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NONCROSSING PARTITIONS FOR THE GROUP D_n*

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Dedicated to the memory of Rodica Simion

Abstract. The poset of noncrossing partitions can be naturally defined for any finite Coxeter group W. It is a self-dual, graded lattice which reduces to the classical lattice of noncrossing partitions of $\{1, 2, \ldots, n\}$ defined by Kreweras in 1972 when W is the symmetric group S_n, and to its type B analogue defined by the second author in 1997 when W is the hyperoctahedral group. We give a combinatorial description of this lattice in terms of noncrossing planar graphs in the case of the Coxeter group of type D_n, thus answering a question of Bessis. Using this description, we compute a number of fundamental enumerative invariants of this lattice, such as the rank sizes, number of maximal chains, and Möbius function.

We also extend to the type D case the statement that noncrossing partitions are equidistributed to nonnesting partitions by block sizes, previously known for types A, B, and C. This leads to a (case-by-case) proof of a theorem valid for all root systems: the noncrossing and nonnesting subspaces within the intersection lattice of the Coxeter hyperplane arrangement have the same distribution according to W-orbits.

Key words. noncrossing partition, nonnesting partition, reflection group, root poset, antichain, Catalan number, Narayana numbers, type D, Garside structure

AMS subject classifications. Primary, 06A07; Secondary, 05A18, 05E15, 20F55

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1. Introduction and results. The lattice $NC^A(n)$ of noncrossing partitions is a well-behaved and well-studied subposet inside the lattice $\Pi(n)$ of partitions of the set $[n] := \{1, 2, \ldots, n\}$. It consists of all set partitions π of $[n]$ such that if $a < b < c < d$ and a, c are contained in a block B of π while b, d are contained in a block B' of π, then $B = B'$. The lattice of noncrossing partitions arises naturally in such diverse areas of mathematics as combinatorics, discrete geometry, representation theory, group theory, probability, combinatorial topology, and mathematical biology; see the survey [21] by Simion. This paper concerns analogues of this lattice for Coxeter groups and, specifically, for the Coxeter group of type D_n.

Such analogues were suggested for the Coxeter groups of types B_n and D_n in [20] and were shown to have enumerative and order theoretic properties similar to those of $NC^A(n)$. Reiner [20, section 6] asked for a natural definition of the lattice of noncrossing partitions for any finite Coxeter group W. Although the main idea may be described as folklore (cf. [7]), only fairly recently, and in particular after the work of Bessis [4] and Brady and Watt [12], it has become apparent that such a definition is both available and useful. More precisely, for $u, w \in W$, let $u \leq w$ if there is a shortest factorization of u as a product of reflections in W which is a prefix of such a shortest factorization of w. This partial order turns W into a graded poset T^W having the identity 1 as its unique minimal element, where the rank of w is the length of the shortest factorization of w into reflections. Let γ be a Coxeter element of W. Since all Coxeter elements in W are conjugate to each other, the interval $[1, \gamma]$ in T^W

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is independent, up to isomorphism, of the choice of γ. We denote this interval by NC^W or by NC^{X_n}, where X_n is the Cartan–Killing type of W. The poset NC^W plays a crucial role in the construction of new monoid structures and $K(\pi,1)$ spaces for Artin groups associated with finite Coxeter groups [4, 11, 12] and shares many of the fundamental properties of $NC^A(n)$. For instance, it is self-dual [4, section 2.3] and graded and has been verified case-by-case to be a lattice [4, Fact 2.3.1]; see also [12, section 4].

In the case of the symmetric group it is known that the poset $NC^{A_{n-1}}$ is isomorphic to the lattice $NC^A(n)$ of noncrossing partitions; see, for instance, [6, 7, 11]. Similarly, in the case of the hyperoctahedral group, the poset NC^{B_n} is isomorphic to the type B analogue $NC^B(n)$ of $NC^A(n)$ proposed in [20]; see [4, 9, 12]. However, it was observed in [12, section 4] that NC^{D_n} is *not* isomorphic to the type D analogue of $NC^A(n)$ suggested in [20]. Bessis [4, section 4.2] asked for an explicit description of the elements of NC^{D_n} as noncrossing planar graphs, similar to those which appear in the definition of $NC^B(n)$. We give such a description in section 3. Using a construction similar to that of $NC^B(n)$, we define a poset $NC^D(n)$ which we suggest as the type D analogue of $NC^A(n)$ and prove the following theorem.

THEOREM 1.1. *The poset NC^{D_n} is isomorphic to $NC^D(n)$.*

In particular, this gives a different proof that the poset NC^{D_n} is indeed a lattice [12, Theorem 4.14]; see Proposition 3.1. We should mention that, independently, Bessis and Corran [5] have generalized this construction to a class of complex reflection groups that contains D_n.

In our next main result we compute some basic enumerative invariants of NC^{D_n}. Throughout, we use the convention that $\binom{n}{k} = 0$ unless $k \in \{0, 1, 2, \ldots, n\}$.

THEOREM 1.2. (i) *The number of elements of NC^{D_n} of rank k is equal to the type D Narayana number*

$$\mathrm{Nar}(D_n, k) = \binom{n}{k}^2 - \frac{n}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}$$

$$= \binom{n}{k}\left(\binom{n-1}{k} + \binom{n-2}{k-2}\right).$$

In particular, the total number of elements of NC^{D_n} is equal to the type D Catalan number

$$\mathrm{Cat}\,(D_n) = \binom{2n}{n} - \binom{2n-2}{n-1}.$$

(ii) *More generally, for any composition $s = (s_1, s_2, \ldots, s_m)$ of the number n, the number of chains from the minimum to the maximum element in NC^{D_n} with successive rank jumps $s_1, s_2, \ldots, s_m$ is equal to*

$$2\binom{n-1}{s_1}\cdots\binom{n-1}{s_m} + \sum_{i=1}^{m}\binom{n-1}{s_1}\cdots\binom{n-2}{s_i-2}\cdots\binom{n-1}{s_m}.$$

(iii) *The zeta polynomial of NC^{D_n} is given by*

$$Z(NC^{D_n}, m) = 2\binom{m(n-1)}{n} + \binom{m(n-1)}{n-1}.$$

(iv) *In particular,* NC^{D_n} *has* $2(n-1)^n$ *maximal chains, and has Möbius function between the minimum and maximum element equal to*

$$(-1)^n \left(2\binom{2n-2}{n} - \binom{2n-3}{n-1}\right).$$

The Narayana and Catalan numbers which appear in part (i) of Theorem 1.2 can be defined for any finite Coxeter group; see [2, 3] for a number of interesting combinatorial and algebraic-geometric interpretations. It is known [16, 20] that the number of elements of a given rank in NC^W and the total number of elements are equal to the corresponding Narayana and Catalan numbers, respectively, in the cases of types A and B. Thus part (i) of the theorem extends this fact to the case of type D. The statement on the cardinality of NC^{D_n} is also claimed to have been checked by Picantin [19]. We note that the type D analogue of $NC^A(n)$ suggested in [20] has the same cardinality and rank sizes as NC^{D_n} [20, Corollary 10] but different zeta polynomial, number of maximal chains, and Möbius function.

Our definition of the poset $NC^D(n)$ leads naturally to a notion of "block sizes" for noncrossing partitions of type D (see section 2). Such a notion was already suggested in [1] for nonnesting partitions for the classical root systems, which are other families of combinatorial objects counted by the corresponding Catalan numbers; see [1], [20, Remark 2], [25, Exercise 6.19 (uu)], and section 2. Our next result refines Theorem 1.2(i) and extends to the case of type D the main result of [1], stating that noncrossing and nonnesting partitions are equidistributed by block sizes for each of the classical root systems of types A, B, and C.

THEOREM 1.3. *Let* λ *be a partition of* $n-m$ *with* k *parts, where* $m \geq 0$*, and let* $m_\lambda = r_1! \cdot r_2! \cdots$*, where* r_i *is the number of parts of* λ *equal to* i*. The numbers of noncrossing or nonnesting partitions of type* D_n *with block sizes* λ *are equal to each other and are given by the formula*

$$\begin{cases} \dfrac{(n-1)!}{m_\lambda\,(n-k-1)!} & \text{if } m \geq 2, \\ (r_1 + 2(n-k))\ \dfrac{(n-1)!}{m_\lambda\,(n-k)!} & \text{if } m = 0. \end{cases}$$

Note that the type D analogue of $NC^A(n)$ proposed in [20] fails to preserve this similarity between noncrossing and nonnesting partitions [1, section 6].

