

# Strong cocomparability graphs and Slash-free orderings of matrices

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## Abstract

We introduce the class of strong cocomparability graphs, as the class of reflexive graphs whose adjacency matrix can be rearranged by a simultaneous row and column permutation to avoid the submatrix with rows 01, 10, which we call *Slash*.

We provide an ordering characterization, a forbidden structure characterization, and a polynomial-time recognition algorithm, for the class. These results complete the picture in which in addition to, or instead of, the *Slash* matrix one forbids the  $\Gamma$  matrix (which has rows 11, 10). It is well known that in these two cases one obtains the class of interval graphs, and the class of strongly chordal graphs, respectively.

By complementation, we obtain the class of strong comparability graphs, whose adjacency matrix can be rearranged by a simultaneous row and column permutation to avoid the two-by-two identity submatrix. Thus our results give characterizations and algorithms for this class of irreflexive graphs as well. In other words, our results may be interpreted as solving the following problem: given a symmetric 0, 1-matrix with 0-diagonal, can the rows and columns of be simultaneously permuted to avoid the two-by-two identity submatrix?

**Key words:** Forbidden submatrix, strong cocomparability graph, strong comparability graph, characterization, vertex ordering, forbidden substructure, recognition

## 1 Introduction

The problem of deciding whether a given 0, 1-matrix  $M$  can have its rows and columns independently permuted to avoid containing a fixed submatrix  $F$  (or a fixed family  $\mathcal{F}$  of

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such submatrices) has been much studied in the literature, see for instance [2, 7, 12, 14, 16, 20, 21, 22, 24, 27, 28, 29, 32].

Specifically, we say that a matrix  $M$  is  $F$ -free, or *avoids*  $F$ , if it does not include a copy of  $F$  as a submatrix. An  $F$ -free ordering of a matrix  $M$  is an ordering of its rows and an ordering of its columns such that the resulting permuted matrix is  $F$ -free. The research mentioned above is concerned with deciding whether a given  $M$  admits an  $F$ -free ordering. Often these problems have interesting interpretations for graphs, and especially for bipartite graphs (i.e., *bigraphs*). Indeed, any 0,1-matrix is associated with a bigraph as follows. The *biadjacency matrix* of a bigraph  $G$  with parts  $I, J$  is a 0,1,-matrix  $M$  whose rows are indexed by vertices of  $I$  and columns by vertices of  $J$  and  $M(i, j) = 1$  if and only if  $i \in I$  is adjacent to  $j \in J$ .

In the most widely known special case the matrix  $F$  is the so-called  $\Gamma$  matrix  $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ . It is known that a matrix  $M$  admits a  $\Gamma$ -free ordering if and only if it is totally balanced (i.e., does not contain a cycle submatrix) [1, 22, 28]), and efficient algorithms are known to determine whether or not this is the case [28, 29]. Totally balanced matrices have important applications in location theory [22]. They also have a natural graph interpretation. We say that a bigraph is *chordal* if it does not contain an induced cycle of length at least six.

**Theorem 1.** [1, 19, 28] *A matrix admits a  $\Gamma$ -free ordering if and only if it is the biadjacency matrix of a chordal bigraph.*  $\square$

This gives a forbidden structure characterization (in terms of the induced cycles of length at least six in the associated bigraph) of matrices that admit a  $\Gamma$ -free ordering.

We next review the situation when  $F$  is the so-called *Slash* matrix  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

This matrix turns out to be important from the graph theoretic point of view, but we also point out that exchanging 0's and 1's translates this problem into one seeking  $I_2$ -free orderings, where  $I_2$  is the two-by-two identity matrix. This problem has attracted attention in the Russian literature because it yields efficiently solvable cases of a certain plant location problem [24]. It is known that  $M$  admits an  $I_2$ -free ordering if and only if it is connected (i.e., the sequence of differences between the corresponding entries of any two rows changes sign at most once), and efficient recognition algorithms have been reported in the Russian sources [2].

For the *Slash* matrix, we have the following graph-theoretic interpretation. An *edge-asteroid* is a set of edges  $x_0y_0, x_1y_1, \dots, x_{2k}y_{2k}$  such that for each  $i = 0, 1, \dots, 2k$ , there is a walk that begins with the edge  $x_{i+k}y_{i+k}$  and ends with the edge  $x_{i+k+1}y_{i+k+1}$  such that neither  $x_i$  nor  $y_i$  is adjacent to a vertex in the walk. We call a bigraph a *cocomparability* bigraph if it contains no edge-asteroid [14]. This is motivated by an analogous definition and result of Gallai [10], stating that a graph is a comparability graph – i.e., admits a transitive orientation – if and only if its complement contains no asteroid. Here an asteroid is a vertex-focused analogue of an edge-asteroid.; see [14] for details.

**Theorem 2.** [14] *A matrix admits a *Slash*-free ordering if and only if it is the biadjacency matrix of a cocomparability bigraph.*  $\square$

This result also gives a forbidden structure characterization of matrices that admit a *Slash*-free ordering (in terms of the edge-asteroids of the associated bigraph). It implies a robust polynomial-time recognition algorithm, which can also be applied to recognizing matrices that admit  $I_2$ -free orderings. The need for such algorithms is discussed in [31], Section 10.3.

Finally, we review the situation for the family  $\mathcal{F} = \{\Gamma, \textit{Slash}\}$ , i.e., when both  $\Gamma$  and *Slash* are forbidden. (When the order of rows is reversed, the resulting two matrices form a subset of 0, 1-Monge matrices, as discussed in [24], Sections 2.7 and 4.3.) There are several equivalent descriptions of the class of matrices admitting  $\mathcal{F}$ -free orderings, in terms of bigraphs: as complements or circular arc graphs [7], as two-dimensional orthogonal ray graphs [30], or as interval containment bigraphs [15, 20, 21], cf. also [16]. We may state the theorem using already defined terms as follows [7, 20].

**Theorem 3.** [15, 21] *A matrix admits a  $(\Gamma, \textit{Slash})$ -free ordering if and only if it is the biadjacency matrix of a chordal bigraph without edge-asteroids.*  $\square$

Other results give polynomial-time algorithms and characterizations in terms of bigraphs when similar sets of  $2 \times 2$  matrices are forbidden [24, 32].

All the results summarized above, translate to facts about classes of bigraphs. There is a more natural matrix candidate as far as correspondence to graphs is concerned, namely instead of the biadjacency matrix one can consider the usual adjacency matrix of an arbitrary graph. This is the focus of our paper.

For technical reasons, we need to deal with *reflexive graphs* (graphs in which ever vertex has a loop). In matrix terms, this means we have a symmetric 0, 1-matrix with 1's on the main diagonal. In the literature this is often presented without considering loops, but instead of the usual adjacency matrix one takes the *augmented adjacency matrix*, which is just the usual adjacency matrix augmented with 1's on the main diagonal. Note that re-ordering the vertices of a graph corresponds to a simultaneous row and column permutation of its adjacency matrix.

It is also possible, of course, to consider an arbitrary square matrix to be the adjacency matrix of a digraph, and ask similar questions about avoiding given submatrices for simultaneous row and column permutations, see for instance [13]. However, in this paper we focus purely on undirected graphs.

Thus from now on  $M$  is a square symmetric 0, 1-matrix with 1's on the main diagonal. We now examine the situation for the same matrices  $\Gamma$  and *Slash*.

A *symmetric ordering* of a symmetric matrix  $M$  is a matrix obtained from  $M$  by simultaneously permuting the rows and columns of  $M$ . Clearly, a symmetric ordering of a symmetric matrix is again symmetric.

We first describe the situation when the  $\Gamma$  matrix is forbidden.

