_i}. \tag{20}$$

Denote the RHS of Equation (20) by $f_i(p^*,\mu)$. We have proved that $b_i = f_i(p^*,\mu)$ for each ad i. Thus to obtain Equation (19) it suffices to pick $G(p,\mu)$ so that it satisfies

$$\frac{\partial}{\partial p_i} G(p,\mu) = f_i(p,\mu) \quad \text{for each } i \in A. \tag{21}$$

Integrating $f_i(p^*,\mu)$ over p_i, for each ad i, and combining the resulting expressions, we obtain

$$G(p,\mu) = -\sum_{i\in A} p_i \frac{m\beta}{\mu_i} + \frac{m}{2}\sum_{i\in A} p_i^2 \frac{1+\alpha}{\mu_i^2} + \sum_{j\in A\setminus\{i\}} p_i p_j \frac{m\alpha}{\mu_i\mu_j}. \tag{22}$$

It is easy to check that this G satisfies Equation (21), which in turn implies Equation (19). [14] It follows that for this G, $p = p^*$ is a critical point in Equation (18). From here on we will use the G as defined in Equation (22).

We claim that the critical point $p = p^*$ is in fact a local maximum in Equation (18). Equivalently, we claim that $p = p^*$ is a local minimum of the function

$$\lambda(p) = G(p,\mu) - p\cdot b : \Re^k \to \Re.$$

[13] Note that G *can* depend on the CTRs, even though the mechanism does not know them. This is because G is only used for the analysis – to prove `CMON`, and it is not actually used in the mechanism.

[14] Write $f_i(p,\mu) = \phi_i + \sum_{j\in A} p_j\,\gamma_{ij}$ for some numbers ϕ_i and γ_{ij}. Then a function $G(p,\mu)$ satisfying Equation (21) exists if and only if $\gamma_{ij} = \gamma_{ji}$ for all $i \neq j$.

For that, it suffices to prove that the Hessian matrix H of $\lambda(\cdot)$, defined by

$$H_{ij} = \frac{\partial}{\partial p_i\, \partial p_j} \lambda(p) = \frac{\partial}{\partial p_i\, \partial p_j} G(p,\mu),$$

is positive-definite for $p = p^*$. Note that for any $p \in \Re^k$ it holds that

$$H_{ij} = \begin{cases} \tau \rho_i^2, & i = j, \\ \rho_i \rho_j, & i \neq j, \end{cases} \tag{23}$$

where $\rho_i = \frac{\sqrt{\alpha m}}{\mu_i}$ for each $i \in A$, and $\tau = \frac{1+\alpha}{\alpha} = 1 + m - k \geq 2$. By Claim 6.5, such matrix is positive-definite.

To complete the proof, we will show that $p = p^*$ is the *global* maximum in Equation (18) over all $p \in \Re^k$. For that, it suffices to prove that that $p = p^*$ is the unique critical point over the entire $\Re^k$, i.e. the unique solution for the system

$$\frac{\partial}{\partial p_i} G(p,\mu) = b_i \quad \text{for each ad } i \in A. \tag{24}$$

Let us re-write this system using Equation (21). (We find it convenient to use the notation τ and ρ_i, as in Equation (23).) Namely, for each $i \in A$ we have:

$$\begin{aligned} b_i + \tfrac{\beta m}{\mu_i} &= f_i(p,\mu) + \tfrac{\beta m}{\mu_i} \\ &= p_i \left(\tau \rho_i^2\right) + \textstyle\sum_{j \in A \setminus \{i\}} p_j \left(\rho_i \rho_j\right). \end{aligned}$$

It follows that the system in Equation (24) is equivalent to

$$H \cdot p = w,$$

where the $k \times k$ matrix H is defined by Equation (23), and the vector $w \in \Re^k$ is defined by $w_i = b_i + \frac{\beta m}{\mu_i}$ for all i. The matrix H is non-singular (since it is positive-definite), so the system $H \cdot p = w$ has a unique solution p. □

Claim 6.5. *Consider a $k \times k$ matrix H given by Equation (23), where $\rho_1, \ldots, \rho_k$ are arbitrary positive numbers. Assume $\tau \geq 1$. Then H is positive definite.*

