Balanced subdivisions of a large clique in graphs with high average degree

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Abstract

In 1984, Thomassen conjectured that for every constant $k \in \mathbb{N}$, there exists d such that every graph with average degree at least d contains a balanced subdivision of a complete graph on k vertices, i.e. a subdivision in which each edge is subdivided the same number of times. Recently, Liu and Montgomery confirmed Thomassen's conjecture.

We show that for every constant $0 < c < 1/2$, every graph with average degree at least d contains a balanced subdivision of a complete graph of size at least $\Omega(d^c)$. Note that this bound is almost optimal. Moreover, we show that every sparse expander with minimum degree at least d contains a balanced subdivision of a complete graph of size at least $\Omega(d)$.

1 Introduction

Let G be a graph. A *subdivision* of G, denoted by $\mathsf{T}G$, is a graph obtained from G by replacing each of its edges into internally vertex disjoint paths. We call the vertices of $\mathsf{T}G$ corresponding to the vertices of G its *core* vertices. Subdivisions play an important role in topological graph theory. In 1930s, Kuratowski [12] showed that a graph is not planar if and only if it contains a subdivision of a complete graph on five vertices or a subdivision of a complete bipartite graph with three vertices in each partition.

For integer $t > 0$, let $d(t)$ be the minimum number d such that every graph with average degree at least d contains a subdivision of a complete graph K_t. In 1967, Mader [17] showed such $d(t)$ must exist. Mader [17], and independently Erdős and Hajnal [3] conjectured that $d(t) = O(t^2)$. Later Mader [18] improved the upper bound of $d(t)$ to $O(2^t)$. In 1990s, Komlós and Szemerédi [7, 8], and independently, Bollobás and Thomason [2] confirmed this conjecture. Indeed, $d(t) = \Theta(t^2)$. As Jung [6] first observed, the lower bound of $d(t)$ can be achieved by disjoint union of complete regular bipartite graphs.

To guarantee a subdivision of a complete graph of size linear to the average degree, one must impose some additional conditions to eliminate the extremal examples. Minimum girth condition is one of them as complete bipartite graphs contain many short cycles. In fact, Mader [19] conjectured that every C_4-free graph of average degree d contains a subdivision of a complete graph

of size linear to d. Kühn and Osthus [9, 11] showed that every graph with sufficiently large girth contains a subdivision of a complete graph of size larger than its minimum degree. They [10] also showed that every C_4-free graph of average degree d contains a $\mathsf{TK}_{d/\log^{12} d}$. In [1], Balogh, Liu and Sharifzadeh proved Mader's conjecture when the graph is C_6-free. Recently, Liu and Montgomery [13] completely resolved this conjecture. Note that those proofs utilize the technique developed by Komlós and Szemerédi [7, 8].

For $\ell \in \mathbb{N}$, an ℓ-*balanced subdivision* of G, denoted by $\mathsf{T}G^{(\ell)}$, is a graph obtained from G by replacing each of its edges into internally vertex disjoint paths of length exactly ℓ. A *balanced subdivision* is an ℓ-*balanced subdivision* for some $\ell \in \mathbb{N}$. Thomassen [20, 21, 22] conjectured that for every constant $k \in \mathbb{N}$, there exists d such that every graph with average degree at least d contains a $\mathsf{TK}_k^{(\ell)}$ for some $\ell \in \mathbb{N}$. More recently, Liu and Montgomery [14] confirmed Thomassen's conjecture.

In this paper, we study the following question: Given a graph with average degree d, what is the largest size of a balanced subdivision of a complete graph that it contains as subgraph? We will show the following.

Theorem 1.1. *For every constant $0 < c < 1/2$, every graph with average degree at least d contains a $\mathsf{TK}_{\Omega(d^c)}^{(\ell)}$ for some $\ell \in \mathbb{N}$.*

We observe that an upper bound of this problem is $O(d^{1/2})$, given by disjoint union of complete regular bipartite graphs. Therefore, Theorem 1.1 is almost optimal.

The proof of Theorem 1.1 uses the ideas from [8, 13, 14]. By a result of Komlós and Szemerédi, we can find a graph that is as dense as the original graph and has some expansion property. Then we divide into two cases depending on whether the graph is dense or not. The dense case is handled in Lemma 3.1 and the sparse case is covered in Lemma 4.1.

The rest of the paper will be organized as follows: In Section 2, we introduce the notion of graph expanders and adjusters, and give some lemmas that will be used later. We construct large balanced clique subdivisions in dense graphs and show Lemma 3.1 in Section 3. The proof of Lemma 4.1 will be given in Section 4 where we further divide into cases according to whether such graph has many vertices of large degree or not. We conclude in Section 5.

1.1 Notations

Let G be a graph. Let $V(G)$ and $E(G)$ be vertex set and edge set of G respectively. Let $d(G), \delta(G), \Delta(G)$ be average degree, minimum degree and maximum degree of G respectively. For $v \in V(G)$, let $d_G(v)$ denote the degree of v in G. We omit the subscript if there is no confusion.

For two vertices $u, v \in V(G)$, a u,v-path is a path with end vertices u and v. We use $\ell(P)$ to denote the length of a path P. The distance between two set of vertices U, V in a graph G is the minimum length of a u,v-path in G with $u \in U$ and $v \in V$. For a u,v-path P and an integer $0 \le r \le \ell(P)$, let $P(v, r)$ be the subpath of P of length r with one of its end vertex to be v.

Let $X \subseteq V(G)$, we write $G - X$ for the induced subgraph of $G[V(G) \backslash X]$. Denote $N_G(X)$ the (external) neighborhood of X in $G - X$. For integer $i \ge 0$, we define the i-th ball around X in G to be the set of vertices that have distance at most i from X in G, denoted by $B_G^i(X)$. For convenience, $B_G(X) = B_G^1(X) = X \cup N_G(X)$.

We omit the floors and ceilings when they are not crucial. All logarithms are natural.

2 Preliminaries

2.1 Komlós-Szemerédi graph expanders

The well connectedness of a graph can be measured by the expansion property. One form of the expansion property is as follow: For a graph G and for every not too large set $X \subseteq V(G)$, $|N_G(X)| \geq \varepsilon(|X|)|X|$ holds for some function ε depending on $|X|$. A graph that satisfies the expansion property is called an expander graph. A detailed coverage of expander graphs and their applications in theoretical computer science is presented in [5], and their applications in mathematics are given in [16].

While the linear expansion property (when ε is a constant function) has been studied extensively, Komlós and Szemerédi [7, 8] introduced sublinear expansion property, which forms the base of our proof.

Definition 2.1. For each $\varepsilon_1 > 0$ and $k > 0$, a graph G is an (ε_1, k)-*expander* if

$$|N_G(X)| \geq \varepsilon(|X|, \varepsilon_1, k) \cdot |X|$$

for all $X \subseteq V(G)$ with $k/2 \leq |X| \leq |V(G)|/2$, where

$$\varepsilon(x, \varepsilon_1, k) := \begin{cases} 0 & \text{if } x < k/5, \\ \varepsilon_1/\log^2(15x/k) & \text{if } x \geq k/5. \end{cases} \tag{1}$$

Whenever the choices of ε_1, k are clear, we omit them and write $\varepsilon(x)$ for $\varepsilon(x, \varepsilon_1, k)$.