A *trampoline* is a complete graph on  $k$  vertices  $u_1, u_2, \dots, u_k$  with  $k \geq 3$ , together with an independent set of  $k$  vertices  $v_1, v_2, \dots, v_k$  such that each  $v_i$  is adjacent to  $u_i$  and  $u_{i+1}$  (and to no other vertices). A graph is *strongly chordal* if it does not contain, as an induced subgraph, a cycle of length  $\geq 4$  or a trampoline [5].

**Theorem 4.** [5] *A matrix admits a symmetric  $\Gamma$ -free ordering if and only if it is the adjacency matrix of a strongly chordal graph.*  $\square$

This result provides a graph description of the class of matrices, but also a forbidden structure characterization, and a polynomial-time recognition algorithm.

For the family  $\mathcal{F} = \{\Gamma, \text{Slash}\}$ , the symmetric matrices that admit a  $\mathcal{F}$ -free ordering also have an elegant graph description, which results in a well-known forbidden structure characterization and polynomial-time recognition.

A graph  $G$  is an *interval graph* if there is a family of intervals  $I_v, v \in V(G)$  such that for all  $u, v \in V(G)$ ,  $uv \in E(G)$  if and only if  $I_u \cap I_v \neq \emptyset$ .

**Theorem 5.** [8, 15] *A matrix admits a symmetric  $(\Gamma, \text{Slash})$ -free ordering if and only if it is the adjacency matrix of an interval graph.*  $\square$

The purpose of our paper is to complete the picture by introducing the “last piece of the puzzle”, that is, the class of reflexive graphs whose adjacency matrices have symmetric *Slash*-free orderings. We call such graphs *strong cocomparability graphs*. We provide a natural ordering characterization of strong cocomparability graphs, as well as a characterization by forbidden substructures that is analogous to similar characterizations of interval graphs and other graph and digraph classes. This latter characterization implies a polynomial-time algorithm for the recognition of strong cocomparability graphs.

We apply our results to prove that the class of interval graphs is the intersection of the classes of strongly chordal graphs and strong cocomparability graphs, enhancing a well known theorem of Gilmore and Hoffman [11].

These results can be applied by complementation to the class of *strong comparability graphs*, which correspond to adjacency matrices that have a symmetric  $I_2$ -free ordering. We ask if the domination problem, NP-complete for comparability graphs, can be solved in polynomial time for strong comparability graphs, in analogy to the situation for chordal versus strongly chordal graphs.

Unfortunately, we do not have a forbidden subgraph characterization of strong cocomparability graphs, although such characterizations are known for interval graphs, cocomparability graphs, chordal graphs, and strongly chordal graphs. We do, however, provide a simple description of the class of bipartite strong cocomparability graphs, and relate it to the class of cocomparability bigraphs and to the class of bipartite permutation graphs.

## 2 Symmetric *Slash*-free orderings

Let  $\prec$  be a vertex ordering of a reflexive graph  $G$  and  $M$  be the adjacency matrix of  $G$  whose rows and columns are ordered according to  $\prec$ . Then  $M$  contains the *Slash* matrix as a submatrix if and only if  $\prec$  contains, as ordered induced subgraphs, a pattern in Figure 1. This can be easily seen by considering the possible relative positions of the row numbers and the column numbers of the occurrence of *Slash*.



Figure 1: Forbidden patterns for strong cocomparability orderings.

Thus we have the following ordering characterization of strong cocomparability graphs.

**Proposition 6.** *A reflexive graph is a strong cocomparability graph if and only if it admits a vertex ordering that does not contain any of the four patterns in Figure 1.  $\square$*

We call such a vertex ordering of  $G$  a *strong cocomparability ordering*. It follows by comparison with the cocomparability ordering mentioned in [4] that a strong cocomparability graph is also a cocomparability graph.

We now proceed to give a structural characterization of strong cocomparability graphs. Our result parallels other structural characterizations of various classes of graphs and digraphs [3, 8, 9, 16, 17, 18], as it is given in terms of absence of so-called invertible pairs, defined below. We point out that, for example, interval graphs have a very similar structural characterization [8]. To prove that the absence of invertible pairs implies the existence of a strong cocomparability ordering, we use the method of “complementary circuits” pioneered in [8], and also used in [3, 17, 18].

In particular, our characterization will imply a polynomial-time recognition algorithm.

We first introduce a useful “forcing” relation  $\Lambda$ .

Let  $G$  be a reflexive graph with vertex  $V(G)$  and edge set  $E(G)$ . Note that  $E(G)$  includes all *loops*  $vv$ ,  $v \in V(G)$ . Denote by  $Z(G)$  the set of ordered pairs  $(u, v)$  of distinct vertices of  $G$ . For  $(u, v), (u', v') \in Z(G)$ , we say that  $(u, v)$  *forces*  $(u', v')$ , denoted by  $(u, v)\Lambda(u', v')$ , if  $u = u'$  and  $v = v'$  or  $uu', vv' \in E(G)$  and  $uw', vu' \notin E(G)$ . Clearly,  $(u, v)\Lambda(u', v')$  if and only if  $(v, u)\Lambda(v', u')$ . We also say that  $(u, v)$  *implies*  $(u', v')$ , denoted by  $(u, v) \sim (u', v')$ , if there exist walks  $u_1u_2 \dots u_k$  and  $v_1v_2 \dots v_k$  in  $G$  where  $(u_1, v_1) = (u, v)$  and  $(u_k, v_k) = (u', v')$  such that  $(u_i, v_i)\Lambda(u_{i+1}, v_{i+1})$  for each  $i = 1, 2, \dots, k - 1$ . It is easy to verify that  $\sim$  is an equivalence relation on  $Z(G)$ . An equivalence class of  $\sim$  will be called an *implication class*. We shall use  $[(u, v)]$  to denote the implication class of  $(u, v)$  and call  $[(u, v)]$  *non-trivial* if  $(u, v)$  is not the only pair in  $[(u, v)]$ .

The next proposition follows immediately from the definitions of strong cocomparability orderings and the relation  $\Lambda$ .

**Proposition 7.** *Let  $G$  be a reflexive graph. Then the following statements hold.*

1. *If  $abc$  is an induced  $P_3$  in  $\overline{G}$ , then  $(a, b)\Lambda(c, b)$ .*
2. *If  $abcd$  is an induced  $P_4$  in  $G$ , then  $(a, c)\Lambda(a, d)\Lambda(b, d)\Lambda(a, d)\Lambda(b, c)$ .*
3. *If  $abcd$  is an induced  $C_4$  in  $G$ , then  $(a, d)\Lambda(b, c)$ . □*

Each vertex ordering  $\prec$  of  $G$  gives rise to the set  $\{(u, v) : u \prec v\}$  which we denote by  $U_\prec$ . The set  $U_\prec$  contains either  $(u, v)$  or  $(v, u)$  for all distinct vertices  $u, v$  of  $G$ . Suppose that  $\prec$  is a strong cocomparability ordering. Then for all  $(u, v)\Lambda(u', v')$ ,  $U_\prec$  either contains both  $(u, v)$  and  $(u', v')$  or both  $(v, u)$  and  $(v', u')$ . Hence  $U_\prec$  contains either  $[(u, v)]$  or  $[(v, u)]$  for all distinct vertices  $u, v$  of  $G$  and in particular, it is a union of implication classes. Conversely, if  $U_\prec$  is a union of implication classes then  $\prec$  is a strong cocomparability ordering of  $G$ .

