Many cliques in bounded-degree hypergraphs

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Abstract

Recently Chase determined the maximum possible number of cliques of size t in a graph on n vertices with given maximum degree. Soon afterward, Chakraborti and Chen answered the version of this question in which we ask that the graph have m edges and fixed maximum degree (without imposing any constraint on the number of vertices). In this paper we address these problems on hypergraphs. For s-graphs with $s \geq 3$ a number of issues arise that do not appear in the graph case. For instance, for general s-graphs we can assign degrees to any i-subset of the vertex set with $1 \leq i \leq s - 1$.

We establish bounds on the number of t-cliques in an s-graph $\mathcal{H}$ with i-degree bounded by Δ in three contexts: $\mathcal{H}$ has n vertices; $\mathcal{H}$ has m (hyper)edges; and (generalizing the previous case) $\mathcal{H}$ has a fixed number p of u-cliques for some u with $s \leq u \leq t$. When Δ is of a special form we characterize the extremal s-graphs and prove that the bounds are tight. These extremal examples are the shadows of either Steiner systems or partial Steiner systems. On the way to proving our uniqueness results, we extend results of Füredi and Griggs on uniqueness in Kruskal-Katona from the shadow case to the clique case.

1   Introduction

There has been recent interest in generalized Turán problems: determining the maximum (or minimum) number of copies of a fixed graph T that a graph G can contain, subject to a variety of constraints. The roots of this problem go back to Turán's theorem [23] and its extension by Zykov [25] which determine, respectively, the maximum number of copies of K_2 and K_t in a graph on n vertices containing no K_{r+1}. The paper of Alon and Shikhelman [1] proved many foundational results and introduced the general problem to a wider audience.

1.1   Many cliques in bounded-degree graphs

We will focus on hypergraph versions of three generalized Turán problems: determining the maximum number of cliques in graphs of bounded degree, using either vertices, edges, or cliques as a "resource." We discuss the graph problems below; for a more complete history see [2, 4, 5, 6, 12, 17, 18]. The first phase of progress in these problems consisted of "signpost" results: estimates that are best possible infinitely often, but not for all values of the parameters.

We write $k^t(G)$ for the number of cliques of size t (and always insist that $t \geq 1$). Similarly $k^{\geq t}(G)$ is the number of cliques of size at least t in G. The next two theorems are versions of results due to Wood, phrased to match the hypergraph results we prove later.

Theorem 1 (Wood [24]). *If G is a graph on n vertices with $\Delta(G) \leq r-1$ then*

$$k^t(G) \leq \frac{n}{r}\binom{r}{t} \quad and \quad k^{\geq 1}(G) \leq \frac{n}{r}\left(2^r - 1\right)$$

with equality when $G = aK_r$.

Theorem 2 (Wood [24]). *If G is a graph having m edges with $\Delta(G) \leq r-1$ then*

$$k^t(G) \leq \frac{m}{\binom{r}{2}}\binom{r}{t} \quad and \quad k^{\geq 2}(G) \leq \frac{m}{\binom{r}{2}}\left(2^r - r - 1\right),$$

with equality when $G = aK_r$.

Quite recently results in this direction were proved that are best possible for all values of the parameters. The vertex problem was solved by Chase [4]. He proved a conjecture of Gan, Loh, and Sudakov [12] using their reduction of the problem to the case $t = 3$. Later Chao and Dong [3] gave a new proof of Theorem 3 that proves the result for all t simultaneously.

Theorem 3 (Chase [4], Chao and Dong [3]). *Let G be a graph with $\Delta(G) \leq r-1$ on n vertices. Let a and b satisfy $n = ar + b$ with $0 \leq b < r$. Then*

$$k^t(G) \leq a\binom{r}{t} + \binom{b}{t},$$

with equality for the graph $G = aK_r \cup K_b$, the disjoint union of a copies of K_r and one copy of K_b.

Using Theorem 3, Chakraborti and Chen [2] solved the edge problem.

Theorem 4 (Chakraborti and Chen [2]). *Let G be a graph with $\Delta(G) \leq r-1$ having m edges. Let a and b satisfy $m = a\binom{r}{2} + b$ with $0 \leq b < \binom{r}{2}$. Then*

$$k^t(G) \leq a\binom{r}{t} + k^t(\mathcal{C}_2(b)),$$

with equality for the graph $G = aK_r \cup \mathcal{C}_2(b)$. Here, $\mathcal{C}_2(b)$ is the colex graph having b edges: the graph on vertex set $\mathbb{N}$ whose edges are the first b pairs in colexicographic order.

In this paper we are concerned with hypergraph versions of these problems. To state the questions we need to introduce our notation for hypergraphs and discuss the issue of degrees in hypergraphs. This we do next.

In Section 2 we discuss various versions of the Kruskal-Katona theorem, which is central in this area. In Section 3 we prove general results for arbitrary degree bounds. In Section 4 we introduce constructions which, in some cases, give optimal examples, and prove some results about optimality and asymptotic optimality. Finally, in Section 5 we mention some open problems.

1.2 Hypergraph definitions and questions

Our notation is mostly standard.

Definition 5. An *s-graph* $\mathcal{H}$ is a pair $(V, \mathcal{E})$ consisting of a set of vertices V together with a subset $\mathcal{E} \subseteq \binom{V}{s}$. Frequently we'll suppress mention of the vertex set and simply use $\mathcal{H}$ to refer to the edge set. If $I \subseteq V$ has size i then we define the *neighborhood* $\mathcal{H}(I)$ of I to be the $(s-i)$-graph with edge set

$$\mathcal{E}\big(\mathcal{H}(I)\big) = \{E \setminus I \,:\, I \subseteq E \in \mathcal{E}(\mathcal{H})\}.$$

The *degree of I in $\mathcal{H}$* is the number of these edges, i.e.,

$$d_{\mathcal{H}}(I) = \big|\{E \in \mathcal{E}(\mathcal{H}) \,:\, I \subseteq E\}\big|.$$

We let the vertex set of $\mathcal{H}(I)$ be the union of all the edges in $\mathcal{E}(\mathcal{H}(I))$, i.e., we omit all vertices not contained in an edge of $\mathcal{H}(I)$. The *maximum i-degree* of $\mathcal{H}$ is simply

$$\Delta_i(\mathcal{H}) = \max\Big\{d_{\mathcal{H}}(I) \,:\, I \in \binom{V}{i}\Big\}.$$

We now define shadows and cliques in hypergraphs.

Definition 6. Suppose that $\mathcal{A}$ is an s-graph. The *shadow of $\mathcal{A}$ on level q* (where $q < s$) is given by

$$\partial_q(\mathcal{A}) = \Big\{B \,:\, |B| = q \text{ and } \exists\, A \in \mathcal{A} \text{ s.t. } B \subseteq A\Big\} = \bigcup_{A \in \mathcal{A}} \binom{A}{q}.$$

The *set of cliques on level t* (where $t > s$) is

$$K^t(\mathcal{A}) = \Big\{C \,:\, |C| = t \text{ and } \binom{C}{s} \subseteq \mathcal{A}\Big\}.$$

We let $k^t(\mathcal{A}) = \big|K^t(\mathcal{A})\big|$.

We can now state the questions we address in this paper.

Question 1. Suppose that an s-graph $\mathcal{H}$ has n vertices, and that for some $1 \le i \le s - 1$ and $D > 0$ we have $\Delta_i(\mathcal{H}) \le D$. Given $t \ge s$, what is the maximum possible value of $k^t(\mathcal{H})$? In other words we aim to determine

$$\max\{k^t(\mathcal{H}) \,:\, \mathcal{H} \text{ an s-graph with } n \text{ vertices and } \Delta_i(\mathcal{H}) \le D\}.$$

Question 2. Suppose that an s-graph $\mathcal{H}$ has m edges, and that for some $1 \le i \le s - 1$ and $D > 0$ we have $\Delta_i(\mathcal{H}) \le D$. Given $t \ge s$, what is the maximum possible value of $k^t(\mathcal{H})$? In other words, what is

$$\max\{k^t(\mathcal{H}) \,:\, \mathcal{H} \text{ an s-graph with } m \text{ edges and } \Delta_i(\mathcal{H}) \le D\}?$$

Question 3. Suppose that an s-graph $\mathcal{H}$ has $k^u(\mathcal{H}) = p$ for some $u \ge s$, and that for some $1 \le i \le s-1$ and $D > 0$ we have $\Delta_i(\mathcal{H}) \le D$. Given $t \ge u$, what is the maximum possible value of $k^t(\mathcal{H})$? I.e., determine

$$\max\{k^t(\mathcal{H}) \,:\, \mathcal{H} \text{ an s-graph with } k^u(\mathcal{H}) = p \text{ and } \Delta_i(\mathcal{H}) \le D\}.$$

1.3 Related extremal problems

The area of extremal problems for hypergraphs is rich and deep. The Kruskal-Katona theorem, which we discuss in Section 2, is an upper bound on the number of t-cliques in an s-graph with a given number of edges. Moreover, it implies a bound on the number of t-cliques in an s-graph having a given number of u-cliques for some $s < u \leq t$. In [9], Frohmader improved this bound in the case $s = 2$.

The Kruskal-Katona theorem puts few restrictions on the s-graphs involved. A substantial amount of work has been done when we forbid large cliques in our s-graphs. The earliest such result is by Zykov [25]. He proved the following result for graphs.

