

Quantum Pushdown Automata with a Garbage Tape

Masaki Nakanishi

Faculty of Education, Art and Science, Yamagata University,
Yamagata, 990-8560, Japan
masaki@cs.e.yamagata-u.ac.jp

Abstract. Several kinds of quantum pushdown automaton models have been proposed, and their computational power is investigated intensively. However, for some quantum pushdown automaton models, it is not known whether quantum models are at least as powerful as classical counterparts or not. This is due to the reversibility restriction. In this paper, we introduce a new quantum pushdown automaton model that has a garbage tape. This model can overcome the reversibility restriction by exploiting the garbage tape to store popped symbols. We show that the proposed model can simulate any quantum pushdown automaton with a classical stack as well as any probabilistic pushdown automaton. We also show that our model can solve a certain promise problem exactly while deterministic pushdown automata cannot. These results imply that our model is strictly more powerful than classical counterparts in the setting of exact, one-sided error and non-deterministic computation.

Keywords: quantum pushdown automata, deterministic pushdown automata, quantum computation models

1 Introduction

One important question in quantum computing is whether a computational gap exists between models that are allowed to use quantum effects and models that are not. Several types of quantum computation models have been proposed, including quantum finite automata, quantum counter automata, and quantum pushdown automata. Quantum finite automata are the simplest model of quantum computation, and have been investigated intensively [3,4,5,7,8,13,14,15,17,21,22,23,25,26,27]. Several quantum automata augmented with additional computational resources have also been proposed, including quantum counter automata and quantum pushdown automata [6,12,16,17,18,19,22,28,29].

It might be a surprising result that some of simple quantum computation models can be less powerful than classical counterparts [15,28,29] due to the reversibility restriction. Thus, it is a natural question what kinds of restrictions make quantum models less powerful than classical counterparts, and what kinds of computational resources make quantum models more powerful. Motivated by those questions, quantum pushdown automata have been investigated. Quantum pushdown automata were first proposed in [17], but their model is the

generalized quantum pushdown automata whose evolution does not have to be unitary. Then Golovkins proposed quantum pushdown automata including unitarity criteria[12], and he showed that quantum pushdown automata can recognize every regular language and some non-context-free languages. However, it is still open whether Golovkins's model of quantum pushdown automata are more powerful than probabilistic pushdown automata or not. In [18], it is shown that a certain promise problem can be solved exactly by Golovkins's model of quantum pushdown automata while it cannot be solved by deterministic pushdown automata. However, it is not known whether Golovkins's model can simulate any deterministic pushdown automaton or not. This is because quantum computation models must be reversible while pop operation deletes the stack-top symbol, which is not a reversible operation. In [19], a quantum pushdown automaton model that has a classical stack is proposed, and it is shown that the model is strictly more powerful than classical counterparts in the setting of one-sided error as well as non-deterministic computation.

The above mentioned results are for the models whose state transitions are described by unitary operators. It is known that by allowing more general operators such as trace preserving completely positive (TPCP) maps, quantum finite automata can simulate classical counterparts as well as several quantum finite automata mentioned above[13,14]. These results were generalized and it was shown how to define general quantum operators for other models in [27]. For counter automata and pushdown automata, it is also known that generalized quantum models (i.e., the models that can use TPCP maps) can simulate classical counterparts[22,23].

In this paper, we focus on the restricted quantum computation models (i.e., the models whose state transitions are described by unitary operators) rather than the general models (i.e., the models whose state transitions are described by TPCP maps). As mentioned above, it is known that the generalized quantum computation models can simulate classical counterparts and sometimes can be strictly more powerful than classical counterparts. Nevertheless, studying restricted models is important. That is, our goal is to investigate what kinds of restriction makes quantum models less powerful and under what kinds of restrictions quantum models are still more powerful than classical counterparts. This could lead to understand the source of the power of quantum computation in architecturally restricted models such as quantum automata.

Motivated by these discussions, we introduce a new model of quantum pushdown automata, called quantum pushdown automata with a garbage tape. This model has a garbage tape on which popped symbols are stored, and thus, we can pop the stack-top symbol preserving reversibility. The garbage tape is a write-only memory, and thus, classical pushdown automata cannot exploit it. A quantum computation model that has a write-only memory was proposed in [24]. The model uses a write-only memory in order to control interference between distinct computation paths. In our model, the write-only garbage tape is restricted to store popped symbols. Also the similar notion of garbage tapes were proposed in [8,21]. In those models, a garbage tape is used to make transi-

tions reversible. Our model is constructed so as to take advantages of both of a write-only tape and a garbage tape.

Another motivation is that it is expected that investigating quantum pushdown automata reveals how last-in first-out manner of memory access affects (or limits) quantum computation. However, for this purpose, Golovkins's model[12] is too restrictive on pop operation, i.e., we can pop a stack-top symbol only if we can delete stack-top symbol preserving reversibility. Thus, we cannot identify from which the impossibilities come from, reversibility or last-in first-out manner of memory access. In contrast, our model is useful for this purpose since pop operations can always be executed preserving reversibility.

In this paper, we show that the proposed model can simulate any quantum pushdown automaton with a classical stack, which is proposed in [19], as well as any classical pushdown automaton. It is known that quantum pushdown automata with a classical stack are strictly more powerful than classical counterparts in the setting of one-sided error and non-deterministic computation[19]. Thus, so is our model. We also show that our model can solve a certain promise problem (Problem I) exactly while deterministic pushdown automata cannot. This implies that our model is strictly more powerful than classical counterparts also in the setting of exact computation. It is a common technique to apply the pumping lemma (or Ogden's lemma[20], which is a generalization of the pumping lemma) in order to show that a language is not context-free, i.e., pushdown automata cannot recognize the language. However, our problem is a promise problem. Thus, we cannot apply the pumping lemma to our case.¹ In [2], the pumping lemma is proved through the analysis of pushdown automata. We modify their notion of *full state*, and use it to show the impossibility by directly analyzing time evolution of pushdown automata. This is a new technique to prove that a certain promise problem cannot be computed by pushdown automata. For OBDD models, an impossibility proof for a certain partial function, which is a function counterpart of promise problems, was shown recently in [1].

This paper is organized as follows: In Sect. 2, we define quantum pushdown automata with a garbage tape. In Sect. 3, we show how to simulate classical pushdown automata and quantum pushdown automata with a classical stack by quantum pushdown automata with a garbage tape. In Sect. 4, we show there is a promise problem that quantum pushdown automata with a garbage tape can solve exactly while deterministic pushdown automata cannot. In Sect. 5, we discuss comparison between quantum pushdown automata with and without a garbage tape.

2 Preliminaries

A quantum pushdown automaton with a garbage tape (QPAG) has an input tape, a stack and a garbage tape. A QPAG also has a finite state control. The

¹ As far as the author knows, [18] is the only exception in which the pumping lemma is used for a promise problem. The technique in [18] can be applied only to the limited cases.

input tape contains a classical input string, and its tape head is implemented by qubits that represent the position of the tape head. The stack and the garbage tape are implemented by qubits that represent contents of the stack and the garbage tape, respectively. The finite state control is also implemented by qubits that represents the current state. A QPAG reads the stack top symbol and the input symbol pointed by the input tape-head, and then evolves as follows: The tape head can move to the right or stay at the same position, the finite state control moves to the next state, and a stack symbol is pushed to the stack or popped from the stack. When we pop a symbol from the stack, the popped symbol is written on the garbage tape, moving the garbage tape head to the right. This allows a QPAG to pop the stack top symbol preserving reversibility. We define QPAGs formally as follows.

