LENS RIGIDITY IN SCATTERING BY UNIONS OF STRICTLY CONVEX BODIES IN $\mathbb{R}^2$*

LYLE NOAKES† AND LUCHEZAR STOYANOV‡

Abstract. It was proved in [NS1] that obstacles K in $\mathbb{R}^d$ that are finite disjoint unions of strictly convex domains with C^3 boundaries are uniquely determined by the travelling times of billiard trajectories in their exteriors and also by their so called scattering length spectra. However the case $d = 2$ is not covered in [NS1]. In the present paper we give a separate different proof of this result in the case $d = 2$.

Key words. scattering by obstacles, billiard flow, scattering ray, travelling times spectrum, trapped trajectory

AMS subject classifications. 37D20, 37D40, 53D25, 58J50

1. Introduction. In scattering by an obstacle in $\mathbb{R}^d$ ($d \geq 2$) the obstacle K is a compact subset of $\mathbb{R}^d$ with a C^3 boundary ∂K such that $\Omega_K = \overline{\mathbb{R}^d \setminus K}$ is connected. A *scattering ray* in Ω_K is an unbounded in both directions generalized geodesic (in the sense of Melrose and Sjöstrand [MS1], [MS2]). Most of these scattering rays are billiard trajectories with finitely many reflection points at ∂K. In this paper we consider the case when K has the form

$$K = K_1 \cup K_2 \cup \ldots \cup K_{k_0}, \tag{1.1}$$

where K_i are strictly convex disjoint domains in $\mathbb{R}^d$ with C^3 smooth boundaries ∂K_i. Then all scattering rays in Ω_K are billiard trajectories, and the so called generalized Hamiltonian (or bicharacteristic) flow

$$\mathcal{F}_t^{(K)} : \mathbb{S}^*(\Omega_K) \longrightarrow \mathbb{S}^*(\Omega_K)$$

coincides with the billiard flow (see [CFS]).

Given an obstacle K in $\mathbb{R}^d$, consider a large ball M containing K in its interior, and let $S_0 = \partial M$ be its boundary sphere. For any $q \in \partial K$ let $\nu_K(q)$ the *outward unit normal* to ∂K. For $q \in S_0$ we will denote by $\nu(q)$ the *inward unit normal* to S_0 at q. Set

$$\mathbb{S}^*_+(S_0) = \{x = (q, v) : q \in S_0 \ , \ v \in \mathbb{S}^{d-1} \ , \ \langle v, \nu(q)\rangle \geq 0\}.$$

Given $x \in \mathbb{S}^*_+(S_0)$, define the *travelling time* $t_K(x) \geq 0$ as the maximal number (or ∞) such that $\mathrm{pr}_1(\mathcal{F}_t^{(K)}(x))$ is in the interior of $\Omega_K \cap M$ for all $0 < t < t_K(x)$, where $\mathrm{pr}_1(p, w) = p$ (see Figure 1). For $x = (q, v) \in \mathbb{S}^*_+(S_0)$ with $\langle \nu(q), v\rangle = 0$ set $t(x) = 0$. The set

$$\{(x; t_K(x)) : x \in \mathbb{S}^*_+(S_0)\}$$

will be called the *travelling times spectrum* of K.

It is natural to ask what information about the obstacle K can be derived from its travelling times spectrum. For example: what is the relationship between two obstacles K and L in $\mathbb{R}^d$ if they have (almost) the same travelling times spectra? We

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say that K and L *have almost the same travelling times* if there exists a subset R of full Lebesgue measure in $\mathbb{S}^*_+(\mathbb{S}_0)$ such that $t_K(x) = t_L(x)$ for all $x \in R$.