Theorem 7 (Zykov [25])**.** *If $\mathcal{H}$ is a graph on n vertices containing no $(r + 1)$-clique then $k^t(\mathcal{H}) \leq k^t(T_r(n))$. Here $T_r(n)$ is the* Turán graph*, that is to say it is the complete r-partite graph on n vertices whose parts are of sizes as equal as possible.*

The analogous result where we constrain G to have m edges is much more recent. The following result is due to Frohmader [8]. To describe the result we need to define the r-partite colex Turán graph. Let r be a positive integer. The *r-partite colex order* is the restriction of the colex order on $\binom{\mathbb{N}}{2}$ to $\{ij : i \not\equiv j \pmod{r}\}$. The *$r$-partite colex Turán graph with m edges*, $CT_r(m)$, is the graph on vertex set $\mathbb{N}$ whose edge set consists of the first m edges in r-partite colex order. (Note that if $m = t_r(n)$, then the unique non-trivial component of $CT_r(m)$ is isomorphic to $T_r(n)$.)

Theorem 8 (Frohmader [8])**.** *If G is a K_{r+1}-free graph with m edges and $2 \leq t \leq r$, then $k^t(G) \leq k^t\big(CT_r(m)\big)$.*

In stark contrast to these positive results about graphs, even the Turán problem for s-graphs with $s > 2$ is apparently intractable. For no $r > s \geq 3$ is the problem of determining

$$\max\{|\mathcal{H}| : \mathcal{H} \text{ is an } s\text{-graph on vertex set } [n] \text{ not containing an } (r+1)\text{-clique}\}$$

solved for all n, even asymptotically. (See Keevash's survey [15] for extensive discussion of this problem.) The hypergraph analogue of Theorem 8 seems no easier.

In a recent paper, Liu and Wang [20] determined the maximum number of t-cliques in an s-graph on n vertices containing at most k disjoint edges (for n sufficiently large).

In the context of hypergraphs with bounded degree, Jung [13] considered the question of minimizing the ratio $|\partial_{s-1}(\mathcal{H})|/|\mathcal{H}|$ for s-graphs $\mathcal{H}$ having bounded 1-degree. Jung's results have a similar spirit to ours, but are not directly comparable. In an opposite direction Füredi and Zhao [11] considered 3-graphs $\mathcal{H}$ with large minimum degree and gave asymptotically best possible lower bounds on the size of $\partial_2(\mathcal{H})$.

2 The Kruskal-Katona Theorem

The fundamental theorem given in Theorem 10 below was proved independently by Kruskal [19] and Katona [14]. It shows that for a given number of edges m, the s-graph with the most t-cliques and the smallest q-shadow is the *colex hypergraph*, denoted $\mathcal{C}_s(m)$, whose edges form an initial segment in the *colexicographic* (or *colex*) *order*. Colex order is defined on finite subsets of $\mathbb{N}$ by $A < B$ iff $\max(A \triangle B) \in B$. The original version of the Kruskal-Katona theorem discussed only shadows, but the version below describes also a closely related version, giving bounds on the number of cliques in s-graphs. For completeness we prove these versions (and slightly more) in Section A.

Definition 9. We define the following functions mapping a number of edges m to the the size of the q-shadow and the number of t-cliques of $\mathcal{C}_s(m)$.

$$\partial_q^s(m) = |\partial_q(\mathcal{C}_s(m))| \quad \text{and} \quad k_s^t(m) = k^t(\mathcal{C}_s(m)).$$

Theorem 10 (The Kruskal-Katona Theorem [14, 19])**.** *For all $0 \le q < s < t \le n$, if $\mathcal{A}$ is an s-graph on vertex set V with $|V| = n$ then we have*

$$|\partial_q(\mathcal{A})| \ge \partial_q^s(m), \qquad and \qquad k^t(\mathcal{A}) \le k_s^t(m),$$

where $m = |\mathcal{A}|$. In other words, the colex s-graph $\mathcal{C}_s(m)$ has the smallest q-shadow and the largest number of t-cliques among all s-graphs of size m.

We also record here the following relationship between the functions k_s^t and ∂_{n-t}^{n-s}.

Lemma 11. *For all $0 \le s \le t \le n$ and $0 \le m \le \binom{n}{s}$,*

$$k_s^t(m) = \binom{n}{t} - \partial_{n-t}^{n-s}\left(\binom{n}{s} - m\right).$$

2.1 Cascade notation

The standard way of describing initial segments of the colex order is *cascade* notation, introduced by Kruskal in [19]. A good reference for the material in this subsection is Chapter 6 of the book [7] by Frankl and Tokushige.

Definition 12. We will say that an integer sequence $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ is a *cascade* if it is strictly decreasing. We will define, for $s \ge 1$ and arbitrary cascades $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ of length $\ell \ge 0$,

$$[n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s = \sum_{k=0}^{\ell-1} \binom{n_{s-k}}{s-k}.$$

We say that a cascade is a *strict s-cascade* if $n_{s-k} \ge s - k$ for all $0 \le k \le \ell - 1$, and also $\ell \le s$. In that case every term in (the sum defining) $[n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$ is positive.

Remark 13. In checking that a cascade $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ is strict it is sufficient to check that $n_{s-k} \ge s - k$ for $k = \ell - 1$, because if so then for every $k < \ell - 1$ we have

$$n_{s-k} \ge n_{s-\ell+1} + (\ell - 1 - k) \ge s - \ell + 1 + (\ell - 1 - k) = s - k.$$

Definition 14. If $\mathcal{B}$ is a family of sets, each disjoint from a fixed set A, we write $A + \mathcal{B}$ for the family

$$A + \mathcal{B} = \{A \cup B \ : \ B \in \mathcal{B}\}.$$

Lemma 15. *For all $m \ge 0$ and all $s \ge 1$ there exists a unique strict s-cascade such that $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$. Indeed $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ is the unique strictly decreasing sequence*

of length $\ell \geq 0$ satisfying

$$\binom{n_s}{s} < m < \binom{n_s + 1}{s}$$

$$\binom{n_s}{s} + \binom{n_{s-1}}{s-1} < m < \binom{n_s}{s} + \binom{n_{s-1} + 1}{s-1}$$

$$\vdots$$

$$\binom{n_s}{s} + \binom{n_{s-1}}{s-1} + \cdots + \binom{n_{s-\ell+2}}{s-\ell+2} < m < \binom{n_s}{s} + \binom{n_{s-1}}{s-1} + \cdots + \binom{n_{s-\ell+2} + 1}{s-\ell+2}$$

$$\binom{n_s}{s} + \binom{n_{s-1}}{s-1} + \cdots + \binom{n_{s-\ell+1}}{s-\ell+1} = m.$$

If $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ has length 1 then the first of these inequalities is satisfied with equality on the left. If $m = 0$ then we get the unique sequence of length 0 for all $s \geq 1$. Moreover, for all $m \geq 0$ and $s \geq 1$ the colex initial segment of $\binom{\mathbb{N}}{s}$ of length m is

$$\mathcal{C}_s(m) = \bigcup_{k=0}^{\ell-1} \left(\{ n_{s-j} + 1 : 0 \leq j < k \} + \binom{[n_{s-k}]}{s-k} \right)$$

where $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ is the unique s-cascade such that $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$.

Definition 16. For all $m \geq 0$ and all $s \geq 1$, we denote by $i_s(m)$ the unique s-cascade such that $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$, guaranteed by Lemma 15.

Using cascade notation, we can exhibit lovely expressions for the number of cliques and the size of the shadow of a colex initial segment.

Lemma 17. *If $(n_s, n_{s-1}, \ldots, n_{s-\ell+1})$ is a strict s-cascade and $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$ then*

$$k_s^t(m) = k^t(\mathcal{C}_s(m)) = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_t, \text{ and}$$

$$\partial_q^s(m) = |\partial_q(\mathcal{C}_s(m))| = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_q.$$

Proof. Straightforward. See [7] for a proof of the shadow case when $q = s - 1$. The general shadow result and the proof for cliques are similar. Note that neither the t-cascade nor the q-cascade need be strict. $\square$

2.2 Lovász Kruskal-Katona

Cascades have the merit of giving the precise values of $\partial_q^s(m)$ and $k_s^t(m)$, but are somewhat unwieldy to work with. There is a simpler form of the Kruskal-Katona theorem, due to Lovász [21], that is often strong enough. We work with the natural polynomial generalization of the binomial coefficient $\binom{n}{k}$ to real values of n.

Definition 18. For a real number x and natural number k, the generalized binomial coefficient is defined as $\binom{x}{k} = (x)(x-1)\cdots(x-k+1)/k!$. Note that $\binom{x}{k}$ is strictly increasing for $x \geq k-1$ and all $y \geq 0$ can be represented in the form $y = \binom{x}{k}$ for some $x \geq k-1$.

Lemma 19 (Lovász [21]). *Let $\mathcal{H}$ be an r-graph. If $|\mathcal{H}| = \binom{u}{r}$, where $u \geq r$ is real, then $|\partial_k(\mathcal{H})| \geq \binom{u}{k}$ for all $k \in [r]$.*

The clique version of this result is a straightforward consequence.

Theorem 20. *Let $s, t \in \mathbb{N}$ with $t \geq s$. Let $\mathcal{H}$ be an s-graph with $|\mathcal{H}| = \binom{x}{s}$, where $x \geq s - 1$ is real. Then if $x < t$ we have $k^t(\mathcal{H}) = 0$ and otherwise $k^t(\mathcal{H}) \leq \binom{x}{t}$.*

Proof. If $x < t$ then $|\mathcal{H}| < \binom{t}{s}$ and in particular $\mathcal{H}$ does not have enough edges to contain a t-clique, i.e., $k^t(\mathcal{H}) = 0$. If $x \geq t$, then let $\mathcal{T} = K^t(\mathcal{H})$, so $\mathcal{T}$ is a t-graph. We define $u \geq t$ by $|\mathcal{T}| = \binom{u}{t}$. By Lemma 19, the number of s-sets (edges of $\mathcal{H}$) contained in edges of $\mathcal{T}$ (t-cliques of $\mathcal{H}$) is at least $\binom{u}{s}$. The number of edges of $\mathcal{H}$ contained in t-cliques of $\mathcal{H}$ is at most the number of edges of $\mathcal{H}$, so we have $\binom{x}{s} = |\mathcal{H}| \geq \binom{u}{s}$. Since $\binom{x}{s}$ is strictly increasing in x for $x \geq s - 1$ we must have $x \geq u \geq t$, so $k^t(\mathcal{H}) = |\mathcal{T}| = \binom{u}{t} \leq \binom{x}{t}$. $\qquad\square$

3 Signpost Results for Hypergraphs

In this section we prove "signpost" versions of Theorems 3 and 4 for hypergraphs. We solve three related problems, fixing the numbers of vertices, edges, and cliques. For each problem we prove an upper bound on the number of t-cliques.