Definition 1. *A quantum pushdown automaton with a garbage tape (QPAG) is defined as the following 7-tuple: $M = (Q, \Sigma, \Gamma, \delta, q_0, Q_{acc}, Q_{rej})$, where Q is a set of states, Σ is a set of input symbols including the left and the right endmarkers $\{\$, \#\}$, respectively, Γ is a set of stack symbols including the bottom symbol Z , δ is a quantum state transition function ($\delta : (Q \times \Sigma \times \Gamma \times Q \times G \cup \{\varepsilon, pop\}) \times \{0, 1\} \rightarrow \mathbb{C}$), where $G (\subseteq (\Gamma \setminus \{Z\})^+)$ is a finite set and $(\Gamma \setminus \{Z\})^+$ is the set of all nonempty strings of finite length from alphabet $\Gamma \setminus \{Z\}$, q_0 is the initial state, $Q_{acc} (\subseteq Q)$ is the set of accepting states, and $Q_{rej} (\subseteq Q)$ is the set of rejecting states, where $Q_{acc} \cap Q_{rej} = \emptyset$. \square*

$\delta(q, a, b, q', b', D) = \alpha$ means that the amplitude of the transition from q to q' updating the input tape head to D ($D = 1$ means ‘right’ and $D = 0$ means ‘stay’) and pushing b' to the stack (or popping the stack-top symbol if $b' = pop$) is α when reading input symbol a and stack symbol b . A configuration of a QPAG is (q, k, w_s, w_g) , where $q \in Q$ is the current state of the finite state control, k is the position of the input tape head, and w_s and w_g are the strings on the stack and the garbage tape, respectively. We store a configuration of a QPAG in a quantum register, where a basis state is described as $|q, k, w_s, w_g\rangle$. For input string \mathbf{x} , we define the time evolution operator $U^{\mathbf{x}}$ as follows:

$$U^{\mathbf{x}}(|q, k, w_s, w_g\rangle) = \sum_{q' \in Q, b' \in G \cup \{\varepsilon, pop\}, D \in \{0, 1\}} \delta(q, x(k), b, q', b', D) |q', k + D, w'_s, w'_g\rangle,$$

where $x(k)$ is the k -th input symbol of input \mathbf{x} ,

$$w'_s = \begin{cases} w_s b b' & \text{if } b' \neq pop \\ w_s & \text{if } b' = pop \end{cases}$$

and $w'_g = \begin{cases} w_g & \text{if } b' \neq pop \\ w_g b & \text{if } b' = pop \end{cases}$ (b is the popped stack-top symbol).

If $U^{\mathbf{x}}$ is unitary (for any input string \mathbf{x}), then the corresponding QPAG is well-formed. A well-formed QPAG is considered valid in terms of the quantum theory. We consider only well-formed QPAGs. Let the initial quantum state and

the initial position of the input tape head be q_0 and ‘0’, respectively. We define $|\psi_0\rangle$ as $|\psi_0\rangle = |q_0, 0, Z, \varepsilon\rangle$. We also define E_{non} , E_{acc} and E_{rej} as follows:

$$\begin{aligned} E_{non} &= \text{span}\{|q, k, w_s, w_g\rangle \mid q \notin Q_{acc} \text{ and } q \notin Q_{rej}\}, \\ E_{acc} &= \text{span}\{|q, k, w_s, w_g\rangle \mid q \in Q_{acc}\}, \quad E_{rej} = \text{span}\{|q, k, w_s, w_g\rangle \mid q \in Q_{rej}\}. \end{aligned}$$

We define observable \mathcal{O} as $\mathcal{O} = E_{non} \oplus E_{acc} \oplus E_{rej}$. For notational simplicity, we define the outcome of a measurement corresponding to E_j as j for $j \in \{non, acc, rej\}$. A QPAG computation proceeds as follows:

- (a) $U^{\mathbf{x}}$ is applied to $|\psi_i\rangle$, and we obtain $|\psi_{i+1}\rangle = U^{\mathbf{x}}|\psi_i\rangle$.
- (b) $|\psi_{i+1}\rangle$ is measured with respect to \mathcal{O} . Let $|\phi_j\rangle$ be the projection of $|\psi_{i+1}\rangle$ to E_j . Then each outcome j is obtained with probability $|\langle \phi_j | \psi_{i+1} \rangle|^2$. Note that this measurement causes $|\psi_{i+1}\rangle$ to collapse to $\frac{1}{\|\phi_j\|} |\phi_j\rangle$, where j is the obtained outcome.
- (c) If the outcome of the measurement is *acc* or *rej*, the automaton outputs the measurement result and halts. Otherwise, go to (a).

To check well-formedness, we show the following theorem.

Theorem 1. *A QPAG M is well-formed if the quantum state transition function of M satisfies the following conditions.*

1. $\forall (q, a, b) \in Q \times \Sigma \times \Gamma,$

$$\sum_{q' \in Q, b' \in G \cup \{\varepsilon, pop\}, D \in \{0,1\}} |\delta(q, a, b, q', b', D)|^2 = 1,$$

2. $\forall (q_1, a, b) \neq (q_2, a, b) \in Q \times \Sigma \times \Gamma,$

$$\sum_{q' \in Q, b' \in G \cup \{\varepsilon, pop\}, D \in \{0,1\}} \delta^*(q_1, a, b, q', b', D) \delta(q_2, a, b, q', b', D) = 0,$$

3. (a) $\forall (q_1, a, b_1) \neq (q_2, a, b_2) \in Q \times \Sigma \times \Gamma, \forall b_3 \in G \cup \{\varepsilon\},$

$$\sum_{q' \in Q, D \in \{0,1\}} \delta^*(q_1, a, b_1, q', \varepsilon, D) \delta(q_2, a, b_2, q', b_3 b_1, D) = 0,$$

- (b) $\forall (q_1, a, b_1) \neq (q_2, a, b_2) \in Q \times \Sigma \times \Gamma, \forall b_3 \in G \cup \{\varepsilon\},$

$$\sum_{q' \in Q, D \in \{0,1\}} \delta^*(q_1, a, b_1, q', pop, D) \delta(q_2, a, b_2, q', b_3, D) = 0,$$

4. $\forall (q_1, a_1, b) \neq (q_2, a_2, b) \in Q \times \Sigma \times \Gamma,$

$$\sum_{q' \in Q, b \in G \cup \{\varepsilon, pop\}} \delta^*(q_1, a_1, b, q', b', 0) \delta(q_2, a_2, b, q', b', 1) = 0,$$

5. (a) $\forall (q_1, a_1, b_1) \neq (q_2, a_2, b_2) \in Q \times \Sigma \times \Gamma, \forall D_1 \neq D_2 \in \{0,1\}, \forall b_3 \in G \cup \{\varepsilon\},$

$$\sum_{q' \in Q} \delta^*(q_1, a_1, b_1, q', \varepsilon, D_1) \delta(q_2, a_2, b_2, q', b_3 b_1, D_2) = 0,$$

- (b) $\forall (q_1, a_1, b_1) \neq (q_2, a_2, b_2) \in Q \times \Sigma \times \Gamma, \forall D_1 \neq D_2 \in \{0,1\}, \forall b_3 \in G \cup \{\varepsilon\},$

$$\sum_{q' \in Q} \delta^*(q_1, a_1, b_1, q', pop, D_1) \delta(q_2, a_2, b_2, q', b_3, D_2) = 0,$$

Proof. The matrix $U^{\mathbf{x}}$ is unitary if and only if the columns of $U^{\mathbf{x}}$ are orthonormal. The condition (1) implies that each column of $U^{\mathbf{x}}$ is normalized. The rest of the conditions implies any two distinct columns are orthogonal. The condition (2) is for the columns corresponding to the configurations in which only the state is different. The conditions (3-a) and (3-b) are for the columns corresponding to the configurations in which the position of the tape-head is the same but the stack contents are different. The condition (4) is for the columns corresponding to the configurations in which the position of the tape-head is different but the stack contents are the same. The conditions (5-a) and (5-b) are for the columns corresponding to the configurations in which the position of the tape head is different and the stack contents are also different. Note that in the case of (3-b) and (5-b), the contents of the garbage tape are different between the two configurations; one is shorter than the other by one symbol. Also note that in the case of the rest, the contents of the garbage tape are the same between the two configurations. \square

3 Simulation of QCPDAs

In this section, we show that a QPAG can simulate a quantum pushdown automaton with a classical stack (QCPDA). Since QCPDAs can simulate any probabilistic pushdown automata [19], QPAGs can simulate any probabilistic pushdown automata as well. We describe the definitions of probabilistic pushdown automata and QCPDAs in Appendices A and B, respectively, or readers may refer to [19]. A quantum pushdown automaton with a classical stack (QCPDA) is a quantum pushdown automaton whose classical stack operations are determined by measurement results. We can use the garbage tape so that if we measure the garbage tape, the stack contents will be identical among all the basis states contained in the resulting superposition. Therefore, we can simulate a QCPDA by a QPAG.