Finally, we show that this equidistribution of noncrossing and nonnesting partitions for the classical types A, B, C, D leads to a case-by-case proof of a result (Theorem 6.3) valid for all (finite, crystallographic) root systems: there are embeddings of the sets of noncrossing and nonnesting partitions into the intersection lattice Π^W of the Coxeter hyperplane arrangement, and the two distributions according to W-orbits coincide.

This paper is organized as follows. Section 2 collects the necessary background and definitions related to the Coxeter group of type D_n, noncrossing partitions, and nonnesting partitions. In particular, the poset NC^{D_n} is explicitly described. We also include a few enumerative results from [1] which are used in the following sections. Theorem 1.1 is proved in section 3 after the poset $NC^D(n)$ is defined. Theorem 1.2 is proved in section 4 using Theorem 1.1 and bijective methods similar to those employed in [13, 20] in the case of $NC^A(n)$ and $NC^B(n)$. Theorem 1.3 is proved in section 5. Section 6 describes the embeddings of the sets of noncrossing and nonnesting partitions into the intersection lattice Π^W and proves Theorem 6.3 on the equidistribution of their W-orbits. Section 7 concludes with a few remarks.

2. Background and definitions. This section includes notation, definitions, and some basic background related to Coxeter groups as well as noncrossing and nonnesting partitions of types B and D.

We will mostly follow notation introduced in [1, 12, 20]. We refer the reader to the texts by Humphreys [15] and Stanley [24] for any undefined terminology related to Coxeter groups and partially ordered sets, respectively. Throughout the paper we let

$$[n] := \{1, 2, \dots, n\},$$
$$[n]^{\pm} := \{-1, -2, \dots, -n, 1, 2, \dots, n\}$$

for any positive integer n.

The Coxeter group D_n. Let S_{2n} denote the symmetric group on the set $[n]^{\pm}$. For any cycle $c = (i_1, i_2, \dots, i_k)$ in S_{2n}, we let $\bar{c} = (-i_1, -i_2, \dots, -i_k)$. If c is the transposition (i, j) and $i \neq -j$, we denote by $((i, j))$ the product $c\bar{c} = (i, j)(-i, -j)$ and call $((i, j))$ a *D_n-reflection*, or simply a *reflection*. The Coxeter group W^{D_n} is the subgroup of S_{2n} generated by the reflections $((i, j))$. Any element of W^{D_n} can be expressed uniquely (up to reordering) as a product of disjoint cycles

$$c_1\bar{c}_1 \cdots c_k\bar{c}_k \, d_1 \cdots d_r, \tag{2.1}$$

each having at least two elements, where $\bar{d}_j = d_j$ for $j = 1, 2, \dots, r$ and r is even; see, for instance, [12, Proposition 3.1]. Following [12], for a cycle $c = (i_1, i_2, \dots, i_k)$ in S_{2n} we write

$$((i_1, i_2, \dots, i_k)) = c\bar{c} = (i_1, i_2, \dots, i_k)(-i_1, -i_2, \dots, -i_k)$$

and call $c\bar{c}$ a *paired cycle* if c is disjoint from $\bar{c}$. We also write

$$c = [i_1, i_2, \dots, i_k]$$

if $c = \bar{c} = (i_1, \dots, i_k, -i_1, \dots, -i_k)$ and call c a *balanced cycle*. Note that $[i]$ denotes both the balanced cycle $(i, -i)$ and the set $\{1, 2, \dots, i\}$. We will leave it to the reader to decide which notation is meant each time, hoping that no confusion will arise.

For $w \in W^{D_n}$ we denote by $l(w)$ the minimum number r for which w can be written as a product of r reflections and call it the *length* of w. (Note: this is *not* the usual Coxeter group length function, which is defined with respect to the *simple* reflections as generating set.) The cycle $((i_1, i_2, \dots, i_k))$ has length $k - 1$. The length of any element of W^{D_n} in the form (2.1) can be written as a sum over its paired and balanced cycles, where the contribution of $((i_1, i_2, \dots, i_k))$ and $[i_1, i_2, \dots, i_k]$ to this sum is $k - 1$ and k, respectively [12, section 3]. We denote by T^{D_n} the partial order on the set W^{D_n} defined by letting $u \leq w$ if $l(w) = l(u) + l(u^{-1}w)$. The poset T^{D_n} is graded by length and has the identity element 1 as its unique minimal element. For a choice γ of a Coxeter element of W^{D_n}, which we fix as $\gamma = [1, 2, \dots, n-1][n]$ for convenience, we denote by NC^{D_n} the interval $[1, \gamma]$ in the poset T^{D_n}. The poset NC^{D_n} is a self-dual, graded lattice of rank n [4, section 2], [12, section 4], where the rank function is the restriction of the rank function from T^{D_n}.

Noncrossing partitions. A *B_n-partition* is a partition π of the set $[n]^{\pm}$ into blocks such that (i) if B is a block of π, then its negative $-B$ is also a block of π, and (ii) there is at most one block, called the *zero block* if present, which contains both i and $-i$ for some i. The *type* of π is the integer partition λ which has a part equal to the

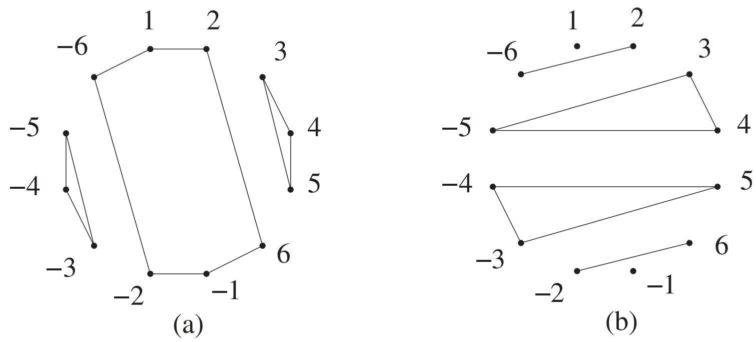

FIG. 1. *Two elements of* $NC^B(n)$ *for* $n = 6$ *with blocks* (a) $\{3,4,5\}$, $\{-3,-4,-5\}$, $\{1,2,6,-1,-2,-6\}$ *and* (b) $\{3,4,-5\}$, $\{-3,-4,5\}$, $\{2,-6\}$, $\{-2,6\}$, $\{1\}$, $\{-1\}$.

cardinality of B for each pair $\{B, -B\}$ of nonzero blocks of π. Thus λ is a partition of $n-m$, where m is half the size of the zero block of π, if present, and $m = 0$ otherwise. We refer to the parts of λ as the *block sizes* of π. A *D_n-partition* is a B_n-partition π with the additional property that the zero block of π, if present, does not consist of a single pair $\{i, -i\}$. The set of all B_n-partitions, ordered by refinement, is denoted by $\Pi^B(n)$. Its subposet consisting of all D_n-partitions is denoted by $\Pi^D(n)$. The posets $\Pi^B(n)$ and $\Pi^D(n)$ are geometric lattices which are isomorphic to the intersection lattices of the B_n and D_n Coxeter hyperplane arrangements, respectively, and hence they can be considered as type B and D analogues of the partition lattice $\Pi(n)$. In particular they are graded of rank n, and the corank of an element π in either poset is the number of pairs $\{B, -B\}$ of nonzero blocks of π.

Let us label the vertices of a convex $2n$-gon as $1, 2, \ldots, n, -1, -2, \ldots, -n$ clockwise, in this order. Given a B_n-partition π and a block B of π, let $\rho(B)$ denote the convex hull of the set of vertices labeled with the elements of B. We call π *noncrossing* if $\rho(B)$ and $\rho(B')$ have void intersection for any two distinct blocks B and B' of π. Two noncrossing partitions are depicted in Figure 1 for $n = 6$. The subposet of $\Pi^B(n)$ consisting of the noncrossing B_n-partitions is a self-dual, graded lattice of rank n which is denoted by $NC^B(n)$ [20, section 2].

Nonnesting partitions. Let $e_1, e_2, \ldots, e_n$ be the unit coordinate vectors in $\mathbb{R}^n$ and let Φ be a root system of one of the types B_n, C_n, or D_n. In what follows, we identify Φ with its type X_n and fix the choices

$$\Phi^+ = \begin{cases} \{e_i \pm e_j : 1 \le i < j \le n\} & \text{if } \Phi = D_n, \\ D_n^+ \bigcup \{e_i : 1 \le i \le n\} & \text{if } \Phi = B_n, \\ D_n^+ \bigcup \{2e_i : 1 \le i \le n\} & \text{if } \Phi = C_n \end{cases}$$

of positive roots for Φ. The *root poset* of Φ is the set Φ^+ of positive roots partially ordered by letting $\alpha \le \beta$ if $\beta - \alpha$ is a nonnegative linear combination of the elements of Φ^+. An *antichain* in Φ^+ is a subset of Φ^+ consisting of pairwise incomparable elements.