In the above definition, note that $\varepsilon(x, \varepsilon_1, k)$ decreases as $x \geq k/2$ increases, so the rate of expansion decreases as the size of X grows. However, $\varepsilon(x, \varepsilon_1, k) \cdot x$ increases as x increases, so the number of vertices that X expands increases as the size of X grows.

Komlós and Szemerédi [8] showed that every graph G contains an expander subgraph with average degree and minimum degree linear to the average degree of G.

Lemma 2.2 ([8]). *There exists some $\varepsilon_1 > 0$ such that the following holds for every $k > 0$. Every graph G has an (ε_1, k)-expander subgraph H with $d(H) \geq d(G)/2$ and $\delta(H) \geq d(H)/2$.*

Note that, in Theorem 2.2, the expander subgraph H can be much smaller than the original graph G in size. To see this, one can take G to be the disjoint union of many copies of such H.

The expansion property allows us to connect vertex sets with a short path even after removing a small set of vertices (see Lemma 3.4 from [13]).

Lemma 2.3 ([13]). *For each $0 < \varepsilon_1, \varepsilon_2 < 1$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2)$ such that the following holds for each $n \geq d \geq d_0$ and $x \geq 1$. Let G be an n-vertex $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d - 1$.*

Let $A, B \subseteq V(G)$ with $|A|, |B| \geq x$, and let $W \subseteq V(G) \setminus (A \cup B)$ satisfy $|W|\log^3 n \leq 10x$. Then, there is a path from A to B in $G - W$ with length at most $\frac{40}{\varepsilon_1}\log^3 n$.

It is well known that every graph G has a bipartite subgraph H with $d(H) \geq d(G)/2$. The next corollary follows immediately from this fact and Lemma 2.2.

Corollary 2.4. *There exists some $\varepsilon_1 > 0$ such that the following holds for every $\varepsilon_2 > 0$ and $d \in \mathbb{N}$. Every graph G with $d(G) \geq 8d$ has a bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander subgraph H with $\delta(H) \geq d$.*

The following notation (also see [14]) is convenient as we often work in a bipartite graph.

Definition 2.5 ([14]). For any connected bipartite graph H and $u, v \in V(H)$, let

$$\pi(u, v, H) = \begin{cases} 0 & \text{if } u = v, \\ 1 & \text{if } u \text{ and } v \text{ are in different vertex classes in the (unique) bipartition of } H, \\ 2 & \text{if } u \text{ and } v \text{ are in the same vertex class and } u \neq v. \end{cases}$$

2.2 Liu-Montgomery adjusters

First, we need the definition below from Liu and Montgomery [14].

Definition 2.6 ([14]). Given a vertex v in a graph F, F is a (D, m)-*expansion of* v if $|F| = D$ and $v \in V(F)$ is at distance at most m in F from any other vertex of F.

Expansion has the following trimming property.

Proposition 2.7 ([14]). *Let $D, m \in \mathbb{N}$ and $1 \leq D' \leq D$. Then, any graph F which is a (D, m)-expansion of v contains a subgraph which is a (D', m)-expansion of v.*

The following is a technical lemma that allows us to find one set within a collection that expands to larger size. We remark that the same proof (by setting $\alpha = 2^{-10}$) of the original Lemma 3.7 in [14] could imply a stronger conclusion that there exists a set that expands to size $\exp((\log \log n)^{200})$, instead of $\log^k n$.

Lemma 2.8 ([14]). *For each $0 < \varepsilon_1 < 1$ and $0 < \varepsilon_2 < 1/5$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2)$ such that the following holds for each $n \geq d \geq d_0$. Suppose that G is an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$. Let $U \subseteq V(G)$ satisfy $|U| \leq \exp((\log \log n)^{200})$. Let $r = n^{1/8}$ and $\ell_0 = (\log \log n)^{10^6}$. Let $m = \frac{800}{\varepsilon_1} \log^3 n$. Suppose $(A_i, B_i, C_i), i \in [r]$ are such that the following holds for each $i \in [r]$.*

A1 $|A_i| \geq d_0$.

A2 $B_i \cup C_i$ *and* A_i *are disjoint sets in* $V(G) \setminus U$, *with* $|B_i| \leq |A_i| / \log^{1000} |A_i|$.

A3 A_i *has 4-limited contact with* C_i *in* $G - U - B_i$.

A4 *Each vertex in* $B_{G-U-B_i-C_i}^{\ell_0}(A_i)$ *has at most* $d/2$ *neighbours in* U.

A5 *For each* $j \in [r] \setminus \{i\}$, A_i *and* A_j *are at least at distance* $2\ell_0$ *apart in* $G - U - B_i - C_i - B_j - C_j$.

Then, for some $i \in [r]$, $|B_{G-U-B_i-C_i}^{\ell_0}(A_i)| \geq \exp((\log \log n)^{200})$.

Liu and Montgomery [14] introduced a structure called *adjuster* which contains paths of lengths that belong to a long arithmetic progression of difference 2. We can use this structure to adjust a path to the desired length.

Definition 2.9 ([14]). A (D, m, k)-*adjuster* $\mathcal{A} = (v_1, F_1, v_2, F_2, A)$ in a graph G consists of vertices $v_1, v_2 \in V(G)$, graphs $F_1, F_2 \subseteq G$ and a vertex set $A \subseteq V(G)$ such that the following hold for some $\ell \in \mathbb{N}$.

B1 A, $V(F_1)$ and $V(F_2)$ are pairwise disjoint.

B2 For each $i \in [2]$, F_i is a (D, m)-expansion of v_i.

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B3 $|A| \leq 10mk$.

B4 For each $i \in \{0, 1, \ldots, k\}$, there is a v_1, v_2-path in $G[A \cup \{v_1, v_2\}]$ with length $\ell + 2i$.

We call the smallest such ℓ for which these properties hold the *length of the adjuster* and denote it $\ell(\mathcal{A})$. Note that it immediately follows that $\ell(\mathcal{A}) \leq |A| + 1 \leq 10mk + 1$. We call a $(D, m, 1)$-adjuster a *simple adjuster*. Let $V(\mathcal{A}) = V(F_1) \cup V(F_2) \cup A$.

Note that if $m > m'$, then a (D, m', k)-adjuster is also a (D, m, k)-adjuster. Lemma 4.3 in [14] showed that there exists a simple adjuster in every expander without $\mathsf{TK}_{d/2}^{(2)}$ even after removing a moderate size of vertices. We remark that Lemma 2.8 allows us to prove the following strengthening variant. (The same proof works by setting $\Delta = \exp((\log \log n)^{400}), \ell_0 = (\log \log n)^{10^6}$ and $|Z| = \exp((\log \log n)^{400})$ in the original proof.)

Lemma 2.10 ([14]). *There exists some $\varepsilon_1 > 0$ such that, for any $0 < \varepsilon_2 < 1/5$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2)$ such that the following holds for each $n \geq d \geq d_0$. Suppose that G is an n-vertex $\mathsf{TK}_{d/2}^{(2)}$-free bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$. Let $m = \frac{200}{\varepsilon_1} \log^3 n$ and $|D| \leq \exp((\log \log n)^{200})$. Let $U \subseteq V(G)$ such that $|U| \leq \exp((\log \log n)^{200})$.*
Then, $G - U$ contains a $(D, m, 1)$-adjuster.

Therefore, as Lemma 4.7 in [14], we obtain the following variant that connects simple adjusters into a larger adjuster robustly.