Theorem 2. *Let $M_{qc} = (Q, \Sigma, \Gamma, \delta, q_0, \sigma, Q_{acc}, Q_{rej})$ be a QCPDA. Then, there exists a QPAG M_q such that for any input, the acceptance probability of M_q is the same as that of M_{qc} .*

Proof. For a transition of M_{qc} from state q to q' moving the input tape head to D , we construct the corresponding transitions of M_q , which consist of three successive transitions, as follows: Note that the stack operation of M_{qc} is determined solely by the state q' to which it transits, denoted by $\sigma(q')$. We add two new states q_a and q_b to Q and also add $\sigma(q')$ to Γ . Then, we replace the original transition with the transition from q to q_a such that the stack operation is the same as the original transition ($\sigma(q')$), the direction of the tape head is D and the transition probability is also the same. We define the transition from q_a to q_b , whose probability is one, as a transition pushing the label $\sigma(q')$ to the stack, the input tape head staying at the same position. We also define the transition from q_b to q' , whose probability is one, as a transition popping $\sigma(q')$ from the stack and moving it to the garbage tape, the input tape head staying at the

same position. This records the history of stack operations in the garbage tape. Thus, if the history of stack operations are different between two computation paths, they do not interfere with each other since the contents of the garbage tape are different. This means that if we measure the garbage tape, the contents of the stack are identical between any basis states contained in the resulting superposition at any moment of computation. In other words, if we trace out the garbage tape, then, the stack configuration is not in a superposition but in a classical mixture of basis states. Thus, it can be regarded as a classical stack, and the resulting QPAG M_q simulates the original QCPDA M_{qc} . \square

It is known that QCPDAs can recognize a certain non-context-free language with one-sided error[19]. This means that QPAGs are strictly more powerful than classical pushdown automata in the setting of one-sided error as well as non-deterministic computation.

Corollary 1. *The class of languages recognized by one-sided error QPAGs properly includes the class of languages recognized by one-sided error probabilistic pushdown automaton as well as by non-deterministic pushdown automaton.* \square

4 Possibility and Impossibility of Solving a Certain Promise Problem

We say that two strings, u and v , have even (resp. odd) distinctions, denoted by $u \overset{e}{\sim} v$ (resp. $u \overset{o}{\sim} v$), if $|u| = |v|$ and u and v are different at even (resp. odd) number of positions. For example, $1100 \overset{e}{\sim} 1111$ since the third and the fourth bits are different between the two strings, and $1000 \overset{o}{\sim} 1111$ since the second, the third and the fourth bits are different between the two strings. We define a promise problem, Problem I, as follows:

Problem I *The input is promised to be of the form $w_1\#w_2\#w_3$, where $w_1, w_2 \in \{a, b, c\}^n$ and $w_3 \in \{a, b, c, d\}^n$.*

Yes-instances *are formed by the strings $w_1\#w_2\#w_3$ such that*

$$((w_1 \overset{e}{\sim} w_2^R) \text{ xor } (w_1 \overset{e}{\sim} w_3^R)) = 1.$$

No-instances *are formed by the strings $w_1\#w_2\#w_3$ such that*

$$((w_1 \overset{e}{\sim} w_2^R) \text{ xor } (w_1 \overset{e}{\sim} w_3^R)) = 0.$$

\square

We show that QPAGs can solve Problem I exactly while deterministic pushdown automata cannot solve it. This result combined with Theorem 2 implies that QPAGs are strictly more powerful than classical pushdown automata in the setting of exact computation.

Theorem 3. *There exists a QPAG that solves Problem I exactly.*

Proof. We use the same technique as in Theorem 3.1 of [18]. We construct a QPAG, M , that solves Problem I as follows: We consider two sub-automata, M_1 and M_2 , such that M_1 (resp. M_2) computes whether $w_1 \stackrel{e}{\sim} w_2^R$ (resp. $w_1 \stackrel{e}{\sim} w_3^R$), and run them in a superposition. It is straightforward to see that M_1 and M_2 can be implemented by reversible deterministic pushdown automata with a garbage tape, which is a special case of QPAGs, and we can construct M_1 and M_2 so that the contents of the garbage tape at the moment of reading the right-endmarker can be the same between the two sub-automata. Then, we utilize the algorithm in [9] (the improved Deutsch-Jozsa algorithm[11]) to compute the exclusive-or exactly using the two sub-automata as the oracle for Deutsch's problem[10]. We show the transition function of M in Appendix C. \square

The reason why we can use the same technique as in Theorem 3.1 of [18] even though our model and the model used in [18] seems incomparable is the following. When the stack-top symbol is popped, it is always written in the garbage tape in our model. This makes an entanglement between the stack contents and the garbage tape. Sometimes, this can be an unwanted behavior and make our model weaker than the model in [18]. However, in our algorithm shown in the proof of Theorem 3, the contents of the garbage tape at the moment of reading the right-endmarker can be the same between the two sub-automata. Therefore, the stack contents and the garbage tape are separable at the moment of reading the right-endmarker, which causes no problem when using the same technique in Theorem 3.1 of [18].

In the following, we show that no deterministic pushdown automata can solve Problem I.

Theorem 4. *No deterministic pushdown automata can solve Problem I.* \square

We introduce several lemmas in order to prove Theorem 4. We divide w_1 into two segments $w_1 = w_{1L}w_{1R}$. Similarly, we divide w_2 and w_3 as $w_2 = w_{2L}w_{2R}$ and $w_3 = w_{3L}w_{3R}$, respectively. In the following discussion, we assume that there exists a deterministic pushdown automaton that solves Problem I. Let $h_{max}(k)$ be the maximum height of the stack over all w_1 's at the moment of reading the k -th symbol of w_1 . Note that stack height can increase at most $O(1)$ when reading each symbol². Then, it is obvious that there is a constant, c , for which the following holds:

$$h_{max}\left(\frac{n}{c}\right) < \log_{|\Gamma|} \left(3^{\frac{c-1}{c}n} / (\#states \cdot n(n+1)) \right),$$

where $\#states$ denotes the number of states of the finite state control, and $n = |w_1|$. We fix such a constant c , and also fix the length of w_{1L} to be n/c .

We say that pushdown automaton M is in a *state-configuration* of (q, a) if M is in the state q and the stack-top symbol is a . In other words, a state-configuration is a configuration of a pushdown automaton ignoring the position of the tape head and the stack contents except for the stack-top. The

² Note that, on the other hand, stack height may decrease more than $\omega(1)$ when reading each input symbol.

notion of a state-configuration is a modification of the notion of *full state* in [2]. Note that the tape head can be stationary at a transition. Thus, the stack height can increase (or decrease) multiple times with multiple transitions during reading one symbol. Let $h_b(i)$ and $c_b(i)$ denote the stack height and the state-configuration, respectively, immediately before reading the i -th symbol of the input. Also let $h_a(i)$ denote the set of stack heights that consists of the stack height after reading the i -th symbol and the stack heights during reading the $(i + 1)$ -th symbol with the tape head being stationary on the $(i + 1)$ -th symbol. For each $h \in h_a(i)$, let $c_a(i, h)$ be the corresponding state-configuration. We define the notations “ $h_b(i) > h_a(j)$ ” and “ $h_b(i) - h_a(j)$ ” as follows: $h_b(i) > h_a(j)$ iff $h_b(i) > \min_{h' \in h_a(j)} h'$. $h_b(i) - h_a(j) = h_b(i) - \min_{h' \in h_a(j)} h'$. A *zero-stack pair* is a pair (l, r) ($1 \leq l < r \leq n$) such that $h_b(l) \in h_a(r)$ and $h_b(l) \not\in h_a(t)$ for any t ($l < t < r$). Then, we have the following lemma.