Proof. We will use the *Gram matrix* characterization of positive-definite matrices. Namely, to prove that H is positive-definite, it suffices to construct finite-dimensional vectors $w_1, \ldots, w_k$ such that $H_{ij} = w_i \cdot w_j$ for all i, j and the vectors are linearly independent. Consider vectors $w_1, \ldots, w_k \in \Re^{k+1}$ defined as follows:

$$w_i(\ell) = \begin{cases} \sqrt{\tau - 1}\, \rho_i, & \ell = i,\ \ell \leq k \\ 0, & \ell \neq i,\ \ell \leq k \\ \rho_i, & \ell = k+1, \end{cases}$$

It is easy to see that these vectors satisfy the desired properties. □

6.3 An impossibility result for stochastically MIDR allocation rules

Let us consider stochastically MIDR allocation rules for multi-parameter MAB mechanisms. We show that any such allocation rule (with a significant but reasonable restriction) is essentially trivial.

Let us formulate what it means for a given allocation rule $\mathcal{A}$ to be stochastically `MIDR` in our setting, in a specific way that is convenient for us to work with. For a given bid vector $b \in (0,\infty)^m$, and CTR vector $\mu \in [0,1]^m$, let the *allocation-vector* be a vector $a = (a_1\,,\,\dots\,,a_m)$ such that a_j is the expected number of times ad $j \in [m]$ is allocated by $\mathcal{A}$. Note that the expected welfare corresponding to a given allocation vector a is simply $\sum_j a_j b_j \mu_j$. Let $\mathcal{F}_0 = \{a \in [0,T]^m : \sum_j a_j \leq T\}$ be the set of all feasible allocation-vectors. (The sum of the entries can be less than T because skips are allowed.) Then $\mathcal{A}$ is stochastically `MIDR` if and only if for all bid vectors b and all CTR vectors μ it holds that

$$W(\mathcal{A}(b)) = \max_{a\in\mathcal{F}} \textstyle\sum_j \; a_j b_j \mu_j \tag{25}$$

for some $\mathcal{F} \subset \mathcal{F}_0$ that does not depend on b, but can depend on μ.

Note that Equation (25) does not immediately provide a stochastically truthful mechanism via VCG payments, because the computation of VCG payments is not immediately feasible without knowing the CTRs. In fact, Equation (25) does not even provide an immediate way to compute the allocation (assuming $|\mathcal{F}| \geq 2$), again because of the issue of not knowing the CTRs. This is in stark contrast with the prior work on `MIDR` (which studied settings without the ”no-simulation” constraint) where the `MIDR` property immediately gave rise to a truthful mechanism via the VCG payment rule.

However, if an allocation rule satisfies Equation (25) then a truthful mechanism can be obtained, with an arbitrarily small loss in welfare, via the transformation in Wilkens and Sivan [2012].

We consider a restricted version of Equation (25) where the range $\mathcal{F}$ cannot depend on the CTRs (we will call such range $\mathcal{F}$ *CTR-independent*). We prove that any such allocation rule is welfare-equivalent to a time-invariant allocation rule. Here an allocation rule is called *time-invariant* if in each round, it picks an ad independently from the same distribution over ads (this distribution may depend on the bids). Note that time-invariant allocation rules ignore the feedback that they receive (i.e., the clicks), and thus cannot adjust to the CTRs.

Lemma 6.6. *Consider a multi-parameter MAB domain. Let $\mathcal{A}$ be a stochastically* `MIDR` *allocation rule with CTR-invariant range. For each bid vector b there exists an allocation-vector $a = a(b) \in \mathcal{F}_0$ such that $W(\mathcal{A}(b)) = \sum_j a_j b_j \mu_j$ for all CTR vectors μ. So $\mathcal{A}$ is welfare-equivalent to a time-invariant allocation rule (where, letting T be the time horizon, each ad j is chosen with probability $a_j(b)/T$). The approximation ratio of $\mathcal{A}$ (compared to the welfare of the best ad) is at least m on some problem instances.*