Lemma 2.11 ([14]). *There exists some $\varepsilon_1 > 0$ such that, for any $0 < \varepsilon_2 < 1/5$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2)$ such that the following holds for each $n \geq d \geq d_0$. Suppose that G is an n-vertex $\mathsf{TK}_{d/2}^{(2)}$-free bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$.*
Let $m = \frac{800}{\varepsilon_1} \log^3 n$. Suppose $\log^{10} n \leq D \leq \exp((\log \log n)^{100})$, $1 \leq r \leq 20m$ and $U \subseteq V(G)$ with $|U| \leq \exp((\log \log n)^{100})$.
Then, there is a (D, m, r)-adjuster in $G - U$.

2.3 Connecting vertices by paths of specific lengths

Liu and Montgomery (see Corollary 3.15 in [14]) proved the existence of two vertex disjoint paths in an expander graph so that the sum of their lengths is close to the desired length while avoiding a set of moderate size.

Lemma 2.12 ([14]). *For any $0 < \varepsilon_1, \varepsilon_2 < 1$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2)$ such that the following holds for each $n \geq d \geq d_0$. Suppose that G is an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$.*
Let $\log^{10} n \leq D \leq n/\log^{10} n$, $\frac{100}{\varepsilon_1} \log^3 n \leq m \leq \log^4 n$ and $\ell \leq n/\log^{10} n$. Let $A \subseteq V(G)$ satisfy $|A| \leq D/\log^3 n$. Let $F_1, \ldots, F_4 \subseteq G - A$ be vertex disjoint subgraphs and $v_1, \ldots, v_4$ be vertices such that, for each $i \in [4]$, F_i is a (D, m)-expansion of v_i.
Then, $G - A$ contains vertex disjoint paths P and Q with $\ell \leq \ell(P) + \ell(Q) \leq \ell + 20m$ such that both P and Q connect $\{v_1, v_2\}$ to $\{v_3, v_4\}$.

Using adjuster structure, they also showed the existence of a path of specific length connecting two given vertices while avoiding a set of moderate size (see Lemma 4.8 in [14]).

Lemma 2.13 ([14])**.** *There exists some $\varepsilon_1 > 0$ such that, for any $0 < \varepsilon_2 < 1/5$ and $k \geq 10$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, k)$ such that the following holds for each $n \geq d \geq d_0$. Suppose that G is an n-vertex $\mathsf{TK}_{d/2}^{(2)}$-free bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$.*

Suppose $\log^{10} n \leq D \leq \log^k n$, and $U \subseteq V(G)$ with $|U| \leq D/2\log^3 n$, and let $m = \frac{800}{\varepsilon_1} \log^3 n$. Suppose $F_1, F_2 \subseteq G - U$ are vertex disjoint such that F_i is a (D, m)-expansion of v_i, for each $i \in [2]$. Let $\log^7 n \leq \ell \leq n/\log^{10} n$ be such that $\ell = \pi(v_1, v_2, G) \mod 2$.

Then, there is a v_1, v_2-path with length ℓ in $G - U$.

2.4   Expansion of vertices in sparse graphs

When maximum degree is bounded, there exist many vertices that are pairwise far apart in the graph (see Proposition 5.3 in [13]).

Lemma 2.14 ([13])**.** *Let $s \geq 1$. There exists n_0 such that the following holds for all $n \geq n_0$. Suppose G is an n-vertex graph with maximum degree at most $\log^{30s} n$. Then G contains at least $n^{1/5}$ vertices which are pairwise at distance at least $\log n/(50s \log \log n)$ apart.*

We need the following definition.

Definition 2.15 ([13])**.** We say that paths $P_1, \cdots, P_q$, each starting with the vertices v and contained in the vertex set W, are *consecutive shortest paths* from v in W if, for each i, $1 \leq i \leq q$, the path P_i is a shortest path between its endpoints in the set $W - \cup_{j<i} P_j + v$.

We can expand a vertex v to a set of moderate size while avoiding internal vertices of a family of paths if those paths do not intersect balls around v much (see Lemma 5.5 in [13]). Note that in the following lemma, the condition that $P_1, \cdots, P_q$ are consecutive shortest paths from v in $B_H^r(v)$, is used to guarantee that for each $p < r$, only the first $p + 2$ vertices of each of the paths P_i including the vertex v, can belong in $N_H(B_{H-P+v}^p(v))$.

Lemma 2.16 ([13])**.** *Let $0 < \varepsilon_1 < 1, 0 < \varepsilon_2 < 1/20$ and $s \geq 1$. Then there is some $c > 0$ and $d_0 \in \mathbb{N}$ for which the following holds for any n and d with $d_0 \leq d \leq \log^{20s} n$. Suppose H is an n-vertex $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(H) \geq d/16$. Let $r = (\log \log n)^5$ and $P = \cup_i V(P_i)$, if $q \leq cd$ and $P_1, \cdots, P_q$ are consecutive shortest paths from v in $B_H^r(v)$, then $|B_{H-P+v}^r(v)| \geq 2d^2 \log^{10} n$.*

We can further expand a set of moderate size to a large set avoiding a set of vertices (see Proposition 5.6 in [13]).

Lemma 2.17 ([13])**.** *Let $0 < \varepsilon_1 < 1$, $0 < \varepsilon_2 < 1/20$ and $s \geq 1$. There exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ such that the following is true for each $d \geq d_0$. Suppose that H is an $(\varepsilon_1, \varepsilon_2 d)$-expander with n vertices and let $k = \log n/100s \log \log n$. If $Y, W \subseteq V(G)$ are disjoint sets with $|Y| \geq d^2 \log^{10} n$ and $|W| \leq d^2 \log^7 n$, then $|B_{G-W}^k(Y)| \geq \exp((\log n)^{1/4})$.*

Note that the exponents of $\log n$ in the above two lemmas are altered to adapt to our proof. The proofs are similar to the original proofs so we omit here.

3   Constructing balanced clique subdivisions in dense graphs

In this section, we deal with the case when graph is dense, that is, $d \geq \log^s n$ for some large s, and prove the following.

Lemma 3.1. *There exists $0 < \varepsilon_1 < 1$ such that for any $0 < \varepsilon_2 < 1$ and $s \geq 20$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ such that the following holds for each $n \geq d \geq d_0$ and $d \geq \log^s n$. Suppose that G is an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$. Then G contains a $\mathsf{TK}^{(\ell)}_{\sqrt{d}/2 \log^{10} n}$ for some $\ell \in \mathbb{N}$.*

Note that when such graph G is dense, it contains a balanced clique subdivision of size at least $\Omega(d^{\frac{1}{2} - \frac{10}{s}})$ for arbitrarily large s.

We adopt the idea in [14] that uses adjuster structure to alter the length of a path, and refine the analysis when graph is dense. In any dense expander graph with minimum degree d, we can find a simple adjuster with expansion size linear to d.

Lemma 3.2. *For any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $s \geq 20$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ such that the following is true for each $n \geq d \geq d_0$ and $d \geq \frac{1}{10} \log^s n$. Suppose that G is an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$. Let $D \leq d/3$.*
Then, G contains a $(D, \log n, 1)$-adjuster.

Proof. Let C be a shortest cycle in G. Since G is bipartite, C must have even length, say $2\ell_0$. Since $\delta(G) \geq d$, we have $\ell_0 \leq \log n / \log(d-1) < \log n$. Let x_1, x_2 be two vertices of distance $\ell_0 - 1$ on C.

