

# Consistent digital line segments

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## Abstract

We introduce a novel and general approach for digitalization of line segments in the plane that satisfies a set of axioms naturally arising from Euclidean axioms. In particular, we show how to derive such a system of digital segments from any total order on the integers. As a consequence, using a well-chosen total order, we manage to define a system of digital segments such that all digital segments are, in Hausdorff metric, optimally close to their corresponding Euclidean segments, thus giving an explicit construction that resolves the main question of [1].

## 1 Introduction

One of the most fundamental challenges in digital geometry is to define a “good” digital representation of a geometric object. Of course, the meaning of the word “good” here heavily depends on particular conditions we may impose. Looking at the problem of digitalization in the plane, the goal is to find a set of points on the integer grid  $\mathbb{Z}^2$  that approximates well a given object. The topology of the grid  $\mathbb{Z}^2$  is commonly defined by the graph whose vertices are all the points of the grid, and each point is connected by an edge to each of the four points that are either horizontally or vertically adjacent to it.

Knowing that a straight line segment is one of the most basic geometric objects and a building block for many other objects, defining its digitalization in a satisfying manner is vital. Hence, it is no wonder that this has been a hot scientific topic in the last few decades, see [4] for a recent survey and [3], [2], and [6] for related work, dealing with the problem of representing objects in digital geometry without causing topological and combinatorial inconsistencies.

For any pair of points  $p$  and  $q$  in the grid  $\mathbb{Z}^2$  we want to define the digital line segment  $S(p, q)$  connecting them, that is,  $\{p, q\} \subseteq S(p, q) \subseteq \mathbb{Z}^2$ . Chun et al. in [1] put forward the following four axioms that arise naturally from properties of line segments in Euclidean geometry.

(S1) *Grid path property:* For all  $p, q \in \mathbb{Z}^2$ ,  $S(p, q)$  is the vertex set of a path from  $p$  to  $q$  in the grid graph.

(S2) *Symmetry property:* For all  $p, q \in \mathbb{Z}^2$ , we have  $S(p, q) = S(q, p)$ .

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- (S3) *Subsegment property*: For all  $p, q \in \mathbb{Z}^2$  and every  $r \in S(p, q)$ , we have  $S(p, r) \subseteq S(p, q)$ .
- (S4) *Prolongation property*: For all  $p, q \in \mathbb{Z}^2$ , there exists  $r \in \mathbb{Z}^2$ , such that  $r \notin S(p, q)$  and  $S(p, q) \subseteq S(p, r)$ .

First, note that (S3) is not satisfied by the usual way a computer visualizes a segment. A natural definition of the digital straight segment between  $p = (p_x, p_y)$  and  $q = (q_x, q_y)$ , where  $p_x \leq q_x$  and  $0 \leq q_y - p_y < q_x - p_x$  is  $\left\{ \left( x, \left\lfloor (x - p_x) \frac{q_y - p_y}{q_x - p_x} + p_y + 0.5 \right\rfloor \right) : p_x \leq x \leq q_x \right\}$ . This does not satisfy (S1), but it could be easily fixed by a slight modification of the definition. Still, it also does not satisfy (S3), for example, for  $p = (0, 0), r = (1, 0), q = (4, 1)$ , the subsegment from  $r$  to  $q$  is not contained in the segment from  $p$  to  $q$ .

Even though the set of axioms (S1)-(S4) seems rather natural, there are still some fairly exotic examples of digital segment systems that satisfy all four of them. For example, let us fix a double spiral  $\mathcal{D}$  centered at an arbitrary point of  $\mathbb{Z}^2$ , traversing all the points of  $\mathbb{Z}^2$ . As it is a spanning path of the grid graph, we can set  $S(p, q)$  to be the path between  $p$  and  $q$  on  $\mathcal{D}$ , for every  $p, q \in \mathbb{Z}^2$ . It is easy to verify that this system satisfies axioms (S1)-(S4).

Another condition was introduced in [1] to enforce the monotonicity of the segments, ruling out pathological examples like the one above. Here, we phrase this monotonicity axiom differently, but still, the system of axioms (S1)-(S5) remains equivalent to the one given in [1].

- (S5) *Monotonicity property*: If both  $p, q \in \mathbb{Z}^2$  lie on a line that is either horizontal or vertical, then the whole segment  $S(p, q)$  belongs to this line.

We call a system of digital line segments that satisfies the system of axioms (S1)-(S5) a *consistent digital line segments system (CDS)*. It is straightforward to verify that every CDS also satisfies the following three conditions.

- (C1) If the slope of the line going through  $p$  and  $q$  is non-negative, then the slope of the line going through any two points of  $S(p, q)$  is non-negative. The same holds for non-positive slopes.
- (C2) For all  $p, q \in \mathbb{Z}^2$ , the grid-parallel box spanned by points  $p$  and  $q$  contains  $S(p, q)$ .
- (C3) If the intersection of two digital segments contains two points  $p, q \in \mathbb{Z}^2$ , then their intersection also contains the whole digital segment  $S(p, q)$ .

We give a simple example of a CDS, where the segments follow the boundary of the grid-parallel box spanned by the endpoints. Let  $p, q \in \mathbb{Z}^2$  be two points with coordinates  $p = (p_x, p_y)$  and  $q = (q_x, q_y)$ . If  $p_y \leq q_y$ , we define  $S(p, q) = S(q, p) = \{(x, p_y) : \min\{p_x, q_x\} \leq x \leq \max\{p_x, q_x\}\} \cup \{(q_x, y) : p_y \leq y \leq q_y\}$ . If  $p_y > q_y$ , we swap the points  $p$  and  $q$ , and define the segment as in the previous case.

It can be easily verified that this way we defined a CDS, but the digital segments in this system visually still do not resemble well the Euclidean segments.

One of the standard ways to measure how close a digital segment is to a Euclidean segment is to use the Hausdorff distance. We denote by  $\overline{pq}$  the Euclidean segment between  $p$  and  $q$ , and by  $|\overline{pq}|$  the Euclidean length of  $\overline{pq}$ . For two plane objects  $A$  and  $B$ , by  $H(A, B)$  we denote their Hausdorff distance.

The main question raised in [1] was if it is possible to define a CDS such that a Euclidean segment and its digitalization have a reasonably small Hausdorff distance. More precisely, the goal is to find a CDS satisfying the following condition.

(H) *Small Hausdorff distance property:* For every  $p, q \in \mathbb{Z}^2$ , we have that  $H(\overline{pq}, S(p, q)) = O(\log |\overline{pq}|)$ .

Note that in the CDS example we gave, the Hausdorff distance between a Euclidean segment of length  $n$  and its digitalization can be as large as  $n/\sqrt{2}$ .

While this question was not resolved in [1], a clever construction of a system of digital rays *emanating from the origin of  $\mathbb{Z}^2$*  that satisfy (S1)-(S5) and (H) was presented. Moreover, it was shown using Schmidt's theorem [5] that already for rays emanating from the origin, the log-bound imposed in condition (H) is the best bound we can hope for, directly implying the following theorem.

**Theorem 1** [1] *There exists a constant  $c > 0$ , such that for any CDS and any  $d > 0$ , there exist  $p, q \in \mathbb{Z}^2$  with  $|\overline{pq}| > d$ , such that  $H(\overline{pq}, S(p, q)) > c \log |\overline{pq}|$ .*

In this paper, we introduce a novel and general approach for the construction of a CDS. Namely, for any total order  $\prec$  on  $\mathbb{Z}$ , we show how to derive a CDS from  $\prec$ . (By total order we always mean a strict total order.) This process is described in Section 2. As a consequence, in Section 3, we manage to define a CDS that satisfies (H), deriving it from a specially chosen order on  $\mathbb{Z}$ , and thus giving the explicit construction that resolves the main question of [1].