**Proposition 8.** *Let  $\prec$  be a vertex ordering of a reflexive graph  $G$  and let  $U_\prec = \{(u, v) \in Z(G) : u \prec v\}$ . The following statements are equivalent:*

1.  *$\prec$  is a strong cocomparability ordering;*
2. *for all  $(u, v) \sim (u', v')$ ,  $u \prec v$  if and only if  $u' \prec v'$ ;*
3.  *$U_\prec$  is a union of implication classes. □*

An *invertible pair* in  $G$  is a pair of distinct vertices  $u, v$  such that  $(u, v) \sim (v, u)$ . Clearly, if  $u, v$  are an invertible pair then  $[(u, v)] = [(v, u)]$  and for any  $(x, y) \in [(u, v)]$ ,  $x, y$  are an invertible pair. Invertible pairs are absent in a strong cocomparability graph.

**Proposition 9.** *A strong cocomparability graph cannot contain an invertible pair.*

**Proof:** If  $G$  is a strong cocomparability graph which contains an invertible pair  $u, v$ , then by Proposition 8 any strong cocomparability ordering  $\prec$  of  $G$  would have  $u \prec v$  and  $v \prec u$ , a contradiction. □

An invertible pair in a reflexive graph  $G$  (if exists) can be easily discovered via an auxiliary graph of  $G$  defined next.

Define the *auxiliary graph*  $G^+$  of reflexive graph  $G$  as follows. The vertex set of  $G^+$  is the set  $Z(G)$ . Two vertices  $(u, v)$  and  $(v', u')$  are adjacent in  $G^+$  if and only if  $(u, v)\Lambda(u', v')$ , that is,  $u = u'$  and  $v = v'$ , or  $uu', vv' \in E(G)$  and  $uv', vu' \notin E(G)$ .

**Proposition 10.** *A reflexive graph  $G$  has an invertible pair if and only if its auxiliary graph  $G^+$  is not bipartite.*

**Proof:** Suppose that  $u_1, v_1$  are an invertible pair in  $G$ . Then there exist walks  $u_1 u_2 \dots u_{k-1} u_k$  and  $v_1 v_2 \dots v_{k-1} v_k$  where  $u_k = v_1$  and  $v_k = u_1$  such that  $(u_i, v_i) \Lambda (u_{i+1}, v_{i+1})$  for each  $i = 1, 2, \dots, k-1$ . Hence

$$(u_1, v_1)(v_1, u_1)(u_2, v_2)(v_2, u_2) \cdots (u_{k-1}, v_{k-1})(v_{k-1}, u_{k-1})(u_k, v_k)$$

is a closed walk of odd length in  $G^+$ , showing that  $G^+$  is not bipartite.

Conversely, suppose that  $G^+$  is not bipartite. Let  $(u_1, v_1)(u_2, v_2) \cdots (u_{2k+1}, v_{2k+1})$  be a cycle of odd length in  $G^+$ . Then

$$(u_1, v_1) \Lambda (u_3, v_3) \Lambda \cdots \Lambda (u_{2k+1}, v_{2k+1}) \Lambda (v_1, u_1)$$

showing that  $u_1, v_1$  are an invertible pair in  $G$ . □

In the next section we prove that the absence of invertible pairs in a reflexive graph in fact ensures that the graph is a strong cocomparability graph.

### 3 The main result

Let  $G$  be a reflexive graph. For each implication class  $I$  of  $G$ , denote by  $I^{-1}$  the *inverse* class of  $I$ , that is,

$$I^{-1} = \{(u, v) : (v, u) \in I\}.$$

A *circuit* of length  $\ell$  is a sequence of pairs

$$(u_1, u_2), (u_2, u_3), \dots, (u_{\ell-1}, u_\ell), (u_\ell, u_1).$$

Note that if  $G$  has no invertible pair then, for any implication class  $I$  of  $G$ ,  $I \cap I^{-1} = \emptyset$ . Also note that having an invertible pair is equivalent to having a circuit of length two, and hence if  $G$  has no invertible pair then, for any implication class  $I$  of  $G$  neither  $I$  nor  $I^{-1}$  contains a circuit of length two.

**Theorem 11.** *If a reflexive graph  $G$  does not contain an invertible pair, then it is a strong cocomparability graph.*

Before proving the theorem, we give an overview of the proof. To show that  $G$  is a strong cocomparability graph we prove that it has a strong cocomparability ordering  $\prec$  by constructing the set  $U_\prec$ . The set  $U_\prec$  is constructed by adding for each implication class  $I$  either  $I$  or  $I^{-1}$ . A necessary property for  $U_\prec$  to have is that it does not contain a *circuit*. We will show that it is always possible to add either  $I$  or  $I^{-1}$  for each implication class  $I$  without creating a circuit. For the sake of contradiction, suppose that circuits are created for the first time when adding an implication class  $I$  or its inverse  $I^{-1}$ . By considering a shortest circuit and two cases depending on the implication classes of the pairs in the circuit we will show that the circuit has either one or two pairs from the newly added implication class. This will allow us to deduce that a circuit already exists even before adding either of  $I$  and  $I^{-1}$ , which is a contradiction.

**Proof:** Suppose that  $G$  contains no invertible pair. In view of Proposition 8, it suffices to show that there exists a set  $U$  of pairs of distinct vertices of  $G$  which satisfies the following two properties:

- $U$  contains either  $I$  or  $I^{-1}$  for all implication classes  $I$ ;
- $U$  contains no circuit.

In fact, we will show such a set  $U$  can be constructed by adding implication classes one at a time, in no particular order except priority will be given to non-trivial implication classes. Initially,  $U = \emptyset$ . At each step, we consider an implication  $I$  and its inverse class  $I^{-1}$ , no pairs of which have been added to  $U$ . We will add all pairs of  $I$  to  $U$  (that is, replace  $U$  by  $U \cup I$ ) if  $U \cup I$  contains no circuit, otherwise we add all pairs of  $I^{-1}$  to  $U$ . It remains to show that if  $U$  does not contain a circuit, then at least one of  $U \cup I$  and  $U \cup I^{-1}$  does not contain a circuit.

Suppose that  $U \cup I$  contains circuits. Let  $C$  be a shortest circuit contained in  $U \cup I$ . Suppose that  $I$  is trivial, i.e.,  $I = \{(u, v)\}$  for some pair  $(u, v)$ . Since there is no circuit in  $U$ ,  $(u, v)$  is in  $C$ . Since  $C$  is a shortest circuit in  $U \cup I$ ,  $(u, v)$  occurs exactly once in  $C$ . Similarly, if  $U \cup I^{-1}$  also contains a circuit, then it has a circuit  $C'$  in which  $(v, u)$  occurs exactly once in  $C'$ . Hence  $(C \cup C') \setminus \{(u, v), (v, u)\}$  is a circuit in  $U$ , contradicting the assumption that  $U$  has no circuit. Hence  $I$  is non-trivial. This implies that, prior to considering  $I$  and  $I^{-1}$ , no pair in  $U$  forms an trivial implication class. It follows that for each  $(u, v) \in U \cup I$  there exists a pair  $(u', v') \in U \cup I$  such that  $(u', v') \neq (u, v)$  and  $(u', v') \wedge (u, v)$ ; in particular,  $u'$  is adjacent to  $u$  but not to  $v$  and  $v'$  is adjacent to  $v$  but not to  $u$ .

Denote  $C : (x_1, x_2), (x_2, x_3), \dots, (x_{a-1}, x_a), (x_a, x_1)$ . Note that  $a \geq 3$  because  $G$  contains no invertible pair. Since  $C$  is a shortest circuit in  $U \cup I$ , for all  $i, j \in \{1, 2, \dots, a\}$ ,  $(x_i, x_j) \in U \cup I$  if and only if  $j = i + 1$  (here and in the rest of proof the subscripts of  $x$ 's are taken modulo  $a$ ). This implies in particular that the pairs on  $C$  are pairwise distinct. We know from the above that for each  $i$  there is a vertex adjacent to  $x_i$  but not to  $x_{i-1}$  and there is also a vertex adjacent to  $x_i$  but not to  $x_{i+1}$ . Below we distinguish two cases depending on whether such vertices can be chosen to be the same one for some  $i$ .