Choose $F_i \subseteq N(x_i)$ such that $|F_i| = D - 1$ and F_i is disjoint from $F_{3-i} \cup V(C) \cup \{x_1, x_2\}$ for $i \in [2]$. This is possible because we may first choose $F_1 \subseteq N(x_1) \setminus (V(C) \cup \{x_2\})$ and then choose $F_2 \subseteq N(x_2) \setminus (F_1 \cup V(C) \cup \{x_1\})$, noting that $|N(x_2)| - |F_1| - |V(C)| > d - d/3 - \log n > d/3$. Thus $F_i \cup \{x_i\}$ is $(D, 2)$-expansion of x_i for $i \in [2]$. Let $A = C \setminus \{x_1, x_2\}$, so $|A| < |C| < 2 \log n$. Therefore, $(x_1, F_1 \cup \{x_1\}, x_2, F_2 \cup \{x_2\}, A)$ is a desired $(D, \log n, 1)$-adjuster. $\square$

Moreover, such a simple adjuster exists robustly in dense expander graph upon removal of any subset of vertices of moderate size.

Lemma 3.3. *There exists some $\varepsilon_1 > 0$ such that for every $\varepsilon_2 > 0$ and $s \geq 20$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ such that the following is true for each $n \geq d \geq d_0$ and $d \geq \log^s n$. Suppose that G is an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$. Let $D \leq d/30$ and $U \subseteq V(G)$ such that $|U| \leq d/10$.*
Then, $G - U$ contains a $(D, \log n, 1)$-adjuster.

Proof. Let $\varepsilon_1 > 0$ be such that Corollary 2.4 holds. Note that $d(G-U) = 2|E(G-U)|/|V(G-U)| \geq (dn - 2n|U|)/n = d - 2|U| \geq 4d/5$.

By Corollary 2.4, $G - U$ has a bipartite $(\varepsilon_1, \varepsilon_2 d/10)$-expander subgraph H with $\delta(H) \geq d/10$. Also note that $d(H) \geq \delta(H) \geq d/10 \geq \frac{1}{10} \log^s n \geq \frac{1}{10} \log^s |V(H)|$. Apply Lemma 3.2 with $G_{3.2} = H$, we obtain a $(D, \log n, 1)$-adjuster in H, and thus in $G - U$. $\square$

We chain simple adjuster together into a larger adjuster.

Lemma 3.4. *There exists some $0 < \varepsilon_1 < 1$ such that for every $0 < \varepsilon_2 < 1$ and $s \geq 20$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ such that the following is true for each $n \geq d \geq d_0$ and $d \geq \log^s n$. Suppose that G is an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$. Let $c = 1/100$, $m = \frac{800}{\varepsilon_1} \log^3 n$, $D = cd$, $1 \leq r \leq 20m$ and $U \subseteq V(G)$ such that $|U| \leq cd/10$.*
Then, $G - U$ contains a (D, m, r)-adjuster.

Proof. Let $\varepsilon_1 > 0$ be such that Lemma 3.3 holds and $d_0 = d(\varepsilon_1, \varepsilon_2, s)$ be large. We prove the property by induction on r. When $r = 1$, Lemma 3.3 gives the desired $(D, \log n, 1)$-adjuster.

Now assume that for some $1 \leq r < 20m$, $G - U$ contains a (D, m, r)-adjuster, say $\mathcal{A}_1 := (v_1, F_1, v_2, F_2, A_1)$. We aim to show the existence of a $(D, m, r+1)$-adjuster. Let $U' = U \cup F_1 \cup F_2 \cup A_1$. So $|U'| = |U| + |F_1| + |F_2| + |A_1| \leq cd/10 + cd + cd + 10mr < 3cd$. Applying Lemma 3.3 with $(G, U)_{3.3} = (G, U')$, we have that $G - U'$ contains a $(D, \log n, 1)$-adjuster, say $\mathcal{A}_2 := (v_3, F_3, v_4, F_4, A_2)$. Since $|F_1 \cup F_2| = |F_3 \cup F_4| = 2D$ and $|A_1 \cup A_2| < 20mr < \log^7 n < 2D/(10 \log^3 n)$, there exists a path P of length at most m from $V(F_1) \cup V(F_2)$ to $V(F_3) \cup V(F_4)$ avoiding $A_1 \cup A_2$ by Lemma 2.3.

Without loss of generality, we may assume that P is from $V(F_1)$ to $V(F_3)$. By definition of F_1 and F_3, there exists a path Q from v_1 to v_3 in $F_1 \cup P \cup F_3$ of length at most $2 \log n + 3m$. We claim that $\mathcal{A}_3 := (v_2, F_2, v_4, F_4, A_1 \cup A_2 \cup V(Q))$ is a desired $(D, m, r+1)$-adjuster. In fact, by construction F_2, F_4 and $A_1 \cup A_2 \cup V(Q)$ are pairwise disjoint, and F_i is a (D, m)-expansion of v_i for $i \in \{2, 4\}$. Note that $|A_1 \cup A_2 \cup V(Q)| \leq |A_1| + |A_2| + |V(Q)| \leq 10mr + 10 \log n + (2 \log n + 3m) \leq 10m(r+1)$. Finally, let $\ell := \ell(\mathcal{A}_1) + \ell(\mathcal{A}_2) + |V(Q)|$. We show that for each $i \in \{0, 1, \cdots, r+1\}$, there is a v_2, v_4-path in $G[A_1 \cup A_2 \cup V(Q) \cup \{v_2, v_4\}]$ with length $\ell + 2i$. If $i \in \{0, 1, \cdots, r\}$, then let $i_1 = i$ and $i_2 = 0$; otherwise, let $i_1 = r$ and $i_2 = 1$. Let P_1 be a v_2, v_1-path of length $\ell(\mathcal{A}_1) + 2i_1$ in $G[A_1 \cup \{v_1, v_2\}]$ and P_2 be a v_3, v_4-path of length $\ell(\mathcal{A}_2) + 2i_2$ in $G[A_2 \cup \{v_3, v_4\}]$. Therefore, $P_1 \cup Q \cup P_2$ is a desired v_2, v_4-path in $G[A_1 \cup A_2 \cup V(Q) \cup \{v_2, v_4\}]$ with length $\ell + 2i$. $\square$

In the follow lemma, we show that there exists a path of certain length connecting two given vertices.

Lemma 3.5. *There exists some $0 < \varepsilon_1 < 1$ such that, for any $0 < \varepsilon_2 < 1$ and $s \geq 20$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ such that the following holds for each $n \geq d \geq d_0$ and $d \geq \log^s n$. Suppose that G is an n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \geq d$.*

Suppose $D = d/\log^{10} n$, and $U \subseteq V(G)$ with $|U| \leq D/2 \log^3 n$, and let $m = \frac{800}{\varepsilon_1} \log^3 n$. Suppose $F_1, F_2 \subseteq G - U$ are vertex disjoint subgraphs such that F_i is a (D, m)-expansion of v_i, for each $i \in [2]$. Let $\log^7 n \leq \ell \leq n/\log^{10} n$ be such that $\ell = \pi(v_1, v_2, G) \mod 2$.

Then, there is a v_1, v_2-path with length ℓ in $G - U$.

Proof. Let $0 < \varepsilon_1 < 1$ be such that Lemma 3.4 holds and $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ be large.