**Theorem 2** *There is a CDS that satisfies condition (H).*

Note that Theorem 1 ensures that such a CDS is optimal up to a constant factor in terms of the Hausdorff distance from the Euclidean segments. In Section 4 we make a step towards a characterization of CDSes, demonstrating their natural connection to total orders on  $\mathbb{Z}$ , while in Section 5 we make use of the digital line segment definition to introduce digital lines. Finally, in Section 6 we discuss prospects of using a similar approach to define a CDS in higher dimensions.

## 2 Digital line segments derived from a total order on $\mathbb{Z}$

Let  $\prec$  be a total order on  $\mathbb{Z}$ . We are going to define a CDS  $\mathcal{S}_\prec$ , deriving it from  $\prec$ .

Let  $p, q \in \mathbb{Z}^2$ ,  $p = (p_x, p_y)$  and  $q = (q_x, q_y)$ . If  $p_x > q_x$ , we swap  $p$  and  $q$ . Hence, from now on we may assume that  $p_x \leq q_x$ .

If  $p_y \leq q_y$ , then  $S_\prec(p, q)$  is defined as follows. We start at the point  $p = (p_x, p_y)$  and we repeatedly go either up or to the right, collecting the points from  $\mathbb{Z}^2$ , until we reach  $q$ . Note that the sum of the coordinates  $x + y$  increases by 1 in each step. In total we have to make  $q_x + q_y - p_x - p_y$  steps and in exactly  $q_y - p_y$  of them we have to go up. The decision whether to go up or to the right is made as follows: if we are at the point  $(x, y)$  for which  $x + y$  is among the  $q_y - p_y$  greatest elements of the interval  $[p_x + p_y, q_x + q_y - 1]$  according to  $\prec$ , we go up, otherwise we go to the right. We will refer to this interval as the *segment interval*.

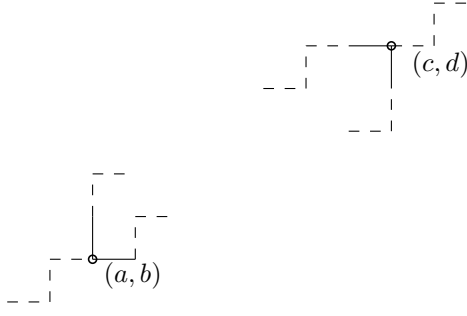


Figure 1: Two paths splitting up at  $(a, b)$  and meeting again at  $(c, d)$ .

If  $p_y > q_y$ , that is, if  $p$  is the top-left and  $q$  the bottom-right corner of the grid-parallel box spanned by  $p$  and  $q$ , then we define  $S_{\prec}(p, q)$  as the mirror reflection of  $S_{\prec}((-q_x, q_y), (-p_x, p_y))$  over the  $y$ -axis.

**Example.** Suppose  $p = (0, 0)$  and  $q = (2, 2)$ . Their segment interval consists of four numbers,  $0, 1, 2, 3$ . If  $\prec$  is the natural order on  $\mathbb{Z}$ , then the two greatest elements of the segment interval are 2 and 3. Since  $0 + 0$  is not one of these, at  $(0, 0)$  we go right, to  $(1, 0)$ . At  $(1, 0)$  we again go to right, to  $(2, 0)$ , from there to  $(2, 1)$  (since  $2 + 0$  is one of the greater elements) and finally to  $(2, 2)$ . In fact, it can be easily seen that using the natural order on  $\mathbb{Z}$  we get the CDS mentioned in Section 1, the one that always follows the boundary of the box spanned by the endpoints.

**Theorem 3**  $S_{\prec}$ , defined as above, is a CDS.

**Proof.** We will verify that  $S_{\prec}$  satisfies the axioms (S1)-(S5).

- (S1) The condition (S1) follows directly from the definition of  $S_{\prec}$ .
- (S2) Let  $p, q$  be two points from  $\mathbb{Z}^2$ . If the first coordinates of  $p$  and  $q$  are different, then condition (S2) follows directly. Otherwise,  $p$  and  $q$  belong to the same vertical line, and from the construction we see that both  $S_{\prec}(p, q)$  and  $S_{\prec}(q, p)$  consist of all the points on that line between  $p$  and  $q$ .
- (S3) For a contradiction, assume that there are points  $p = (p_x, p_y)$ ,  $q = (q_x, q_y)$  and  $r = (r_x, r_y)$ , with  $r \in S_{\prec}(p, q)$ , such that  $S_{\prec}(p, r) \not\subseteq S_{\prec}(p, q)$ . W.l.o.g. we may assume that  $\overline{pq}$  has a non-negative slope.

*Case 1.*  $p_x \leq q_x$  and  $p_y \leq q_y$ . We also have  $p_x \leq r_x$  and  $p_y \leq r_y$ , and going on each of the segments  $S_{\prec}(p, r)$  and  $S_{\prec}(p, q)$  point-by-point starting from  $p$ , we move either up or right. By assumption, these two segments separate at some point  $(a, b)$  and then meet again, for the first time after this separation, at some other point  $(c, d)$ , see Figure 1. One of the segments goes up at  $(a, b)$  and enters  $(c, d)$  horizontally coming from the left, which implies that  $a + b$  is among the greater numbers of the segment interval of this segment, while  $c + d - 1$  is not, thus  $c + d - 1 \prec a + b$ . But the other segment goes horizontally at  $(a, b)$  and enters  $(c, d)$  vertically coming from below, which similarly implies  $a + b \prec c + d - 1$ , a contradiction.

*Case 2.*  $q_x \leq p_x$  and  $q_y \leq p_y$ . We also have  $q_x \leq r_x$  and  $q_y \leq r_y$ . By assumption, the two segments starting at  $q$  and  $r$ ,  $S_{\prec}(q, p)$  and  $S_{\prec}(r, p)$ , separate at some point  $(a, b)$  and then meet again, for the first time after this separation, at some other point  $(c, d)$ . Using the same argument as before, we get a contradiction. Hence, (S3) holds.

(S4) To show that condition (S4) holds, consider the segment from  $p = (p_x, p_y)$  to  $q = (q_x, q_y)$ . W.l.o.g. we can assume that  $p_x \leq q_x$  and  $p_y \leq q_y$ . We distinguish two cases.

*Case 1.* If  $q_x + q_y$  is among the  $q_y - p_y + 1$  greatest numbers of  $[p_x + p_y, q_x + q_y]$  according to  $\prec$ , then we can prolong the segment going one step vertically up, that is, the segment  $S_{\prec}((p_x, p_y), (q_x, q_y + 1))$  contains the segment  $S_{\prec}((p_x, p_y), (q_x, q_y))$  as a subsegment.

*Case 2.* If, on the other hand,  $q_x + q_y$  is not among the  $q_y - p_y$  greatest numbers of  $[p_x + p_y, q_x + q_y]$ , we can prolong the segment horizontally to the right, that is,  $S_{\prec}((p_x, p_y), (q_x, q_y)) \subset S_{\prec}((p_x, p_y), (q_x + 1, q_y))$ .

Note that if  $q_x + q_y$  is exactly the  $(q_y - p_y + 1)^{th}$  number in  $[p_x + p_y, q_x + q_y]$ , then the conclusions of both cases are true, and indeed the rays emanating from  $(p_x, p_y)$  split at  $(q_x, q_y)$ .

(S5) The condition (S5) follows directly from the definition of  $S_{\prec}$ . □

Apparently in the definition of  $S_{\prec}(p, q)$  only the sum of the coordinates of the points plays a role, so if we translate  $p$  and  $q$  by a vector  $(t, -t)$ , for any integer  $t$ , the digital line segment will look the same.