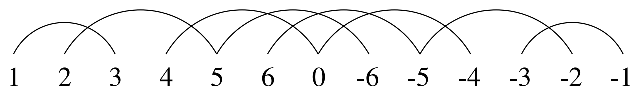

FIG. 2. *A picture of the* B_6*-nonnesting partition with blocks* $\{1,3\}$*,* $\{-1,-3\}$*,* $\{2,5,-6\}$*,* $\{-2,-5,6\}$*, and* $\{4,-4\}$*.*

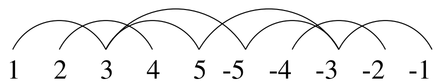

FIG. 3. *A picture of the* D_5*-nonnesting partition with blocks* $\{2,4\}$*,* $\{-2,-4\}$*, and* $\{1,3,5,$ $-1,-3,-5\}$*.*

Given an antichain A in Φ^+, we define an equivalence relation on the set $[n]^{\pm}\cup\{0\}$ if $\Phi = B_n$ or C_n, and on the set $[n]^{\pm}$ if $\Phi = D_n$, as follows. For $1 \le i < j \le n$, let

$$\begin{aligned} i \sim j \text{ and } -i \sim -j &\text{ if } e_i - e_j \in A,\\ i \sim -j \text{ and } -i \sim j &\text{ if } e_i + e_j \in A. \end{aligned}$$

Moreover, in the cases $\Phi = B_n$ or C_n, let

$$i \sim 0 \sim -i \quad \text{if } e_i \in A \text{ or } 2e_i \in A, \text{ respectively.}$$

Let $\pi_0(A)$ be the set of equivalence classes of the transitive closure of $\sim$. Let $\pi(A)$ be the partition of $[n]^{\pm}$ obtained from $\pi_0(A)$ by removing 0 from its class if $\Phi = B_n$ or C_n, and let $\pi(A) = \pi_0(A)$ if $\Phi = D_n$. Observe that $\pi(A)$ is a B_n-partition. Moreover, in the case $\Phi = D_n$, $\pi(A)$ has a zero block if and only if A contains both $e_i - e_n$ and $e_i + e_n$ for some $i < n$ and hence, in this event, the zero block contains $\{n, -n\}$ and at least one more pair $\{i, -i\}$. Thus in general $\pi(A)$ is a Φ-partition, where a C_n-partition is defined to be the same as a B_n-partition. A Φ*-nonnesting partition* is a Φ-partition of the form $\pi(A)$ for some antichain A in Φ^+. We denote by NN^{Φ} the set of Φ-nonnesting partitions and refer the reader to [1, section 2] and Figures 2 and 3 for the motivation behind the terminology "nonnesting," suggested by Postnikov [1], [20, Remark 2]. By definition, NN^{Φ} is in bijection with the set of antichains in the root poset Φ^+.

Block size enumeration. For an integer partition λ, we denote by $NC^B_\lambda(n)$ the set of elements of $NC^B(n)$ of type λ. Similarly, for $\Phi = B_n, C_n$, or D_n we denote by NN^{Φ}_λ the set of Φ-nonnesting partitions of type λ. The following theorem is the main result of [1].

THEOREM 2.1 (see [1]). *Let* λ *be a partition of* $n-m$ *with* k *parts, where* $m \ge 0$*, and let* $m_\lambda = r_1! \cdot r_2! \cdots$*, where* r_i *is the number of parts of* λ *equal to* i*.*

(i)

$$\#NC^B_\lambda(n) = \frac{n!}{m_\lambda\,(n-k)!}.$$

(ii) *The same formula holds for* Φ*-nonnesting partitions if* $\Phi = B_n$ *or* C_n*:*

$$\#NN^{B_n}_\lambda = \#NN^{C_n}_\lambda = \frac{n!}{m_\lambda\,(n-k)!}.$$

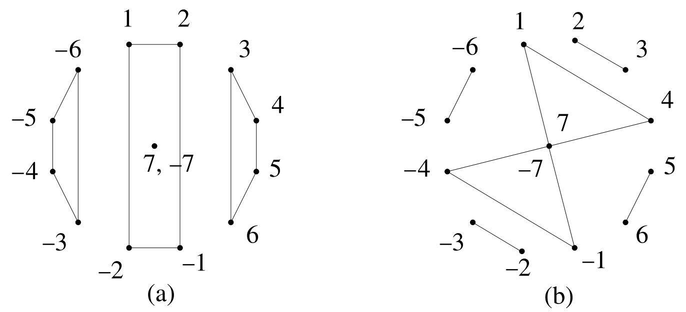

FIG. 4. *Two elements of* $NC^D(n)$ *for* $n = 7$ *with blocks* (a) $\{3, 4, 5, 6\}$, $\{-3, -4, -5, -6\}$, $\{1, 2, 7, -1, -2, -7\}$ *and* (b) $\{1, 4, 7\}$, $\{-1, -4, -7\}$, $\{2, 3\}$, $\{-2, -3\}$, $\{5, 6\}$, $\{-5, -6\}$.

(iii) *For* $m \geq 2$,

$$\#NN_{\lambda}^{D_n} = \frac{(n-1)!}{m_{\lambda}\,(n-k-1)!}.$$

3. Noncrossing partitions of type *D*. In this section we define our type D analogue of the noncrossing partition lattice $NC^A(n)$ and prove Theorem 1.1. Let us label the vertices of a regular $(2n-2)$-gon as $1, 2, \ldots, n-1, -1, -2, \ldots, -(n-1)$ clockwise, in this order, and label its centroid with both n and $-n$. Given a D_n-partition π and a block B of π, let $\rho(B)$ denote the convex hull of the set of points labeled with the elements of B. Two distinct blocks B and B' of π are said to *cross* if $\rho(B)$ and $\rho(B')$ do not coincide and one of them contains a point of the other in its relative interior. Observe that the case $\rho(B) = \rho(B')$, which we have allowed, can occur only when B and B' are the singletons $\{n\}$ and $\{-n\}$, and that if π has a zero block B, then B and the block containing n cross unless $\{n, -n\} \subseteq B$.

The poset $NC^D(n)$ is defined as the subposet of $\Pi^D(n)$ consisting of those D_n-partitions π with the property that no two blocks of π cross. Figure 4 shows two elements of $NC^D(n)$ for $n = 7$, one with a zero block and one with no zero block. Figure 5 shows the Hasse diagram of $NC^D(n)$ for $n = 3$.

PROPOSITION 3.1. *The poset* $NC^D(n)$ *is a graded lattice of rank* n *in which the corank of* π *is equal to the number of pairs* $\{B, -B\}$ *of nonzero blocks of* π.

Proof. Since $NC^D(n)$ is finite with a maximum and minimum element, to prove that it is a lattice, it suffices to show that meets in $NC^D(n)$ exist. Indeed, given elements x, y of $NC^D(n)$, one can check that the meet z of x and y in $\Pi^D(n)$ is an element of $NC^D(n)$ and hence z is also the meet of x and y in $NC^D(n)$.

As was the case for $NC^B(n)$ [20, Proposition 2], the rest of the proposition follows from the observation that given any two elements $\pi_1 \leq \pi_2$ of $NC^D(n)$, there exists a maximal chain in the interval $[\pi_1, \pi_2]$ of $\Pi^D(n)$ which passes only through elements of $NC^D(n)$, so that the grading of $NC^D(n)$ is inherited from that of $\Pi^D(n)$. ∎

To prove Theorem 1.1 we need to describe the covering relations in the posets T^{D_n} and $NC^D(n)$. In the case of T^{D_n}, the result of multiplying any element of D_n with a reflection $(\!(i, j)\!)$ is described explicitly in [12, Example 3.6]. From the computations given there we can conclude that y covers x in T^{D_n} if and only if x can be obtained from y by replacing one or two balanced cycles of y or one paired cycle of y with one