Let $U' = U \cup V(F_1) \cup V(F_2)$. So $|U'| = |U| + |V(F_1)| + |V(F_2)| \leq D/(2 \log^3 n) + D + D \leq 3D < d/1000$. By Lemma 3.4 with $(G, U, r)_{3.4} = (G, U', 20m)$, there is a $(d/100, m, 20m)$-adjuster, and thus a $(D, m, 20m)$-adjuster, say $\mathcal{A} = (v_3, F_3, v_4, F_4, A)$, in $G - U'$ with length $\ell(\mathcal{A}) \leq |A| + 1 \leq 400m^2$. Let $\bar{\ell} = \ell - 20m - \ell(\mathcal{A})$, so that $0 \leq \bar{\ell} \leq n/\log^{10} n$. As $|A \cup U| \leq 400m^2 + D/2 \log^3 n \leq D/\log^3 n$, by Lemma 2.12 with $(G, D, A)_{2.12} = (G, D, A \cup U)$, there are paths P and Q in $G - U - A$ which are vertex disjoint, both connect $\{v_1, v_2\}$ to $\{v_3, v_4\}$ and so that $\bar{\ell} \leq \ell(P) + \ell(Q) \leq \bar{\ell} + 20m$. Note that we can assume, without loss of generality, that P is a v_1, v_3-path and Q is a v_2, v_4-path.

Now, $0 \leq \ell - \ell(P) - \ell(Q) - \ell(\mathcal{A}) \leq 20m$. As $\mathcal{A}$ is a $(D, m, 20m)$-adjuster, there is a v_3, v_4-path in $G[A \cup \{v_3, v_4\}]$ with length $\ell(\mathcal{A})$, and therefore $\ell(\mathcal{A}) = \pi(v_3, v_4, G) \mod 2$. Then, as $\ell(P) = \pi(v_1, v_3, G) \mod 2$, $\ell(Q) = \pi(v_2, v_4, G) \mod 2$, $\ell = \pi(v_1, v_2, G) \mod 2$ and $\pi(v_1, v_2, G) = \pi(v_1, v_3, G) + \pi(v_3, v_4, G) + \pi(v_4, v_2, G) \mod 2$, we have $\ell - \ell(P) - \ell(Q) - \ell(\mathcal{A}) = 0 \mod 2$. That is, there is some $i \in \mathbb{N}$ with $2i = \ell - \ell(P) - \ell(Q) - \ell(\mathcal{A})$, where $i \leq 10m$.

Therefore, by the definition of the adjuster, there is a v_3, v_4-path R with length $\ell(\mathcal{A}) + 2i = \ell - \ell(P) - \ell(Q)$ in $G[A \cup \{v_3, v_4\}]$. Then, $P \cup R \cup Q$ is a v_1, v_2-path with length ℓ in $G - U$. $\square$

Finally, we are ready to show Lemma 3.1.

Proof of Lemma 3.1. Let $0 < \varepsilon_1 < 1$ be such that Lemma 3.5 holds and $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ be large. Let $t = \lfloor \sqrt{d}/(2\log^{10} n) \rfloor$ and $\ell = 2\lceil \log^7 n \rceil \equiv 0 \pmod 2$. Let $v_1, \cdots, v_t$ be t distinct vertices in the same class of the partition of G that will serve as t core vertices in the final balanced clique subdivision construction. Let $\mathcal{P} = \{P_1, \cdots, P_K\}$ be a maximal collection of pairwise internally disjoint paths such that

C1 For each $k \in [K]$, P_k is a v_i, v_j-path of length ℓ for some distinct $i, j \in [t]$.

C2 For distinct $i, j \in [t]$, there is at most one path in $\mathcal{P}$ with v_i and v_j as end vertices.

One can verify that if $K = \binom{t}{2}$, then the graph formed by all the paths in $\mathcal{P}$ is a desired $\mathsf{TK}_t^{(\ell)}$. Hence, we may assume that there exist distinct $i, j \in [t]$ such that $\mathcal{P}$ contains no such v_i, v_j-path of length ℓ.

Let $U = (\bigcup_{k\in[K]} V(P_k)) \bigcup \{v_q : q \in [t]\} \setminus \{v_i, v_j\}$. So $|U| \le K\ell + t \le t^2\ell \le d/2\log^{13} n$. Choose $F_m \subseteq N(v_m)$ such that $|F_m| = d/\log^{10} n - 1$ and F_m is disjoint from $F_{i+j-m} \cup U \cup \{v_i, v_j\}$ for $m \in \{i, j\}$. This is possible because we may first choose $F_i \subseteq N(v_i) \setminus (U \cup \{v_j\})$ and then choose $F_j \subseteq N(v_j) \setminus (F_i \cup U \cup \{v_i\})$, noting that $|N(v_j)| - |F_i| - |U| > d - d/\log^{10} n - d/\log^{13} n > d/\log^{10} n$. By Lemma 3.5, there is a v_i, v_j-path P_{K+1} of length ℓ in $G - U$. Therefore, $\{P_1, P_2, \cdots, P_{K+1}\}$ contradicts the maximality of $\mathcal{P}$. This completes the proof. $\qquad\square$

4 Constructing balanced clique subdivisions in sparse graphs

In this section, we handle the case when graph is sparse, that is $d < \log^s n$ for some large s, and prove the following.

Lemma 4.1. *There exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon_2 < 1/5$ and $s \ge 20$, there exist $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ and some constant $t > 0$ such that the following holds for each $n \ge d \ge d_0$ and $d < \log^s n$. Suppose that G is a $\mathsf{TK}_{d/2}^{(2)}$-free n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \ge d$. Then G contains a $\mathsf{TK}_{td}^{(\ell)}$ for some $\ell \in \mathbb{N}$.*

Note that when such graph G is sparse, we are able to find a balanced clique subdivision of size linear to its average degree. This may be of independent interest.

We discuss two cases depending on whether there are many vertices with degree at least $\Delta(G) \ge c^2 d^2 \log^{10} n$ or not.

4.1 When many vertices have large degree

When many vertices have large degree, we could use the neighbourhood of those vertices to construct expansion and find paths of specific length between them.

Lemma 4.2. *There exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon_2 < 1/5$, $c > 0$ and $s \ge 20$, there exist $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ and a constant $0 < t_3 < c/3$ such that the following holds for each $n \ge d \ge d_0$ and $d < \log^s n$. Suppose that G is a $\mathsf{TK}_{d/2}^{(2)}$-free n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander subgraph G with $\delta(G) \ge d$. Moreover, suppose at least $2t_3 d$ vertices have degree at least $\Delta = c^2 d^2 \log^{10} n$. Then G contains a $\mathsf{TK}_{t_3 d}^{(\ell)}$ for some $\ell \in \mathbb{N}$.*

Proof. Let $\varepsilon_1 > 0$ be such that Lemma 2.13 holds and $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ be large. Let $0 < t_3 < c/3$, $t = \lceil t_3 d \rceil$ and $\ell = 2\lceil \log^7 n \rceil$. Let $v_1, \cdots, v_t$ be t distinct vertices in the same partition of G of degree at least Δ that will serve as t core vertices in the final balanced clique subdivision construction. Let $\mathcal{P} = \{P_1, \cdots, P_K\}$ be a maximal collection of pairwise internally disjoint paths such that

D1 For each $k \in [K]$, P_k is a v_i, v_j-path of length ℓ for some distinct $i, j \in [t]$.

D2 For distinct $i, j \in [t]$, there is at most one path in $\mathcal{P}$ with v_i and v_j as end vertices.

One can verify that if $K = \binom{t}{2}$, then the graph formed by all the paths in $\mathcal{P}$ is a desired $\mathsf{TK}_t^{(\ell)}$. Hence, we may assume that there exist distinct $i, j \in [t]$ such that $\mathcal{P}$ contains no such v_i, v_j-path of length ℓ.