**Observation 4** *Let  $t \in \mathbb{Z}$  be an integer.  $S_{\prec}(p + (t, -t), q + (t, -t)) = S_{\prec}(p, q) + (t, -t) = \{(x + t, y - t) \in \mathbb{Z}^2 : (x, y) \in S_{\prec}(p, q)\}$ .*

### 3 Digital segments with small Hausdorff distance to Euclidean segments

For integers  $k$  and  $l \geq 2$ , let  $|k|_l$  denote the number of times  $k$  is divisible by  $l$ , that is,

$$|k|_l = \sup \{m : l^m \mid k\}.$$

We define a total order on  $\mathbb{Z}$  as follows. Let  $a \prec b$  if and only if there exists a non-negative integer  $i$  such that  $|a - i|_2 < |b - i|_2$ , and for all  $j \in \{0, \dots, i - 1\}$  we have  $|a - j|_2 = |b - j|_2$ . In plain words, for two integers  $a$  and  $b$ , we say that the one that contains a higher power of 2 is greater under  $\prec$ . In case of a tie, we repeatedly subtract 1 from both  $a$  and  $b$ , until at some point one of them contains a higher power of 2 than the other. Thus, for example,  $-1 \prec -5 \prec 3 \prec -3 \prec 5 \prec 1 \prec -2 \prec 6 \prec -6 \prec 2 \prec -4 \prec 4 \prec 0$ .

Note that if we take the elements of an interval of the form  $(-2^n, 2^n)$  in  $\prec$ -decreasing order and we apply the function  $0.5 - x2^{-n-1}$  to them, then we get the first few elements of the Van der Corput sequence [7].

We will prove that using this total order to define the system of digital line segments  $S_{\prec}$ , as described in the previous section, we obtain a CDS which satisfies condition (H). In Figure 2 we give some examples of digital segments in this CDS, and Figure 3 shows the segments emanating from  $(0, 0)$  to some neighboring points, as well as the segments from  $(2, 3)$  to the neighboring points.

At first sight it may be surprising to observe that all the segments emanating from the origin in our construction coincide with the ones given in the construction of digital rays in [1]. However, it is not a coincidence, as the construction from [1] also relies on the same total order on integers.

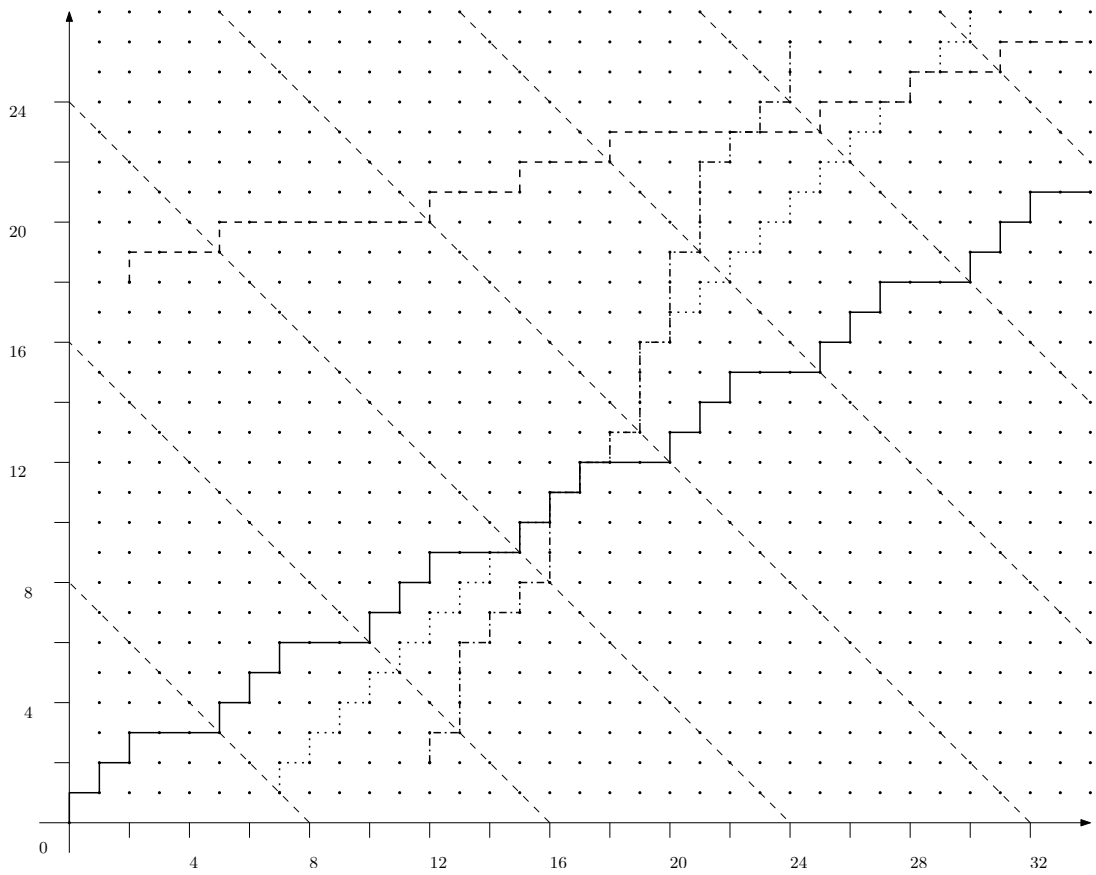


Figure 2: Some line segments.

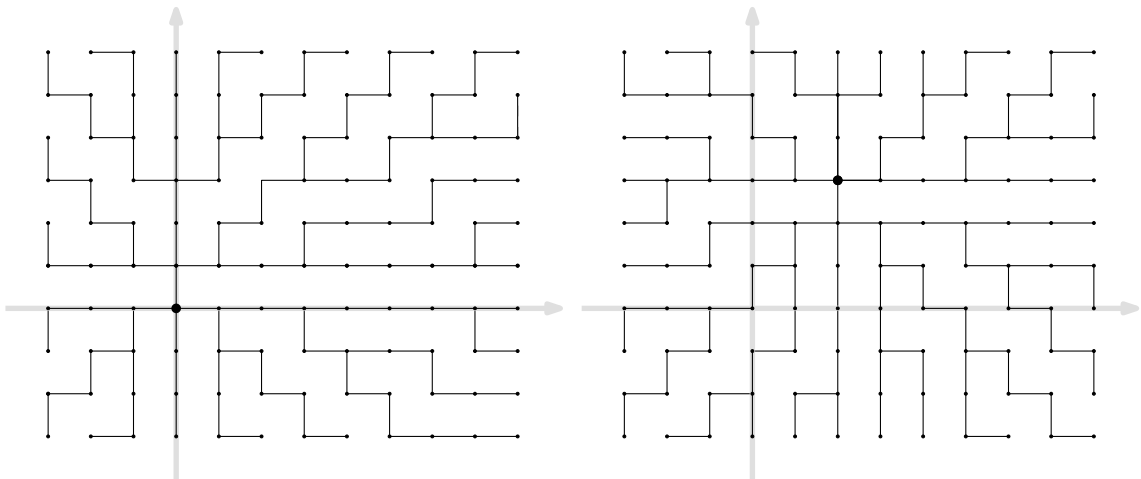


Figure 3: Digital line segments emanating from  $(0,0)$  and from  $(2,3)$ .

For points  $v, w \in \mathbb{R}^2$  and  $A \subseteq \mathbb{R}^2$ , let  $d(v, w) = |v - w|$  and  $d(v, A) = \inf_{a \in A} d(v, a)$  denote the usual Euclidean distances between two points, and between a point and a set. For  $p, q, r, s \in \mathbb{Z}^2$ , by  $\overline{pqr\bar{s}}$  we denote the union of Euclidean linear line segments from  $p$  to  $q$ , from  $q$  to  $r$ , and from  $r$  to  $s$ .