3.1 Hypergraphs with a fixed number of vertices

We start with a bound on the number of t-cliques in an s-graph on n vertices with maximum degree at most Δ. The argument bounds the number of cliques that can contain a fixed i-set I, and deduces a bound on the total number of t-cliques.

Theorem 21. *Let $1 \leq i < s$ and suppose that $\mathcal{H}$ is an s-graph on n vertices such that $\Delta_i(\mathcal{H}) \leq \Delta$. Then*

$$k^t(\mathcal{H}) \leq \binom{n}{i} \frac{k_{s-i}^{t-i}(\Delta)}{\binom{t}{i}}.$$

If equality holds then for each $I \in \binom{[n]}{i}$ the neighborhood $\mathcal{H}(I)$ contains $k_{s-i}^{t-i}(\Delta)$ $(t-i)$-cliques.

Proof. We count pairs (I, K) where $I \in \binom{[n]}{i}$, $K \in K^t(\mathcal{H})$, and $I \subseteq K$. Counting by t-cliques in $\mathcal{H}$ we have a total of $\binom{t}{i} k^t(\mathcal{H})$. On the other hand consider $I \in \binom{[n]}{i}$. For cliques K that contain I all s-sets E such that $I \subseteq E \subseteq K$ must be in $\mathcal{H}$. Thus $|\{K : I \subseteq K \in K^t(\mathcal{H})\}| \leq k^{t-i}(\mathcal{H}(I))$. Since by hypothesis $|\mathcal{H}(I)| = d_{\mathcal{H}}(I) \leq \Delta$ we have

$$|\{K : I \subseteq K \in K^t(\mathcal{H})\}| \leq k^{t-i}(\mathcal{H}(I)) \leq k_{s-i}^{t-i}(\Delta)$$

by Theorem 10. Thus, summarizing, we have

$$\binom{t}{i} k^t(\mathcal{H}) \leq \binom{n}{i} k_{s-i}^{t-i}(\Delta)$$

$$k^t(\mathcal{H}) \leq \binom{n}{i} \frac{k_{s-i}^{t-i}(\Delta)}{\binom{t}{i}}.$$

If we have equality then $k^{t-i}(\mathcal{H}(I)) = k_{s-i}^{t-i}(\Delta)$ for every $I \in \binom{[n]}{i}$. $\qquad\square$

From this result the following corollary is immediate from our known bounds on k_{s-i}^{t-i}.

Corollary 22. *Let $1 \leq i < s$ and suppose that $\mathcal{H}$ is an s-graph on n vertices such that $\Delta_i(\mathcal{H}) \leq \Delta$.*

a) If the cascade representation of Δ is given by $[n_{s-i}, n_{s-i-1}, \ldots, n_{s-i-\ell+1}]_{s-i}$ then

$$k^t(\mathcal{H}) \leq \binom{n}{i} \frac{[n_{s-i}, n_{s-i-1}, \ldots, n_{s-i-\ell+1}]_{t-i}}{\binom{t}{i}}.$$

b) If $\Delta = \binom{x-i}{s-i}$ for some (not necessarily integral) $x \geq s$ then we have

$$k^t(\mathcal{H}) \leq \binom{n}{i} \frac{\binom{x-i}{t-i}}{\binom{t}{i}} = \binom{n}{i} \frac{\binom{x}{t}}{\binom{x}{i}},$$

if $x \geq t$ and $k^t(\mathcal{H}) = 0$ for $s \leq x < t$.

Proof. The two parts follow from Theorem 21 together with Lemma 17 and Theorem 20 respectively. $\qquad\square$

3.2 Hypergraphs with a fixed number of edges

We switch now to considering hypergraphs with a fixed number of edges.

We write $K^t_{\mathcal{H}}(E)$ for the set of t-cliques in $\mathcal{H}$ containing the edge E and $k^t_{\mathcal{H}}(E)$ for $|K^t_{\mathcal{H}}(E)|$.

Lemma 23. *For any s-graph $\mathcal{H}$ and $t \geq s$,*

$$k^t(\mathcal{H}) \binom{t}{s} = \sum_{E \in \mathcal{H}} k^t_{\mathcal{H}}(E).$$

Proof. Count the pairs (E, K), where $E \subseteq K \in K^t(\mathcal{H})$, in two ways. $\qquad\square$

Lemma 24. *Let $\mathcal{H}$ be an s-graph containing an edge $E \in \mathcal{H}$, and let $I \subsetneq E$ with $|I| = i$. Let K be a t-clique of $\mathcal{H}$ containing E. Then $K \setminus I$ is a $(t-i)$-clique in $\mathcal{H}(I)$, and $k^t_{\mathcal{H}}(E) \leq k^{t-i}_{\mathcal{H}(I)}(E \setminus I)$.*

Proof. We'll show that $K \mapsto K \setminus I$ is map from $K^t_{\mathcal{H}}(E)$ to $K^{t-i}_{\mathcal{H}(I)}(E \setminus I)$ from which it is clear that the map is an injection. We have $|K \setminus I| = t - i$ since $I \subsetneq E \subseteq K$. Consider then an $(s-i)$-subset $F \subseteq K \setminus I$. We have $F \cup I \in \binom{K}{s} \subseteq \mathcal{H}$, hence $F = (F \cup I) \setminus I \in \mathcal{H}(I)$. Therefore $K \setminus I \in K^{t-i}(\mathcal{H}(I))$ and $K \setminus I \in K^{t-i}_{\mathcal{H}(I)}(E \setminus I)$. $\qquad\square$

Lemma 25. *Let $1 \leq i < s < t$ and suppose that $\mathcal{H}$ is an s-graph such that $\Delta_i(\mathcal{H}) \leq \binom{x-i}{s-i}$ for some (not necessarily integral) $x \geq t - 1$. If $I \subsetneq E \in \mathcal{H}$ and $\mathcal{J} = \mathcal{H}(I)$, then*

$$\frac{k^{t-i}(\mathcal{J})}{|\mathcal{J}|} \leq \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}},$$

where $k^{t-i}(\mathcal{J})$ is the number of $(t-i)$-cliques in the $(s-i)$-graph $\mathcal{J}$. If equality is achieved then $|\mathcal{J}| = \binom{x-i}{s-i}$ and $k^{t-i}(\mathcal{J}) = \binom{x-i}{t-i}$.

Proof. The number of edges in the neighborhood is $|\mathcal{J}| = d_{\mathcal{H}}(I) \leq \Delta_i(\mathcal{H}) \leq \binom{x-i}{s-i}$, so $|\mathcal{J}| = \binom{y}{s-i}$ for some $s - i - 1 \leq y \leq x - i$. If $y < t - i$, then $k^{t-i}(\mathcal{J}) = 0$, so the lemma holds. Otherwise, $y \geq t - i$. By Theorem 20, $k^{t-i}(\mathcal{J}) \leq \binom{y}{t-i}$, so

$$\frac{k^{t-i}(\mathcal{J})}{|\mathcal{J}|} \leq \frac{\binom{y}{t-i}}{\binom{y}{s-i}} \tag{1}$$

$$= \frac{y(y-1)\cdots(y-s+i+1)(y-s+i)\cdots(y-t+i+1)}{y(y-1)\cdots(y-s+i+1)} \frac{(s-i)!}{(t-i)!}$$

$$= \frac{(y-s+i)_{(t-s)}}{(t-i)_{(t-s)}} \quad \text{using } s < t$$

$$\leq \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}}, \tag{2}$$

since $(x-s)_{(t-s)}$ is a strictly increasing function of x for $x \geq t - 1$, and we have $y + i > t - 1$. If $\frac{k^{t-i}(\mathcal{J})}{|\mathcal{J}|} = \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}}$, then equality holds in (2), so $y + i = x$, and $|\mathcal{J}| = \binom{x-i}{s-i}$. Then equality in (1) implies that $k^{t-i}(\mathcal{J}) = \binom{x-i}{t-i}$. $\square$

Remark 26. The expression $k^{t-i}_{s-i}(m)/m$ is not an increasing function of m, whereas $\frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}}$ is an increasing function of x. For values of m where

$$\frac{k^{t-i}_{s-i}(m)}{m} = \max_{m' \leq m} \frac{k^{t-i}_{s-i}(m')}{m'} \tag{3}$$

we can improve Lemma 25 to say that if $\Delta_i(\mathcal{H}) \leq m$ then

$$\frac{k^{t-i}(\mathcal{J})}{|\mathcal{J}|} \leq \frac{k^{t-i}_{s-i}(m)}{m}.$$

For $m = \binom{x-i}{s-i}$ where x is an integer, it is easy to check that (3) holds. It is an interesting question to determine which values of m satisfy (3).

Theorem 27. *Let $1 \leq i < s$ and suppose that $\mathcal{H}$ is an s-graph having m edges such that $\Delta_i(\mathcal{H}) \leq \binom{x-i}{s-i}$ for some (not necessarily integral) $x \geq s$. Then, for all $t \geq s + 1$,*

$$k^t(\mathcal{H}) \leq m\frac{\binom{x}{t}}{\binom{x}{s}}.$$

If equality holds then for each $I \in \partial_i(\mathcal{H})$ we have $k^{t-i}(\mathcal{H}(I)) = \binom{x-i}{t-i}$.