Let $U = (\bigcup_{k \in [K]} V(P_k)) \bigcup \{v_q : q \in [t]\} \backslash \{v_i, v_j\}$. So $|U| \le K\ell + t \le t^2 \ell \le t_3^2 d^2 \log^7 n$. Choose $F_m \subseteq N(v_m)$ such that $|F_m| = \frac{c^2}{4} d^2 \log^{10} n - 1$ and F_m is disjoint from $F_{i+j-m} \cup U \cup \{v_i, v_j\}$ for $m \in \{i, j\}$. This is possible because we may first choose $F_i \subseteq N(v_i) \backslash (U \cup \{v_j\})$ and then choose $F_j \subseteq N(v_j) \backslash (F_i \cup U \cup \{v_i\})$, noting that $|N(v_j)| - |F_i| - |U| > c^2 d^2 \log^{10} n - \frac{c^2}{4} d^2 \log^{10} n - t_3^2 d^2 \log^7 n > \frac{c^2}{4} d^2 \log^{10} n$. By Lemma 2.13 with $(G, U, D, k)_{2.13} = (G, U, \frac{c^2}{4} d^2 \log^{10} n, 2s + 10)$, there is a v_i, v_j-path P_{K+1} of length ℓ in $G - U$. Therefore, $\{P_1, P_2, \cdots, P_{K+1}\}$ contradicts the maximality of $\mathcal{P}$. This completes the proof. $\square$

4.2 When all vertices have bounded maximum degree

We adopt the idea from [8] and [13]. We find two balls of radius r_1 and r_2 respectively ($r_1 \ll r_2$) around each core vertex, and try to connect two core vertices by a path of specific length avoiding balls of radius r_1 of all other core vertices. The existence of such paths is guaranteed by the expander property and large expansion around each core vertex (namely, the ball of radius r_2). Since the ball of radius r_1 of each core vertex is only used by the paths leading to it, we can also grow it to form a ball of radius r_2 in each step.

First, we prove a strengthened version of Lemma 2.13.

Lemma 4.3. *There exists some $\varepsilon_1 > 0$ such that, for any $0 < \varepsilon_2 < 1/5$ and $s \ge 20$, there exists $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ such that the following holds for each $n \ge d \ge d_0$ and $d < 2\log^s n$. Suppose that G is an n-vertex $\mathsf{TK}_{d/2}^{(2)}$-free bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander with $\delta(G) \ge d$.*

Moreover, let $r = \lceil (\log\log n)^5 \rceil$ and $V_i = B_G^r(v_i)$ with $|V_i| \le \exp((\log\log n)^7)$ for $i \in [2]$. Suppose that $\mathcal{P}$ is a family of paths such that there is an ordering $\mathcal{P}_i$ of consecutive shortest paths $\mathcal{P}(v_i, r) := \{P(v_i, r) : P \in \mathcal{P}, v_i \in P\}$ from v_i in V_i for $i \in [2]$.

Suppose $U \subseteq V(G)$ such that $U \cap V_i = (V(\mathcal{P}) \cap V_i) \backslash \{v_i\}$ for $i \in [2]$ and $|U| \le \exp((\log\log n)^{10})$, and let $m = \frac{800}{\varepsilon_1} \log^3 n$. Suppose that $F_1, F_2 \subseteq G - U$ are vertex disjoint subsets such that $B_{G-V(\mathcal{P})+v_i}^r(v_i) \subseteq F_i$ and F_i is an $(\exp((\log\log n)^{90}), m)$-expansion of v_i in $G - U$ for $i \in [2]$. Let $\log^7 n \le \ell \le n/\log^{10} n$ be such that $\ell = \pi(v_1, v_2, G) \mod 2$.

Then, there is a v_1, v_2-path P' with length ℓ in $G - U$. Moreover, we have $\mathcal{P}_i, P'(v_i, r)$ is an ordering of consecutive shortest paths from v_i in V_i and $(V(P') \backslash V(P'(v_i, r))) \cap B_{G-V(\mathcal{P})+v_i}^r(v_i) = \emptyset$ for $i \in [2]$.

Proof. Let $0 < \varepsilon_1 < 1$ be such that Lemma 2.11 holds and $d_0 = d_0(\varepsilon_1, \varepsilon_2)$ be large.

For $i \in [2]$, choose $v_i' \in B_{G-V(\mathcal{P})+v_i}^r(v_i)$ and $F_i' \subseteq (F_i \backslash B_{G-V(\mathcal{P})+v_i}^r(v_i)) \cup \{v_i'\}$ such that F_i' is an $(\exp((\log\log n)^{80}), m)$-expansion of v_i' in F_i. This is possible because since F_i is connected,

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$F_i \setminus B^r_{G-V(\mathcal{P})+v_i}(v_i)$ has at most $|B^r_{G-V(\mathcal{P})+v_i}(v_i)| + 1$ components. By Pigeonhole Principle, one of them has size at least $|F_i \setminus B^r_{G-V(\mathcal{P})+v_i}(v_i)|/(|B^r_{G-V(\mathcal{P})+v_i}(v_i)| + 1) \geq (|F_i| - |V_i|)/(|V_i| + 1) \geq (\exp((\log\log n)^{90}) - \exp((\log\log n)^7))/(\exp((\log\log n)^7) + 1) > \exp((\log\log n)^{80})$. For $i \in [2]$, let L_i be a shortest path from v_i to v_i' in $B^r_{G-V(\mathcal{P})+v_i}(v_i)$. Note that in fact $\ell(L_i) = r < m$ by our choice.

Let $U' = U \cup V(F_1) \cup V(F_2) \cup V_1 \cup V_2$. So $|U'| = |U| + |V(F_1)| + |V(F_2)| + |V_1| + |V_2| \leq \exp((\log\log n)^{10}) + \exp((\log\log n)^{90}) + \exp((\log\log n)^{90}) + \exp((\log\log n)^7) + \exp((\log\log n)^7) \leq \exp((\log\log n)^{100})$. By Lemma 2.11 with $(G, U, D, r)_{2.11} = (G, U', \exp((\log\log n)^{100}), 30m)$, there is a $(\exp((\log\log n)^{100}), m, 30m)$-adjuster, and thus a $(\exp((\log\log n)^{80}), m, 30m)$-adjuster, say $\mathcal{A} = (v_3, F_3, v_4, F_4, A)$, in $G - U'$ with length $\ell(\mathcal{A}) \leq |A| + 1 \leq 600m^2$. Let $\bar{\ell} = \ell - 20m - \ell(\mathcal{A}) - \ell(L_1) - \ell(L_2)$, so that $0 \leq \bar{\ell} \leq n/\log^{10} n$. As $|A \cup U \cup B^r_{G-V(\mathcal{P})+v_1}(v_1) \cup B^r_{G-V(\mathcal{P})+v_2}(v_2)| \leq 600m^2 + \exp((\log\log n)^{10}) + \exp((\log\log n)^7) + \exp((\log\log n)^7) \leq \exp((\log\log n)^{80})/\log^3 n$, by Lemma 2.12 with $(G, D, A)_{2.12} = (G, \exp((\log\log n)^{80}), A \cup U \cup B^r_{G-V(\mathcal{P})+v_1}(v_1) \cup B^r_{G-V(\mathcal{P})+v_2}(v_2) \setminus \{v_1', v_2'\})$, there are paths P and Q in $G - (A \cup U \cup B^r_{G-V(\mathcal{P})+v_1}(v_1) \cup B^r_{G-V(\mathcal{P})+v_2}(v_2) \setminus \{v_1', v_2'\})$ which are vertex disjoint, both connect $\{v_1', v_2'\}$ to $\{v_3, v_4\}$ and so that $\bar{\ell} \leq \ell(P) + \ell(Q) \leq \bar{\ell} + 20m$. Note that we can assume, without loss of generality, that P is a v_1', v_3-path and Q is a v_2', v_4-path.