**Observation 5** For any  $p, q \in \mathbb{Z}^2$ ,  $H(S_{\prec}(p, q), \overline{pq}) = \max\{d(r, \overline{pq}) : r \in S_{\prec}(p, q)\}$ .

We proceed by proving three statements that we will use to ultimately prove Theorem 2.

**Lemma 6** If  $p, q \in \mathbb{Z}^2$  and  $r, s \in S_{\prec}(p, q)$ . Then

$$H(\overline{rs}, S_{\prec}(r, s)) \leq 2H(\overline{pq}, S_{\prec}(p, q)).$$

**Proof.** We know that  $d(r, \overline{pq}) \leq H(\overline{pq}, S_{\prec}(p, q)) =: h$  and  $d(s, \overline{pq}) \leq h$ , therefore  $H(\overline{prsq}, \overline{pq}) \leq h$ . Hence, for all  $v \in \overline{pq}$ ,  $d(v, \overline{prsq}) \leq h$ . Let  $t \in S_{\prec}(r, s) \subseteq S_{\prec}(p, q)$  and  $v \in \overline{pq}$  be such that  $d(t, v) = d(t, \overline{pq}) \leq h$ . Using the triangle inequality we conclude  $d(t, \overline{prsq}) \leq d(t, v) + d(v, \overline{prsq}) \leq 2h$ . Because of (C2), we have  $d(t, \overline{rs}) = d(t, \overline{prsq}) \leq 2h$ , and therefore  $H(S_{\prec}(r, s), \overline{rs}) \leq 2h$ .  $\square$

**Lemma 7** Let  $p, q, r, r' \in \mathbb{Z}^2$ , such that  $r_x - p_x = q_x - r'_x + \varepsilon$ ,  $r_y - p_y = q_y - r'_y - \varepsilon$ , with  $\varepsilon \in \{0, 1, -1\}$ ,  $r'_x = r_x$  and  $r'_y = r_y + 1$ . Then  $H(\overline{pq}, \overline{pr'r'q}) \leq c = \sqrt{5}/2$ .

**Proof.** Without loss of generality  $p = (0, 0)$ . We have

$$H(\overline{pq}, \overline{pr'r'q}) = \max\{d(r, \overline{pq}), d(r', \overline{pq})\}.$$

By assumption  $2r_x - \varepsilon = q_x$  and  $2r_y + \varepsilon + 1 = q_y$ . So we get

$$\begin{aligned} d(r, \overline{pq}) &= \frac{q_y r_x - q_x r_y}{\sqrt{q_x^2 + q_y^2}} \\ &= \frac{q_y \frac{q_x + \varepsilon}{2} - q_x \frac{q_y - \varepsilon - 1}{2}}{\sqrt{q_x^2 + q_y^2}} \\ &= \frac{1}{2\sqrt{q_x^2 + q_y^2}} (q_y \varepsilon + q_x \varepsilon + q_x). \end{aligned}$$

Similarly,

$$d(r', \overline{pq}) = \frac{1}{2\sqrt{q_x^2 + q_y^2}} (q_y \varepsilon + q_x \varepsilon - q_x).$$

Setting  $x := q_x/q_y$ , we observe

$$\begin{aligned} H(\overline{pq}, \overline{pr'r'q}) &\leq \frac{1}{2\sqrt{(xq_y)^2 + q_y^2}} (q_y + 2xq_y) \\ &= \frac{x+1/2}{\sqrt{x^2+1}} \\ &\leq \sqrt{5}/2, \end{aligned}$$

as the function

$$f(x) = \frac{x+1/2}{\sqrt{x^2+1}}$$

attains its global maximum at  $x = 2$ .  $\square$

The following lemma is a statement about the order  $\prec$  and will be the key ingredient of the proof of Theorem 2.

**Lemma 8** Let  $\{x \in \mathbb{Z} | A \leq x < B\}$  be an interval of integers with the following properties:

- (i) Its number of elements is  $B - A = 2^{k+1} - 1$  for some number  $k$ .
- (ii)  $|(A + B - 1)/2|_2 \geq k$  and  $|x|_2 < k$ , for any other  $A \leq x < B$ ,  $x \neq (A + B - 1)/2$ .

Let  $x_1 \prec x_2 \prec \dots \prec x_{2^{k+1}-1}$  be the elements of the interval sorted in increasing order according to  $\prec$ .

Then elements from the left half and elements from the right half of the interval alternate, that is, if  $x_i < (A + B - 1)/2$  for some  $1 \leq i < 2^{k+1} - 1$ , then  $x_{i+1} \geq (A + B - 1)/2$ .

**Proof.** Define  $M := (A+B-1)/2$ . Assume for a contradiction that there is an  $1 \leq i < 2^{k+1} - 1$  such that both  $x_i < M$  and  $x_{i+1} < M$ . First we look at the case that  $|A - 1|_2 < |M|_2$ . Then on one hand,  $x_i \prec x_i + 2^k$ , because  $|x_i - j|_2 = |x_i + 2^k - j|_2$  for all  $0 \leq j \leq x_i - A$  and  $|x_i - j|_2 = |A - 1|_2 < |M|_2 = |x_i + 2^k - j|_2$  for  $j = x_i - (A - 1)$ . (We use the simple observation that if  $|x|_2 < k$  then  $|x + 2^k|_2 = |x|_2$ .) But on the other hand,  $x_i + 2^k \prec x_{i+1}$ , because there is a  $0 \leq j_0 \leq x_i - A$  such that  $|x_i + 2^k - j_0|_2 = |x_i - j_0|_2 < |x_{i+1} - j_0|_2$  and  $|x_i + 2^k - j|_2 = |x_i - j|_2 = |x_{i+1} - j|_2$  for all  $0 \leq j < j_0$ . So  $x_i \prec x_i + 2^k \prec x_{i+1}$ , a contradiction.

In the case  $|A - 1|_2 > |M|_2$ , we can argue similarly that  $x_i \prec x_{i+1} + 2^k \prec x_{i+1}$ . (Note that equality never occurs, as  $|A - 1|_2 = |M|_2$  implies  $|(A - 1 + M)/2|_2 \geq |M|_2 \geq k$ , but  $|(A - 1 + M)/2|_2 < k$  by assumption.)

We have shown that if  $x_i < M$ , then  $x_{i+1} \geq M$ . Similarly we can show that  $x_i > M$  implies  $x_{i+1} \leq M$ . So the elements to the left and to the right of  $M$  alternate.  $\square$

**Proof.** (of Theorem 2) Let  $p, q \in \mathbb{Z}^2$ . We may assume that  $p_x < q_x$  and  $p_y < q_y$ . We are going to prove that  $H(\overline{pq}, S_{\prec}(p, q)) \leq 2c \log(p_x + p_y - q_x - q_y)$  for  $c = \sqrt{5}/2$ . Let  $r \in S_{\prec}(p, q)$  be the point with the property that  $r_x + r_y$  is the greatest element of the segment interval, that is,  $r_x + r_y \succ s$  for all  $s \in [p_x + p_y, q_x + q_y)$ ,  $s \neq r_x + r_y$ , see Figure 4 for an example. Now let  $s'$  be the second greatest element of the segment interval according to  $\prec$ . Define  $k := |s'|_2 + 1$ .

We can extend the segment  $S_{\prec}(p, q)$  over both endpoints, moving both  $p$  and  $q$  such that  $|p_x + p_y - 1|_2 \geq k$  and  $|q_x + q_y|_2 \geq k$ , that is, we extend the segment as far as we can, so that  $k$ , defined as above, remains unchanged. From Lemma 6 we get that by this extension we decreased the Hausdorff distance by at most a factor of 2. Now the segment interval contains exactly  $2^{k+1} - 1$  elements and  $r_x + r_y$  is the element in the very middle. We call such a segment *normalized*.