Similar questions can be asked about the so called scattering length spectrum (SLS) of an obstacle. Given a scattering ray γ in Ω_K, if $\omega \in \mathbb{S}^{d-1}$ is the incoming direction of γ and $\theta \in \mathbb{S}^{d-1}$ its outgoing direction, γ will be called an (ω, θ)*-ray*. For any vector $\xi \in \mathbb{S}^{d-1}$ denote by Z_ξ the hyperplane in $\mathbb{R}^d$ orthogonal to ξ and tangent to S_0 such that S_0 is contained in the open half-space R_ξ determined by Z_ξ and having ξ as an inner normal. For an (ω, θ)-ray γ in Ω, the *sojourn time* T_γ of γ is defined by $T_\gamma = T'_\gamma - 2a$, where T'_γ is the length of the part of γ which is contained in $R_\omega \cap R_{-\theta}$ and a is the radius of S_0. It is known that this definition does not depend on the choice of the sphere S_0. The *scattering length spectrum* of K is defined to be the family of sets of real numbers

$$\mathcal{SL}_K = \{\mathcal{SL}_K(\omega, \theta)\}_{(\omega,\theta)}$$

where (ω, θ) runs over $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ and $\mathcal{SL}_K(\omega, \theta)$ is the set of sojourn times T_γ of all (ω, θ)-rays γ in Ω_K. It is known (cf. [PS]) that for $d \geq 3$, d odd, and C^∞ boundary ∂K, we have

$$\mathcal{SL}_K(\omega, \theta) = \text{sing supp } s_K(t, \theta, \omega)$$

for almost all (ω, θ). Here s_K is the *scattering kernel* related to the scattering operator for the wave equation in $\mathbb{R} \times \Omega_K$ with Dirichlet boundary condition on $\mathbb{R} \times \partial\Omega_K$ (cf. [LP], [M], [PS]). Following [St3], we will say that two obstacles K and L have *almost the same SLS* if there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ such that $\mathcal{SL}_K(\omega, \theta) = \mathcal{SL}_L(\omega, \theta)$ for all $(\omega, \theta) \in \mathcal{R}$.

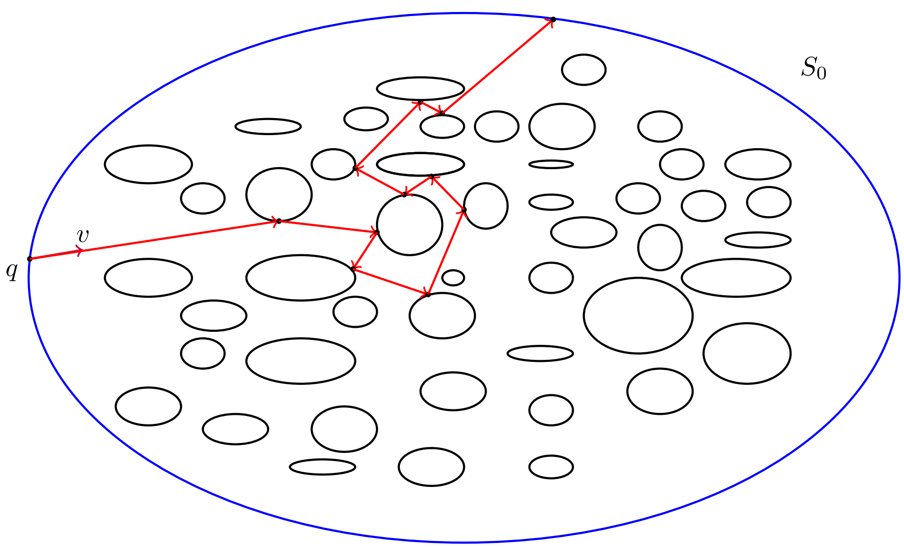

Figure 1

It is a natural and rather important problem in inverse scattering by obstacles to get information about the obstacle K from its SLS. It is known that various kinds of information about K can be recovered from its SLS, and for some classes of obstacles K is completely recoverable (see [St3] for more information) – for example star-shaped obstacles are in this class.

Similar inverse problems concerning metric rigidity have been studied for a very long time in Riemannian geometry – see [SU], [SUV] and the references there for more information. It appears that some of the methods used in this area, e.g. those in [Gu], [DGu], could be applied to obstacle scattering as well.