Lemma 1. *We fix w_1 arbitrarily. Let (i, j) be a zero-stack pair such that the maximum of $h_b(k) - h_a(l)$ for $i < k < l < j$ is $\omega(1)$. Then, for any zero-stack pair (i', j') ($1 \leq i' < j' < i$ or $j < i' < j' \leq n$), the maximum of $h_b(k) - h_a(l)$ for $i' < k < l < j'$ is $O(1)$.*

Proof. We consider a zero-stack pair (i, j) ($1 \leq i < j \leq n$) such that the maximum of $h_b(k) - h_a(l)$ for $i < k < l < j$ is $\omega(1)$. Let the maximum (resp. minimum) height of the stack during processing from the i -th symbol to the j -th symbol be h_{max} (resp. h_{min}). Note that $h_{max} - h_{min} > \omega(1)$. For each $h \in \{h_{min}, \dots, h_{max}\}$, let ZS_h be the set of zero-stack pairs such that $ZS_h = \{(l, r) | h_b(l) = h, i \leq l < r \leq j\}$. Note that for at least a constant fraction of $\{h_{min}, \dots, h_{max}\}$, ZS_h is nonempty. For each $h \in \{h_{min}, \dots, h_{max}\}$, we choose at most one $(l_h, r_h) \in ZS_h$ such that $l_h < l_{h+1}$ and $r_{h+1} \leq r_h$. It is obvious that we can have such (l_h, r_h) 's for at least a constant fraction of $\{h_{min}, \dots, h_{max}\}$. Let $(c_a(k, h), t)$ be a pair of a state-configuration and an input symbol where $t \in \Sigma$ is the input symbol pointed by the tape head at the moment when the automaton is in the state-configuration $c_a(k, h)$ with the k and h . Then, there exists two distinct zero-stack pairs (l_{h_1}, r_{h_1}) and (l_{h_2}, r_{h_2}) ($h_1 < h_2$) such that $c_b(l_{h_1}) = c_b(l_{h_2})$ and $(c_a(r_{h_1}, h_1), t) = (c_a(r_{h_2}, h_2), t)$ for some t since $|\Sigma|$ and the number of possible state-configurations are both $O(1)$ while we have $\omega(1)$ pairs of (l_h, r_h) 's. We divide w_1 as $w_1 = uvxyz$ where $u = w_1(1) \cdots w_1(l_{h_1} - 1)$, $v = w_1(l_{h_1}) \cdots w_1(l_{h_2} - 1)$, $x = w_1(l_{h_2}) \cdots w_1(r_{h_2})$, $y = w_1(r_{h_2} + 1) \cdots w_1(r_{h_1})$, and $z = w_1(r_{h_1} + 1) \cdots w_1(n)$, where $w_1(i)$ denotes the i -th symbol of w_1 . Then, for any $i \geq 0$, the configuration after reading $uv^i xy^i z$ and the configuration after reading $uvxyz$ are the same, including the contents of the stack.

We assume that there exists two zero-stack pairs (i_1, j_1) and (i_2, j_2) ($1 \leq i_1 < j_1 < i_2 < j_2 \leq n$) such that the maximum of $h_b(k) - h_a(l)$ for $i_1 < k < l < j_1$ and the maximum for $i_2 < k < l < j_2$ are both $\omega(1)$. Then, we can divide w_1 in two ways: $w_1 = u_k v_k x_k y_k z_k$ with (i_k, j_k) ($k \in \{1, 2\}$). It is obvious that there exist p and q such that $|u_1 v_1^p x_1 y_1^p z_1| = |u_2 v_2^q x_2 y_2^q z_2|$. Thus, there exist two inputs $u_1 v_1^p x_1 y_1^p z_1$ and $u_2 v_2^q x_2 y_2^q z_2$ for which the configurations after reading the two inputs are the same, including the contents of the stack. This implies that for

any completion of the inputs, both of $u_1 v_1^p x_1 y_1^p z_1$ and $u_2 v_2^q x_2 y_2^q z_2$ leads to the same answer, which is a contradiction. \square

Let w_{pref} be a string for which there is a zero-stack pair (i, j) and the maximum of $h_b(k) - h_a(l)$ for $i < k < l < j$ is $\omega(1)$ where $|w_{pref}| = c|w_{1L}|$ for some constant c ($0 < c < 1$). If there is no such zero-stack pair for any long enough w_{pref} , we define w_{pref} to be an empty string. We fix such a w_{pref} . We define $a^+ = b, b^+ = c, c^+ = a$. For two strings $u, v \in \{a, b, c\}^n$, we say $u \leq v$ iff $[(u(k) = x) \rightarrow (v(k) = x \text{ or } v(k) = x^+)]$, where $x \in \{a, b, c\}$ and $u(k)$ (resp. $v(k)$) represents the k -th symbol of u (resp. v). Let WL_{all} be the set of w_{1L} 's such that $WL_{all} = \{w_{pref} a^{|w_{1L}| - |w_{pref}| - k} b^k \mid 0 \leq k \leq |w_{1L}| - |w_{pref}| - 1\}$ ($= \{w_{pref} aaaa \dots aaaa, w_{pref} aaaa \dots aab, w_{pref} aaaa \dots abb, w_{pref} aaaa \dots bbb, \dots, w_{pref} abb \dots bbb\}$). Note that for any two distinct $u, v \in WL$, $u \leq v$ or $v \leq u$. Then, we have the following lemma.

Lemma 2. *There exists $WL \subseteq WL_{all}$ satisfying the following conditions: (1) Any $w \in WL$ leads to the same state-configuration, say C_{WL} . (2) Given a constant c , after reading w , the stack contents between the top and the c -th from the top are the same among all $w \in WL$. (3) $|WL| = \Theta(n)$.* \square

Proof. There exists a constant fraction of WL_{all} , which is WL , satisfying the first and the second conditions of the lemma since the number of possible state-configurations is a constant and the number of possible stack contents between the top and the c -th from the top is also a constant. It is obvious that $|WL| = \Theta(n)$ since $|WL_{all}| = |w_{1L}| - |w_{pref}| = \Theta(n)$. \square

We consider the case that the following Condition I holds:

Condition I *There exists $w_{1L} \in WL$ and w_{2L} such that for at least $1/(n+1)$ fraction of $\{w_{1R}\}$, stack height is less than $\log_{|\Gamma|}(3^{n-|w_{1L}|}/(\#states \cdot n(n+1)))$ at the moment of reading the last symbol of $w_{1L}w_{1R}w_{2L}$.* \square

In this case, at the moment when stack height is less than $\log_{|\Gamma|}(3^{n-|w_{1L}|}/(\#states \cdot n(n+1)))$, the number of possible configurations (including stack contents and the position of the input tape head) is less than $\frac{1}{n+1}3^{n-|w_{1L}|}$, which means there exist at least two distinct partial inputs $w_{1L}w_{1R}w_{2L}$ and $w_{1L}w'_{1R}w_{2L}$ that result in the same configuration (including stack contents and the position of an input tape head) since $|\{w_{1R}\}| = 3^{n-|w_{1L}|}$. Thus both of $w_{1L}w_{1R}w_{2L}$ and $w_{1L}w'_{1R}w_{2L}$ lead to the same answer for any completion of the rest of the input. This is a contradiction. Thus, we can say the negation of Condition I holds. In this case, given w_2 , at every step of processing w_2 , for at most $1/(n+1)$ fraction of $\{w_{1R}\}$, stack height becomes less than $\log_{|\Gamma|}(3^{n-|w_{1L}|}/(\#states \cdot n(n+1)))$. Thus, for at most $n/(n+1)$ fraction of $\{w_{1R}\}$, stack height becomes less than $\log_{|\Gamma|}(3^{n-|w_{1L}|}/(\#states \cdot n(n+1)))$ while processing w_2 ; for at least $1/(n+1)$ fraction of $\{w_{1R}\}$, stack height is always more than or equal to $\log_{|\Gamma|}(3^{n-|w_{1L}|}/(\#states \cdot n(n+1)))$ while processing w_2 . We consider the case that the following Condition II holds:

Condition II For any $w_{1L} \in WL$ and w_2 , at least $1/(n+1)$ fraction of $\{w_{1R}\}$, stack height is always greater than or equal to $\log_{|T|}(3^{n-|w_{1L}|}/(\#states \cdot n(n+1)))$ while processing w_2 . \square

We define w_{3L} as the prefix of w_3 such that stack height is always higher than $\hat{h} - O(1)$ ($= \hat{h}'$) during reading $w_{1R}w_2w_{3L}$ and it becomes \hat{h}' when reading the last symbol of w_{3L} , where \hat{h} denotes the stack height after reading the last symbol of w_{1L} . If stack height is always higher than \hat{h}' during reading w_3 , we define $w_{3L} = w_3$.