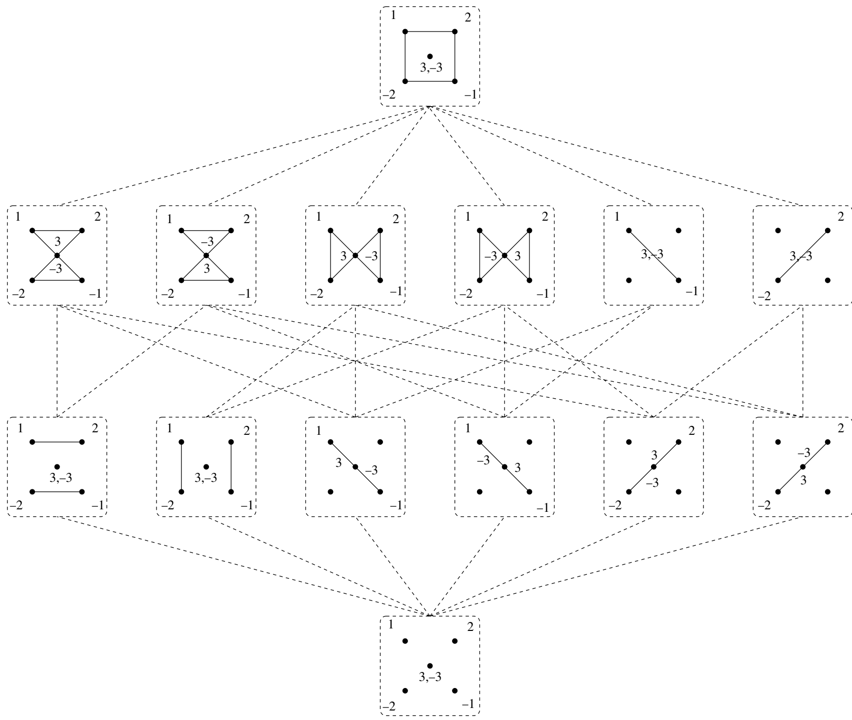

FIG. 5. *The lattice $NC^D(n)$ for $n = 3$.*

or more cycles as follows:

$$
\begin{aligned}
&[i_1, i_2, \ldots, i_k] \longrightarrow [i_1, \ldots, i_j]\, (\!(i_{j+1}, \ldots, i_k)\!),\\
&(\!(i_1, i_2, \ldots, i_k)\!) \longrightarrow (\!(i_1, \ldots, i_j)\!)\, (\!(i_{j+1}, \ldots, i_k)\!),\\
&[i_1, \ldots, i_j]\, [i_{j+1}, \ldots, i_k] \longrightarrow (\!(i_1, i_2 \ldots, i_k)\!).
\end{aligned}
\tag{3.1}
$$

In the case of $NC^D(n)$, it follows directly from the definition that y covers x in $NC^D(n)$ if and only if x can be obtained from y by one of the following:

(i) splitting the zero block of y into the zero block of x and a pair $\{B, -B\}$ of nonzero blocks,

(ii) splitting a pair of nonzero blocks $\{B, -B\}$ of y into two such pairs of x, or

(iii) splitting the zero block of y into one pair $\{B, -B\}$ of nonzero blocks of x (so that one of $B, -B$ contains n and the other contains $-n$).

Proof of Theorem 1.1. For $x \in NC^{D_n}$, let $f(x)$ denote the partition of $[n]^{\pm}$ whose nonzero blocks are formed by the paired cycles of x, and whose zero block is the union of the elements of all balanced cycles of x if such exist. We first observe that $f(x) \in NC^D(n)$. Indeed, this is clear if x is the top element $\gamma = [1, 2, \ldots, n-1]\,[n]$ of NC^{D_n}. If not, then x is covered by some element y of NC^{D_n} and we may assume, by induction on the corank of x, that y has either zero or two balanced cycles, one of which must be $[n]$ in the latter case, and that $f(y) \in NC^D(n)$. Since x can be obtained from y by one of the moves in the list (3.1), it follows with a case-by-case check that x has zero or two balanced cycles as well, one of which must be $[n]$ in the latter case, and that $f(x) \in NC^D(n)$. The map

$$f : NC^{D_n} \to NC^D(n)$$

is thus well-defined and order-preserving, since $f(x) \le f(y)$ in $NC^D(n)$ follows from (3.1) when x is covered by y in NC^{D_n}.

To define the inverse map, for $x \in NC^D(n)$, let $g(x)$ be the element of NC^{D_n}

- whose paired cycles are formed by the nonzero blocks of x, each ordered with respect to the cyclic order

$$-1, -2, \ldots, -n, 1, 2, \ldots, n, -1,$$

  and
- whose balanced cycles are $[n]$ and the cycle formed by the entries of the zero block of x other than n and $-n$, ordered in the same way, if the zero block is present in x.

We claim that $g(x) \in NC^{D_n}$. This is clear if x is the top element of $NC^D(n)$. If not, let y be any element of $NC^D(n)$ which covers x. We may assume, by induction on the corank of x, that $g(y) \in NC^{D_n}$, in other words, that $g(y) \le \gamma$ holds in T^{D_n}. It follows from the possible types of covering relations in the posets $NC^D(n)$ and T^{D_n} that $g(y)$ covers $g(x)$ in T^{D_n}. This implies that $g(x) \le \gamma$ holds in T^{D_n} or, equivalently, that $g(x) \in NC^{D_n}$. Thus the map

$$g : NC^D(n) \to NC^{D_n}$$

is well-defined and order-preserving, since $g(x) \le g(y)$ in NC^{D_n} follows from (3.1) when x is covered by y in $NC^D(n)$. Since f and g are clearly inverses of each other, they are poset isomorphisms. ∎

4. The zeta polynomial and chain enumeration. In this section we use bijective methods similar to those employed in [13, 20] for $NC^A(n)$ and $NC^B(n)$ to prove Theorem 1.2. We first recall a few constructions from [20, section 3]. After setting

$$P_n^B := \{(L, R) : L, R \subseteq [n], \#L = \#R\},$$

a map $\tau^B : P_n^B \to NC^B(n)$ is constructed in [20, section 3] as follows. Given $x = (L, R) \in P_n^B$, place a left parenthesis before each occurrence of i and $-i$ in the infinite cyclic sequence

$$\ldots, -1, -2, \ldots, -n, 1, 2, \ldots, n, -1, -2, \ldots \tag{4.1}$$

for $i \in L$ and a right parenthesis after each occurrence of i and $-i$ for $i \in R$. Let the strings of integers inside the lowest level matching pairs of parentheses form blocks of $\tau^B(x)$. Remove these lowest level parentheses from (4.1) and the integers they enclose and continue similarly with the remaining parenthesization until all parentheses have been removed. The remaining integers, if any, form the zero block of $\tau^B(x)$. We have the following proposition.

PROPOSITION 4.1 (see [20, Proposition 6]). *The map τ^B is a bijection from the set P_n^B to $NC^B(n)$. Moreover, for any pair $x = (L, R) \in P_n^B$, the number of pairs $\{B, -B\}$ of nonzero blocks of $\tau^B(x)$ is equal to $\#R$.*

To extend the previous proposition to the type D case, let

$$P_n^D = P_{n-1}^B \bigcup \{(L, R, \varepsilon) : L, R \subseteq [n-1], \#R = \#L + 1,\ \varepsilon = \pm 1\}.$$

For $x \in P_n^D$, we define a partition $\pi = \tau^D(x) \in \Pi^D(n)$ as follows. If $x \in P_{n-1}^B$, then π is the partition obtained from $\tau^B(x)$ by adding n and $-n$ to the zero block of $\tau^B(x)$, if such a block exists, and by adding the singletons $\{n\}$ and $\{-n\}$ to $\tau^B(x)$ otherwise. Suppose that x is not in P^B_{n-1}, say, $x = (L, R, \varepsilon)$. We parenthesize the infinite cyclic sequence

$$\ldots, -1, -2, \ldots, -(n-1), 1, 2, \ldots, n-1, -1, -2, \ldots \tag{4.2}$$

as in the type B case and form blocks of π with the same procedure, until a right parenthesis remains after each occurrence of i and $-i$ for a unique $i \in [n-1]$. Then let B and $-B$ be blocks of π, where B consists of the integers in

$$\{-i-1, \ldots, -n+1, 1, 2, \ldots, i\}$$

which have not been removed from the infinite sequence together with n or $-n$, if $\varepsilon = 1$ or $\varepsilon = -1$, respectively. For instance, if $n = 9$, $L = \{2, 5, 6\}$, $R = \{1, 3, 7, 8\}$, and $\varepsilon = -1$, then π has blocks $\{2, 3\}$, $\{5, 8\}$, $\{6, 7\}$, $\{1, -4, -9\}$, and their negatives.

It is clear from the previous construction that $\pi \in NC^D(n)$; thus we have a well-defined map $\tau^D : P^D_n \to NC^D(n)$.