**Case 1.** For some  $i$ , there is a vertex adjacent to  $x_i$  but not to either of  $x_{i-1}$  and  $x_{i+1}$ . By symmetry we may assume that  $i = 1$  and  $x'_1$  is such a vertex (i.e., adjacent to  $x_1$  but not to  $x_a$  or  $x_2$ ).

**Claim 1.1.** For each  $i = 2, \dots, a$ , let  $x'_i$  be a vertex adjacent to  $x_i$  but not to  $x_{i-1}$ . Let  $z'$  be a vertex adjacent to  $x_a$  but not to  $x_1$ . Then, for each  $i = 1, 2, \dots, a$ ,  $x'_i$  is not adjacent to any of  $x_a, x_1, \dots, x_{i-1}$  and  $x_i$  is not adjacent to any of  $z', x'_1, x'_2, \dots, x'_{i-1}$  for each  $i = 1, 2, \dots, a$ , except when  $i = a$  in which case  $x_a$  is adjacent both  $x'_a$  and  $z'$  by definition. Moreover,

$$C' : (x'_1, x'_2), (x'_2, x'_3), \dots, (x'_{a-1}, x'_a), (x'_a, x'_1)$$

is a shortest circuit in  $U \cup I$ .



**Proof of Claim 1.1.** The statement is clearly true for  $i = 1$ . By definition  $x'_2$  is not adjacent to  $x_1$  and by assumption  $x_2$  is not adjacent to  $x'_1$ . If  $x'_2$  is adjacent to  $x_a$  then  $(x_1, x_2)\Lambda(x'_1, x'_2)\Lambda(x_1, x_a)$ , which implies  $(x_1, x_a) \in U \cup I$ , a contradiction to the observation above. So  $x'_2$  is not adjacent to  $x_a$ . If  $x_2$  is adjacent to  $z'$  then  $(x_1, x_2)\Lambda(x'_1, z')\Lambda(x_1, x_a)$ , a contradiction to the observation above. So  $x_2$  is not adjacent to  $z'$ . Hence the statement is also true for  $i = 2$ . Assume that  $2 \leq k < a$  and the statement is true for  $i = 1, 2, \dots, k$ . Suppose that  $x_{k+1}$  is adjacent to  $x'_k$ . The vertex  $x_{k+1}$  must be adjacent to  $x'_{k-1}$  as otherwise  $(x_{k-1}, x_k)\Lambda(x'_{k-1}, x'_k)\Lambda(x_{k-1}, x_{k+1})$ , which implies  $(x_{k-1}, x_{k+1}) \in U \cup I$ , a contradiction. Similarly,  $x_{k+1}$  must be adjacent to  $x'_{k-2}$  as otherwise  $(x_{k-2}, x_{k-1})\Lambda(x'_{k-2}, x'_{k-1})\Lambda(x_{k-2}, x_{k+1})$ , which implies  $(x_{k-2}, x_{k+1}) \in U \cup I$ , a contradiction. Continuing this way we have that  $x_{k+1}$  is adjacent to all of  $z', x'_1, x'_2, \dots, x'_k$ . Let  $x'$  be a vertex adjacent to  $x_k$  but not to  $x_{k+1}$ . We know from above such a vertex  $x'$  exists. Then  $x'$  must be adjacent to  $x_{k-1}$  as otherwise  $(x_{k-1}, x_k)\Lambda(x'_{k-1}, x')\Lambda(x_{k+1}, x_k)$ , which implies  $(x_{k+1}, x_k) \in U \cup I$ , a contradiction. Similarly,  $x'$  must be adjacent to  $x_{k-2}$  as otherwise  $(x_{k-2}, x_{k-1})\Lambda(x'_{k-2}, x')\Lambda(x_{k+1}, x_{k-1})$ , which implies  $(x_{k+1}, x_{k-1}) \in U \cup I$ , a contradiction. Continuing this way we have  $x'$  is adjacent to all of  $x_a, x_1, x_2, \dots, x_k$ . But then  $(x_k, x_{k+1})\Lambda(x', z')\Lambda(x_1, x_{k+1})$ , which implies  $(x_1, x_{k+1}) \in U \cup I$ , a contradiction. Hence  $x_{k+1}$  is not adjacent to  $x'_k$ . It cannot be adjacent to  $x'_{k-1}$  as otherwise  $(x_k, x_{k-1})\Lambda(x'_k, x'_{k-1})\Lambda(x_k, x_{k-1})$ , which implies  $(x_k, x_{k-1}) \in U \cup I$ , a contradiction. Similarly, it cannot be adjacent to  $x'_{k-2}$  as otherwise  $(x_k, x_{k+1})\Lambda(x'_k, x'_{k-2})\Lambda(x_k, x_{k-2})$ , which implies  $(x_k, x_{k-2}) \in U \cup I$ , a contradiction. Continuing this way we have that  $x_{k+1}$  is not adjacent to any of  $z', x'_1, x'_2, \dots, x'_k$ , except when  $k+1 = a$  in which case  $x_a$  is by definition adjacent to  $z'$ .

If  $x'_{k+1}$  is adjacent to  $x_{k-1}$ , then  $(x_{k-1}, x_k)\Lambda(x'_{k+1}, x'_k)\Lambda(x_{k+1}, x_k)$ , implying  $(x_{k+1}, x_k) \in U \cup I$ , a contradiction. So  $x'_{k+1}$  is not adjacent to  $x_{k-1}$ . If  $x'_{k+1}$  is adjacent to  $x_{k-2}$ , then  $(x_{k-2}, x_{k-1})\Lambda(x'_{k+1}, x'_{k-1})\Lambda(x_{k+1}, x_{k-1})$ , which implies  $(x_{k+1}, x_{k-1}) \in U \cup I$ , a contradiction. So  $x'_{k+1}$  is not adjacent to  $x_{k-2}$ . Continuing this way we see that  $x'_{k+1}$  is not adjacent to any of  $x_a, x_1, \dots, x_k$ , except when  $k+1 = a$  in which case  $x'_a$  is by definition adjacent to  $x_a$ .

Since  $x_i$  is adjacent to  $x'_i$  but not to  $x'_{i+1}$  and  $x_{i+1}$  is adjacent to  $x'_{i+1}$  but not to  $x'_i$ ,  $(x_i, x_{i+1})\Lambda(x'_i, x'_{i+1})$  for each  $i = 1, 2, \dots, a$ . Therefore  $C'$  is a circuit in  $U \cup I$ . Since  $C'$  has the same length as  $C$ , it is a shortest circuit in  $U \cup I$ .  $\square$

It follows from Claim 1.1 that the edges  $x_i x'_i$  form an independent set of edges in  $G$ , that is,  $x_i$  is not adjacent to  $x'_j$  for all  $i \neq j$ .

**Claim 1.2.** For any vertex  $x'$  that is adjacent to  $x_i$  but not to  $x_{i+1}$ ,  $x'$  is not adjacent to any of  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_a$ . Moreover,

$$(x'_1, x'_2), \dots, (x'_{i-2}, x'_{i-1}), (x'_{i-1}, x'), (x', x_{i+1}), \dots, (x'_{a-1}, x'_a), (x'_a, x'_1)$$

is a shortest circuit in  $U \cup I$ .