More recently various results have been established concerning inverse scattering by obstacles – see [St2], [St3], [NS1] - [NS3], [St5]. It turns out that some kind of obstacles are uniquely recoverable from their travelling times spectra and also from their scattering length spectra. For example, it was shown in [NS1] that if K and L are finite disjoint unions of strictly convex bodies in $\mathbb{R}^d$ with C^3 boundaries and K and L have almost the same travelling times spectra (or almost the same SLS), then $K = L$. However the argument in [NS1] does not work in the case $d = 2$. We are grateful to Antoine Gansemer who pointed this to us. As he showed in [Gan], when $d = 2$ and $k_0 > 1$ the set $\mathbb{S}^*_+(S_0) \setminus \mathrm{Trap}(\Omega_K)$ is disconnected, and then the argument in [NS1] does not work. Here $\mathrm{Trap}(\Omega_K)$ is the *set of all trapped points* in $\mathbb{S}^*(\Omega_K)$, i.e. points $x = (q, v) \in \mathbb{S}^*(\Omega_K)$ such that either the forward billiard trajectory

$$\gamma^+_K(x) = \{\mathrm{pr}_1(\mathcal{F}^{(K)}_t(x)) : t \geq 0\}$$

or the backward trajectory $\gamma^-_K(q, v) = \gamma^+_K(q, -v)$ is infinitely long. That is, either the billiard trajectory in the exterior of K issued from q in the direction of v is bounded (contained entirely in M) or the one issued from q in the direction of $-v$ is bounded. The obstacle K is called *non-trapping* if $\mathrm{Trap}(\Omega_K) = \emptyset$.

Here we prove the following.

THEOREM 1.1. *Let K and L be obstacles in $\mathbb{R}^2$ such that each of them is a finite disjoint union of strictly convex compact domains with C^3 boundaries. If K and L have almost the same travelling times or almost the same scattering length spectra, then $K = L$.*

The argument we use is completely different from that in [NS1]. A result similar to that in [NS1] concerning non-trapping obstacles satisfying certain non-degeneracy conditions was proved recently in [St5].

The set of trapped points plays a rather important role in various inverse problems in scattering by obstacles, and also in problems on metric rigidity in Riemannian geometry. It is known that $\mathrm{Trap}(\Omega_K) \cap \mathbb{S}^*_+(S_0)$ has Lebesgue measure zero in $\mathbb{S}^*_+(S_0)$ (see Sect. 4 for more information about this). However, as an example of M. Livshits shows (see Ch. 5 in [M] or Figure 1 in [NS1]), in general the set of points $x \in \mathbb{S}^*(\Omega_K)$ for which

$$\gamma_K(x) = \gamma^+_K(x) \cup \gamma^-_K(x)$$

is trapped in both directions may contain a non-trivial open set. In the latter case the obstacle cannot be recovered from travelling times (and also from the SLS). Similar examples in higher dimensions are given in [NS3].

DEFINITION 1.2. *Let K, L be two obstacles in $\mathbb{R}^d$. We will say that Ω_K and Ω_L have conjugate flows if there exists a homeomorphism*

$$\Phi : \mathbb{S}^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K) \longrightarrow \mathbb{S}^*(\Omega_L) \setminus \mathrm{Trap}(\Omega_L)$$

which is C^1 on an open dense subset of $\mathbb{S}^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K)$ and satisfies

$$\mathcal{F}^{(L)}_t \circ \Phi = \Phi \circ \mathcal{F}^{(K)}_t \quad , \quad t \in \mathbb{R},$$

and $\Phi = id$ on $\mathbb{S}^*(\mathbb{R}^d \setminus M) \setminus \mathrm{Trap}(\Omega_K) = \mathbb{S}^*(\mathbb{R}^d \setminus M) \setminus \mathrm{Trap}(\Omega_L)$.

For K, L in a generic class of obstacles in $\mathbb{R}^d$ $(d \geq 2)$, which includes the type of obstacles considered here, it is known that if K and L have almost the same SLS or almost the same travelling times, then Ω_K and Ω_L have conjugate flows ([St3] and [NS2]). Thus, Theorem 1.1 is an immediate consequence of the following.