Proposition 4.2. *The map τ^D is a bijection from the set P^D_n to $NC^D(n)$. Moreover, for any $x \in P^D_n$, the number of pairs $\{B, -B\}$ of nonzero blocks of $\tau^D(x)$ is equal to*

$$\begin{cases} \#R & \text{if } x \in P^B_{n-1} \text{ and } \tau^B(x) \text{ has a zero block}, \\ \#R + 1 & \text{if } x \in P^B_{n-1} \text{ and } \tau^B(x) \text{ has no zero block}, \\ \#R & \text{if } x \notin P^B_{n-1}. \end{cases}$$

Proof. The inverse of τ^D can be defined as in the proof of [20, Proposition 6] for the map τ^B. More precisely, given $\pi \in NC^D(n)$, find a nonzero block B of π such that the elements of $B \backslash \{n, -n\}$ form a nonempty, consecutive string of integers in the sequence (4.2). If B does not contain n or $-n$, then place the absolute values of the first and last element of B, with respect to (4.2), in L and R, respectively. If it does, then place the absolute value i of the last element of $B \backslash \{n, -n\}$, with respect to (4.2), in R and let $\varepsilon = 1$ or $\varepsilon = -1$ if n or $-n$ is in the same block as i, respectively. Remove the elements of B and $-B$ from π and (4.2) and continue similarly until the zero block, or the singletons $\{n\}$ and $\{-n\}$, or no block of π remains. We leave it to the reader to check that this map is indeed the inverse of τ^D. The second statement is obvious. ∎

It is shown in [20, Proposition 7] that the bijection τ^B of Proposition 4.1 extends to a bijection from the set

$$P^B_{n,m} = \left\{ (L, R_1, \ldots, R_{m-1}) : L, R_j \subseteq [n], \sum_{j=1}^{m-1} \#R_j = \#L \right\}$$

to the set of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{m-1}$ in $NC^B(n)$. This bijection is defined as follows. Given $(L, R_1, \ldots, R_{m-1}) \in P^B_{n,m}$, place a left parenthesis before each occurrence of i and $-i$ in the infinite cyclic sequence (4.1) for $i \in L$ and a right parenthesis labeled $)^j$ after each occurrence of i and $-i$ for $i \in R_j$. Observe that more than one right parenthesis with different labels may have been placed after some integers in (4.1). In this case order these right parentheses as

$$)^{j_1} \,)^{j_2} \, \cdots \,)^{j_t},$$

where $j_1 < j_2 < \cdots < j_t$. Read this parenthesization as in the case of the map τ^B to obtain $\pi_1 \in NC^B(n)$. Next, remove from the parenthesization all right parentheses labeled $)^1$ and their corresponding left parentheses to obtain $\pi_2 \in NC^B(n)$, and continue the process until all parentheses have been removed to obtain the multichain $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{m-1}$.

The type D analogue of this construction is given in the following proposition. To state it we introduce the following notation. Think of a multichain $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{m-1}$ in $\Pi^B(n)$ as a multichain from $\hat{0}$ to $\hat{1}$ which has m steps; in other words, set $\pi_0 := \hat{0}$ and $\pi_m := \hat{1}$. The *rank jump vector* for such a multichain c is the composition $s = (s_1, \ldots, s_m)$ of the number n (denoted $s \models n$) defined by $s_i := \mathrm{r}(\pi_i) - \mathrm{r}(\pi_{i-1})$. There is a unique step i at which a zero block is first created, meaning that π_{i-1} has no zero block but π_i does. Define $\mathrm{ind}(c)$ to be this index i.

PROPOSITION 4.3. *The bijection τ^D extends to a bijection from the union $P^D_{n,m}$ of $P^B_{n-1,m}$ with the set*

$$\left\{(L, R_1, \ldots, R_{m-1}, \varepsilon) : L, R_j \subseteq [n-1], \sum_{j=1}^{m-1} \#R_j = \#L + 1,\ \varepsilon = \pm 1\right\}$$

to the set of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{m-1}$ in $NC^D(n)$. Moreover, for $x \in P^D_{n,m}$, one has that

- *if $x \notin P^B_{n-1,m}$ and $x = (L, R_1, \ldots, R_{m-1}, \varepsilon)$, then the multichain in $NC^D(n)$ corresponding to x has rank jump vector*

$$s = (n - 1 - \#L, \#R_1, \ldots, \#R_{m-1}),$$

- *if $x \in P^B_{n-1,m}$ and the multichain c in $NC^B(n-1)$ corresponding to x under the generalized map τ^B has rank jump vector $s = (s_1, \ldots, s_m)$ and $\mathrm{ind}(c) = i$, then the multichain in $NC^D(n)$ corresponding to x has rank jump vector*

$$(s_1, \ldots, s_{i-1}, s_i + 1, s_{i+1}, \ldots, s_m).$$

Proof. Given $x \in P^D_{n,m}$, we construct a multichain $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{m-1}$ in $NC^D(n)$ as follows. If $x \in P^B_{n-1,m}$, let $\pi'_1 \leq \pi'_2 \leq \cdots \leq \pi'_{m-1}$ be the multichain in $NC^B(n-1)$ corresponding to x under the bijection of [20, Proposition 7]. Let π_i be the partition obtained from π'_i by adding n and $-n$ to the zero block of π'_i if such a block exists, and by adding the singletons $\{n\}$ and $\{-n\}$ to π'_i otherwise. It is then clear that $\pi_i \in NC^D(n)$ and that $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{m-1}$ is a multichain in $NC^D(n)$. Suppose now that x is not in $P^B_{n-1,m}$, say, $x = (L, R_1, \ldots, R_{m-1}, \varepsilon)$. Place a left parenthesis before each occurrence of i and $-i$ in the infinite cyclic sequence (4.2) for $i \in L$ and a right parenthesis labeled $)^j$ after each occurrence of i and $-i$ for $i \in R_j$, using the same rules as in the type B case described earlier for placing multiple right parentheses. Read this parenthesization as in the case of the map τ^D to obtain $\pi_1 \in NC^D(n)$. Observe that the singletons $\{n\}$ and $\{-n\}$ may be blocks of π_1 if $m - 1 \geq 2$. Next, remove from the parenthesization all right parentheses labeled $)^1$ and their corresponding left parentheses, if any, to obtain $\pi_2 \in NC^D(n)$, and continue the process until all parentheses have been removed. This results in a multichain $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{m-1}$ in $NC^D(n)$ in which n belongs to a nonsingleton, nonzero block of π_j for at least one index j.

To define the inverse of this map, let $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{m-1}$ be a multichain in $NC^D(n)$. Parenthesize the sequence (4.2) by applying the inverse of τ^D to π_{m-1} and label all right parentheses by $)^{m-1}$. Repeat the process with π_{m-2} and label all right parentheses by $)^{m-2}$, but include neither the new pairs of parentheses that would produce more than one left parenthesis before the occurrence of a single integer in (4.2), nor a new unmatched right parenthesis, if one exists already. Continue similarly with the remaining elements of the multichain to get an element of $P^D_{n,m}$. We leave it again to the reader to check that this map is well-defined and that the two maps are indeed inverses of each other. The "moreover" statement is obvious from the construction. ∎

In [20, Proposition 7], the bijection from $P^B_{n,m}$ to multichains in $NC^B(n)$ was used to deduce that there are $\binom{n}{s_1}\cdots\binom{n}{s_m}$ chains in $NC^B(n)$ with rank jump vector $s=(s_1,\ldots,s_m)$. In order to perform the analogous chain enumeration for $NC^D(n)$, we will need the following refinement of this type B result, keeping track of the extra statistic $\mathrm{ind}(c)$.

LEMMA 4.4. *Let $s=(s_1,\ldots,s_m)\models n$. Then among the $\binom{n}{s_1}\cdots\binom{n}{s_m}$ chains c in $NC^B(n)$ having rank jump vector s, the fraction of those having* $\mathrm{ind}(c)=i$ *is equal to $\frac{s_i}{n}$.*

Unlike our other enumerative results, our proof of Lemma 4.4 is not bijective. For this reason, we have relegated it to the appendix.

Proof of Theorem 1.2. In view of Theorem 1.1, it suffices to prove the theorem for the poset $NC^D(n)$ instead.

(i) Clearly, the set P^D_n has

$$\binom{2n-2}{n-1}+2\binom{2n-2}{n}=\binom{2n}{n}-\binom{2n-2}{n-1}$$

elements. Hence the statement on the total number of elements of $NC^D(n)$ follows from the first statement in Proposition 4.2. By an easy computation, the statement on the number of elements of rank k is equivalent to the case $m=2$ in (ii).