Proof. If $t > x$ then $k^t(\mathcal{H}) = 0$ because any i-set I contained in a t-clique would have $d_{\mathcal{H}}(I) \geq \binom{t-i}{s-i} > \binom{x-i}{s-i}$. Therefore we may assume $t \leq x$. We will count

$$S = \{(I, E, K) : I \subsetneq E \subseteq K \in K^t(\mathcal{H}), |I| = i, |E| = s\}$$

in two ways. Counting by K, then E, then I, we obtain

$$|S| = k^t(\mathcal{H})\binom{t}{s}\binom{s}{i}.$$

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Counting by I, then E, then K, and letting $\mathcal{J} = \mathcal{H}(I)$, we obtain

$$|S| = \sum_{I \in \partial_i(\mathcal{H})} \sum_{E \supseteq I} k_{\mathcal{H}}^t(E)$$

$$\leq \sum_{I \in \partial_i(\mathcal{H})} \sum_{E \supseteq I} k_{\mathcal{J}}^{t-i}(E \setminus I) \quad \text{by Lemma 24}$$

$$= \sum_{I \in \partial_i(\mathcal{H})} k^{t-i}(\mathcal{J}) \binom{t-i}{s-i} \quad \text{by Lemma 23}$$

$$\leq \binom{t-i}{s-i} \sum_{I \in \partial_i(\mathcal{H})} \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}} |\mathcal{J}| \quad \text{by Lemma 25}$$

$$= \binom{t-i}{s-i} \frac{(x-s)_{(t-s)}}{(t-i)_{(t-s)}} \sum_{I \in \partial_i(\mathcal{H})} d_{\mathcal{H}}(I)$$

$$= \binom{t-i}{t-s} \frac{\binom{x-s}{t-s}}{\binom{t-i}{t-s}} \sum_{I \in \partial_i(\mathcal{H})} d_{\mathcal{H}}(I)$$

$$= \binom{x-s}{t-s} \binom{s}{i} m.$$

Therefore, $k^t(\mathcal{H}) \binom{t}{s} \binom{s}{i} = |S| \leq \binom{x-s}{t-s} \binom{s}{i} m$, and

$$k^t(\mathcal{H}) \leq \frac{\binom{x-s}{t-s}}{\binom{t}{s}} m = \frac{\binom{x}{t}}{\binom{x}{s}} m.$$

The last equation follows from the fact that $\binom{x}{t}\binom{t}{s} = \frac{(x)(x-1)\cdots(x-t+1)}{s!(t-s)!} = \binom{x}{s}\binom{x-s}{t-s}$.

If $k^t(\mathcal{H}) = m \frac{\binom{x}{t}}{\binom{x}{s}}$ then we have equality in the above application of Lemma 25 for every $I \in \partial_i(\mathcal{H})$. By Lemma 25, $k^{t-i}(\mathcal{H}(I)) = \binom{x-i}{t-i}$ for every $I \in \partial_i(\mathcal{H})$. $\qquad \square$

3.3 Hypergraphs with a fixed number of cliques

In this section we consider s-graphs that have a fixed number of u-cliques, for some $u > s$. The numbers of vertices and edges are not specified. We will use the following lemma to connect this problem to our previous results.

Lemma 28. *Let $1 \leq i < s \leq u$ and suppose that $\mathcal{H}$ is an s-graph such that $\Delta_i(\mathcal{H}) \leq \binom{x-i}{s-i}$ for some (not necessarily integral) $x \geq s$. If $x < u$ then $\mathcal{H}$ has no u-cliques, and otherwise the u-graph $\mathcal{U} := K^u(\mathcal{H})$ satisfies $\Delta_i(\mathcal{U}) \leq \binom{x-i}{u-i}$.*

Proof. For any i-set I of vertices of $\mathcal{H}$, let $\mathcal{K} = \mathcal{U}(I)$ and let $\mathcal{F} = \mathcal{H}(I)$. We prove first that $\mathcal{K} \subseteq K^{u-i}(\mathcal{F})$. Consider an arbitrary $(u-i)$-edge E_I of $\mathcal{K}$. By definition it satisfies $E_I \cup I \in K^u(\mathcal{H})$, so every s-set in $E_I \cup I$ is an edge of $\mathcal{H}$, and every $(s-i)$-set in E_I is an edge of $\mathcal{H}(I)$. Therefore E_I is a $(u-i)$-clique in $\mathcal{H}(I) = \mathcal{F}$, as required.

We are given that $|\mathcal{F}| = d_{\mathcal{H}}(I) \leq \binom{x-i}{s-i}$, so we have $|\mathcal{F}| = \binom{y}{s-i}$ for some $s-i-1 \leq y \leq x-i$. By Theorem 20, if $y < u-i$ then $k^{u-i}(\mathcal{F}) = 0$, i.e., $\mathcal{K}$ is empty, and otherwise $k^{u-i}(\mathcal{F}) \leq \binom{y}{u-i} \leq \binom{x-i}{u-i}$. If $x < u$ then we are always in the first case. Otherwise we have $d_{\mathcal{U}}(I) = |\mathcal{K}| \leq k^{u-i}(\mathcal{F}) \leq \binom{x-i}{u-i}$. $\qquad \square$

We generalize Theorem 27 as follows. The $s = u$ case is exactly Theorem 27.

Theorem 29. *Let $1 \leq i < s \leq u$ and suppose that $\mathcal{H}$ is an s-graph such that $k^u(\mathcal{H}) = p$ and $\Delta_i(\mathcal{H}) \leq \binom{x-i}{s-i}$ for some (not necessarily integral) $x \geq s$. Then, for all $t \geq u$,*

$$k^t(\mathcal{H}) \leq p \frac{\binom{x}{t}}{\binom{x}{u}}.$$

If equality holds then for each $I \in \partial_i(\mathcal{U})$ we have $k^{t-i}(\mathcal{U}) = \binom{x-i}{t-i}$, where $\mathcal{U} = K^u(\mathcal{H})$.

Proof. By Lemma 28, we can apply Theorem 27 to the u-graph $\mathcal{U} := K^u(\mathcal{H})$. Since $\mathcal{U}$ is a u-graph with p edges and $\Delta_i(\mathcal{U}) \leq \binom{x-i}{u-i}$, Theorem 27 implies that for all $t \geq u$ we have $k^t(\mathcal{U}) \leq p \frac{\binom{x}{t}}{\binom{x}{u}}$, with equality only if for each $I \in \partial_i(\mathcal{U})$ we have $k^{t-i}(\mathcal{U}(I)) = \binom{x-i}{t-i}$. Recall $s \leq u \leq t$. Given a t-clique T in the s-graph $\mathcal{H}$, every u-set in T is a u-clique of $\mathcal{H}$, so T is also a t-clique in the u-graph $\mathcal{U}$. Therefore $k^t(\mathcal{H}) \leq k^t(\mathcal{U}) \leq p \frac{\binom{x}{t}}{\binom{x}{u}}$. $\qquad\square$

4 Extremal Hypergraphs and Asymptotic Tightness

In this section we discuss the extent to which the signpost results from the previous section are tight. We begin in Section 4.1 by discussing cases where colex hypergraphs are the unique examples achieving the bounds in Theorem 10. In Section 4.2 we then introduce some constructions that we use to produce cases of equality in our theorems. In the later subsections we discuss the three signpost results in relation to asymptotic tightness and uniqueness of examples.

4.1 Uniqueness in Kruskal-Katona

We introduce two definitions from [10] by Füredi and Griggs.

Definition 30. Given $1 \leq q < s \leq n$ we say that m is a *jumping number* (or (s, q)-jumping number if we want to be more explicit) if $\partial_q^s(m+1) > \partial_q^s(m)$. We say that m is a *colex-unique number* if all s-graphs with m edges satisfying $|\partial_q(\mathcal{H})| = \partial_q^s(m)$ are isomorphic to $\mathcal{C}_s(m)$.

The following two theorems are proved in [10].

Theorem 31 (Füredi and Griggs [10]). *Suppose that $1 \leq q < s \leq n$ and that $0 \leq m \leq \binom{n}{s}$ is represented by the strict s-cascade $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$. Then m is an (s, q)-jumping number if and only if $\ell \leq q$.*

Theorem 32 (Füredi and Griggs [10]). *Suppose that $1 \leq q < s \leq n$ and that $0 \leq m \leq \binom{n}{s}$ is represented by the strict s-cascade $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$. Then m is a colex-unique number for all $m \leq s + 1$. If $m > s + 1$ then m is a colex-unique number if and only if one of the following is true:*

a) m is a jumping number, i.e. $\ell \leq q$, or

b) there exists $n' \leq n$ such that $m = \binom{n'}{s} - 1$.

For $m > s + 1$ conditions a) and b) are mutually exclusive.

The next lemma and the subsequent corollary will help us in the process of tracing the criterion for uniqueness through the steps of the proof of Theorem 10.

Lemma 33. *Suppose that $u, v \geq 1$ and the cascade representations*

$$N = [n_u, n_{u-1}, \ldots, n_{u-k+1}]_u \qquad and \qquad M = [m_v, m_{v-1}, \ldots, m_{v-\ell+1}]_v$$

satisfy $n_{u-k+1} = m_{v-\ell+1}$. Let $b = n_{u-k+1} = m_{v-\ell+1}$. Suppose moreover that

$$\{b, n_{u-k+2}, \ldots, n_{u-1}, n_u\} \cup \{b, m_{v-\ell+2}, \ldots, m_{v-1}, m_v\} = \{b, b+1, \ldots, u+v-1\},$$

and

$$\{b, n_{u-k+2}, \ldots, n_{u-1}, n_u\} \cap \{b, m_{v-\ell+2}, \ldots, m_{v-1}, m_v\} = \{b\}.$$

Then $N + M = \binom{u+v}{u} = \binom{u+v}{v}$.