We are going to proceed by induction on  $k$  to prove that for all normalized digital line segments  $H(\overline{pq}, S_{\prec}(p, q)) \leq ck$  with  $c = \sqrt{5}/2$ . This will prove the theorem, as  $k + 1 = \log(p_x + p_y - q_x - q_y + 1)$ , the distance of the unnormalized original segment (we started from) is at most  $2ck = 2c(\log(p_x + p_y - q_x - q_y + 1) - 1) \leq 2c \log(p_x + p_y - q_x - q_y)$ .

In the base case  $k = 1$ , the segment interval consists of 3 numbers, so  $S_{\prec}(p, q)$  is a path of length 3 and by checking all possibilities we see that  $H(\overline{pq}, S_{\prec}(p, q)) < c$ .

If  $k > 1$ , the idea is to split the segment at  $r$  into two subsegments which are similar in some sense and apply induction. Let  $r' = (r_x, r_y + 1)$  be the point that comes after  $r$  in the segment  $S_{\prec}(p, q)$ . (We know that we go up at  $r$ , because we go up at least once and  $r_x + r_y$  is the greatest element of the segment interval). Consider the subsegments  $S_{\prec}(p, r)$  and  $S_{\prec}(r', q)$  and partition the segment interval accordingly. The key observation is that picking the elements of the interval according to  $\prec$  starting with the greatest, we first get  $r$ , and then alternately



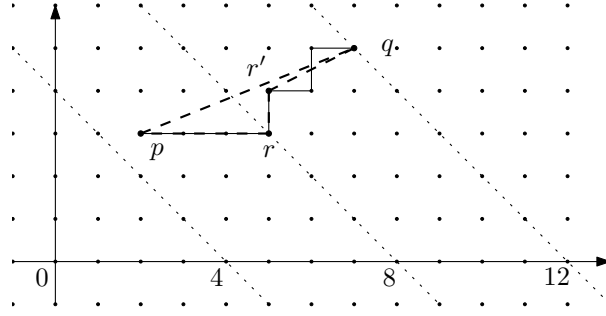


Figure 4: The digital line segment from  $p = (2, 3)$  to  $q = (7, 5)$ , with the ordered segment interval  $8 \succ 10 \succ 6 \succ 9 \succ 5 \succ 11 \succ 7$ . The Euclidean segments  $\overline{pq}$  and  $\overline{pr r' q}$  (dashed) can have Hausdorff distance at most  $c$ , as shown in Lemma 7.

an element of the left and the right subsegment interval. This is shown in Lemma 8 setting  $A = p_x + p_y$ ,  $B = q_x + q_y$ . Therefore, up to a difference of at most one, half of the  $q_y - p_y - 1$  greatest elements (after  $r_x + r_y$ ) belong to  $[p_x + p_y, r_x + r_y)$  and half of them to  $[r_x + r_y + 1, q_x + q_y)$ . This implies that  $p, q, r, r'$  meet the conditions of Lemma 7, leading to  $H(\overline{pq}, \overline{pr r' q}) \leq c$ . By the induction hypothesis we have  $H(\overline{pr}, S_{\prec}(p, r)) \leq c(k - 1)$  and  $H(\overline{r'q}, S_{\prec}(r', q)) \leq c(k - 1)$ . Now  $H(\overline{pr r' q}, S_{\prec}(p, q)) = \max\{H(\overline{pr}, S_{\prec}(p, r)), H(\overline{r'q}, S_{\prec}(r', q))\} \leq c(k - 1)$ . Using the triangle inequality we conclude  $H(\overline{pq}, S_{\prec}(p, q)) \leq ck$ .  $\square$

## 4 A step towards a characterization of CDSes

Now we approach the same problem from a different angle, taking arbitrary CDSes and trying to find some common patterns in their structure. Knowing that condition (C1) holds for all CDSes, it is easy to verify that we can analyze the segments with non-positive and non-negative slopes separately, as they are completely independent. More precisely, the union of any CDS on segments with non-positive slopes and another CDS on segments with non-negative slopes is automatically a CDS. Having this in mind, in this section we will proceed with the analysis of only one half of a CDS, namely of segments with non-negative slope.

We will show that, in a CDS, all the segments with non-negative slope emanating from a fixed point must be derived from a total order. However, as we will show later, these orders may differ for different points.

**Theorem 9** *For any CDS and for any point  $p = (p_x, p_y) \in \mathbb{Z}^2$ , there is a total order  $\prec_p$  that is uniquely defined on both  $(-\infty, p_x + p_y - 1]$  and  $[p_x + p_y, +\infty)$ , such that the segments with non-negative slope emanating from  $p$  are derived from  $\prec_p$  (in the way described in Section 2).*

**Proof.** We fix a CDS  $\mathcal{S}$  and a point  $p$ . The segments with non-negative slope with  $p = (p_x, p_y)$  as their upper-right point will induce an order on the integers smaller than  $p_x + p_y$ , and the segments for which  $p$  is the lower-left endpoint will induce an order on the rest of the integers. In the following we will just look at the latter type of segments. First, we prove an auxiliary statement.

**Lemma 10** *In a CDS, it cannot happen that for two segments with non-negative slope having the same lower-left endpoint  $p$ , one of them goes up at  $(a, C - a)$  and the other goes right at  $(b, C - b)$ , for some  $C$  and  $a > b$ .*

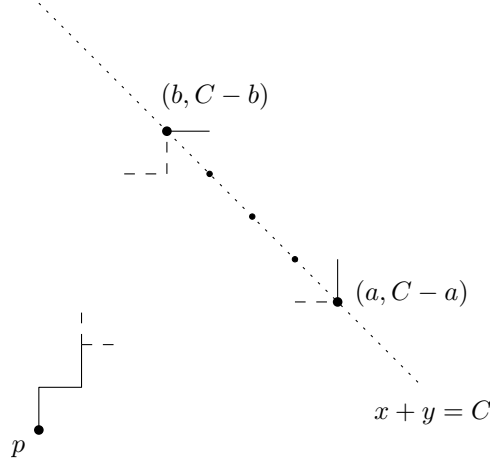


Figure 5: Two segments having  $p$  as their lower-left endpoint, one of them going up at  $(a, C - a)$  and the other going right at  $(b, C - b)$ , with  $a > b$ .

**Proof.** (of Lemma 10) Let us, for a contradiction, assume the opposite, see Figure 5. Now, we look at the  $a - b + 1$  segments between the point  $p$  and each of the points on the line  $x + y = C$  between the points  $(a, C - a)$  and  $(b, C - b)$ . It is possible to extend all of them through their upper-right endpoints, applying (S4). Note that each of the extended segments goes through a different point on the line  $x + y = C$ , and hence, because of condition (C3), no two of them can go through the same point on the line  $x + y = C + 1$ . But, there are only  $a - b$  available points on the line  $x + y = C + 1$  between the points  $(a, C - a + 1)$  and  $(b + 1, C - b)$ , one less than the number of segments, a contradiction.  $\square$

Now, we define the relation  $\prec_p$  in the following way. Whenever there is a segment in  $\mathcal{S}$  with non-negative slope starting at  $p$ , going right at a point  $(x, D - x)$  and going up at a point  $(x', E - x')$ , for some  $x$  and  $x'$ , we set  $D \prec_p E$ . This way we defined a relation on  $[p_x + p_y, +\infty)$ , which is obviously irreflexive. To show that  $\prec_p$  is a total order, it remains to prove that it is asymmetric, transitive and total.

To show asymmetry, assume for a contradiction that for some integers  $D$  and  $E$  we have both  $D \prec_p E$  and  $E \prec_p D$ . That can happen only when there are two segments with non-negative slope having  $p$  as their lower-left endpoint, such that on the line  $x + y = D$  one of them goes up, the other right, and then on  $x + y = E$  they both go in different direction than at  $x + y = D$ . But then the situation described in Lemma 10 must occur on one of the two lines, a contradiction.