Proof. Let us fix the bid vector b and consider both sides of Equation (25) as functions of μ. First, we note that the expected welfare $W(\mathcal{A}(b))$ is a finite-degree polynomial in variables $\mu_1\,,\,\dots\,,\mu_m$. [15] This is because, letting $\mathcal{A}_j(b,\rho,t)$ be the probability that ad j is displayed at round j given click-realization ρ, it holds that

$$W(\mathcal{A}(b)) = \textstyle\sum_\rho \Pr[\rho] \; \sum_{j,t} \rho(t,j)\, \mathcal{A}_j(b,\rho,t). \tag{26}$$

Here the outer sum is over all click-realizations ρ, and the inner sum is over all rounds t and all ads j. $\Pr[\rho]$ is the probability that ρ is realized for the given CTR vector. Equation (26) is a polynomial in the CTRs

[15] Namely, the degree is at most T, the time horizon.

because for each click-realization ρ, the inner sum is a fixed number, and $\Pr[\rho]$ is a polynomial in the CTRs of degree T.

Let us re-write Equation (25) as follows:

$$W(\mathcal{A}(b)) = \max_{\beta \in \mathcal{F}_b} \beta \cdot \mu, \text{ where } \mathcal{F}_b = \{\beta \in \Re^m : \beta_j = a_j b_j \text{ for each } j, a \in \mathcal{F}\}. \tag{27}$$

Since $\mathcal{F}_b$ is fixed, the right-hand side of Equation (27) is uniquely determined by μ, denote it $W(\mu)$. Note that for each $\beta \in \mathcal{F}_b$, it holds that

$$W(\mu) = \beta \cdot \mu \text{ if and only if } (\beta - \beta') \cdot \mu \geq 0 \text{ for all } \beta' \in \mathcal{F}_b.$$

For each $\beta \in \Re^k$, consider the half-space $H_\beta = \{\mu \in [0,1]^m : \beta \cdot \mu \geq 0\}$. Then

$$W(\mu) = \beta \cdot \mu \text{ if and only if } \mu \in S_\beta, \text{ where } S_\beta = \bigcap_{\beta' \in \mathcal{F}_b} H_{\beta - \beta'}.$$

Note that S_β is a convex set, as an intersection of convex sets. Moreover, all half-spaces in the intersection contain the 0-vector, and hence so does S_β. Therefore if $W(\mu) = \beta\mu$ for some $\mu \neq 0$ and $\beta \in \mathcal{F}_b$ then by convexity for any $z \in [0,1]$ it holds that $z\mu \in S_\beta$, and therefore $W(z\mu) = z\,(\beta \cdot \mu) = z\,W(\mu)$. We have proved the following:

$$W(z\mu) = z\,W(\mu) \text{ for every } z \in [0,1] \text{ and } \mu \in [0,1]^m. \tag{28}$$

Now recall that $W(\mu)$ is a finite-degree polynomial in μ. A known fact about multi-variate polynomials is that any finite-degree polynomial in μ which satisfies Equation (28) is in fact of the form $W(\mu) = \gamma \cdot \mu$ for some $\gamma \in \Re^m$.

Now, let $A = \{j : b_j > 0\}$ be the set of ads with non-zero bids. Define a vector $a \in \Re^m$ by $a_j = \gamma_j / b_j$ for each ad $j \in A$, and $a_j = 0$ otherwise. To complete the proof, it remains to show that, letting T be the time horizon, a/T is a valid distribution over the ads (assuming skips are allowed). That is, we need to show that $a_j \geq 0$ and $\sum_{j \in A} a_j \leq T$. We use the fact that for any allocation rule, the expected welfare is at least 0 and at most that of always playing the best ad:

$$W(\mu) = \textstyle\sum_{j \in A} a_j b_j \mu_j \in [0, T\ \max_j b_j \mu_j]. \tag{29}$$

Applying Equation (29) with μ being the unit vector in the direction $j \in A$, it follows that $W(\mu) = a_j b_j \geq 0$, so $a_j \geq 0$. Now, let $B = (\max_{j \in A} b_j^{-1})^{-1}$ and define a CTR vector μ by $\mu_j = B/b_j$ for $j \in A$ and $\mu_j = 0$ otherwise.[16] Plugging this μ into Equation (29), we obtain $W(\mu) = \sum_{j \in A} B a_j \leq BT$, which implies $\sum_{j \in A} a_j \leq T$, completing the proof. □

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