Lemma 3. We assume that there exists a deterministic pushdown automaton that solves Problem I. Then, there exist w_{1R} , k ($1 \leq k \leq n$) and a set W_2 of w_2 's such that, starting from C_{WL} , $w_{1R}w_2w_{3L}$ leads to the same state-configuration for all $w_2 \in W_2$ where $w_{3L} = d^k$, stack height is always greater than or equal to $\hat{h} - O(1)$, and $|W_2| = \Omega(\frac{1}{n^2}3^n)$, where C_{WL} is as in Lemma 2.

Proof. Note that for each w_2 , there are more than $\frac{1}{n+1}3^{|w_{1R}|}$ of w_{1R} 's for which stack height is always greater than or equal to $\log_{|T|}(3^{n-|w_{1L}|}/(\#states \cdot n(n+1)))$ while processing w_2 by Condition II. This means that for some w_{1R} , there are $\Omega(\frac{1}{n}3^n)$ of w_2 's for which stack height is always greater than or equal to $\log_{|T|}(3^{n-|w_{1L}|}/(\#states \cdot n(n+1)))$ while processing w_2 . By Lemma 1 and the fact that $\hat{h} < \log_{|T|}(3^{|w_{1R}|}/(\#states \cdot n(n+1)))$, the lemma follows immediately. \square

We fix w_{1R} , k and W_2 as those in Lemma 3 in the following. For WL in Lemma 2, the following lemma holds.

Lemma 4. We assume that there exists a deterministic pushdown automaton that solves Problem I. For w_{1R} , k and W_2 in Lemma 3, there exist $w_{1L} \in WL$ and two distinct partial inputs $w_{1L}w_{1R}w_2w_{3L}$ and $w_{1L}w_{1R}w'_2w_{3L}$ ($w_2, w'_2 \in W_2$ and $w_{3L} = d^k$) such that $w_1 \stackrel{e}{\sim} w_2^R$ and $w_1 \stackrel{o}{\sim} w'_2{}^R$.

Proof. Let $WL = \{w_{1L}^1, w_{1L}^2, \dots, w_{1L}^m\}$ where $w_{1L}^i \leq w_{1L}^{i+1}$. $W_{2,even}^1$ denotes the set of $w_2 \in \{a, b, c\}^n$ such that $w_{1L}^1 w_{1R} \stackrel{e}{\sim} w_2^R$. Also $W_{2,even}^2$ denotes the set of $w_2 \in W_2^1$ such that $w_{1L}^2 w_{1R} \stackrel{e}{\sim} w_2^R$. Similarly, $W_{2,even}^i$ denotes the set of $w_2 \in W_{2,even}^{i-1}$ such that $w_{1L}^i w_{1R} \stackrel{e}{\sim} w_2^R$. In other words, for all $w_2 \in W_{2,even}^i$ and $j \leq i$, $w_{1L}^j w_{1R} \stackrel{e}{\sim} w_2^R$. We show that $|W_{2,even}^i| \leq c|W_{2,even}^{i-1}|$ for some constant $c < 1$ in the following. We consider the positions at which w_{1L}^{i-1} and w_{1L}^i differ. We define the set of such positions to be D^i . Note that $w_{1L}^{i-1}(k) = a$ and $w_{1L}^i(k) = b$ for $k \in D^i$, where $w(k)$ represents the k -th symbol of w . We define $S = \{w_2 \in W_{2,even}^{i-1} | \exists k \in D^i w_2^R(k) = b \text{ or } w_2^R(k) = c.\}$, where $w_2(k)$ denotes the k -th symbol of w_2 . It is obvious that $|S| \geq c_1|W_{2,even}^{i-1}|$ for some constant $c_1 < 1$. For $w_2 \in S$, let l be the largest position in D^i such that $w_2^R(l) = b$ or $w_2^R(l) = c$. We assume that $w_2^R(l) = b$ without loss of generality. We consider w'_2 such that $w_2^R(i) = w_2^R(i)$ for $i \neq l$ and $w_2^R(i) = c$. It is obvious that w'_2 is also in S . Then,

either $w_{1L}^i w_{1R} \stackrel{o}{\sim} w_2^R$ or $w_{1L}^i w_{1R} \stackrel{o}{\sim} w_2'^R$ holds. This implies that a half of elements in S cannot belong to $W_{2,even}^i$. Thus, $|W_{2,even}^i| \leq |W_{2,even}^{i-1}| - \frac{c_1}{2}|W_{2,even}^{i-1}| = c_2|W_{2,even}^{i-1}|$, where $c_2 = 1 - \frac{c_1}{2}$. Similar to $W_{2,even}^1$, we define $W_{2,odd}^1$, and then, similarly, it can be shown that $|W_{2,odd}^i| \leq c|W_{2,odd}^{i-1}|$ for some constant $c < 1$. Therefore, $|W_{2,even}^i|$ and $|W_{2,odd}^i|$ can be smaller than $|W_2|$ for $i \in \Theta(n)$. The lemma follows. \square

(Proof of Theorem 4)

We assume that there exists a classical deterministic pushdown automaton that solves Problem I. Then, by Lemma 4, we have two input strings, $w_a = w_{1L}w_{1R}w_2w_{3L}$ and $w_b = w_{1L}w_{1R}w_2'w_{3L}$ ($w_2, w_2' \in W_2$ and $w_{3L} = d^k$), such that $w_1 \stackrel{o}{\sim} w_2^R$ and $w_1 \stackrel{o}{\sim} w_2'^R$. We fix $w_{3R} = d^{n-k}$. Then, the answer only depends on the number of distinctions between w_1 and w_2^R (or $w_2'^R$). Thus, one is YES and the other is NO for w_a and w_b . However, the configurations (including the contents of the stack and the position of the input tape head) at the moment of reading the last symbol of w_{3L} are the same between w_a and w_b if $k \neq n$. On the other hand, if $k = n$, the state-configuration at the moment of reading the last symbol of w_a and w_b are the same. Thus, both of w_a and w_b lead to the same answer. This is a contradiction. \square

5 Comparison between Quantum Pushdown Automata with and without a Garbage Tape – Concluding Remarks

In this paper, we showed that QPAGs are strictly more powerful than classical pushdown automata in the setting of exact, one-sided error and nondeterministic computation. In this section, we discuss comparison between quantum pushdown automata with and without a garbage tape. Our conjecture is that Problem I cannot be solved exactly by quantum pushdown automata without a garbage tape, which is Golovkins's model[12], since it seems to be impossible to compute $w_1 \stackrel{e}{\sim} w_2^R$ or $w_1 \stackrel{e}{\sim} w_3^R$ without a garbage tape. On the other hand, in the QPAG model, popped symbols are always stored in the garbage tape. Thus, if the contents of the garbage tape are different between two computation paths, they no longer interfere with each other. In other words, only the two computation paths that have the same contents in the garbage tape can interfere with each other. This might make the QPAG model less powerful than Golovkins's model. Therefore, we conjecture that the class of languages recognized by the two models are incomparable. We also conjecture that even the generalized quantum pushdown automata without a garbage tape constructed by the technique in [27] cannot solve Problem I. At least, the generalized model of quantum pushdown automata without a garbage tape cannot execute the algorithm in Theorem 3. This is because, although the garbage tape is in a superposition in the middle of the computation of the algorithm, the generalized quantum pushdown automaton cannot represent such a superposition without a garbage tape. Thus, our model and the generalized model without a garbage tape might also be incomparable.

Acknowledgments This work was supported by JSPS KAKENHI Grant Nos. 24500003 and 24106009.