Next, if  $C \prec_p D$  and  $D \prec_p E$ , then we also have  $C \prec_p E$  – we just take a segment starting from  $p$  that goes right at  $C$  and up at  $D$ , and (if necessary) extend it until it passes the line  $x + y = E$ . It must also go up at  $E$ , because of  $D \prec_p E$  and the asymmetry of  $\prec_p$ . Hence, the relation  $\prec_p$  is transitive.

It remains to prove the totality of  $\prec_p$ . That is, for any pair of integers  $p_x + p_y \leq D < E$ , either  $D \prec_p E$  or  $E \prec_p D$  holds. Consider a segment from  $p$  to some point  $q$  on the line  $x + y = E$ , such that this segment splits at  $q$ , that is, there are two extensions of the segment, one going up and another one going right. Such segment exists since in the upper-right quadrant of  $p$ , the line  $x + y = E + 1$  contains one more point than the line  $x + y = E$ . If we look at all the segments between  $p$  and the points on  $x + y = E + 1$ , the pigeonhole principle ensures that two of them will contain the same point  $q$  on the line  $x + y = E$ . Now the segment  $S(p, q)$

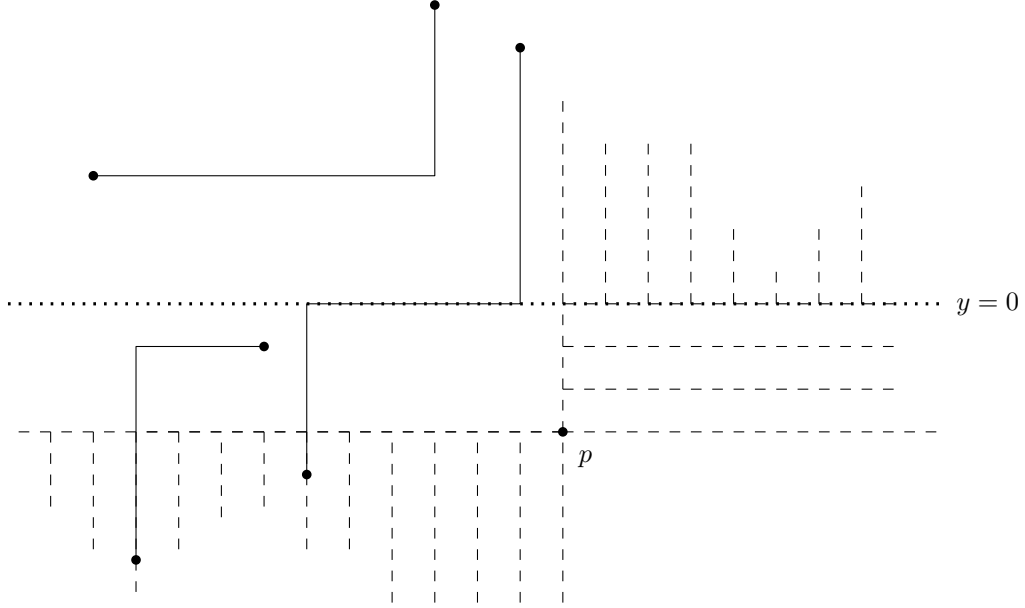


Figure 6: The waterline example: Examples of three characteristic segments, and the rays emanating from a point  $p$ , which is below the waterline

crosses the line  $x + y = D$  at some point  $q'$ . Depending on whether it goes up or right at the point  $q'$ , either  $E \prec_p D$  or  $D \prec_p E$  holds.  $\square$

To see that these orders can differ for different points, consider the following example of a CDS, which we call the *waterline example* because of the special role of the  $x$ -axis. To connect two points with a segment, we do the following. Above the  $x$ -axis we go first right, then up, below the  $x$ -axis we go first up, then right, and when we have to traverse the  $x$ -axis, we go straight up to it, then travel on it to the right, and finally continue up, see Figure 6. It is easy to check that this construction satisfies all five axioms.

Now, if we consider a point  $p = (a, b)$  below the waterline,  $b < 0$ , the induced total order  $\prec_p$  on  $[a + b, +\infty)$  is  $a \prec_p a + 1 \prec_p \dots \prec_p (+\infty) \prec_p a - 1 \prec_p a - 2 \prec_p \dots \prec_p a + b$ , and the order on  $(-\infty, a + b - 1]$  is  $a + b - 1 \prec_p a + b - 2 \prec_p \dots \prec_p -\infty$ . If  $p$  is above the waterline,  $b \geq 0$ , the induced total orders are  $a - 1 \prec a - 2 \prec \dots \prec (-\infty) \prec a \prec a + 1 \prec \dots \prec a + b - 1$  and  $a + b \prec a + b + 1 \prec \dots \prec +\infty$ . Obviously, there is no total order on  $\mathbb{Z}$  compatible with these orders for all possible choices of  $p$ .

The special role played by the  $x$ -axis in the waterline example can be fulfilled by any other monotone digital line with a positive slope; above the line go right, then up, below the line go up, then right, and whenever the line is hit, follow it until either the  $x$ - or the  $y$ -coordinate matches that of the final destination, see Figure 7. Again, it is straightforward to show that this way we obtain a CDS.

A way to see that such a CDS cannot be derived from a total order is to observe that now the diagonal translation of a digital line segment does not always yield another digital line segment, that is, these examples do not satisfy the condition from Observation 4. Actually, we can prove that it suffices to add this condition to force a unifying total order on all integers.

**Theorem 11** *Let  $\mathcal{S}$  be a CDS, such that for any  $t \in \mathbb{Z}$  and any  $p, q \in \mathbb{Z}^2$ ,*

$$S(p + (t, -t), q + (t, -t)) = S(p, q) + (t, -t).$$

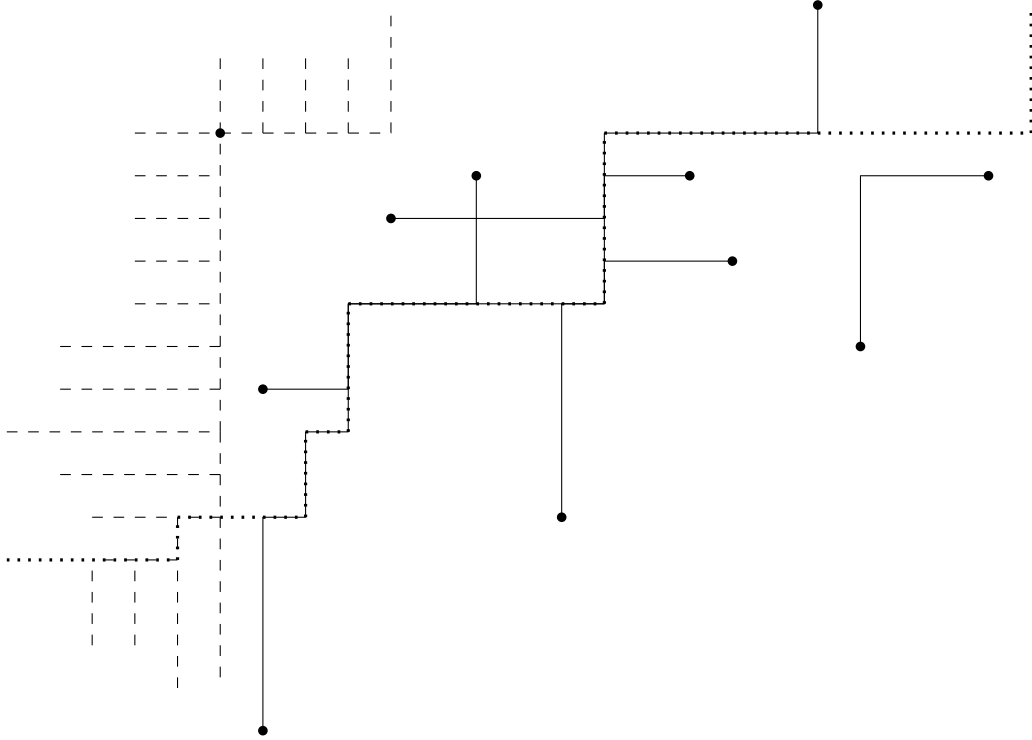


Figure 7: A more exotic example of a CDS with an arbitrary “special” line (bold and dotted).