**Proof of Claim 1.2.** If  $x'$  is adjacent to  $x_{i-1}$ , then  $(x_i, x_{i+1})\Lambda(x', x'_{i+1})\Lambda(x_{i-1}, x_{i+1})$ , which implies  $(x_{i-1}, x_{i+1}) \in U \cup I$ , a contradiction. So  $x'$  is adjacent to  $x_i$  but not to  $x_{i-1}$  or  $x_{i+1}$ . By Claim 1.1,  $x'$  is not adjacent to any of  $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_a$ . Since

$(x_{i-1}, x_i)\Lambda(x'_{i-1}, x')$ ,  $(x_i, x_{i+1})\Lambda(x', x'_{i+1})$  and  $(x_j, x_{j+1})\Lambda(x'_j, x'_{j+1})$  for all  $j \neq i-1$  or  $i$ ,  $(x'_1, x'_2), \dots, (x'_{i-2}, x'_{i-1}), (x'_{i-1}, x')$ ,  $(x', x_{i+1}), \dots, (x'_{a-1}, x'_a), (x'_a, x'_1)$  is a shortest circuit in  $U \cup I$ .  $\square$

**Claim 1.3.** For any vertex  $x$  that is adjacent to  $x'_i$  but not to  $x'_{i+1}$ ,  $x$  is not adjacent to any of  $x'_1, \dots, x'_{i-1}, x'_{i+1}, \dots, x'_a$ . Moreover,

$$(x_1, x_2), \dots, (x_{i-2}, x_{i-1}), (x_{i-1}, x), (x, x_{i+1}), \dots, (x_{a-1}, x_a), (x_a, x_1)$$

is a shortest circuit in  $U \cup I$ .

**Proof of Claim 1.3.** The proof is the same as for Claim 1.2 when the roles of  $x_i$  and  $x'_i$  are switched for each  $i$ .  $\square$

**Claim 1.4.** The circuit  $C$  contains a unique pair from  $I$ .

**Proof of Claim 1.4.** Since  $U$  contains no circuit, there is at least one pair on  $C$  which is from  $I$ . Without loss of generality assume  $(x_a, x_1) \in I$ . Suppose that  $(x_j, x_{j+1}) \in I$  for some  $j \neq a$ . Then there is a sequence

$$(u_1, v_1)\Lambda(u_2, v_2)\Lambda \cdots \Lambda(u_k, v_k)$$

of pairs in  $I$  such that  $(u_1, v_1) = (x_a, x_1)$  and  $(u_k, v_k) = (x_j, x_{j+1})$ . Moreover, we assume that  $(u_i, v_i) \neq (u_{i+1}, v_{i+1})$  for each  $i$ . This guarantees that  $u_{i+1}$  is adjacent to  $u_i$  but not to  $v_i$  and  $v_{i+1}$  is adjacent to  $v_i$  but not to  $u_i$ . It follows from Claims 1.3 and 1.4, for each even  $i$  with  $1 \leq i \leq k$ ,  $(u_i, v_i), (v_i, x'_2), (x'_2, x'_3), \dots, (x'_{a-2}, x'_{a-1}), (x'_{a-1}, u_i)$  is a shortest circuit in  $U \cup I$ . Moreover,  $u_i$  is not adjacent to any of  $x_1, x_2, \dots, x_{a-1}$ . In particular, when  $k$  is even,  $u_k$  is not adjacent to  $x_j = u_k$ , which is a contradiction. So  $k$  is odd. By Claims 1.3 and 1.4,  $(u_k, v_k), (v_k, x_2), (x_2, x_3), \dots, (x_{a-2}, x_{a-1}), (x_{a-1}, u_k)$  is a shortest circuit in  $U \cup I$ . This is a contradiction because  $u_k = x_j$  and  $(u_k, v_k), (v_k, x_2), (x_2, x_3), \dots, (x_{j-1}, x_j)$  is a shorter circuit in  $U \cup I$ .  $\square$

**Claim 1.5.** For each pair  $(u, v) \in I$ ,  $U \cup I$  contains a circuit in which  $(u, v)$  is the only pair from  $I$ .

**Proof of Claim 1.5.** By Claim 1.4, we may again assume that  $(x_a, x_1)$  is the only pair in  $C$  that is from  $I$ . Since  $(x_a, x_1)$  and  $(u, v)$  are both from  $I$ , there is a sequence

$$(u_1, v_1)\Lambda(u_2, v_2)\Lambda \cdots \Lambda(u_k, v_k)$$

of pairs in  $I$  such that  $(u_1, v_1) = (x_a, x_1)$ ,  $(u_k, v_k) = (u, v)$ , and  $(u_i, v_i) \neq (u_{i+1}, v_{i+1})$  for each  $i$ . The proof of Claim 1.4 also shows that if  $k$  is even then substituting  $(u, v)$  for the pair  $(x'_a, x'_1)$  in  $C'$  we obtain a circuit in which  $(u, v)$  is the only pair from  $I$ . Similarly, if  $k$  is odd then substituting  $(u, v)$  for the pair  $(x_a, x_1)$  in  $C$  we obtain a circuit in which  $(u, v)$  is the only pair from  $I$ .  $\square$

**Case 2.** For each  $i$ , any vertex adjacent to  $x_i$  must be adjacent to at least one of  $x_{i-1}$  and  $x_{i+1}$ .

**Claim 2.1.** For each  $i = 1, 2, \dots, a$ , let  $x'_i$  be a vertex adjacent to  $x_i$  but not to  $x_{i-1}$  and  $x''_i$  be a vertex adjacent to  $x_i$  but not to  $x_{i+1}$ . Then

- $x'_i$  is adjacent to all of  $x_a, x_1, x_2, \dots, x_{i-2}$ , except when  $i = 1$  in which case it is not adjacent to  $x_a$ .
- $x''_i$  is adjacent to all of  $x_a, x_1, \dots, x_i$ , except when  $i = a - 1$  in which case it is not adjacent to  $x_a$  and when  $i = a$  in which case it is not adjacent to  $x_1$ .
- $x_i$  is adjacent to all of  $x'_1, x'_2, \dots, x'_{i-1}, x''_a, x''_1, x''_2, \dots, x''_{i-2}$ , except when  $i = 1$  in which case it is not adjacent to  $x'_2$  or  $x''_a$ .