(ii) This follows from Proposition 4.3 and Lemma 4.4. The summand $2\binom{n-1}{s_1}\cdots\binom{n-1}{s_m}$ counts the chains coming from $x\in P^D_{n,m}-P^B_{n-1,m}$. Within the summation, the ith term

$$\begin{aligned}\binom{n-1}{s_1}&\cdots\binom{n-2}{s_i-2}\cdots\binom{n-1}{s_m}\\ &=\frac{s_i-1}{n-1}\binom{n-1}{s_1}\cdots\binom{n-1}{s_i-1}\cdots\binom{n-1}{s_m}\end{aligned}$$

counts the chains coming from $x\in P^B_{n-1,m}$ that correspond to chains c in $NC^B(n-1)$ with rank jump vector $(s_1,\ldots,s_{i-1},s_i-1,s_{i+1},\ldots,s_m)$ and $\mathrm{ind}(c)=i$.

(iii) Observe (as for $P^B_{n,m}$ in the proof of [20, Proposition 7]) that the set $P^D_{n,m}$ has

$$2\binom{m(n-1)}{n}+\binom{m(n-1)}{n-1}$$

elements, and recall that the value $Z(P,m)$ of the zeta polynomial for a poset P is defined to be the number of multichains in P of cardinality $m-1$. The formula in (iii) for the zeta polynomial of $NC^D(n)$ then follows from Proposition 4.3.

(iv) Both assertions follow from the zeta polynomial calculated in (iii), via [24, Proposition 3.11.1]. ∎

Finally, we briefly discuss how Theorem 1.2(ii) leads to a nice expression for $F_{NC^D(n)}$, where F_P denotes Ehrenborg's quasi-symmetric function associated with a ranked poset P; we refer the reader to [14, 26] for the definitions.

As mentioned in section 1, the posets $NC^A(n), NC^B(n)$, and $NC^D(n)$ are self-dual by virtue of a result of Bessis [4, section 2.3], stating that the poset NC^W is always self-dual. A case-free proof of a stronger statement, namely, that all intervals of NC^W are self-dual, was outlined by McCammond [17, section 3]. Alternatively, it is easy to check from the explicit descriptions of $NC^A(n), NC^B(n)$, and $NC^D(n)$ that any interval in one of these posets is isomorphic to a Cartesian product of posets lying in the union of these three families; see also [18], [20, Remark 1], and the appendix. Hence their intervals are also self-dual, which implies that the posets themselves are *locally rank-symmetric* and their quasi-symmetric functions are actually *symmetric* functions.

In [27], Stanley used the known explicit expressions for the numbers of chains in $NC^A(n)$ and $NC^B(n)$ with given rank jump vector to compute nice formulas for $F_{NC^A(n)}$ and $F_{NC^B(n)}$ (and to connect them with symmetric group actions on parking functions of types A and B; see also Biane [8]). He proved that

$$F_{NC^A(n)} = \frac{1}{n}\left[t^{n-1}\right] E(t)^n$$

and

$$F_{NC^B(n)} = [t^n]\, E(t)^n,$$

where $E(t) := \prod_{i\geq 1}(1+x_i t)$ and $[t^n]\,\psi(t)$ denotes the coefficient of t^n in a formal power series $\psi(t)$ in the variable t. An equally easy computation (which we omit) shows that Theorem 1.2(ii) is equivalent to the following proposition.

PROPOSITION 4.5. *We have*

$$F_{NC^D(n)} = [t^n]\, E(t)^{n-1}\left(2+\sum_{i\geq 1}\frac{x_i^2 t^2}{1+x_i t}\right).$$

5. Enumeration by block sizes. For an integer partition λ, let $NC^D_\lambda(n)$ denote the set of elements of $NC^D(n)$ with block sizes λ. To enumerate the elements of $NC^D(n)$ by block sizes we define a map

$$\tau : NC^D(n) \to NC^B(n-1)$$

as follows. Let $\pi \in NC^D(n)$. If π has a zero block B, then simply remove n and $-n$ from B to obtain $\tau(\pi)$. Otherwise n and $-n$ are in distinct blocks B and $-B$ of π. Then either remove B and $-B$ from π if they are singletons, or if not, replace them with the zero block $B \cup (-B) \setminus \{n, -n\}$ to obtain $\tau(\pi)$. It should be clear that $\tau(\pi) \in NC^B(n-1)$.

LEMMA 5.1. *The map $\tau : NC^D(n) \to NC^B(n-1)$ has the following property: if $x \in NC^B_\lambda(n-1)$, with $\lambda \vdash n-m-1$, then the set $\tau^{-1}(x)$ consists of $2m+1$ elements. Moreover, $2m$ of these have type $\lambda \uplus \{m+1\}$, and the remaining element has type*

$$\begin{cases} \lambda \uplus \{1\} & \text{if } \ m=0, \\ \lambda & \text{if } \ m \geq 1. \end{cases}$$

Proof. If $x \in NC^B_\lambda(n-1)$ has no zero block, then $\tau^{-1}(x)$ consists of a single element π, obtained from x by adding the singletons $\{n\}$ and $\{-n\}$. Suppose that x has a zero block B of size $2m$. A partition in $\tau^{-1}(x)$ is obtained either by adding $n, -n$ to the zero block B, or by splitting B into two parts C and $-C$ and replacing B with the pair of blocks $C \cup \{n\}$ and $-C \cup \{-n\}$. There are $2m$ ways to do the latter so that the resulting partition is in $NC^D(n)$. ◻

COROLLARY 5.2. *If λ is a partition of $n-m$, where $m \geq 0$, then $\#NC^D_\lambda(n)$ is given by the formula in Theorem* 1.3.

Proof. Lemma 5.1 implies that

$$\#NC^D_\lambda(n) = \begin{cases} \#NC^B_\lambda(n-1) & \text{if } m \geq 2, \\ \#NC^B_{\lambda\backslash 1}(n-1) + \sum_{p\geq 2}(2p-2)\ \#NC^B_{\lambda\backslash p}(n-1) & \text{if } m = 0, \end{cases}$$

and the result follows from Theorem 2.1(i). Note that, in the above formula, we interpret $NC^B_{\lambda\backslash p}(n-1)$ as empty for any integer $p \geq 1$ that does not appear as a part of λ. ◻

In the remainder of this section we show that nonnesting partitions of type D have the same distribution by block sizes as noncrossing partitions of the same type.

Assume that $\lambda \vdash n$, so that $NN^{B_n}_\lambda \subseteq NN^{D_n}_\lambda$, and observe that the inclusion is strict for $n \geq 3$ since $\{e_i + e_n, e_j - e_n\}$ is an antichain in D^+_n for $i < j < n$ but not in B^+_n. Let $\pi \in NN^{D_n}_\lambda$. Since π does not have a zero block, n and $-n$ belong to distinct blocks B and $-B$ of π, respectively. Let π' denote the partition obtained from π by exchanging n and $-n$ in the blocks B and $-B$, and let

$$\sigma : NN^{D_n}_\lambda \to NN^{B_n}_\lambda$$

be defined by

$$\sigma(\pi) = \begin{cases} \pi & \text{if } \pi \in NN^{B_n}, \\ \pi' & \text{otherwise.} \end{cases}$$

One can check directly from the definitions that σ is well-defined. Let $T_\lambda(n)$ be the set of partitions $\pi \in NN^{B_n}_\lambda$ such that if B is the block of π containing n, then $B \setminus \{n\}$ contains both positive and negative elements, and let $T^+_\lambda(n)$, $T^-_\lambda(n)$ be the sets of those $\pi \in NN^{B_n}_\lambda$ for which $B \setminus \{n\}$, if nonempty, contains only positive elements and only negative elements, respectively.

LEMMA 5.3. *Let $\lambda \vdash n$.*

(i) *The map $\sigma : NN^{D_n}_\lambda \to NN^{B_n}_\lambda$ induces a bijection between $NN^{D_n}_\lambda \setminus NN^{B_n}_\lambda$ and $T_\lambda(n)$.*

(ii) *We have*

$$\#T^+_\lambda(n) = \#T^-_\lambda(n) = \sum_{p\geq 1} \#NN^{B_{n-1}}_{\lambda\backslash p}$$

and

$$\#(T^+_\lambda(n) \bigcap T^-_\lambda(n)) = \#NN^{B_{n-1}}_{\lambda\backslash 1}.$$

Proof.

(i) If $\pi \in NN^{D_n}_\lambda \setminus NN^{B_n}_\lambda$, then there exist integers $i < j < n$ such that j and n are in a block B of π, while i and $-n$ are in a different block, which must be $-B$. Thus

$\{i, -j, -n\}$ is contained in a block of π and hence $\{i, -j, n\}$ is contained in a block of $\sigma(\pi)$. This implies that $\sigma(\pi) \in T_\lambda(n)$, so that the map $\sigma : NN_\lambda^{D_n} \setminus NN_\lambda^{B_n} \to T_\lambda(n)$ is well-defined. The inverse map again switches n and $-n$ in a partition in $T_\lambda(n)$ and is checked to be well-defined by reversing the previous argument.