Then there is a unique total order  $\prec$  on  $\mathbb{Z}$  such that  $\mathcal{S} = \mathcal{S}_\prec$ .

**Proof.** Let  $p, q \in \mathbb{Z}^2$ . By Theorem 9 the segments starting at  $p$  define a unique total order  $\prec_p$  on  $[p_x + p_y, \infty)$  and similarly the segments starting at  $q$  define a unique total order  $\prec_q$  on  $[q_x + q_y, \infty)$ . Let  $q_x + q_y \geq p_x + p_y$ . We want to check whether these two orders agree on  $[q_x + q_y, \infty)$ . Assume for contradiction there are integers  $q_x + q_y \leq A < B$  such that there is a segment  $S$  starting at  $p$  going to some point  $r$  implying  $B \prec_p A$  and another segment  $T$  starting at  $q$  implying  $A \prec_q B$ , see Figure 8 for an example. We translate the segment  $S$  diagonally by a vector  $(t, -t)$  until  $q$  lies on the translated segment  $S'$  from  $p'$  to  $r'$ . Then the subsegment from  $q$  to  $r'$  still goes up at level  $A$  and to the right at level  $B$ , implying  $B \prec_q A$ , a contradiction.

Therefore we can define a unique total order  $\prec$  as follows. To compare two integers  $A$  and  $B$  take any point  $p$  with  $p_x + p_y \leq A$  and define  $A \prec B$  if and only if  $A \prec_p B$ . By the argument above, this definition is independent of the choice of  $p$  and the arguments in the proof of Theorem 9 carry over to  $\prec$ , which shows it is a total order.  $\square$

## 5 Digital lines

Even though our focus is on the digitalization of line segments, the present setup can be conveniently extended to a definition of digital lines. We say that a *digital line* is the vertex set of a path infinite in both directions in the  $\mathbb{Z}^2$  base graph, such that the digital line segment between any two points on the digital line belongs to the digital line.

In this section we restrict our attention to CDSes that are derived from a total order as described in Section 2. Furthermore, for simplicity, we are only going to consider lines with

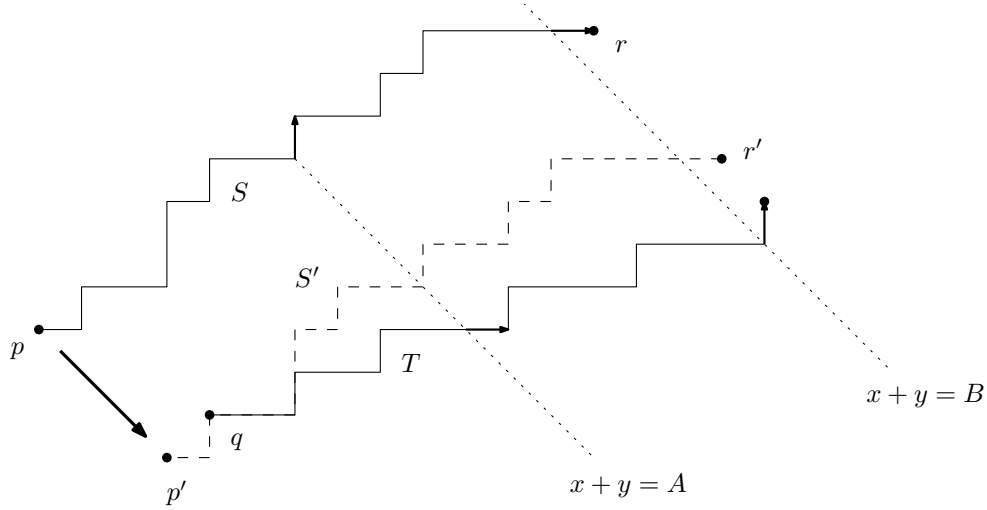


Figure 8: Two line segments leading to a inconsistency in the derived orders. After translating one of them diagonally (dashed) we find a contradiction to the subsegment property.

non-negative slope, meaning that the Euclidean segment between any two points of the line has non-negative slope (including zero and infinity).

Consider a digital line  $\ell$  derived from  $\mathcal{S}_{\prec}$ . We define  $A_{\ell} \subseteq \mathbb{Z}$  to be the set of numbers  $x + y$  for which  $\ell$  goes upward at  $(x, y)$  and call it the *slope* of  $\ell$ . Note that the slope  $A_{\ell}$  is an interval in  $(\mathbb{Z}, \prec)$  which is unbounded in the increasing direction, i.e., if  $x \in A_{\ell}$ , then for any  $y \succ x$  we have  $y \in A_{\ell}$ . This implies that there is a natural total order on the set of possible slopes given by inclusion.

The following observation follows directly from the definition of  $\mathcal{S}_{\prec}$ . Starting at a point  $p$  we can construct a line with slope  $A$ .

**Observation 12** *Every line  $\ell$  can be described by its slope  $A_{\ell}$  and a point  $p$  it contains. Also, given a point  $p$ , every  $\prec$ -interval  $A$  which is unbounded in the increasing direction is a valid slope of a line through  $p$ , that is, there is a line  $\ell$  such that  $p \in \ell$  and  $A_{\ell} = A$ .*

It follows directly from Theorem 3 that two lines having a point in common cannot split and then meet later. Using similar arguments we can see that two lines cannot “touch” without crossing.

**Observation 13** *If two different lines intersect, then they either cross (having a common segment), or they have a common half-line.*

It is straightforward to verify that the following lemma holds.

**Lemma 14** *Consider two different slopes  $A$  and  $B$  with  $A \subset B$ . Let  $I = B \setminus A$  be the difference of the slopes. We distinguish three cases.*

- 1) *If  $I$  is finite, then there are lines  $l$  and  $s$  such that  $A_l = A$ ,  $A_s = B$  and  $l$  and  $s$  intersect in a lower-left halfline and there are lines  $l'$ ,  $s'$  with slopes  $A_{l'} = A$ ,  $A_{s'} = B$  intersecting in an upper-right halfline.*

- 2) If  $I$  is infinite and bounded in one direction in the natural order on  $\mathbb{Z}$ , then we can find lines  $l$  and  $s$  with slope  $A$  and  $B$  intersecting in a lower-left or an upper-right halfline depending on whether  $I$  is lower- or upper-bounded with respect to the natural order.
- 3) If  $I$  is unbounded in both directions, then all lines  $l$  and  $s$  with slope  $A$  and  $B$  do intersect in a finite segment.

We define two lines to be *parallel* if they do not cross, that is, according to Observation 13, two lines are parallel if they are disjoint or if they agree on a halfline. We distinguish two possible cases how the slope  $A_\ell$  of a line  $\ell$  may look like. If there is a  $c \in \mathbb{Z}$  such that  $A_\ell = [c, \infty)_< = \{a \in \mathbb{Z} | a > c\} \cup \{c\}$  or  $A_\ell = (c, \infty)_< = \{a \in \mathbb{Z} | a > c\}$ , that is, if its boundary can be described by the smallest element, either including or excluding this element, then we call the slope  $A_\ell$  *rational*. In the special cases  $A_\ell = \mathbb{Z}$  and  $A_\ell = \emptyset$  we define  $A_\ell$  as rational, too. If no such  $c$  exists, or, in other words, if  $A_\ell$  does not have a smallest element and its complement  $\mathbb{Z} \setminus A_\ell$  does not have a greatest element (and they are not empty), then we call  $A_\ell$  *irrational*.