**Proof of Claim 2.1.** By definition,  $x'_1$  is not adjacent to  $x_a$  and  $x_1$  is not adjacent to  $x'_2$  or  $x''_a$ . The vertex  $x'_1$  is adjacent to  $x_1$  but not to  $x_2$  so it is adjacent to  $x_a$  by assumption. Thus the statement holds for  $i = 1$ . The vertex  $x'_2$  must be adjacent to  $x_a$  as otherwise  $(x_1, x_2)\Lambda(x'_1, x'_2)\Lambda(x_a, x_2)$ , which implies  $(x_a, x_2) \in U \cup I$ , a contradiction to the choice of  $C$ . By assumption  $x_2$  is adjacent to  $x'_1$ . If  $x_2$  is not adjacent to  $x''_a$  then  $(x_a, x_1)\Lambda(x''_a, x'_1)\Lambda(x_a, x_2)$ , a contradiction. So  $x_2$  is adjacent to both  $x'_1$  and  $x''_a$ . Note that  $x'_2$  is adjacent to  $x_2$  but not to  $x_1$  so it must be adjacent to  $x_3$  by assumption. Now consider  $x''_2$ . It is adjacent to  $x_2$  by definition. It must be adjacent to  $x_a$  as otherwise  $(x_a, x_1)\Lambda(x'_2, x''_2)\Lambda(x_3, x_1)$ , a contradiction. Hence the statement holds also for  $i = 2$ . Assume that  $k \geq 2$  and the statement holds for each  $i = 1, 2, \dots, k$ . We show that it holds for  $i = k + 1$ . Consider first the vertex  $x'_{k+1}$ . It must be adjacent to  $x_{k-1}$  as otherwise  $(x_k, x_{k+1})\Lambda(x'_k, x'_{k+1})\Lambda(x_{k-1}, x_{k+1})$ , a contradiction. It must also be adjacent to  $x_{k-2}$  as otherwise  $(x_k, x_{k+1})\Lambda(x'_k, x'_{k+1})\Lambda(x_{k-2}, x_{k+1})$ , a contradiction. Continuing this way, we see that  $x'_{k+1}$  is adjacent to all of  $x_a, x_1, x_2, \dots, x_{k-1}$ . Now consider  $x_{k+1}$ . It must be adjacent to  $x'_1$  as otherwise  $(x_k, x_{k+1})\Lambda(x'_1, x'_{k+1})\Lambda(x_k, x_a)$ , a contradiction. It must also be adjacent to  $x'_2$  as otherwise  $(x_k, x_{k+1})\Lambda(x'_2, x'_{k+1})\Lambda(x_k, x_1)$ , a contradiction. Continuing this way we see that  $x_{k+1}$  is adjacent to all of  $x'_1, x'_2, \dots, x'_k$ . A similar proof shows that  $x_{k+1}$  is adjacent to all of  $x''_a, x''_1, x''_2, \dots, x''_{k-1}$ . Note that  $x'_{k+1}$  is adjacent to  $x_{k+1}$  but not to  $x_k$  so it must be adjacent to  $x_{k+2}$  by assumption. The vertex  $x_{k+2}$  must also be adjacent to  $x'_k$  as otherwise  $(x_{k-1}, x_k)\Lambda(x'_{k+1}, x'_k)\Lambda(x_{k+2}, x_k)$ , a contradiction. Continuing this way we see that  $x_{k+2}$  is adjacent to all of  $x'_1, x'_2, \dots, x'_{k+1}$ . Finally we consider  $x''_{k+1}$ . It is adjacent to  $x_{k+1}$  by definition. It must be adjacent to  $x_{k-1}$  as otherwise  $(x_{k-1}, x_k)\Lambda(x'_{k+1}, x''_{k+1})\Lambda(x_k, x_{k+2})$ , a contradiction. It must be adjacent to  $x_{k-2}$  as otherwise  $(x_{k-2}, x_{k-1})\Lambda(x'_k, x''_{k+1})\Lambda(x_{k-1}, x_{k+2})$ . Continuing this manner we see that  $x''_{k+1}$  is adjacent to all of  $x_a, x_1, \dots, x_{k+1}$ .  $\square$

Note that for each  $i = 1, 2, \dots, a$ ,  $x_i x'_{i+1}$  and  $x_i x''_{i-1}$  are edges in the complement  $\overline{G}$  of  $G$ . It follows from Claim 2.1 that in  $\overline{G}$ , the edges  $x_i x'_{i+1}$  form an independent set of edges, that is,  $x_i$  is not adjacent to  $x'_j$  for all  $i \neq j - 1$ ; similarly, the edges  $x_i x''_{i-1}$  form an independent set of edges, that is,  $x_i$  is not adjacent to  $x''_j$  for all  $i \neq j + 1$ .

**Claim 2.2.** For each  $i = 1, 2, \dots, a$ , let  $x'_i$  be a vertex adjacent to  $x_i$  but not to  $x_{i-1}$  and  $x''_i$  be a vertex adjacent to  $x_i$  but not to  $x_{i+1}$ . Then

$$C'' : (x'_a, x'_{a-1}), (x'_{a-1}, x'_{a-2}), \dots, (x'_2, x'_1), (x'_1, x'_a)$$

is a shortest circuit in  $U \cup I$ .

**Proof of Claim 2.2.** By Claim 2.1, each  $x_i$  is adjacent to  $x'_{i+2}$  and  $x_{i+1}$  is adjacent to  $x'_{i+1}$ . So  $(x_i, x_{i+1})\Lambda(x'_{i+2}, x'_{i+1})$  for each  $i$ . Hence  $C''$  is a circuit in  $U \cup I$ . It is the shortest because it has the same length as  $C$ .  $\square$

**Claim 2.3.** Let  $i$  be fixed. If  $(x_i, x_{i+1})\Lambda(u, v)$  for some  $(u, v) \neq (x_i, x_{i+1})$ , then substituting  $(u, v)$  for  $(x'_{i+2}, x'_{i+1})$  in  $C''$  results in a shortest circuit in  $U \cup I$ . Similarly, if  $(x'_{i+2}, x'_{i+1})\Lambda(u', v')$  for some  $(u', v') \neq (x'_{i+2}, x'_{i+1})$ , then substituting  $(u', v')$  for  $(x_i, x_{i+1})$  in  $C$  results in a circuit in  $U \cup I$ .

**Proof of Claim 2.3.** If  $(x_i, x_{i+1})\Lambda(u, v)$ , then  $u$  can play the role of  $x'_{i+2}$  and  $v$  can play role of  $x'_{i+1}$  in Claim 2.1. Hence by Claim 2.2 substituting  $(u, v)$  for  $(x'_{i+2}, x'_{i+1})$  in  $C''$  results in a circuit in  $U \cup I$ . The new circuit is a shortest one in  $U \cup I$  because it has the same length as  $C$ . The proof of the second part is the same when the roles of  $C$  and  $C''$  are switched in both Claims 2.1 and 2.2.  $\square$

**Claim 2.4.**  $C$  contains one or two pairs from  $I$ .

**Proof of Claim 2.4.** Since there is no circuit in  $U$ ,  $C$  must contain at least one pair from  $I$ . Suppose that  $(x_r, x_{r+1}), (x_s, x_{s+1}) \in I$  for some  $r \neq s$ . Since both  $(x_r, x_{r+1}), (x_s, x_{s+1})$  are from  $I$ , there is a sequence

$$(u_1, v_1)\Lambda(u_2, v_2)\Lambda \cdots \Lambda(u_k, v_k)$$

of pairs in  $I$  such that  $(u_1, v_1) = (x_r, x_{r+1})$ ,  $(u_k, v_k) = (x_s, x_{s+1})$ , and  $(u_i, v_i) \neq (u_{i+1}, v_{i+1})$  for each  $i$ . If  $k$  is odd, then it follows from Claim 2.3 substituting  $(x_s, x_{s+1})$  for  $(x_r, x_{r+1})$  in  $C$  results in a shortest circuit in  $U \cup I$ . But the new circuit has equal vertices  $x_s$  (and  $x_{s+1}$ ) so it is not the shortest, a contradiction. Hence  $k$  must be even. Therefore any two pairs in  $C$  which are from  $I$  can only be connected by such a sequence consisting of an even number of pairs. Suppose that  $C$  contains three pairs from  $I$ . Then there is a sequence as above consisting of an even number of pairs connecting the first and the second pairs, and there is a sequence consisting consisting of an even number of pairs connecting the second and the third pairs. By concatenating the two sequences we obtain a sequence consisting of an odd number of pairs connecting the first and the third pairs, which contradicts the above fact.  $\square$

**Claim 2.5.** If  $C$  contains exactly one pair from  $I$ , then for any  $(u, v) \in I$ ,  $U \cup I$  contains a circuit in which  $(u, v)$  is the only pair from  $I$ . If  $C$  contains exactly two pairs from  $I$ , then for any two pairs  $(u_1, v_1), (u_k, v_k) \in I$  connected by a sequence

$$(u_1, v_1)\Lambda(u_2, v_2)\Lambda \cdots \Lambda(u_k, v_k)$$

where  $(u_i, v_i) \neq (u_{i+1}, v_{i+1})$  and  $k$  is odd,  $U \cup I$  contains a circuit in which  $(u_1, v_1), (u_k, v_k)$  are the only pairs from  $I$ .