Now, $0 \leq \ell - \ell(P) - \ell(Q) - \ell(\mathcal{A}) - \ell(L_1) - \ell(L_2) \leq 20m$. As $\mathcal{A}$ is a $(\exp((\log\log n)^{80}), m, 30m)$-adjuster, there is a v_3, v_4-path in $G[A \cup \{v_3, v_4\}]$ with length $\ell(\mathcal{A})$, and therefore $\ell(\mathcal{A}) = \pi(v_3, v_4, G) \mod 2$. Then, as $\ell(L_1) = \pi(v_1, v_1', G) \mod 2$, $\ell(L_2) = \pi(v_2, v_2', G) \mod 2$, $\ell(P) = \pi(v_1', v_3, G) \mod 2$, $\ell(Q) = \pi(v_2', v_4, G) \mod 2$, $\ell = \pi(v_1, v_2, G) \mod 2$ and $\pi(v_1, v_2, G) = \pi(v_1, v_1', G) + \pi(v_1', v_3, G) + \pi(v_3, v_4, G) + \pi(v_4, v_2', G) + \pi(v_2', v_2, G) \mod 2$, we have $\ell - \ell(P) - \ell(Q) - \ell(\mathcal{A}) - \ell(L_1) - \ell(L_2) = 0 \mod 2$. That is, there is some $i \in \mathbb{N}$ with $2i = \ell - \ell(P) - \ell(Q) - \ell(\mathcal{A}) - \ell(L_1) - \ell(L_2)$, where $i \leq 10m$.

Therefore, by the definition of adjuster, there is a v_3, v_4-path R with length $\ell(\mathcal{A}) + 2i = \ell - \ell(P) - \ell(Q) - \ell(L_1) - \ell(L_2)$ in $G[A \cup \{v_3, v_4\}]$. Then, $P' = L_1 \cup P \cup R \cup Q \cup L_2$ is a v_1, v_2-path with length ℓ in $G - U$. Moreover, $\mathcal{P}_i, L_i$ is an ordering of consecutive shortest paths from v_i in V_i for $i \in [2]$. By construction, $V(P \cup R \cup Q) \setminus \{v_1', v_2'\}$ is disjoint from $B^r_{G-V(\mathcal{P})+v_i}(v_i)$ for $i \in [2]$. This completes the proof. $\square$

Now we are ready to show the following.

Lemma 4.4. *There exists $\varepsilon_1 > 0$ such that for any $0 < \varepsilon_2 < 1/5$ and $s \geq 20$, there exist $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ and a constant $0 < t_4 < 1/3$ such that the following holds for each $n \geq d \geq d_0$ and $d < 2\log^s n$. Suppose that G is a $\mathsf{TK}^{(2)}_{d/2}$-free n-vertex bipartite $(\varepsilon_1, \varepsilon_2 d)$-expander subgraph G with $\delta(G) \geq d$ and $\Delta(G) \leq d^2 \log^{10} n$. Then G contains a $\mathsf{TK}^{(\ell)}_{t_4 d}$ for some $\ell \in \mathbb{N}$.*

Proof. Let $\varepsilon_1 > 0$ be such that Lemma 4.3 holds and $d_0 = d_0(\varepsilon_1, \varepsilon_2, s)$ be large. Let $0 < t_4 < 1/3$ be such that $t = \lceil t_4 d \rceil$, $l = 2\lceil \log^7 n \rceil$, $r_1 = \lceil (\log\log n)^5 \rceil$ and $r_2 = \lceil \log n/(300s \log\log n) \rceil$.

By Lemma 2.14 with $s_{2.14} = s$, let $v_1, \cdots, v_{2t}$ be $2t$ distinct vertices that are at distance at least $\log n/(50s \log\log n) > 3r_2$ apart. At least half of them are in the same partition of G. Without loss of generality, let $v_1, \cdots, v_t$ be t vertices in the same partition of G that will serve as t core vertices in the final balanced clique subdivision construction. By Lemma 2.16 with $q = 0$, for each $i \in [t]$, let $V_i = B^{r_1}_G(v_i)$ be the ball of radius r_1 around v_i. Note that $|V_i| \leq \Delta(G)^{r_1} \leq (d^2 \log^{10} n)^{(\log\log n)^5 + 1} < \exp((\log\log n)^7)$.

Let $K \subseteq \binom{[t]}{2}$ be maximal such that there exists a family of pairwise internally disjoint paths $\mathcal{P} = \{P_k : k \in K\}$ such that

E1 For each $\{i,j\} \in K$, $P_{\{i,j\}}$ is a v_i,v_j-path of length ℓ. $P_{\{i,j\}}$ is disjoint from V_q for $q \in [t]\backslash\{i,j\}$.

E2 For each $i \in [t]$, there is some ordering of $\mathcal{P}(v_i,r_1) := \{P_k(v_i,r_1) : k \in K, i \in k\}$, that is, all the subpaths of length r_1 with one end vertex v_i of the paths in $\mathcal{P}$ incident to v_i, so that they form consecutive shortest paths from v_i in V_i.

E3 For each $i \in [t]$, $(V(\mathcal{P}) \setminus V(\mathcal{P}(v_i,r_1))) \cap B^{r_1}_{G-V(\mathcal{P}(v_i,r_1))+v_i}(v_i) = \emptyset$.

One can verify that if $K = \binom{[t]}{2}$, then the graph formed by all the paths in $\mathcal{P}$ is a desired $\mathsf{TK}_t^{(\ell)}$. Hence, we may assume that there exist distinct $i,j \in [t]$ such that $\mathcal{P}$ contains no such v_i,v_j-path of length ℓ.

Let $W = V(\mathcal{P})\backslash\{v_i,v_j\}$. So $|W| \le |K|\ell < t^2\ell/2 \le d^2\log^7 n$. By Lemma 2.16 with $\{P_1,\ldots,P_q\}_{2.16} = \mathcal{P}(v_i,r_1)$, we have $|B^{r_1}_{G-V(\mathcal{P}(v_i,r_1))+v_i}(v_i)| \ge d^2\log^{10} n$. Since $(V(\mathcal{P})\backslash V(\mathcal{P}(v_i,r_1)))\cap B^{r_1}_{G-V(\mathcal{P}(v_i,r_1))+v_i}(v_i) = \emptyset$, we have $B^{r_1}_{G-W}(v_i) = B^{r_1}_{G-V(\mathcal{P}(v_i,r_1))+v_i}(v_i)$ and $|B^{r_1}_{G-W}(v_i)| \ge d^2\log^{10} n$. By Lemma 2.17 with $(Y,W,s)_{2.17} = (B^{r_1}_{G-W}(v_i),W,3s)$, we have $|B^{r_1+r_2}_{G-W}(v_i)| \ge \exp((\log n)^{1/4}) > \exp((\log\log n)^{90})$. Similarly, $B^{r_1}_{G-W}(v_j) = B^{r_1}_{G-V(\mathcal{P}(v_j,r_1))+v_j}(v_j)$ and $|B^{r_1+r_2}_{G-W}(v_j)| > \exp((\log\log n)^{90})$. Since for any distinct $p,q \in [t]$, v_p and v_q are at distance at least $3r_2$ by our choice, $B^{r_1+r_2}_{G-W}(v_i)$ (respectively $B^{r_1+r_2}_{G-W}(v_j)$) is disjoint from V_q for $q \in [t]\backslash\{i\}$ (respectively $q \in [t]\backslash\{j\}$). Let $U = W \cup (\cup_{q\in[t]\backslash\{v_i,v_j\}}V_q)$ and $m = \frac{800}{\varepsilon_1}\log^3 n$. Hence, there exist vertex disjoint $B^{r_1}_{G-W}(v_i) \subseteq F_i \subseteq B^{r_1+r_2}_{G-W}(v_i)$ and $B^{r_1}_{G-W}(v_j) \subseteq F_j \subseteq B^{r_1+r_2}_{G-W}(v_j)$ such that F_i is $(\exp((\log\log n)^{90}),m)$-expansion of v_i in $G-U$ and F_j is $(\exp((\log\log n)^{90}),m)$-expansion of v_j in $G-U$.