We proceed by analyzing the digital lines derived from the special total order that we defined in Section 3. With the help of the previous lemma, we can characterize which lines do have unique parallels and which do not.

**Theorem 15** *Let  $<$  be the total order on  $\mathbb{Z}$  defined in Section 3 using the powers of 2,  $\mathcal{S}_<$  the CDS obtained from it, and  $\ell$  a line with respect to  $\mathcal{S}_<$ . Let  $p \in \mathbb{Z}^2$  such that  $p \notin \ell$ .*

- 1) *If the slope  $A_\ell$  is irrational, then there is a unique line  $\ell'$  through  $p$  that is parallel to  $\ell$ . Furthermore,  $A_{\ell'} = A_\ell$ .*
- 2) *If  $A_\ell$  is rational, then  $\ell$  has exactly two parallels  $\ell'$  and  $\ell''$  through  $p$ , one of them with the same slope as  $\ell$ , the other with a slope that differs by one element. Consequently  $\ell'$  and  $\ell''$  either have a common halfline to the left and split at one point to run parallel at distance one thereafter, or the other way round.*

**Proof.** Note that  $<$  is a dense order on  $\mathbb{Z}$ , that is, for any integers  $a < b$ , we find a  $c \in \mathbb{Z}$  such that  $a < c < b$ . Equivalently, this means that every  $<$ -interval  $[a, b)_<$  with  $a \neq b$  is infinite. Let  $s$  be an arbitrary line through  $p$  parallel to  $\ell$ . Consider the symmetric difference of the slopes  $I = A_\ell \triangle A_s$ . If  $I$  is empty, the slopes are equal and  $s$  is the diagonal translate of  $\ell$  going through  $p$ . If  $I \neq \emptyset$ , by density of the order,  $I$  is either infinite or consists only of one element  $a$ . If  $I$  is infinite, then it is unbounded according to the natural order in both directions. (Beside density, this is the only property of  $<$  that we use.) According to Lemma 14,  $s$  and  $\ell$  do intersect in a finite segment in this case, which is a contradiction because  $s$  is parallel to  $\ell$ . Therefore,  $I$  consists of one element. If  $A_\ell$  is irrational, by adding one element to  $A_\ell$  or removing one element from  $A_\ell$  we do not get an interval. Hence, in this case, there is only one possible slope for any line parallel to  $\ell$ , namely,  $A_\ell$ . If  $A_\ell$  is rational, then either  $A_\ell = \{a \in \mathbb{Z} | a > c\} \cup \{c\}$  or  $A_\ell = \{a \in \mathbb{Z} | a > c\}$ , for some  $c \in \mathbb{Z}$ , so removing  $c$  or adding  $c$ , respectively, yields the only different possible slope for a parallel through  $p$ . As the slopes of these two parallels only differ by one element, they either have a common halfline to the left or a common halfline to the right.  $\square$

## 6 Higher dimensions

It is natural to ask whether there is a CDS in more than two dimensions. The definition of a CDS directly carries over to the higher dimensional spaces. Instead of  $\mathbb{Z}^2$  we now consider  $\mathbb{Z}^d$ , for a fixed integer  $d \geq 3$ , with the usual graph structure, that is, two points  $p$  and  $q$  are adjacent if they differ in exactly one coordinate by exactly one. The axioms (S1)-(S4) stay the same, verbatim, and the monotonicity axiom (S5) now reads as follows: If for  $p = (p_1, \dots, p_d), q = (q_1, \dots, q_d) \in \mathbb{Z}^d$  there is an  $i$  such that  $p_i = q_i$ , then for all  $r = (r_1, \dots, r_d) \in S(p, q)$  we have  $r_i = p_i = q_i$ . We define the *slope type* of a pair of points  $(p, q)$  as the sign vector  $(\sigma_1, \dots, \sigma_d) \in \{+1, -1\}^d$ ,  $\sigma_i = +1$  if  $p_i \leq q_i$ , and  $\sigma_i = -1$  if  $p_i > q_i$ . The slope type of a digital segment  $S(p, q)$  is defined as the slope type of  $(p, q)$ . We say a segment has *strictly positive slope*, if its slope type is  $(+1, \dots, +1)$ , that is, the coordinates are monotone increasing in each coordinate.

It is not hard to see that we can again derive a consistent system from an arbitrary total order  $\prec$  on  $\mathbb{Z}$  if we only consider segments that have strictly positive slope, that is, slope type  $(+1, \dots, +1)$ ; the only difference is that now we have to cut the segment interval  $[p_1 + \dots + p_d, q_1 + \dots + q_d - 1]$  into  $d$  parts, according to  $\prec$ . Let  $p, q \in \mathbb{Z}^d$  be two points, such that  $p_i \leq q_i$  for all  $1 \leq i \leq d$ . We can define the segment  $S_{\prec}(p, q)$  in a similar way as in the two dimensional case – starting at  $p$  and repeatedly going in one of the  $d$  possible directions, collecting points from  $\mathbb{Z}^d$ , until reaching  $q$ . If we are at a point  $(r_1, \dots, r_d)$  for which  $r_1 + r_2 + \dots + r_d$  is among the  $q_d - p_d$  greatest elements of the segment interval  $[p_1 + \dots + p_d, q_1 + \dots + q_d - 1]$  according to  $\prec$ , we proceed in direction  $d$ ; if it is among the  $q_{d-1} - p_{d-1}$  greatest elements of the segment interval that remain after removing the  $q_d - p_d$  elements that were the greatest, we proceed in direction  $d - 1$ , and so on. Finally, if  $r_1 + r_2 + \dots + r_d$  it is among the  $q_1 - p_1$  smallest elements of the segment interval, we proceed in the first direction.

**Theorem 16** *The definition above yields a CDS of segments with strictly positive slope.*

**Proof.** The crucial axiom to verify is the subsegment property (S3). Assume there are two segments with strictly positive slope, splitting at some point  $p$  and meeting again for the first time at  $q$ . The two subsegments from  $p$  to  $q$  can be seen as words over the alphabet  $[d] = \{1, 2, \dots, d\}$ , where 1 stands for going in the first direction, 2 for going in the second direction, and so on. So let  $\alpha = a_1 a_2 \dots a_k, \beta = b_1 b_2 \dots b_k \in [d]^k$  be these two words. By assumption they differ at the beginning and at the end and they contain the same number of each of the letters, that is, for any  $l \in [d]$ ,  $|\{i : a_i = l\}| = |\{i : b_i = l\}|$ . Without loss of generality we may assume that  $a_1 > b_1$ . Then there must be an  $1 < i \leq k$  such that  $a_i < a_1$  and  $b_i > b_1$ . (If there were no such  $i$ , then for any  $1 < i \leq k$  with  $a_i < a_1$  we have  $b_i \leq b_1 < a_1$ . Now looking at all letters that are strictly smaller than  $a_1$  in both words, we see that there is at least one such letter more in  $b$ , namely at the first position, a contradiction.) This leads to a contradiction, as translated back to the original setting it implies  $p_1 + \dots + p_d \succ p_1 + \dots + p_d + i - 1$ , if we look at the interval of the first segment, and at the same time  $p_1 + \dots + p_d \prec p_1 + \dots + p_d + i - 1$ , if we look at the interval of the second segment. This proves (S3).

The rest of the axioms can be verified similarly as in the 2-dimensional case.  $\square$

Of course, we can use the same construction to define segments for all the remaining slope types. However, unlike in the 2-dimensional case, putting them all together in an attempt to construct a complete CDS fails, as the axiom (S3) is violated. We are curious if this approach to construction can be modified to yield a CDS.

## Acknowledgments

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