Proof. Consider first the case that $\min(u, v) = 1$. Without loss of generality we suppose that $u = 1$. Then $u + v - 1 = v$ so for some $1 \leq b \leq v$ we have

$$N + M = [b]_1 + [v, v-1, v-2, \ldots, b+1, b]_v$$

$$= b + \sum_{i=b}^{v} \binom{i}{i}$$

$$= b + (v - b + 1) = v + 1 = \binom{u+v}{u}.$$

Now suppose that $u, v > 1$. By symmetry we may suppose that $n_u = u + v - 1$. If $k > 1$ then we let

$$N' = [n_{u-1}, n_{u-2}, \ldots, b].$$

Note that the representations of N' and M satisfy the hypotheses of the lemma, with $u' = u - 1$ and $k' = k - 1$. By induction we get

$$N + M = \binom{u+v-1}{u} + N' + M = \binom{u+v-1}{u} + \binom{u+v-1}{u-1} = \binom{u+v}{u}.$$

On the other hand if $k = 1$ then we're forced to have $N = [u+v-1]_u$ and $M = [u+v-1]_v$, so

$$N + M = \binom{u+v-1}{u} + \binom{u+v-1}{v} = \binom{u+v-1}{u} + \binom{u+v-1}{u-1} = \binom{u+v}{u}. \qquad \square$$

Corollary 34. *Suppose that $1 \leq s < n$ and that $0 < m < \binom{n}{s}$. Let*

$$m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$$

be the s-cascade representation of m. Then the $(n-s)$-cascade representation of $m' = \binom{n}{s} - m$ is

$$m' = [n'_{n-s}, n'_{n-s-1}, \ldots, n'_{n-s-k+1}]_{n-s},$$

where $n_{s-\ell+1} = n'_{n-s-k+1}$ and, writing b for this value,

$$\{b, n_{s-\ell+2}, \ldots, n_{s-1}, n_s\} \cup \{b, n'_{n-s-k+2}, \ldots, n'_{n-s-1}, n'_{n-s}\} = \{b, b+1, \ldots, n-1\} \tag{†}$$
$$\{b, n_{s-\ell+2}, \ldots, n_{s-1}, n_s\} \cap \{b, n'_{n-s-k+2}, \ldots, n'_{n-s-1}, n'_{n-s}\} = \{b\}.$$

In particular $k + \ell - 1 = n - b$, so $k = n - \ell - b + 1$.

Proof. With $n'_{n-s}, n'_{n-s-1}, \ldots, b$ defined to satisfy Eq. (†) it is easy to check that $n'_{n-s-k+1} = b \geq n - s - k + 1$ and $k \leq n - s$. Using Remark 13 we deduce that $(n'_{n-s}, n'_{n-s-1}, \cdots, b)$ is a strict $(n-s)$-cascade. Then, by Lemma 33,

$$[n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s + [n'_{n-s}, n'_{n-s-1}, \ldots, n'_{n-s-k+1}]_{n-s} = \binom{s + (n-s)}{s} = \binom{n}{s}.$$

Thus $[n'_{n-s}, n'_{n-s-1}, \ldots, n'_{n-s-k+1}]_{n-s}$ is the $(n-s)$-cascade representation of $\binom{n}{s} - m$. $\square$

Theorem 35. *Suppose that $1 \leq s < t \leq n$ and that $0 < m < \binom{n}{s}$. Let*

$$m + 1 = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$$

be the s-cascade representation of $m + 1$, having length ℓ. Then m has $k^t_s(m+1) > k^t_s(m)$ if and only if $t \leq \ell + n_{s-\ell+1} - 1$. In this case we say that m is an (s,t)-clique-jumping number.

Proof. From Lemma 11 we have

$$k^t(\mathcal{C}_s(m)) = \binom{n}{t} - \partial^{n-s}_{n-t}\left(\binom{n}{s} - m\right).$$

Thus $k^t(m+1) > k^t(m)$ exactly if we have

$$\partial^{n-s}_{n-t}\left(\binom{n}{s} - m\right) > \partial^{n-s}_{n-t}\left(\binom{n}{s} - m - 1\right)$$

i.e., $\binom{n}{s} - m - 1$ is an $(n-s, n-t)$-jumping number. By Corollary 34, the length of the $(n-s)$-cascade representation of $\binom{n}{s} - m - 1$ is $k = n - \ell - n_{s-\ell+1} + 1$, so by Theorem 31 we need $n - \ell - n_{s-\ell+1} + 1 \leq n - t$, i.e., $t \leq \ell + n_{s-\ell+1} - 1$. $\square$

Theorem 36. *Suppose that $1 \leq s < t \leq n$ and that $0 < m < \binom{n}{s}$. Let*

$$m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s$$

be the s-cascade representation of m, having length ℓ. Then the colex s-graph $\mathcal{H} = \mathcal{C}_s(m)$ is unique up to isomorphism satisfying $|\mathcal{H}| = m$ and $k^t(\mathcal{H}) = k^t_s(m)$ if either $m \geq \binom{n}{s} - n + s - 1$ holds, or $m < \binom{n}{s} - n + s - 1$ and one of the following two (mutually exclusive) conditions holds:
  a) *$t \leq \ell + n_{s-\ell+1} - 1$ (equivalently $m - 1$ is an (s,t)-clique-jumping number), or*
  b) *for some $n - s + 2 \leq n' \leq n$ we have $m = \binom{n}{s} - \binom{n'}{n-s} + 1$.*

Proof. By Lemma 11, the colex s-graph $\mathcal{H} = \mathcal{C}_s(m)$ is unique up to isomorphism satisfying $|\mathcal{H}| = m$ and $k^t(\mathcal{H}) = k^t_s(m)$ if and only if all $(n-s)$-graphs with $\binom{n}{s} - m$ edges satisfying $|\partial_{n-t}(\mathcal{H})| = \partial^{n-s}_{n-t}\left(\binom{n}{s} - m\right)$ are isomorphic to $\mathcal{C}_{n-s}\left(\binom{n}{s} - m\right)$. Applying Theorem 32, and using Corollary 34 and Theorem 35 for condition a), yields the result. In condition b), note that $n' \leq n - s + 1$ and $m = \binom{n}{s} - \binom{n'}{n-s} + 1$ imply $m \geq \binom{n}{s} - n + s - 1$. $\square$

Corollary 37. *If $m = \binom{n'}{s}$ with $n' \geq t$ then $\mathcal{C}_s(m)$ is the unique s-graph $\mathcal{H}$, up to isomorphism, with m edges achieving $k^t(\mathcal{H}) = k^t_s(m)$.*

Proof. By Theorem 36, it suffices to show that either $\binom{n'}{s} \geq \binom{n}{s} - n + s - 1$, or condition a) is satisfied. For that condition note that $m = [n_s, n_{s-1}, \ldots, n_{s-\ell+1}]_s = [n']_s$ has length $\ell = 1$ and final entry $n_{s-\ell+1} = n'$, and we have $t \leq 1 + n' - 1$ by hypothesis. $\square$

4.2 Steiner shadows and packing shadows

Here we define and discuss some important hypergraphs that turn out to be optimal examples in some cases of our problem.

Definition 38. A *Steiner system* with parameters i, r, n (abbreviated as an $S(i, r, n)$) is a collection of r-sets of some n-set V that covers each i-set of V exactly once. That is to say, it is an r-graph $\mathcal{A}$ on vertex set V such that for all $I \in \binom{V}{i}$ there exists a unique $A \in \mathcal{A}$ such that $I \subseteq A$.

It has been known for a long time (by straightforward counting arguments) that in order for a Steiner system with parameters i, r, n to exist it must be the case that certain divisibility conditions are satisfied. In groundbreaking work Peter Keevash [16] showed (among other things) that for sufficiently large n these conditions are also sufficient.

Theorem 39 (Keevash [16]). *For fixed $i \leq r$ and for n sufficiently large, an $S(i, r, n)$ exists if and only if for all $0 \leq j < i$ we have that $(r - j)_{(i-j)}$ divides $(n - j)_{(i-j)}$.*

Corollary 40. *For fixed $i \leq r$, the set of n for which an $S(i, r, n)$ exists has positive lower density.*

Proof. The divisibility conditions are certainly satisfied if $n - i + 1$ is divisible by $r_{(i)}$, so the lower density of $\{n : \text{an } S(i, r, n) \text{ exists}\}$ is at least $1/r_{(i)}$. □

We can weaken the definition of a Steiner system to require only that each i-set is covered at most once (rather than exactly once), giving the following definition.

Definition 41. An *i-packing* of r-sets (abbreviated as a $P(i, r)$), also called a *partial Steiner system*, is a collection of r-sets of some set V that covers each i-set of V at most once. That is to say, it is an r-graph $\mathcal{A}$ on vertex set V such that for all $I \in \binom{V}{i}$ there exists at most one $A \in \mathcal{A}$ such that $I \subseteq A$. Equivalently, any distinct r-sets $A, B \in \mathcal{A}$ have $|A \cap B| < i$.

Existence of $P(i, r)$'s is guaranteed for all values of the parameters. For instance, a disjoint collection of r-sets is a $P(i, r)$ for all $i \geq 1$.

The hypergraphs that will be useful to us are not only Steiner systems and packings themselves, but their shadows on layers intermediate between i and r.

Definition 42. A *Steiner shadow* with parameters i, r, n, s, abbreviated $\partial_s S(i, r, n)$, is the s-shadow of an $S(i, r, n)$. A *packing shadow* with parameters i, r, s, abbreviated $\partial_s P(i, r)$, is the s-shadow of an i-packing of r-sets.