Note that $|U| \le |K|\ell + t\exp((\log\log n)^7) < t^2\ell/2 + t\exp((\log\log n)^7) < \exp((\log\log n)^8)$. By applying Lemma 4.3 with $(F_1,F_2,U,\mathcal{P})_{4.3} = (F_i,F_j,U,\mathcal{P})$, we have that there is a v_i,v_j-path $P_{\{i,j\}}$ of length ℓ in $G-U$. Moreover, previous ordering of $\mathcal{P}(v_i,r_1)$ and $P_{\{i,j\}}(v_i,r_1)$ (respectively, previous ordering of $\mathcal{P}(v_j,r_1)$ and $P_{\{i,j\}}(v_j,r_1)$) are consecutive shortest paths from v_i in V_i (respectively from v_j in V_j). In addition, $(V(P_{\{i,j\}}) \setminus V(P_{\{i,j\}}(v_q,r_1))) \cap B^{r_1}_{G-V(\mathcal{P}(v_q,r_1))+v_q}(v_q) = (V(P_{\{i,j\}}) \setminus V(P_{\{i,j\}}(v_q,r_1))) \cap B^{r_1}_{G-V(\mathcal{P})+v_q}(v_q) = \emptyset$ for $q \in \{i,j\}$. Therefore, $K \cup \{\{i,j\}\}$ contradicts the maximality of K. This completes the proof. $\qquad\square$

We can now prove Lemma 4.1.

Proof of Lemma 4.1. Let $\varepsilon_1 > 0$ be such that Lemmas 4.2 and 4.4 hold. Let $d_0 = d_0(\varepsilon_1,\varepsilon_2,s)$ be large and t_3,t_4 be constants in Lemmas 4.2 and 4.4 respectively.

Let $t_2 = \min\{\frac{\varepsilon_1\varepsilon_2}{8\log^2(15/2)},t_3,\frac{1}{3}t_4\}$ and $c = 1/4$. Let $\Delta = c^2d^2\log^{10} n$ and $L = \{v \in V(G) : d(v) \ge \Delta\}$. If $|L| \ge 2t_2d$, then by Lemma 4.2 G contains a $\mathsf{TK}_{t_2d}^{(\ell_1)}$ for some $\ell_1 \in \mathbb{N}$. So $|L| < 2t_2d$ and let $H = G - L$.

Claim. H is a $\mathsf{TK}_{d/2}^{(2)}$-free bipartite $(\varepsilon_1/2,\varepsilon_2 d)$-expander satisfying $\delta(H) > d/3$, $|V(H)| > n/3$, $\delta(H) < 2\log^s|V(H)|$ and $\Delta(H) < \delta(H)\log^{10}|V(H)|$.

As G is $\mathsf{TK}_{d/2}^{(2)}$-free bipartite, H is also $\mathsf{TK}_{d/2}^{(2)}$-free bipartite. It is easy to see that $\delta(H) \ge \delta(G) - |L| \ge d - 2t_2d > d/3$. Since $n \ge d$ is large, $|V(H)| \ge |V(G)| - |L| \ge n - 2t_2d > n/3$. Note $\delta(H) \le d < \log^s n < 2\log^s|V(H)|$. Moreover, we have $\Delta(H) \le \Delta = c^2d^2\log^{10} n < c^2(3\delta(H))^2\log^{10}(3|V(H)|) < \delta(H)^2\log^{10}|V(H)|$.

To finish the proof of the claim, we show that H is an $(\varepsilon_1/2, \varepsilon_2 d)$-expander. Let $k = \varepsilon_2 d$. For any set X in H of size $x \geq k/2$ with $x \leq |V(H)|/2 \leq |V(G)|/2$, we have

$$|N_G(X)| \geq \varepsilon(x, \varepsilon_1, \varepsilon_2 d) \cdot x \geq \varepsilon(\frac{k}{2}, \varepsilon_1, \varepsilon_2 d) \cdot \frac{k}{2} \geq \frac{\varepsilon_1}{\log^2(15/2)} \cdot \frac{\varepsilon_2 d}{2} \geq 4 t_2 d \geq 2|L|.$$

So $|N_H(X)| \geq |N_G(X)| - |L| \geq |N_G(X)|/2 \geq \varepsilon(x, \varepsilon_1, \varepsilon_2 d)/2 \cdot x = \varepsilon(x, \varepsilon_1/2, \varepsilon_2 d) \cdot x$, as required.

Finally by Lemma 4.4, H contains a $\mathsf{TK}_{t_2 d}^{(\ell_2)}$ for some $\ell_2 \in \mathbb{N}$, and so does G. This completes the proof. $\square$

5 Proof of Theorem 1.1

Now we show that Theorem 1.1 follows from Lemmas 3.1 and 4.1.

Proof of Theorem 1.1. Let $\varepsilon_1 > 0$ be such that Corollary 2.4 and Lemmas 3.1 and 4.1 hold and $\varepsilon_2 = 1/10$. Let G be a graph with average degree $d(G) \geq d$ and $d_0 = d/8$. We may assume that G is $\mathsf{TK}_{d/2}^{(2)}$-free. By Corollary 2.4, G has a bipartite $(\varepsilon_1, \varepsilon_2 d_0)$-expander subgraph H with $\delta(H) \geq d_0$. So H is also $\mathsf{TK}_{d/2}^{(2)}$-free.

Let $n_0 = |V(H)|$ and $s = \frac{20}{1-2c}$. If $d_0 \geq \log^s n_0$, then by Lemma 3.1, G contains a $\mathsf{TK}_{\sqrt{d_0}/(2\log^{10} n_0)}^{(\ell)}$ for some $\ell \in \mathbb{N}$, and thus contains a $\mathsf{TK}_{d_0^c/2}^{(\ell)}$. Otherwise, by Lemma 4.1, G contains a $\mathsf{TK}_{t_2 d_0}^{(\ell)}$ for constant $t_2 > 0$ and some $\ell \in \mathbb{N}$. $\square$

Note added after submission

Recently, the constant c in Theorem 1.1 is improved to $1/2$ by Gil Fernández, Hyde, Liu, Pikhurko and Wu [4], and Luan, Tang, Wang and Yang [15] independently. Note that both proofs use Lemma 4.1.

Acknowledgements

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