(ii) The first two equalities follow from the fact that either $T_\lambda^+(n)$ or $T_\lambda^-(n)$ is in bijection with the set of B_{n-1}-nonnesting partitions whose type is obtained from λ by removing one of its parts, where the bijection removes the blocks B and $-B$ containing n and $-n$ of an element in $T_\lambda^+(n)$ or $T_\lambda^-(n)$ if these blocks are singletons, or replaces them with the zero block $B \cup (-B) \setminus \{n, -n\}$ if they are not. The last equality is obvious. □

COROLLARY 5.4. *If λ is as in Corollary 5.2, then $\# NN_\lambda^{D_n} = \# NC_\lambda^{D_n}$; that is, $\# NN_\lambda^{D_n}$ is given by the formula of Theorem 1.3.*

Proof. For $m \geq 2$ the statement is the content of Theorem 2.1(iii). Suppose that $m = 0$, i.e., $\lambda \vdash n$. Lemma 5.3(i) implies that

$$\# NN_\lambda^{D_n} = \# NN_\lambda^{B_n} + \# T_\lambda(n).$$

Since $T_\lambda(n) = NN_\lambda^{B_n} \setminus (T_\lambda^+(n) \cup T_\lambda^-(n))$, part (ii) of the same lemma gives

$$\# T_\lambda(n) = \# NN_\lambda^{B_n} - \# NN_{\lambda \setminus 1}^{B_{n-1}} - 2 \sum_{p \geq 2} \# NN_{\lambda \setminus p}^{B_{n-1}},$$

and the result follows from Theorem 2.1(ii). □

The next corollary also follows from the main result of [3] and the computations in the type D case carried out there in section 5.

COROLLARY 5.5 (see [3, section 5]). *The number of elements of NN^{D_n} with k pairs $\{B, -B\}$ of nonzero blocks is equal to*

$$\binom{n}{k}^2 - \frac{n}{n-1} \binom{n-1}{k} \binom{n-1}{k-1}.$$

Proof. This follows from Corollary 5.4 and Theorem 1.2(i). □

6. Block sizes and root systems. The goal of this section is to generalize the results of [1] and Theorem 1.3 on the classical root systems to an arbitrary (finite, crystallographic) root system (Theorem 6.3). To this end we begin by recalling some facts about noncrossing and nonnesting partitions for arbitrary finite Coxeter groups and root systems.

For a finite Coxeter group (W, S), acting with its natural reflection representation on a Euclidean space V, we denote by Π^W the poset of all subspaces of V which are intersections of reflecting hyperplanes of W, ordered by reverse inclusion. Thus Π^W is a graded (geometric) lattice of rank $\# S$ which is isomorphic to the lattice $\Pi(n)$, $\Pi^B(n)$, or $\Pi^D(n)$, defined in the first two sections, when W has type A_{n-1}, B_n, or D_n, respectively.

There is a natural embedding of the lattice of noncrossing partitions NC^W into Π^W. Recall from section 1 that NC^W is defined to be the interval $[1, \gamma]$ in a certain partial order T^W on the group W, where γ is any Coxeter element of W. It follows from results of Brady and Watt (see [4, Proposition 1.6.4]) that the map

$$\begin{aligned} &NC^W \to \Pi^W, \\ &\quad w \mapsto V^w := \{v \in V : w(v) = v\} \end{aligned}$$

is a rank- and order-preserving embedding.

Now assume that W is the finite Weyl group associated to a crystallographic root system Φ. Let Φ^+ be a choice of positive roots, equipped with the standard root order, and let Π be the corresponding set of simple roots. Let $\mathcal{A}^\Phi$ be the collection of all antichains in Φ^+, meaning subsets of pairwise incomparable elements. It turns out that, like NC^W, the set $\mathcal{A}^\Phi$ has a natural embedding into Π^W, endowing it with a poset structure. The crucial fact needed is a recent result of Sommers [23].

Theorem 6.1 (see [23, page 1]). *Given an antichain A of positive roots, there exists $w \in W$ such that $w(A) \subseteq \Pi$.*

Corollary 6.2. *If Φ is a crystallographic root system with Weyl group W, then the map*

$$\begin{aligned} \mathcal{A}^\Phi &\to \Pi^W, \\ A &\mapsto \bigcap_{\alpha \in A} \alpha^\perp \end{aligned}$$

is an injection, sending A to an element of rank $\#A$.

Proof. The image of A has rank $\#A$ because Theorem 6.1 implies that any antichain in $\mathcal{A}^\Phi$ is linearly independent (since Π is also). That the map is injective will follow from a stronger assertion about the interaction between the linear independence and convexity structure of Φ^+. Given $B \subseteq \Phi^+$, let $\overline{B}$ denote the *matroid closure* of B, meaning the subset of vectors in Φ^+ lying in the linear span of B. Let $\operatorname{ext}(B)$ denote the set of *extreme vectors* within the convex cone spanned by B. We then claim that for an antichain A in Φ^+,

$$A = \operatorname{ext}(\overline{A}).$$

This will show that the map is injective, since one has the alternate characterization of $\overline{A}$ as

$$\overline{A} = \left\{ \beta \in \Phi^+ : \bigcap_{\alpha \in A} \alpha^\perp \subseteq \beta^\perp \right\}.$$

To prove the claim note that, since A is linearly independent, $\operatorname{ext}(\overline{A})$ has at least as many elements as A. Consequently it suffices to show the inclusion $\operatorname{ext}(\overline{A}) \subseteq A$. To this end, given $\beta \in \operatorname{ext}(\overline{A})$, express β (uniquely) as

$$\beta = \sum_{\alpha \in A} c_\alpha \alpha. \tag{6.1}$$

Since β and all elements of A are positive roots, at least one of the coefficients c_α must be positive. Using Theorem 6.1 again, one can find $w \in W$ so that $w(A) \subset \Pi$. As $w(\beta)$ lies in Φ, it has a unique expression in terms of simple roots with all coefficients of the same sign. Hence the expression

$$w(\beta) = \sum_{\alpha \in A} c_\alpha w(\alpha),$$

obtained by applying w to (6.1), forces all of the other coefficients c_α to be nonnegative. Therefore (6.1) shows that β lies in the convex cone spanned by A. Since β is an extreme vector of the larger cone spanned by $\overline{A}$, it is extreme in this smaller cone, so it must lie in A. □

We require a notion that generalizes the "block sizes" of a type A, B, or D partition to an arbitrary intersection subspace in Π^W. This is supplied by the orbit map, sending a subspace to its W-orbit:

$$\begin{aligned} \Pi^W &\to \Pi^W/W, \\ U &\mapsto W \cdot U := \{w(U) : w \in W\}. \end{aligned}$$

We can now state the main result of this section.

THEOREM 6.3. *Let W be the Weyl group of a crystallographic root system Φ and consider the two composite maps*

$$\begin{aligned} f : NC^W &\hookrightarrow \Pi^W \to \Pi^W/W, \\ g : \mathcal{A}^\Phi &\hookrightarrow \Pi^W \to \Pi^W/W. \end{aligned}$$

Then for each W-orbit $x \in \Pi^W/W$, we have

$$\# f^{-1}(x) = \# g^{-1}(x).$$

Proof. The general statement follows from the corresponding statement for irreducible root systems, so one may proceed case-by-case via the classification.

For types A, B, and C, the statement follows from the results of [1] and the fact that the W-orbits of Π^W are precisely the sets of intersection subspaces whose corresponding type A or B partition has given block sizes. In the type D case, a slight complication arises due to the fact that the W-orbit of an intersection subspace is not always determined by the nonzero block sizes λ of its associated D_n-partition. In fact this occurs exactly when λ is a partition of n having only even parts (so that, in particular, n must be even). In this case the set of intersection subspaces in Π^W whose corresponding D_n-partitions have block sizes λ decomposes further into exactly two W-orbits, determined by one extra (parity) piece of data: pick arbitrarily one block B out of each pair $\{B, -B\}$ of blocks in the partition and compute the parity (even or odd) of the total number of negative elements in the union of these blocks. We claim, however, that the preimages under either f or g of two such parity orbits have the same number of elements, so that the result follows from Corollaries 5.2 and 5.4. To check the claim, simply observe that, for a partition λ of n with even parts, the swap of n and $-n$ gives rise to fixed-point free involutions on both $NC^D_\lambda(n)$ and $NN^{D_n}_\lambda$, which switch parity. This is obvious in the case of $NC^D_\lambda(n)$ and should be clear from the discussion preceding Lemma 5.3 in the case of $NN^{D_n}_\lambda$.