We will show later that Steiner shadows and packing shadows provide examples showing that the signpost results we prove are best possible (at least for some values of the parameters). The following lemma computes relevant parameters of these hypergraphs.

Lemma 43. *If $1 \leq i < s < r$ and $\mathcal{A}$ is a $P(i, r)$, then, if we write $\mathcal{H}$ for the s-graph $\partial_s(\mathcal{A})$, the following hold.*

   a) For all $i \leq j \leq r$ we have $|\partial_j(\mathcal{A})| = \binom{r}{j}|\mathcal{A}|$. In particular, $\mathcal{H}$ has $\binom{r}{s}|\mathcal{A}|$ edges, and for all $s \leq t \leq r$ we have $k^t(\mathcal{H}) = |\partial_t(\mathcal{A})| = \binom{r}{t}|\mathcal{A}|$.

   b) If $I \in \partial_i(\mathcal{H})$ then $\mathcal{H}(I) \cong K_{r-i}^{(s-i)}$, which implies that $d_{\mathcal{H}}(I) = \binom{r-i}{s-i}$ and $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$. In particular $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$.

In particular if $\mathcal{H}$ is a Steiner shadow $\partial_s S(i, r, n)$ then parts a) and b) hold with $|\mathcal{A}| = \binom{n}{i} / \binom{r}{i}$, and $\partial_i(\mathcal{H}) = \binom{[n]}{i}$.

Proof. Straightforward. $\qquad\square$

We use the following lemma to prove the two corollaries following it: that two conditions on clique counts in neighborhoods force a hypergraph to be a packing shadow or a Steiner shadow respectively.

Lemma 44. *Suppose that $i \geq 1$, that $i + 2 \leq s \leq t \leq r$, and that $\mathcal{H}$ is an s-graph with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. If $\mathcal{H}(I) \cong K_{r-i}^{(s-i)}$ for all $I \in \partial_i(\mathcal{H})$, then $\mathcal{H}$ is a packing shadow $\partial_s P(i, r)$.*

Proof. For all sets $I \in \partial_i(\mathcal{H})$ we write A_I for the vertex set of $\mathcal{H}(I)$. Then $R_I = A_I \cup I$ has the property that for all s-sets $S \supseteq I$ we have $S \in \mathcal{H}$ if and only if $S \subseteq R_I$. We let $\mathcal{R} = \{R_I : I \in \partial_i(\mathcal{H})\}$. We'll show that $\mathcal{R}$ is a $P(i, r)$ and that $\mathcal{H} = \partial_s(\mathcal{R})$.

First let's show that if $I \in \partial_i(\mathcal{H})$ and $J \in \binom{R_I}{i}$ then also $J \in \partial_i(\mathcal{H})$ and $R_J = R_I$. We'll first prove the special case where $|J \cap I| = i - 1$. If $R_I \neq R_J$ then we can choose an s-set S in R_I containing $I \cup J$ and an element of $R_I \setminus R_J$, since $s \geq i + 2$. We have $I \subseteq S \subseteq R_I$, so $S \in \mathcal{H}$. Since $J \subseteq S$ we have $J \in \partial_i(\mathcal{H})$. Finally we have $J \subseteq S \not\subseteq R_J$, so $S \notin \mathcal{H}$. This contradiction implies that $R_I = R_J$. For any $J \in \binom{R_I}{i}$ there exists a sequence $I = J_0, J_1, \ldots, J_k = J$ of i-sets of R_I such that $|J_\ell \cap J_{\ell+1}| = i - 1$, and by the argument above we get that $R_{J_\ell} = R_I$ for all ℓ.

From this we can show that if $I \in \partial_i(\mathcal{H})$ then $\binom{R_I}{s} \subseteq \mathcal{H}$. To see this, consider $S \in \binom{R_I}{s}$ and pick $J \in \binom{S}{i}$. Since $J \subseteq S \subseteq R_I = R_J$ we have $S \in \mathcal{H}$.

Finally, set $\mathcal{R} = \{R_I : I \in \partial_i(\mathcal{H})\}$ as above. To show that $\mathcal{R}$ is a $P(i, r)$, suppose R_I and $R_{I'}$ are both in $\mathcal{R}$, and $J \subseteq R_I \cap R_{I'}$ is an i-set. Then by the result in the second paragraph $R_I = R_J = R_{I'}$. The last thing we need to show is that $\mathcal{H} = \partial_s(\mathcal{R})$. If $S \in \mathcal{H}$ then for any i-set of S we have $I \subseteq S \subseteq R_I$, so $S \in \partial_s(\mathcal{R})$. On the other hand if $S \in \partial_s(\mathcal{R})$ then there exists $I \in \partial_i(\mathcal{H})$ with $S \subseteq R_I$ and hence $S \in \mathcal{H}$ by the result in the third paragraph. $\qquad\square$

Corollary 45. *Suppose that $i \geq 1$, that $i + 2 \leq s \leq t \leq r$, and that $\mathcal{H}$ is an s-graph with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. If we have $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$ for every i-set I contained in an edge of $\mathcal{H}$, then $\mathcal{H}$ is a packing shadow $\partial_s P(i, r)$.*

Proof. Corollary 37 implies that for all $I \in \partial_i(\mathcal{H})$ we have $\mathcal{H}(I) \cong K_{r-i}^{(s-i)}$. Lemma 44 completes the proof. $\qquad\square$

The corresponding result for Steiner shadows also follows.

Corollary 46. *Suppose that $i \geq 1$, that $i + 2 \leq s \leq t \leq r$, and that $\mathcal{H}$ is an s-graph with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. If we have $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$ for every i-set I of vertices of $\mathcal{H}$, then $\mathcal{H}$ is a Steiner shadow $\partial_s S(i, r, n)$.*

Proof. Let V be the vertex set of $\mathcal{H}$. Given $I \in \binom{V}{i}$ we have $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$ and $t \leq r$, so $k^{t-i}(\mathcal{H}(I)) \geq 1$. Thus $\partial_i(\mathcal{H}) = \binom{V}{i}$. By Corollary 45, $\mathcal{H}$ is a packing shadow $\partial_s P(i, r)$ with $\partial_i(\mathcal{H}) = \binom{V}{i}$, i.e a Steiner shadow $\partial_s S(i, r, n)$, where $n = |V|$. $\qquad\square$

4.3   Equality cases for the three problems

Here we characterize the extremal hypergraphs for some cases of each of the three problems from Section 3. All the cases we discuss are ones where $\mathcal{H}$ is an s-graph and $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$ for some $r \geq s$ and $i < s$. We also show, for all three problems, that for these particular degree bounds our results are asymptotically tight.

4.3.1 Hypergraphs with a fixed number of vertices

For degree bounds of the form $\binom{r-i}{s-i}$, with r an integer, we show that Steiner shadows achieve the bound from Theorem 21, and that they are the only s-graphs that do when $i \leq s - 2$. We do not know whether other s-graphs achieve the bound when $i = s - 1$.

Theorem 47. *Let $1 \leq i < s \leq t \leq r$, where r is an integer, and suppose that $\mathcal{H}$ is an s-graph on n vertices.*

a) *If $\mathcal{H}$ is a Steiner shadow $\partial_s S(i, r, n)$, then $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = \frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$. I.e., $\mathcal{H}$ achieves the upper bound in Theorem 21.*

b) *If we further assume that $s \neq i+1$, then $\mathcal{H}$ satisfies both $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = \frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$ if and only if $\mathcal{H}$ is a Steiner shadow $\partial_s S(i, r, n)$.*

Note that by Theorem 39 the set of n for which Steiner shadows $\partial_s S(i, r, n)$ exist has positive lower density.

Proof. For both parts, note that $k_{s-i}^{t-i}(\binom{r-i}{s-i}) = \binom{r-i}{t-i}$ and $\binom{r-i}{t-i}\binom{r}{i} = \binom{r}{t}\binom{t}{i}$ (as in Corollary 22), so $\frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t} = \binom{n}{i}\frac{k_{s-i}^{t-i}(\Delta)}{\binom{t}{i}}$.

First, suppose $\mathcal{H} = \partial_s(\mathcal{A})$, where $\mathcal{A}$ is an $S(i, r, n)$. By Lemma 43, $\mathcal{H}$ has $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = \frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$.

Now, suppose $s \neq i + 1$ (so $3 \leq i + 2 \leq s$) and $\mathcal{H}$ is an s-graph on n vertices such that $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = \frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$. By the condition for equality in Theorem 21, for each $I \in \binom{[n]}{i}$ the neighborhood $\mathcal{H}(I)$ contains $\binom{r-i}{t-i}$ $(t - i)$-cliques, and so by Corollary 46, $\mathcal{H}$ is a Steiner shadow $\partial_s S(i, r, n)$. $\square$

Now we show that the upper bounds given by Theorem 21 and Corollary 22 are asymptotically tight. We make use of the famous result where Rödl's nibble was first introduced.

Theorem 48 (Rödl [22])**.** *The maximum number of edges in an i-packing of r-sets in $[n]$ is $(1 - o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}$.*

Theorem 49. *For $1 \leq i < s \leq t \leq r \leq n$, let N be the maximum value of $k^t(\mathcal{H})$ over all s-graphs $\mathcal{H}$ on n vertices with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. Then*

$$N = (1 - o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}.$$

Proof. Let $\mathcal{A}$ be an i-packing of r-sets in V with $|\mathcal{A}| = (1 - o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}$, as guaranteed by Theorem 48. Then $\mathcal{H} = \partial_s(\mathcal{A})$ has $k^t(\mathcal{H}) = |\mathcal{A}|\binom{r}{t} = (1 - o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$ by Lemma 43. For every $I \in \binom{V}{i}$, we have

$$d_{\mathcal{H}}(I) = \begin{cases} \binom{r-i}{s-i} & \text{if } I \in \partial_i(\mathcal{A}) \\ 0 & \text{otherwise,} \end{cases}$$

so $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. Together with Theorem 21 this implies that $N = (1 - o_n(1))\frac{\binom{n}{i}}{\binom{r}{i}}\binom{r}{t}$. $\square$

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In the proof of Theorem 49, $\mathcal{A}$ covers $(1 - o_n(1))\binom{n}{i}$ of the i-sets in V, i.e. almost all of them, so there exists $\mathcal{H}$ that is almost a Steiner shadow and almost attains the upper bound. In particular it seems highly plausible that a stability version of Theorem 47 holds.