**Proof of Claim 2.5.** If  $C$  contains exactly one pair from  $I$ , then  $C''$  also contains exactly one pair from  $I$ . By Claim 2.3, substituting  $(u, v)$  for the pair in  $C$  or the pair in  $C''$  results in a circuit in which  $(u, v)$  is the only pair from  $I$ . Similarly, if  $C$  contains exactly two pairs from  $I$ , then  $C''$  also contains exactly two pairs from  $I$ . By Claim 2.3,

substituting  $(u_1, v_1), (u_k, v_k)$  for the two pairs in  $C$  or the two pairs in  $C''$  results in a circuit in which  $(u_1, v_1), (u_k, v_k)$  are the only two pairs from  $I$ .  $\square$

We know from above that if  $U \cup I$  contains a circuit then Case 1 or Case 2 occurs for any shortest circuit in  $U \cup I$ . In any case, either  $U \cup I$  has a circuit which contains exactly one pair from  $I$  or a circuit which contains exactly two pairs from  $I$ . The same is true for  $U \cup I^{-1}$ . Suppose that both  $U \cup I$  and  $U \cup I^{-1}$  contain circuits. We show that  $U$  has circuits, which contradicts our assumption. Let  $(u, v), (u', v') \in I$  be connected by the sequence

$$(u, v) = (u_1, v_1) \Lambda (u_2, v_2) \Lambda \cdots \Lambda (u_k, v_k) = (u', v')$$

where  $(u_i, v_i) \neq (u_{i+1}, v_{i+1})$  and  $k$  is odd. When each of  $U \cup I$  and  $U \cup I^{-1}$  has a circuit which contains exactly one pair from  $I$ , by Claim 1.5  $U \cup I$  has a circuit  $C$  in which  $(u, v)$  is the only pair from  $I$  and  $U \cup I^{-1}$  has a circuit  $C'$  in which  $(v, u)$  is the only pair from  $I^{-1}$ . Then  $(C \cup C') \setminus \{(u, v), (v, u)\}$  is a circuit in  $U$ . When each of  $U \cup I$  and  $U \cup I^{-1}$  has a circuit which contains exactly two pairs from  $I$ , by Claim 2.5  $U \cup I$  has a circuit  $C$  in which  $(u, v), (u', v')$  are the only pairs from  $I$  and  $U \cup I^{-1}$  has a circuit  $C'$  in which  $(v, u), (v', u')$  are the only pairs from  $I^{-1}$ . Then  $(C \cup C') \setminus \{(u, v), (u', v'), (v, u), (v', u')\}$  is a union of two circuits in  $U$ . Suppose that one of  $U \cup I$  and  $U \cup I^{-1}$  has a circuit which contains exactly one pair from  $I$  and the other has a circuit  $C''$  which contains exactly two pairs from  $I$ . Without loss of generality assume that  $U \cup I$  has a circuit which contains exactly one pair from  $I$ . By Claims 1.5 and 2.5,  $U \cup I$  has a circuit  $C$  in which  $(u, v)$  is the only pair from  $I$  and a circuit  $C'$  in which  $(u', v')$  is the only pair from  $I$ , and  $U \cup I^{-1}$  has a circuit  $C''$  in which  $(v, u), (v', u')$  are the only pairs from  $I$ . Then  $(C \cup C' \cup C'') \setminus \{(u, v), (u', v'), (v, u), (v', u')\}$  is a circuit in  $U$ . Therefore at least one of  $U \cup I$  and  $U \cup I^{-1}$  does not contain a circuit. This completes the proof.  $\square$

Combining Propositions 9, 10, and Theorem 11, we have the following:

**Theorem 12.** *The following statements are equivalent for a reflexive graph  $G$ :*

(i)  $G$  is a strong cocomparability graph;

(ii)  $G$  contains no invertible pairs;

(iii)  $G^+$  is bipartite.  $\square$

Given a reflexive graph  $G$ , in polynomial time one can construct  $G^+$  and test whether it is bipartite. Thus we have the following:

**Corollary 13.** *There is a polynomial-time algorithm to decide whether a reflexive graph is a strong cocomparability graph.  $\square$*

## 4 Concluding remarks and open problems

Strong cocomparability graphs form a proper subclass of cocomparability graphs. Gilmore and Hoffman [11] proved that a reflexive graph is an interval graph if and only if it is

both a chordal graph and a cocomparability graph. The following may be viewed as a strengthening of the theorem of Gilmore and Hoffman.

**Theorem 14.** *A reflexive graph  $G$  is an interval graph if and only if it is both a strongly chordal graph and a strong cocomparability graph.*

**Proof:** Suppose that  $G$  is an interval graph. Then by Theorems 4 and 5 it is strongly chordal. Since  $G$  is an interval graph, it has an interval ordering, which is a vertex ordering  $\prec$  such that for all  $x \prec y \prec z$ ,  $xz \in E(G)$  implies  $xy \in E(G)$ . It is easy to check that  $\prec$  is a strong cocomparability ordering. Hence  $G$  is also a strong cocomparability graph.

On the other hand, suppose that  $G$  is not an interval graph. Then according to [26] it contains an induced cycle of length  $\geq 4$  or an asteroidal triple (i.e., a set of three vertices such that any two of them are joined by a path which contains no neighbour of the third vertex). If it contains an induced cycle of length  $\geq 4$  then it is not strongly chordal. Suppose that  $u, v, w$  form an asteroidal triple in  $G$ . Then there is a path  $P$  joining  $u, v$  which contains no neighbour of  $w$ . For each edge  $xy$  on  $P$ ,  $xwy$  is an induced  $P_3$  in  $\overline{G}$ . By Proposition 7,  $(x, w)\Lambda(y, w)$ . It follows that  $(u, w) \sim (v, w)$ . Similarly,  $(u, v) \sim (w, v)$  and  $(w, u) \sim (v, u)$ . Hence

$$(u, v) \sim (w, v) \sim (w, u) \sim (v, u)$$

which shows that  $u, v$  are an invertible pair in  $G$  and, by Theorem 12,  $G$  is not a strong cocomparability graph.  $\square$

A forbidden induced subgraph characterization of cocomparability graphs has been given by Gallai [10]. It would be very interesting to find a forbidden induced subgraph characterization for strong cocomparability graphs.

**Problem 15.** *Characterize strong cocomparability graphs by forbidden subgraphs.*

Gallai's forbidden induced subgraph characterization of cocomparability graphs has 7 infinite families (including for example cycles with at least five vertices) and 10 of other subgraphs, each on seven vertices. In addition to these, we have found by a computer search that there are many other minimal graphs that are not strong cocomparability graphs (but are cocomparability graphs) – 5 on six vertices, 5 on seven vertices, 12 on eight vertices, and 30 on nine vertices.

Thus the above problem seems difficult at the moment. On the other hand, there is a simple forbidden subgraph characterization for strong cocomparability graphs that are bipartite.

A reflexive graph is *bipartite* if the graph obtained from it by deleting all loops is bipartite in the traditional sense. We use other terminology on bigraphs and bipartite graphs in the same sense, by applying them to the graph with loops removed; notably this is how we define the class of chordal bigraphs for the reflexive case.

Figure 2 depicts the forbidden subgraphs  $S$  and  $T$  for bipartite strong cocomparability graphs.

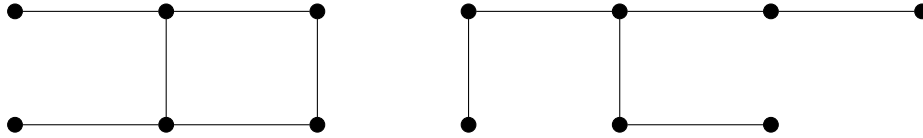


Figure 2: The two minimal forbidden subgraphs,  $S$  on the left and  $T$  on the right.