References

1. Ablayev, F., Gainutdinova, A., Khadiev, K., Yakaryilmaz, A.: Very narrow quantum OBDDs and width hierarchies for classical OBDDs. In: Proceedings of 16th International Workshop on Descriptive Complexity of Formal Systems (DCFS'14). pp. 53–64 (2014)
2. Amarilli, A., Jeanmougin, M.: A proof of the pumping lemma for context-free languages through pushdown automata (2012), coRR, abs/1207.2819
3. Ambainis, A., Freivalds, R.: 1-way quantum finite automata: strengths, weakness and generalizations. In: Proceedings of the 29th Symposium on Foundations of Computer Science (FOCS'98). pp. 332–341 (1998)
4. Ambainis, A., Watrous, J.: Two-way finite automata with quantum and classical states. *Theoretical Computer Science* 287(1), 299–311 (2002)
5. Ambainis, A., Yakaryilmaz, A.: Superiority of exact quantum automata for promise problems. *Information Processing Letters* 112(7), 289–291 (2012)
6. Bonner, R., Freivalds, R., Kravtsev, M.: Quantum versus probabilistic one-way finite automata with counter. In: Proceedings of the 28th Conference on Current Trends in Theory and Practice of Informatics (SOFSEM2001). pp. 181–190 (2001)
7. Brodsky, A., Pippenger, N.: Characterizations of 1-way quantum finite automata. *SIAM Journal on Computing* 31(5), 1456–1478 (2002)
8. Ciamarra, M.P.: Quantum reversibility and a new model of quantum automaton. In: Proceedings of the 13th International Symposium on Fundamentals of Computation Theory (FCT'01). pp. 376–379 (2001)
9. Cleve, R., Ekert, A., Macchiavello, C., Mosca, M.: Quantum algorithms revisited. *Proceedings of the Royal Society A* 454, 339–354 (1998)
10. Deutsch, D.: The Church-Turing principle and the universal quantum computer. *Proceedings of the Royal Society A* 400, 97–117 (1985)
11. Deutsch, D., Jozsa, R.: Rapid solution of problem by quantum computation. *Proceedings of the Royal Society A* 439, 553–558 (1992)
12. Golovkins, M.: Quantum pushdown automata. In: Proceedings of 27th Conference on Current Trends in Theory and Practice of Informatics (SOFSEM2000). pp. 336–346 (2000)
13. Hirvensalo, M.: Various aspects of finite quantum automata. In: Proceedings of Developments in Language Theory 2008 (DLT2008). pp. 21–33 (2008)
14. Hirvensalo, M.: Quantum automata with open time evolution. *International Journal of Natural Computing Research (IJNCR)* 1(1), 70–85 (2010)
15. Kondacs, A., Watrous, J.: On the power of quantum finite state automata. In: Proceedings of the 38th Symposium on Foundations of Computer Science (FOCS'97). pp. 66–75 (1997)
16. Kravtsev, M.: Quantum finite one-counter automata. In: Proceedings of 26th Conference on Current Trends in Theory and Practice of Informatics (SOFSEM1999). pp. 432–442 (1999)
17. Moore, C., Crutchfield, J.P.: Quantum automata and quantum grammars. *Theoretical Computer Science* 237(1–2), 275–306 (2000)
18. Murakami, Y., Nakanishi, M., Yamashita, S., Watanabe, K.: Quantum versus classical pushdown automata in exact computation. *IPSP Journal* 46(10), 2471–2480 (2005)

19. Nakanishi, M., Hamaguchi, K., Kashiwabara, T.: Expressive power of quantum pushdown automata with classical stack operations under the perfect-soundness condition. *IEICE Transactions on Information and Systems* E89-D(3), 1120–1127 (2006)
20. Ogden, W.: A helpful result for proving inherent ambiguity. *Mathematical Systems Theory* 2(3), 191 – 194 (1968)
21. Paschen, K.: Quantum finite automata using ancilla qubits (2000), technical report, University of Karlsruhe, available at <http://digbib.ubka.uni-karlsruhe.de/volltexte/1452000>
22. Say, A.C.C., Yakaryılmaz, A.: Quantum counter automata. *International Journal of Foundations of Computer Science* 23(5), 1099–1116 (2012)
23. Yakaryılmaz, A.: Superiority of one-way and realtime quantum machines. *RAIRO - Theoretical Informatics and Applications* 46(04), 615–641 (2012)
24. Yakaryılmaz, A., Freivalds, R., Say, A.C.C., Agadzanyan, R.: Quantum computation with write-only memory. *Natural Computing* 11(1), 81–94 (2012)
25. Yakaryılmaz, A., Say, A.C.C.: Efficient probability amplification in two-way quantum finite automata. *Theoretical Computer Science* 410(20), 1932–1941 (2009)
26. Yakaryılmaz, A., Say, A.C.C.: Succinctness of two-way probabilistic and quantum finite automata. *Discrete Mathematics and Theoretical Computer Science* 12(4), 19–40 (2010)
27. Yakaryılmaz, A., Say, A.C.C.: Unbounded-error quantum computation with small space bounds. *Information and Computation* 209(6), 873–892 (2011)
28. Yamasaki, T., Kobayashi, H., Imai, H.: Quantum versus deterministic counter automata. *Theoretical Computer Science* 334(1–3), 275–297 (2005)
29. Yamasaki, T., Kobayashi, H., Tokunaga, Y., Imai, H.: One-way probabilistic reversible and quantum one-counter automata. *Theoretical Computer Science* 289(2), 963–976 (2002)

Appendix A: Probabilistic Pushdown Automata

Definition 2. A probabilistic pushdown automaton (PPA) is defined as the following 7-tuple:

$$M = (Q, \Sigma, \Gamma, \delta, q_0, Q_{acc}, Q_{rej}),$$

where Q is a set of states, Σ is a set of input symbols including the left and the right endmarkers $\{\$, \#\}$, respectively, Γ is a set of stack symbols including the bottom symbol Z , δ is a state transition function ($\delta : (Q \times \Sigma \times \Gamma \times Q \times G \cup \{\varepsilon, pop\} \times \{0, 1\}) \rightarrow [0, 1]$), where $G (\subseteq (\Gamma \setminus \{Z\})^+)$ is a finite set and $(\Gamma \setminus \{Z\})^+$ is the set of all nonempty strings of finite length from alphabet $\Gamma \setminus \{Z\}$, q_0 is the initial state, $Q_{acc} (\subseteq Q)$ is the set of accepting states, and $Q_{rej} (\subseteq Q)$ is the set of rejecting states, where $Q_{acc} \cap Q_{rej} = \emptyset$. We restrict that for all q, q', a, D , $\delta(q, a, Z, q', pop, D) = 0$. \square

$\delta(q, a, b, q', w, D) = \alpha$ means that the probability of the transition from q to q' moving the head to D with stack operation w is α when reading input symbol a and stack symbol b . Note that for each input symbol and each stack symbol, the sum of the weights (i.e. the probabilities) of outgoing transitions of a state must be 1. Computation halts when it enters the accepting or rejecting states.

Appendix B: Quantum Pushdown Automata with a Classical Stack

A quantum pushdown automata with a classical stack (QCPDA) has an input tape to which a quantum head is attached and a classical stack to which a classical stack top pointer is attached. A QCPDA also has a quantum finite state control. The quantum finite state control reads the stack top symbol pointed by the classical stack top pointer and the input symbol pointed by the quantum head. Stack operations are determined solely by the results of measurements of the quantum finite state control. We define QCPDAs formally as follows.

Definition 3. A quantum pushdown automaton with a classical stack (QCPDA) is defined as the following 8-tuple:

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \sigma, Q_{acc}, Q_{rej}),$$

where Q is a set of states, Σ is a set of input symbols including the left and the right endmarkers $\{\$, \#\}$, respectively, Γ is a set of stack symbols including the bottom symbol Z , δ is a quantum state transition function ($\delta : (Q \times \Sigma \times \Gamma \times Q \times \{0, 1\}) \rightarrow \mathbb{C}$), q_0 is the initial state, σ is a function by which stack operations are determined ($\sigma : Q \setminus (Q_{acc} \cup Q_{rej}) \rightarrow G \cup \{\varepsilon, pop\}$), where $G (\subseteq (\Gamma \setminus \{Z\})^+)$ is a finite set and $(\Gamma \setminus \{Z\})^+$ is the set of all nonempty strings of finite length from alphabet $\Gamma \setminus \{Z\}$, $Q_{acc} (\subseteq Q)$ is the set of accepting states, and $Q_{rej} (\subseteq Q)$ is the set of rejecting states, where $Q_{acc} \cap Q_{rej} = \emptyset$. We restrict that for all q, q', a, D , if $\sigma(q') = pop$, then $\delta(q, a, Z, q', D) = 0$. \square

$\delta(q, a, b, q', D) = \alpha$ means that the amplitude of the transition from q to q' moving the quantum head to D ($D = 1$ means ‘right’ and $D = 0$ means ‘stay’) is α when reading input symbol a and stack symbol b . The configuration of the quantum portion of a QCPDA is a pair (q, k) , where k is the position of the quantum head and q is in Q . It is obvious that the number of configurations of the quantum portion is $n|Q|$, where n is the input length.