Remark 50. Theorem 39 gives an alternative proof of Theorem 49.

4.3.2 Hypergraphs with a fixed number of edges

For degree bounds of the form $\binom{r-i}{s-i}$, with r an integer, we show that packing shadows achieve the upper bound in Theorem 27, and that for $i \le s - 2$, they are the only s-graphs that achieve this bound. Again, we do not know whether only packing shadows achieve the bound when $i = s - 1$.

Theorem 51. *Let $1 \le i < s \le t \le r$, where r is an integer, and suppose that $\mathcal{H}$ is an s-graph having m edges.*

   a) If $\mathcal{H}$ is a packing shadow $\partial_s P(i,r)$, then $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = m\frac{\binom{r}{t}}{\binom{r}{s}}$. I.e., $\mathcal{H}$ achieves the upper bound in Theorem 27. In particular, if $\binom{r}{s} \mid m$, then $\mathcal{H} = \frac{m}{\binom{r}{s}}K_r^{(s)}$ achieves equality.

   b) If we further assume that $s \ne i+1$, then $\mathcal{H}$ satisfies both $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = m\frac{\binom{r}{t}}{\binom{r}{s}}$ if and only if $\mathcal{H}$ is a packing shadow $\partial_s P(i,r)$.

Proof. First, suppose $\mathcal{H} = \partial_s(\mathcal{A})$, where $\mathcal{A}$ is a $P(i,r)$. By Lemma 43, $\Delta_i(\mathcal{H}) = \binom{r-i}{s-i}$, and we have $m = |\mathcal{A}|\binom{r}{s}$ and $k^t(\mathcal{H}) = |\mathcal{A}|\binom{r}{t}$, so $k^t(H) = m\frac{\binom{r}{t}}{\binom{r}{s}}$.

If $\binom{r}{s} \mid m$, then $\frac{m}{\binom{r}{s}}K_r^{(r)}$ is a $P(i,r)$, and its s-shadow is $\frac{m}{\binom{r}{s}}K_r^{(s)}$. Note that

$$k^t\left(\frac{m}{\binom{r}{s}}K_r^{(s)}\right) = \frac{m}{\binom{r}{s}}k^t\left(K_r^{(s)}\right) = \frac{m}{\binom{r}{s}}\binom{r}{t}$$

and $\Delta_i\left(\frac{m}{\binom{r}{s}}K_r^{(s)}\right) = \binom{r-i}{s-i}$.

Now, suppose $s \ne i + 1$ (so $3 \le i + 2 \le s$) and $\mathcal{H}$ is an s-graph having m edges with $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$ and $k^t(\mathcal{H}) = m\frac{\binom{r}{t}}{\binom{r}{s}}$. We have equality in the statement of Theorem 27. The last sentence of Theorem 27 shows that $k^{t-i}(\mathcal{H}(I)) = \binom{r-i}{t-i}$ for every $I \in \partial_i(\mathcal{H})$. By Corollary 45, $\mathcal{H}$ is a packing shadow $\partial_s P(i,r)$. $\square$

The bound given by Theorem 27 is asymptotically tight.

Theorem 52. *For $1 \le i < s \le t \le r$ and $m \ge 1$, let M be the maximum value of $k^t(\mathcal{H})$ over all s-graphs $\mathcal{H}$ having m edges with $\Delta_i(\mathcal{H}) \le \binom{r-i}{s-i}$. Then*

$$M = (1 - o_m(1))m\frac{\binom{r}{t}}{\binom{r}{s}}.$$

Proof. Given i, s, t, r, m, let $m = a\binom{r}{s} + b$, for $0 \le b < \binom{r}{s}$. Then $M \ge k^t(aK_t^{(s)}) = a\binom{r}{t} = (1 - \frac{b}{m})m\frac{\binom{r}{t}}{\binom{r}{s}}$. Since $0 \le b < \binom{r}{s}$, $\lim_{m\to\infty}\frac{b}{m} = 0$, so $M \ge (1 - o_m(1))m\frac{\binom{r}{t}}{\binom{r}{s}}$. Theorem 27 implies $M \le m\frac{\binom{r}{t}}{\binom{r}{s}}$, completing the proof. $\square$

4.3.3 Hypergraphs with a fixed number of cliques

When the degree bound is of the form $\binom{r-i}{s-i}$, with r an integer, we show that the upper bound given by Theorem 29 is achieved by any s-graph $\mathcal{H}$ for which the edges that contribute to the u-clique count of $\mathcal{H}$ form a packing shadow. By excluding the case $s = u$, which is addressed in Theorem 51, we find that these are the only s-graphs that achieve this bound. The case $s = i + 1$ is included here. In particular, when $s = i + 1$, all degree bounds $\Delta \geq t - i$ are of the form $\binom{r-i}{s-i}$ for some $r \geq t$, so are covered by Theorem 53.

Theorem 53. *Let $1 \leq i < s < u \leq t \leq r$, where r is an integer, and suppose that $\mathcal{H}$ is an s-graph with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. Let $p = k^u(\mathcal{H})$. Then $k^t(\mathcal{H}) = p\dfrac{\binom{r}{t}}{\binom{r}{u}}$ if and only if the set of edges of $\mathcal{H}$ that are contained in a u-clique of $\mathcal{H}$ is a packing shadow $\partial_s P(i,r)$. In particular, if $\binom{r}{u} \mid p$, then $\mathcal{H} = \dfrac{p}{\binom{r}{u}} K_r^{(s)}$ achieves equality.*

Proof. First, let $\mathcal{E} = \{E \in \mathcal{H} : E \subset U \text{ for some } U \in K^u(\mathcal{H})\}$, and suppose $\mathcal{E} = \partial_s(\mathcal{A})$, where $\mathcal{A}$ is a $P(i,r)$. Note $k^u(\mathcal{E}) = k^u(\mathcal{H})$. Any edges in $\mathcal{H} \setminus \mathcal{E}$ are not contained in u-cliques of $\mathcal{H}$ so cannot be contained in t-cliques of $\mathcal{H}$. Therefore $k^t(\mathcal{E}) = k^t(\mathcal{H})$. By Lemma 43, we have $p = k^u(\mathcal{H}) = |\mathcal{A}|\binom{r}{u}$, and $k^t(\mathcal{H}) = |\mathcal{A}|\binom{r}{t}$, so $k^t(\mathcal{H}) = p\dfrac{\binom{r}{t}}{\binom{r}{u}}$.

If $\binom{r}{u} \mid p$, then $\dfrac{p}{\binom{r}{u}} K_r^{(r)}$ is a $P(i,r)$, and its s-shadow is $\dfrac{p}{\binom{r}{u}} K_r^{(s)}$. Note that

$$k^t\left(\frac{p}{\binom{r}{u}} K_r^{(s)}\right) = \frac{p}{\binom{r}{u}} k^t\left(K_r^{(s)}\right) = \frac{p}{\binom{r}{u}}\binom{r}{t}$$

and $\Delta_i\left(\dfrac{p}{\binom{r}{u}} K_r^{(s)}\right) = \binom{r-i}{s-i}$.

Now, suppose $k^t(\mathcal{H}) = p\dfrac{\binom{r}{t}}{\binom{r}{u}}$. By Lemma 28, the u-graph $\mathcal{U} := K^u(\mathcal{H})$ satisfies $\Delta_i(\mathcal{U}) \leq \binom{r-i}{u-i}$. The last sentence of Theorem 29 states that for each $I \in \partial_i(\mathcal{U})$ we have $k^{t-i}(\mathcal{U}) = \binom{r-i}{t-i}$. By Corollary 45, $\mathcal{U}$ is a packing shadow $\partial_u P(i,r)$. Let $\mathcal{A}$ be a $P(i,r)$ such that $\mathcal{U} = \partial_u(\mathcal{A})$. Since $K^u(\mathcal{H}) = \mathcal{U}$, every edge S of $\mathcal{H}$ that is contained in a u-clique U of $\mathcal{H}$ is in $\partial_s(\mathcal{A})$, because there is some r-set $R \in \mathcal{A}$ such that $S \subseteq U \subseteq R$. $\square$

Theorem 29 is asymptotically tight, by a proof very similar to that of Theorem 52.

Theorem 54. *For $1 \leq i < s \leq u \leq t \leq r$ and $p \geq 1$, let P be the maximum value of $k^t(\mathcal{H})$ over all s-graphs $\mathcal{H}$ having $k^u(H) = p$ with $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$. Then*

$$P = (1 - o_p(1))p\frac{\binom{r}{t}}{\binom{r}{u}}.$$

4.3.4 A theorem on 2-graphs

We also obtain the following corollary giving the maximum number of t-cliques among 2-graphs with a fixed number of u-cliques and an arbitrary constant upper bound on the maximum degree.