**Theorem 16.** *A bipartite graph is a strong cocomparability graph if and only if it is a chordal bigraph that does not contain  $S$  or  $T$  as a subgraph.*

Note that the characterization is stated for forbidden (not necessarily induced) subgraphs. It can of course be translated to a forbidden induced subgraph characterization, by forbidding all bipartite supergraphs of  $S$  and  $T$ .

A *caterpillar* is a tree  $G$  such that after the deletion of all its leaves (vertices of degree one)  $G$  becomes a path, called the *spine* of  $G$ . If a vertex  $v$  on the spine of a caterpillar  $G$  has degree two in  $G$ , we say we *replicate* it if we add to  $G$  additional vertices  $v', v'', \dots, v^*$  of degree two, with exactly the same neighbours as  $v$ ; we call  $v$  the *shadow* of all the replicated vertices. Consider a complete bipartite subgraph  $K_{2,s}$  (for any  $s \geq 2$ ) of a graph  $G$ , with the bipartition  $A \cup B$ , where  $|A| = 2, |B| = s \geq 2$ . If all  $s$  vertices in  $B$  have degree two in  $G$ , we call the subgraph a *square in  $G$*  and the two vertices in  $A$  its *connectors*.

**Proof:** It follows from [10] that  $T$  or any cycle with at least six vertices is not a cocomparability (and hence not a strong cocomparability) graph. Moreover, any spanning bipartite supergraph of  $T$  contains a bipartite supergraph of  $S$ . Finally, all bipartite supergraphs of  $S$  fail to be strong cocomparability graphs; this can be seen by a simple but tedious argument. Therefore, each bipartite strong cocomparability graph is a chordal bigraph that does not contain  $S$  or  $T$  as a subgraph.

To prove the converse, let  $G$  be a connected chordal bigraph that does not contain  $S$  or  $T$  as a subgraph. If  $G$  is a tree, then the absence of  $T$  implies that  $G$  is a caterpillar, with spine  $v_1, v_2, \dots, v_k$ . We construct a strong cocomparability ordering of any caterpillar  $G$  as  $v_1 \prec S_1 \prec v_2 \prec S_2 \prec \dots \prec v_k \prec S_k$ , where  $S_i$  is the set of leaves adjacent to  $v_i$ , listed in arbitrary order.

If  $G$  is not a tree, then it must contain a four-cycle. Moreover, each four-cycle must have two opposite vertices of degree two, else we have a copy of  $S$ . Suppose  $abcd$  is a four-cycle with  $a, c$  of degree two. Then any vertex  $x$  adjacent to both  $b$  and  $d$  must also be of degree two, and by adjoining all such vertices  $x$  we obtain a square in  $G$ . This applies even in the case the square is just a four-cycle, as in the above example, where  $b, d$  are the connectors. (Note that in such a case, the connectors can also be of degree two.) Two squares can meet in at most one vertex, which must be a connector in both squares, otherwise  $G$  had a copy of  $S$ . Now consider the graph  $G^*$  obtained from  $G$  by deleting all but one of the degree two vertices that are not connectors in any square. We can view the removed vertices as being the results of replication, and hence the remaining vertex is their shadow. Since the graph is connected and chordal,  $G^*$  is a tree, and hence

a caterpillar. Moreover, each shadow vertex is on the spine, and has no children. The strong cocomparability ordering of the caterpillar  $G^*$ , as described above, can now be modified to a strong cocomparability ordering of  $G$  by inserting each of the replicated vertices next to their shadow vertex from its square. Because the shadow vertices have no children off the spine, this is indeed a strong cocomparability ordering.  $\square$

We observe that the proof gives an implicit description of all bipartite strong cocomparability graphs – they are obtained from caterpillars by replicating an arbitrary set of non-consecutive degree two vertices. This is illustrated in Figure 3.

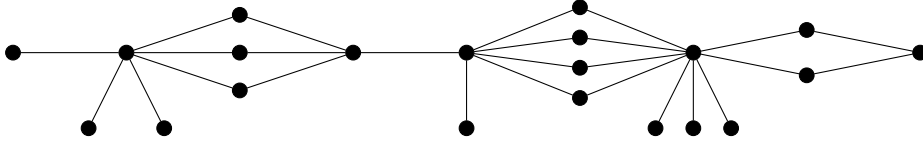


Figure 3: A typical example of a bipartite strong cocomparability graph.

Since bipartite strong cocomparability graphs do not contain asteroidal triples, they are bipartite permutation graphs [32]. On the other hand if a bipartite permutation graph does not contain  $S$  as a subgraph then it does not contain  $T$  as a subgraph and hence by Theorem 16 it is a strong cocomparability graph.

**Corollary 17.** *A bipartite graph is a strong cocomparability graph if and only if it is a bipartite permutation graph which does not contain  $S$  as a subgraph.*  $\square$

We also want point out a relationship between bipartite strong cocomparability graphs and cocomparability bigraphs. Clearly, if a cocomparability bigraph is a strong cocomparability graph then it is a bipartite strong cocomparability graph. Suppose that  $G$  is a bipartite strong cocomparability graph. Then by definition the adjacency matrix of  $G$  has a symmetric *Slash*-free ordering, which implies that the biadjacency matrix of  $G$  has a *Slash*-free ordering and hence  $G$  is a cocomparability bigraph. Therefore bipartite strong cocomparability graphs are precisely those strong cocomparability graphs which are also cocomparability bigraphs.

On a final note, we observe that by complementation we can introduce the class of strong comparability graphs. We define  $G$  to be a *strong comparability graph*, if its complement  $\overline{G}$  is a strong cocomparability graph. The complementation includes the loops, thus strong comparability graphs are irreflexive, and their adjacency matrices have 0-diagonal. A square symmetric 0,1 matrix with 0-diagonal has a symmetric  $I_2$ -free ordering if and only if it is the adjacency matrix of a strong comparability graph. The class of strong comparability graphs has an ordering characterization, a forbidden structure characterization, and a polynomial recognition algorithm, all via the our corresponding results for their complements, i.e., the strong cocomparability graphs.

To further motivate these concepts, we recall from [14] the definition of a *principal*  $\Gamma$  (respectively, *Slash*) submatrix of a symmetric matrix with 1-diagonal: it is a  $\Gamma$  (respectively, *Slash*) submatrix with one of its 1's lying on the main diagonal. Then



- a reflexive graph is chordal if and only if its adjacency matrix admits a symmetric ordering without a principal  $\Gamma$  submatrix; and is strongly chordal if and only if its adjacency matrix admits a symmetric ordering without any  $\Gamma$  submatrix, and
- a reflexive graph is a cocomparability graph if and only if its adjacency matrix admits a symmetric ordering without a principal *Slash* submatrix; and is a strong cocomparability graph if and only if its adjacency matrix admits a symmetric ordering without any *Slash* submatrix.

Similarly, we define a *principal*  $I_2$  submatrix in a symmetric matrix with 0-diagonal: this is an  $I_2$  submatrix with either of the 0's lying on the main diagonal. Then

- an irreflexive graph  $G$  is a comparability graph if and only if its adjacency matrix admits a symmetric ordering without a principal  $I_2$  submatrix, and is a strong comparability graph if and only if its adjacency matrix admits a symmetric ordering without any  $I_2$  submatrix.

These analogies suggest that these concepts are natural and may lead to interesting questions. As a first example, consider the domination problem, which is known to be NP-complete for chordal graphs but polynomial-time solvable for strongly chordal graphs [6]. The domination problem is also NP-complete for comparability graphs [23], so we wonder about strong comparability graphs.

**Problem 18.** *Determine the complexity of the domination problem for strong comparability graphs.*

We note that for cocomparability graphs the domination problem is polynomial-time solvable [25].

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