A superposition of the configurations of the quantum portion of a QCPDA is any element of $l_2(Q \times \mathbb{Z}_n)$ of unit length, where $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$. For each configuration, we define a column vector $|q, k\rangle$ as follows:

- $|q, k\rangle$ is an $n|Q| \times 1$ column vector.
- The row corresponding to (q, k) is 1, and the other rows are 0.

For input word \mathbf{x} (i.e., the string on the input tape between \clubsuit and $\$$) and stack symbol a , we define a time evolution operator $U_a^{\mathbf{x}}$ as follows:

$$U_a^{\mathbf{x}}(|q, k\rangle) = \sum_{q' \in Q, D \in \{0,1\}} \delta(q, x(k), a, q', D) |q', k + D\rangle,$$

where $x(k)$ is the k -th input symbol of input \mathbf{x} . If $U_a^{\mathbf{x}}$ is unitary (for any $a \in \Gamma$ and for any input word \mathbf{x}), that is, $U_a^{\mathbf{x}} U_a^{\mathbf{x}\dagger} = U_a^{\mathbf{x}\dagger} U_a^{\mathbf{x}} = I$, where $U_a^{\mathbf{x}\dagger}$ is the transpose conjugate of $U_a^{\mathbf{x}}$, then the corresponding QCPDA is well-formed. A well-formed QCPDA is considered valid in terms of the quantum theory. We consider only well-formed QCPDAs.

We describe how the quantum portion and the classical stack of a QCPDA work in the following.

Let the initial quantum state and the initial position of the head be q_0 and ‘0’, respectively. We define $|\psi_0\rangle$ as $|\psi_0\rangle = |q_0, 0\rangle$. We also define E_w , E_{acc} and E_{rej} as follows:

$$\begin{aligned} E_w &= \text{span}\{|q, k\rangle \mid \sigma(q) = w\}, \\ E_{acc} &= \text{span}\{|q, k\rangle \mid q \in Q_{acc}\}, \\ E_{rej} &= \text{span}\{|q, k\rangle \mid q \in Q_{rej}\}. \end{aligned}$$

We define observable \mathcal{O} as $\mathcal{O} = \bigoplus_j E_j$, where j is ‘acc’, ‘rej’ or $w \in G \cup \{\varepsilon, pop\}$. For notational simplicity, we define the outcome of a measurement corresponding to E_j as j .

A QCPDA computation proceeds as follows:

For input word \mathbf{x} , the quantum portion works as follows:

- (a) $U_a^{\mathbf{x}}$ is applied to $|\psi_i\rangle$. Let $|\psi_{i+1}\rangle = U_a^{\mathbf{x}} |\psi_i\rangle$, where a is the stack top symbol.
- (b) $|\psi_{i+1}\rangle$ is measured with respect to the observable $\mathcal{O} = \bigoplus_j E_j$. Let $|\phi_j\rangle$ be the projection of $|\psi_{i+1}\rangle$ to E_j . Then each outcome j is obtained with probability $|\langle \phi_j | \psi_{i+1} \rangle|^2$. Note that this measurement causes $|\psi_{i+1}\rangle$ to collapse to $\frac{1}{\|\phi_j\|} |\phi_j\rangle$, where j is the obtained outcome. Then go to (c).

The classical stack works as follows:

- (c) Let the outcome of the measurement be j . If j is ‘acc’ (resp. ‘rej’) then it outputs ‘accept’ (resp. ‘reject’), and the computation halts. If j is ‘ ε ’, then the stack is unchanged. If j is ‘pop’, then the stack top symbol is popped. Otherwise (j is a word in G in this case), word j is pushed. Then, go to (a) and repeat.

Appendix C: State Transition Function of the QPAG that Solves Problem I

We describe the state transition function of the QPAG $M = (Q, \Sigma, \Gamma, \delta, q_0, Q_{acc}, Q_{rej})$ that solves Problem I in the following, where $Q = \{q_0\} \cup \{q_i^{I,j}, q_i^{O,j} | i \in \{1, 2\}, j \in \{0, 1\}\} \cup \{q_f^{acc}, q_f^{rej}, q_f^{-,0}, q_f^{-,1}\}$, $\Sigma = \{a, b, c, d, \#, \$, \}$, $\Gamma = \{a, b, c, Z\}$, the initial state is q_0 , $Q_{acc} = \{q_f^{acc}\}$, and $Q_{rej} = \{q_f^{rej}\}$. Note that sub-automaton M_1 (resp. M_2) consists of the states $\{q_0, q_1^{I,0}, q_1^{I,1}, q_1^{O,0}, q_1^{O,1}\}$ (resp. $\{q_0, q_2^{I,0}, q_2^{I,1}, q_2^{O,0}, q_2^{O,1}\}$). We first describe the outline of the behavior of M . M consists of the following three stages:

Stage I M pushes w_1 into the stack.

Stage II M_1 and M_2 run in a superposition.

M_1 runs in a superposition of states $q_1^{I,0}$ and $q_1^{I,1}$. M_1 reads w_2 and pops the stack-top symbol one by one. If the input symbol is different from the stack-top symbol, the current state changes from $q_1^{I,x}$ to $q_1^{I,x \oplus 1}$ ($x \in \{0, 1\}$), otherwise M_1 stays at the same state. Then, M_1 skips w_3 .

M_2 runs in a superposition of states $q_2^{I,0}$ and $q_2^{I,1}$. M_2 skips w_2 . Then, M_2 reads w_3 and pops the stack-top symbol one by one. If the input symbol is different from the stack-top symbol, the current state changes from $q_2^{O,x}$ to $q_2^{O,x \oplus 1}$ ($x \in \{0, 1\}$), otherwise M_2 stays at the same state.

Stage III M reads the right-endmarker, and then, Hadamard transform is applied to M .

We describe the state transition function below. In the following, * denotes a wild card, which matches any of a, b, c, Z .

Stage I

$$\begin{aligned} \delta(q_0, \$, Z, q_0, \varepsilon, 1) &= 1, \\ \delta(q_0, a, *, q_0, a, 1) &= 1, \quad \delta(q_0, b, *, q_0, b, 1) = 1, \quad \delta(q_0, c, *, q_0, c, 1) = 1, \\ \delta(q_0, \#, *, q_1^{I,0}, \varepsilon, 1) &= \frac{1}{2}, \quad \delta(q_0, \#, *, q_1^{I,1}, \varepsilon, 1) = -\frac{1}{2}, \\ \delta(q_0, \#, *, q_2^{I,0}, \varepsilon, 1) &= \frac{1}{2}, \quad \delta(q_0, \#, *, q_2^{I,1}, \varepsilon, 1) = -\frac{1}{2} \end{aligned}$$

Stage II

$$\delta(q_1^{I,0}, a, a, q_1^{I,0}, pop, 1) = 1, \quad \delta(q_1^{I,0}, b, b, q_1^{I,0}, pop, 1) = 1, \quad \delta(q_1^{I,0}, c, c, q_1^{I,0}, pop, 1) = 1,$$

$$\begin{aligned}
&\delta(q_1^{I,0}, a, b, q_1^{I,1}, pop, 1) = 1, \delta(q_1^{I,0}, a, c, q_1^{I,1}, pop, 1) = 1, \\
&\delta(q_1^{I,0}, b, a, q_1^{I,1}, pop, 1) = 1, \delta(q_1^{I,0}, b, c, q_1^{I,1}, pop, 1) = 1, \\
&\delta(q_1^{I,0}, c, a, q_1^{I,1}, pop, 1) = 1, \delta(q_1^{I,0}, c, b, q_1^{I,1}, pop, 1) = 1, \\
&\delta(q_1^{I,0}, \#, *, q_1^{O,0}, \varepsilon, 1) = 1,
\end{aligned}$$