The exceptional types E_6, E_7, E_8, F_4, G_2 have been checked one by one with computer calculations, using software in Mathematica available from the second author. ∎

7. Remarks.

1. It would be interesting to find a conceptual, case-free proof of Theorem 6.3.

2. The set of maximal chains in the poset NC^W is in bijection with the set of factorizations of shortest possible length, henceforth called *minimal factorizations*, of a Coxeter element of W into reflections. Hence Theorem 1.2(iv) implies the following statement.

COROLLARY 7.1. *The number of minimal factorizations of a Coxeter element of the group W^{D_n} into reflections is equal to $2(n-1)^n$.*

A direct proof of this fact, analogous to the proofs of the corresponding statements by Biane [8] for the symmetric and hyperoctahedral group, is possible. More precisely, let $\gamma = [1, 2, \ldots, n-1]\,[n]$ be as in section 2 and let $\mathcal{M}_n$ be the set of tuples $(t_1, t_2, \ldots, t_n)$ of reflections in W^{D_n} such that $\gamma = t_1 t_2 \cdots t_n$. Let us label the reflections in W^{D_n} as follows:

$$\ell(t) = \begin{cases} i & \text{if } \ t = ((i, \pm n)) \ \text{ and } \ 1 \le i \le n-1, \\ i & \text{if } t = ((i, j)) \ \text{ and } \ 1 \le i < j \le n-1, \\ j & \text{if } t = ((i, -j)) \ \text{ and } \ 1 \le i < j \le n-1. \end{cases}$$

It has been shown by the first author that the map which assigns to any element $(t_1, t_2, \ldots, t_n)$ of $\mathcal{M}_n$ the sequence of labels $(\ell(t_1), \ell(t_2), \ldots, \ell(t_n))$ is a two-to-one map from the set $\mathcal{M}_n$ to $[n-1]^n$.

3. It is natural to conjecture that the poset $NC^D(n)$ is shellable. However, the EL-labellings given by Edelman and Björner [10] in the case of $NC^A(n)$ and Reiner [20] in the case of $NC^B(n)$ do not seem to extend to that of $NC^D(n)$.

4. It has been shown by Eleni Tzanaki (private communication) that the poset $NC^D(n)$ has a symmetric chain decomposition analogous to those of $NC^A(n)$ [22, Theorem 2] and $NC^B(n)$ [20, Theorem 13].

Appendix. Proof of Lemma 4.4. We recall the statement of the lemma.

LEMMA 4.4. *Let $s = (s_1, \ldots, s_m) \models n$. Then among the $\binom{n}{s_1} \cdots \binom{n}{s_m}$ chains c in $NC^B(n)$ having rank jump vector s, the fraction of those having* $\mathrm{ind}(c) = i$ *is equal to $\frac{s_i}{n}$.*

The proof will utilize the type B generalization [20, Theorem 16] of a result of Nica and Speicher [18] on incidence algebras, which we recall here.

Let R be any ring with unit and let P be a poset. The (R-valued) incidence algebra for P consists of R-valued functions f on the set of intervals $[a, b]$ of P, with pointwise addition and multiplication by convolution:

$$(f * g)[a, c] := \sum_{b \in P:\, a \le b \le c} f[a, b]\, g[b, c].$$

In [20, Remark 1] a certain multiplicative subgroup $I^0_{\mathrm{mult}}(NC^B; R)$ of the union of all R-valued incidence algebras of type A and B noncrossing partition lattices was defined. As observed in [18, 20], every interval $[a, b]$ in $NC^A(n)$ or $NC^B(n)$ has a canonical isomorphism to a Cartesian product

$$NC^B(n_0) \times NC^A(n_1) \times NC^A(n_2) \times \cdots \times NC^A(n_r)$$

for some integers $n_0, n_1, \ldots, n_r$, where the factor $NC^B(n_0)$ need not be present. The multiplicative subgroup $I^0_{\mathrm{mult}}(NC^B; R)$ consists of those elements f in the incidence algebra which take the value 1 on $NC^A(1)$ and which are *multiplicative*, in the sense that

$$f[a, b] = f(NC^B(n_0)) \prod_{i=1}^{r} f(NC^A(n_i)).$$

Define a map

$$I^0_{\mathrm{mult}}(NC^B; R) \xrightarrow{\mathcal{F}} R[[t, u]]/(u^2) \ (\cong R[u]/(u^2)[[t]]),$$
$$f \mapsto \mathcal{F}(f) := \frac{\phi_f^{\langle -1 \rangle}}{t},$$

where

$$\phi_f := \sum_{n\geq 1} f(NC^A(n))t^n + f(NC^B(n))t^n u \tag{7.1}$$

and $\phi_f^{\langle -1\rangle}$ denotes the compositional inverse of ϕ_f with respect to the variable t. This map gives an isomorphism of $I^0_{\mathrm{mult}}(NC^B; R)$ onto the multiplicative subgroup of power series in $R[[t,u]]/(u^2)$ whose coefficient of t^0 equals 1 [20, Theorem 16].

Proof of Lemma 4.4. We will perform generating function calculations in these rings, choosing

$$R = \mathbb{Z}[[x_1, x_2, \ldots, y_1, y_2, \ldots]].$$

Define $f \in I^0_{\mathrm{mult}}(NC^B; R)$ by

$$\begin{aligned} f(NC^A(n)) &:= \sum_{s\models n-1} \sum_{\substack{\text{chains in } NC^A(n)\\ \text{with rank jump vector } s}} x^s, \\ f(NC^B(n)) &:= \sum_{s\models n} \sum_{\substack{\text{chains } c \text{ in } NC^B(n)\\ \text{with rank jump vector } s}} x^s \, \frac{y_{\mathrm{ind}(c)}}{x_{\mathrm{ind}(c)}}, \end{aligned} \tag{7.2}$$

where $x^s := x_1^{s_1} \cdots x_m^{s_m}$. A little thought shows that f coincides with the convolution $f = f_1 * f_2 * \cdots$, where

$$\begin{aligned} f_i(NC^A(n)) &:= x_i^{n-1}, \\ f_i(NC^B(n)) &:= x_i^{n-1} y_i. \end{aligned}$$

From this, one calculates that

$$\phi_{f_i} = \sum_{n\geq 1} x_i^{n-1} t^n + x_i^{n-1} y_i t^n u = \frac{t(1+y_i u)}{1-tx_i},$$

and hence, by computing the compositional inverse,

$$\begin{aligned} \phi_{f_i}^{\langle -1\rangle} &= \frac{t}{1+tx_i+uy_i}, \\ \mathcal{F}(f_i) &= \frac{\phi_{f_i}^{\langle -1\rangle}}{t} = \frac{1}{1+tx_i+uy_i}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{F}(f) &= \textstyle\prod_{i\geq 1} \mathcal{F}(f_i) = \prod_{i\geq 1} \frac{1}{1+tx_i+uy_i}, \\ \phi_f^{\langle -1\rangle} &= t\mathcal{F}(f) = \frac{t}{\prod_{i\geq 1}(1+tx_i+uy_i)}. \end{aligned}$$

One can apply the Lagrange inversion formula [25, Theorem 5.4.2] to this last expression. Letting $[t^k]\,\psi(t)$ denote the coefficient of t^k in any formal power series $\psi(t)$ in a variable t, one has that

$$\begin{aligned}
[t^n]\,\phi_f &= \frac{1}{n}[T^{n-1}]\prod_{i\geq 1}(1+x_iT+y_iu)^n \\
&= \frac{1}{n}[T^{n-1}]\prod_{i\geq 1}\sum_{k=0}^{n}\binom{n}{k}(x_iT+y_iu)^k \\
&= \frac{1}{n}[T^{n-1}]\prod_{i\geq 1}\sum_{k=0}^{n}\binom{n}{k}(x_i^kT^k+k\cdot x_i^{k-1}y_iT^{k-1}u) \\
&= \sum_{s\models n-1}\frac{1}{n}\binom{n}{s_1}\cdots\binom{n}{s_m}x^s+u\sum_{s\models n}\binom{n}{s_1}\cdots\binom{n}{s_m}\sum_{i=1}^{m}\frac{s_i}{n}x^s\frac{y_i}{x_i}.
\end{aligned}$$

Comparing (7.1), (7.2) with this last expression gives the result. ∎

The proof of the lemma gives an alternative derivation for [13, Theorem 3.2] and, by setting $y_i = x_i$ for all i, also one for [20, Proposition 7].

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