Theorem 55. *Suppose $3 \leq u \leq t \leq r$ and G is a graph such that $k^u(G) = p$ and $\Delta(G) \leq r - 1$. Then*

  a) $k^t(G) \leq p\binom{r}{t}/\binom{r}{u}$.

b) *The maximum value of $k^t(G)$ over all such graphs is $(1 - o_p(1))p\binom{r}{t}/\binom{r}{u}$.*

c) *We have $k^t(G) = p\binom{r}{t}/\binom{r}{u}$ if and only if G (after removing any edge not contained in a u-clique) is a $(p/\binom{r}{u})K_r$ (possibly together with some isolated vertices). In particular, we have equality if and only if $\binom{r}{u} \mid p$.*

Proof. Apply Theorem 29, Theorem 53, and Theorem 54 with $s = 2$ and $i = 1$. Note that a packing $P(1, r)$ is a set of disjoint r-sets, so its 2-shadow forms a set of disjoint r-cliques. $\qquad\square$

Theorem 55 is a signpost answer to a question in the concluding remarks of [2].

5   Open Problems

Many interesting problems still remain. We list some of them here.

Problem 1. If $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$, where r is an integer, Theorem 47, Theorem 51, and Theorem 53 completely characterize the s-graphs that achieve the upper bounds given by Theorem 21 and Theorem 27 for $i \leq s - 2$, and Theorem 29 for $u \neq s$. In particular, these upper bounds cannot be achieved for some values of the problem parameters.

a) For values of i, r, and n for which Steiner systems $S(i, r, n)$ do not exist (either because they do not satisfy the necessary divisibility conditions or because n is too small—see Theorem 39), Theorem 47 shows that all s-graphs $\mathcal{H}$ on n vertices having $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$ have $k^t(\mathcal{H}) < \binom{n}{i}\binom{r}{t}/\binom{r}{i}$, although by Theorem 49, $\max\{k^t(\mathcal{H})\} = (1 - o_n(1))\binom{n}{i}\binom{r}{t}/\binom{r}{i}$. Which such s-graphs have the maximum number of t-cliques?

b) By Lemma 43, if $\mathcal{H} = \partial_s(\mathcal{A})$, with $\mathcal{A}$ a $P(i, r)$, then $|\mathcal{H}| = k^s(\mathcal{H}) = \binom{r}{s}|\mathcal{A}|$. Therefore, by Theorem 51, when $m \nmid \binom{r}{s}$, all s-graphs $\mathcal{H}$ having m edges and $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$ have $k^t(\mathcal{H}) < m\binom{r}{t}/\binom{r}{s}$, although by Theorem 52, $\max\{k^t(\mathcal{H})\} = (1 - o_m(1))m\binom{r}{t}/\binom{r}{s}$. Which such s-graphs have the maximum number of t-cliques?

c) Similarly, by Theorem 53, when $p \nmid \binom{r}{u}$, all s-graphs having $k^u(\mathcal{H}) = p$ and $\Delta_i(\mathcal{H}) \leq \binom{r-i}{s-i}$ have $k^t(\mathcal{H}) < p\binom{r}{t}/\binom{r}{u}$, although by Theorem 54, $\max\{k^t(\mathcal{H})\} = (1 - o_p(1))p\binom{r}{t}/\binom{r}{u}$. Which such s-graphs have the maximum number of t-cliques?

Problem 2. Among s-graphs with $\Delta_{s-1}(\mathcal{H}) \leq r - s + 1$ (the $s = i + 1$ case) we have determined the exact maximum number of t-cliques and found extremal s-graphs.

a) Are there s-graphs $\mathcal{H}$ on n vertices with $\Delta_{s-1}(\mathcal{H}) \leq r - s + 1$ that have $k^t(\mathcal{H}) = \frac{\binom{n}{s-1}}{\binom{r}{s-1}}\binom{r}{t}$ but are not Steiner shadows $\partial_s S(s - 1, r, n)$?

b) Are there s-graphs $\mathcal{H}$ on m edges with $\Delta_{s-1}(\mathcal{H}) \leq r - s + 1$ that have $k^t(\mathcal{H}) = m\frac{\binom{r}{t}}{\binom{r}{s}}$ but are not packing shadows $\partial_s P(s - 1, r)$?

Problem 3. We have characterized the extremal s-graphs and proved that our upper bounds are asymptotically tight only when the i-degree bound is $\binom{r-i}{s-i}$ for some integer r. Are the upper bounds given by Corollary 22, Theorem 27, and Theorem 29 tight when the i-degree bound does not have this form?

Problem 4. For which values of m does $\frac{k_s^t(m)}{m} = \max_{m' \leq m} \frac{k_s^t(m')}{m'}$? (See Remark 26.)

Appendix A Kruskal-Katona Details

In this appendix we give proof details for some of the results in Section 2. We prove a slightly expanded version of Theorem 10; one that discusses upshadows as well as (down) shadows and cliques. To state this result we define another total order on finite subsets.

Definition 56. The *retrolexicographic* (or *retlex*) order on finite subsets of $\mathbb{N}$ is defined by $A <_R B$ iff $\max(A \triangle B) \in A$. We write $\mathcal{R}_s(n, m)$ for the $<_R$-initial segment of size m in $\binom{[n]}{s}$.

In addition, given a ground set $[n]$ and an s-graph $\mathcal{A}$ on $[n]$, we define

$$\overline{\mathcal{A}} = \{[n] \setminus A \, : \, A \in \mathcal{A}\},$$

an $(n - s)$-graph on $[n]$ with the same size as $\mathcal{A}$.

Remark 57. The definition has the following symmetries with the colex order.
  a) We have $A <_R B$ if and only if $A > B$, i.e., retlex is the reverse of colex order.
  b) If both A and B are subsets of $[n]$ then, since $A \triangle B = ([n] \setminus A) \triangle ([n] \setminus B)$, we have $[n] \setminus A <_R [n] \setminus B$ if and only if $A < B$.

In particular for $0 \leq m \leq \binom{n}{s}$ we have

$$\mathcal{R}_s\left(n, \binom{n}{s} - m\right) = \binom{[n]}{s} \setminus \mathcal{C}_s(m) \qquad \text{and} \qquad \overline{\mathcal{R}_s(n, m)} = \mathcal{C}_{n-s}(m).$$

Note that colex initial segments are independent of n (provided $m \leq \binom{n}{s}$), whereas retlex initial segments depend in an essential way on n. Since there are many nice presentations of the bound on shadows (see for instance [7]) we will only prove the clique and upshadow bounds.

Theorem 10 (The Kruskal-Katona Theorem [14, 19]). *For all $0 \leq q < s < t \leq n$, if $\mathcal{A}$ is an s-graph on vertex set V with $|V| = n$ then we have*

$$|\partial_q(\mathcal{A})| \geq \partial_q^s(m), \qquad k^t(\mathcal{A}) \leq k_s^t(m), \qquad and \qquad |U^t(\mathcal{A})| \geq |U^t(\mathcal{R}_s(n, m))|$$

where $m = |\mathcal{A}|$. In other words, the colex s-graph $\mathcal{C}_s(m)$ has the smallest q-shadow and the largest number of t-cliques among all s-graphs of size m, whereas the smallest upshadow is achieved by the initial segment in the retlex order.

Proof. We may assume without loss of generality that $V = [n]$. We'll start by proving the upshadow bound from the shadow bound. Given $\mathcal{E} \subseteq \binom{[n]}{s}$ and writing $\overline{\mathcal{E}} = \{[n] \setminus E \, : \, E \in \mathcal{E}\} \subseteq \binom{[n]}{n-s}$, we have

$$\partial_{n-t}(\overline{\mathcal{E}}) = \left\{[n] \setminus T \, : \, |T| = t \text{ and } \exists \, ([n] \setminus E) \in \overline{\mathcal{E}} \text{ s.t. } ([n] \setminus T) \subseteq ([n] \setminus E)\right\}$$

$$= \left\{[n] \setminus T \, : \, T \in \binom{[n]}{t} \text{ and } \exists \, E \in \mathcal{E} \text{ s.t. } E \subseteq T\right\} = \overline{U^t(\mathcal{E})}.$$

Thus, by the shadow bound, to minimize $|U^t(\mathcal{E})| = |\overline{U^t(\mathcal{E})}|$ we can take $\overline{\mathcal{E}}$ to be a colex initial segment, i.e., by Remark 57 a), $\mathcal{E}$ to be a retlex initial segment. Now, for the clique bound, note that

$$K^t(\mathcal{A}) = \binom{[n]}{t} \setminus U^t\left(\binom{[n]}{s} \setminus \mathcal{A}\right).$$

Thus to maximize $|K^t(\mathcal{A})|$ we can take $\binom{[n]}{s} \setminus \mathcal{A}$ to be a retlex initial segment, i.e., by Remark 57 b), take $\mathcal{A}$ to be a colex initial segment. $\square$

Using Remark 57 we can immediately read out of the proof of the previous theorem the functions k_s^t and ∂_{n-t}^{n-s}.

Lemma 11. *For all $0 \leq s \leq t \leq n$ and $0 \leq m \leq \binom{n}{s}$,*

$$k_s^t(m) = \binom{n}{t} - \partial_{n-t}^{n-s}\left(\binom{n}{s} - m\right).$$

Proof. We have

$$
\begin{aligned}
K^t(\mathcal{C}_s(m)) &= \binom{[n]}{t} \setminus U^t\left(\binom{[n]}{s} \setminus \mathcal{C}_s(m)\right) \\
&= \binom{[n]}{t} \setminus U^t\left(\mathcal{R}_s\left(n, \binom{n}{s} - m\right)\right) \\
&= \binom{[n]}{t} \setminus \overline{\partial_{n-t}\left(\overline{\mathcal{R}_s\left(n, \binom{n}{s} - m\right)}\right)} \\
&= \binom{[n]}{t} \setminus \overline{\partial_{n-t}\left(\mathcal{C}_{n-s}\left(\binom{n}{s} - m\right)\right)}
\end{aligned}
$$

i.e.,

$$k_s^t(m) = k^t(\mathcal{C}_s(m)) = \binom{n}{t} - \partial_{n-t}^{n-s}\left(\binom{n}{s} - m\right). \qquad \square$$

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