$$\begin{aligned}
&\delta(q_1^{I,1}, a, a, q_1^{I,1}, pop, 1) = 1, \delta(q_1^{I,1}, b, b, q_1^{I,1}, pop, 1) = 1, \delta(q_1^{I,1}, c, c, q_1^{I,1}, pop, 1) = 1, \\
&\delta(q_1^{I,1}, a, b, q_1^{I,0}, pop, 1) = 1, \delta(q_1^{I,1}, a, c, q_1^{I,0}, pop, 1) = 1, \\
&\delta(q_1^{I,1}, b, a, q_1^{I,0}, pop, 1) = 1, \delta(q_1^{I,1}, b, c, q_1^{I,0}, pop, 1) = 1, \\
&\delta(q_1^{I,1}, c, a, q_1^{I,0}, pop, 1) = 1, \delta(q_1^{I,1}, c, b, q_1^{I,0}, pop, 1) = 1, \\
&\delta(q_1^{I,1}, \#, *, q_1^{O,1}, \varepsilon, 1) = 1,
\end{aligned}$$

$$\begin{aligned}
&\delta(q_1^{O,0}, a, Z, q_1^{O,0}, \varepsilon, 1) = 1, \delta(q_1^{O,0}, b, Z, q_1^{O,0}, \varepsilon, 1) = 1, \\
&\delta(q_1^{O,0}, c, Z, q_1^{O,0}, \varepsilon, 1) = 1, \delta(q_1^{O,0}, d, Z, q_1^{O,0}, \varepsilon, 1) = 1,
\end{aligned}$$

$$\begin{aligned}
&\delta(q_1^{O,1}, a, Z, q_1^{O,1}, \varepsilon, 1) = 1, \delta(q_1^{O,1}, b, Z, q_1^{O,1}, \varepsilon, 1) = 1, \\
&\delta(q_1^{O,1}, c, Z, q_1^{O,1}, \varepsilon, 1) = 1, \delta(q_1^{O,1}, d, Z, q_1^{O,1}, \varepsilon, 1) = 1,
\end{aligned}$$

$$\begin{aligned}
&\delta(q_2^{I,0}, a, *, q_2^{I,0}, \varepsilon, 1) = 1, \delta(q_2^{I,0}, b, *, q_2^{I,0}, \varepsilon, 1) = 1, \delta(q_2^{I,0}, c, *, q_2^{I,0}, \varepsilon, 1) = 1, \\
&\delta(q_2^{I,0}, \#, *, q_2^{O,0}, \varepsilon, 1) = 1,
\end{aligned}$$

$$\begin{aligned}
&\delta(q_2^{I,1}, a, *, q_2^{I,1}, \varepsilon, 1) = 1, \delta(q_2^{I,1}, b, *, q_2^{I,1}, \varepsilon, 1) = 1, \delta(q_2^{I,1}, c, *, q_2^{I,1}, \varepsilon, 1) = 1, \\
&\delta(q_2^{I,1}, \#, *, q_2^{O,1}, \varepsilon, 1) = 1,
\end{aligned}$$

$$\begin{aligned}
&\delta(q_2^{O,0}, a, a, q_2^{O,0}, pop, 1) = 1, \delta(q_2^{O,0}, b, b, q_2^{O,0}, pop, 1) = 1, \delta(q_2^{O,0}, c, c, q_2^{O,0}, pop, 1) = 1, \\
&\delta(q_2^{O,0}, a, b, q_2^{O,1}, pop, 1) = 1, \delta(q_2^{O,0}, a, c, q_2^{O,1}, pop, 1) = 1, \delta(q_2^{O,0}, a, d, q_2^{O,1}, pop, 1) = 1, \\
&\delta(q_2^{O,0}, b, a, q_2^{O,1}, pop, 1) = 1, \delta(q_2^{O,0}, b, c, q_2^{O,1}, pop, 1) = 1, \delta(q_2^{O,0}, b, d, q_2^{O,1}, pop, 1) = 1, \\
&\delta(q_2^{O,0}, c, a, q_2^{O,1}, pop, 1) = 1, \delta(q_2^{O,0}, c, b, q_2^{O,1}, pop, 1) = 1, \delta(q_2^{O,0}, c, d, q_2^{O,1}, pop, 1) = 1,
\end{aligned}$$

$$\begin{aligned}
&\delta(q_2^{O,1}, a, a, q_2^{O,1}, pop, 1) = 1, \delta(q_2^{O,1}, b, b, q_2^{O,1}, pop, 1) = 1, \delta(q_2^{O,1}, c, c, q_2^{O,1}, pop, 1) = 1, \\
&\delta(q_2^{O,1}, a, b, q_2^{O,0}, pop, 1) = 1, \delta(q_2^{O,1}, a, c, q_2^{O,0}, pop, 1) = 1, \delta(q_2^{O,1}, a, d, q_2^{O,0}, pop, 1) = 1, \\
&\delta(q_2^{O,1}, b, a, q_2^{O,0}, pop, 1) = 1, \delta(q_2^{O,1}, b, c, q_2^{O,0}, pop, 1) = 1, \delta(q_2^{O,1}, b, d, q_2^{O,0}, pop, 1) = 1, \\
&\delta(q_2^{O,1}, c, a, q_2^{O,0}, pop, 1) = 1, \delta(q_2^{O,1}, c, b, q_2^{O,0}, pop, 1) = 1, \delta(q_2^{O,1}, c, d, q_2^{O,0}, pop, 1) = 1,
\end{aligned}$$

Stage III

$$\begin{aligned}
&\delta(q_1^{O,0}, \$, Z, q_f^{-,0}, \varepsilon, 1) = \frac{1}{2}, \delta(q_1^{O,0}, \$, Z, q_f^{acc}, \varepsilon, 1) = \frac{1}{2}, \\
&\delta(q_1^{O,0}, \$, Z, q_f^{-,1}, \varepsilon, 1) = \frac{1}{2}, \delta(q_1^{O,0}, \$, Z, q_f^{rej}, \varepsilon, 1) = \frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
&\delta(q_1^{O,1}, \$, Z, q_f^{-,0}, \varepsilon, 1) = \frac{1}{2}, \delta(q_1^{O,1}, \$, Z, q_f^{acc}, \varepsilon, 1) = -\frac{1}{2}, \\
&\delta(q_1^{O,1}, \$, Z, q_f^{-,1}, \varepsilon, 1) = \frac{1}{2}, \delta(q_1^{O,1}, \$, Z, q_f^{rej}, \varepsilon, 1) = -\frac{1}{2},
\end{aligned}$$

$$\begin{aligned}\delta(q_2^{O,0}, \$, Z, q_f^{-,0}, \varepsilon, 1) &= -\frac{1}{2}, \quad \delta(q_2^{O,0}, \$, Z, q_f^{acc}, \varepsilon, 1) = -\frac{1}{2}, \\ \delta(q_2^{O,0}, \$, Z, q_f^{-,1}, \varepsilon, 1) &= \frac{1}{2}, \quad \delta(q_2^{O,0}, \$, Z, q_f^{rej}, \varepsilon, 1) = \frac{1}{2},\end{aligned}$$

$$\begin{aligned}\delta(q_2^{O,1}, \$, Z, q_f^{-,0}, \varepsilon, 1) &= -\frac{1}{2}, \quad \delta(q_2^{O,1}, \$, Z, q_f^{acc}, \varepsilon, 1) = \frac{1}{2}, \\ \delta(q_2^{O,1}, \$, Z, q_f^{-,1}, \varepsilon, 1) &= \frac{1}{2}, \quad \delta(q_2^{O,1}, \$, Z, q_f^{rej}, \varepsilon, 1) = -\frac{1}{2},\end{aligned}$$

It is straightforward to see that the corresponding time evolution operator can be extended to be unitary.

Remark

The reason why we can use the same technique as in Theorem 3.1 of [18] even though our model and the model used in [18] seems incomparable is the following. When the stack-top symbol is popped, it is always written in the garbage tape in our model. This makes an entanglement between the stack contents and the garbage tape. Sometimes, this can be an unwanted behavior and make our model weaker than the model in [18]. However, in our algorithm shown in the proof of Theorem 3, the contents of the garbage tape at the moment of reading the right-endmarker can be the same between the two sub-automata. Therefore, the stack contents and the garbage tape are separable at the moment of reading the right-endmarker, which causes no problem when using the same technique in Theorem 3.1 of [18].