THEOREM 1.3. *Let each of the obstacles K and L be a finite disjoint union of strictly convex compact domains in $\mathbb{R}^2$ with C^3 boundaries. If Ω_K and Ω_L have conjugate flows, then $K = L$.*

We prove Theorem 1.3 in Sect. 3 below. In Sect. 2 we state some useful results from [St2] and [St3]. It turns out that billiard trajectories with tangent points to the boundary of the obstacle play an important role in the two-dimensional case considered here. In Sect. 4 we prove that the set of trapped points $\text{Trap}(\Omega_K)$ has Lebesgue measure zero in $\mathbb{S}^*(\Omega_K)$.

2. Preliminaries. Next, we describe some propositions from [St2] and [St3] that are needed in the proof of Theorem 1.3. We state them in the general case $d \geq 2$, although later on we will use them in the special case $d = 2$.

Standing Assumption. K and L are finite disjoint unions of strictly convex domains in $\mathbb{R}^d$ $(d \geq 2)$ with C^3 boundaries and with conjugate flows $\mathcal{F}_t^{(K)}$ and $\mathcal{F}_t^{(L)}$.

PROPOSITION 2.1. *([St2]) (a) There exists a countable family $\{M_i\} = \{M_i^{(K)}\}$ of codimension 1 submanifolds of $\mathbb{S}_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ such that every*

$$\sigma \in \mathbb{S}_+^*(S_0) \setminus (\text{Trap}(\Omega_K) \cup_i M_i)$$

generates a simply reflecting ray in Ω_K. Moreover the family $\{M_i\}$ is locally finite, that is any compact subset of $\mathbb{S}_+^*(S_0) \setminus \text{Trap}(\Omega_K)$ has common points with only finitely many of the submanifolds M_i.

(b) There exists a countable family $\{R_i\}$ of codimension 2 smooth submanifolds of $\mathbb{S}_+^*(S_0)$ such that for any $\sigma \in \mathbb{S}_+^*(S_0) \setminus (\cup_i R_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K.

(c) There exists a countable family $\{Q_i\}$ of codimension 2 smooth submanifolds of $\mathbb{S}_{\partial K}^*(\Omega_K)$ such that for any $\sigma \in \mathbb{S}_+^*(\partial K) \setminus (\cup_i Q_i)$ the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K.

It follows from the conjugacy of flows and Proposition 4.3 in [St3] that the submanifolds M_i are the same for K and L, i.e. $M_i^{(K)} = M_i^{(L)}$ for all i.

The following is Lemma 5.2 in [St2]. In fact the lemma in [St2] assumes C^∞ smoothness for the submanifold X, however its proof only requires C^3 smoothness.

PROPOSITION 2.2. *Let X be a C^3 smooth submanifold of codimension 1 in $\mathbb{R}^d$, and let $x_0 \in X$ and $\xi_0 \in T_x X$, $\|\xi_0\| = 1$, be such that the normal curvature of X at x_0 in the direction ξ_0 is non-zero. Then for every $\epsilon > 0$ there exist an open neighbourhood V of x_0 in X, a smooth map*

$$V \ni x \mapsto \xi(x) \in T_x X$$

and a smooth positive function $t(x) \in [\delta, \epsilon]$ on V for some $\delta > 0$ such that

$$Y = \{y(x) = x + t(x)\xi(x) : x \in V\}$$

is a smooth strictly convex surface with an unit normal field $\nu_Y(y(x)) = \xi(x)$, $x \in V$. That is, the normal field of Y consists of vectors tangent to X at the corresponding points of V. (See Figure 2.)

As one would expect, the case $d = 2$ of the above proposition is rather easy to prove.

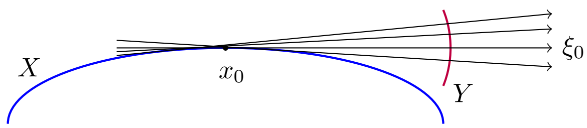

Figure 2

An important consequence of the above is the following proposition which can be proved using part of the argument in the proof of Proposition 5.5 in [St2]. For completeness we sketch the proof in the Appendix.

PROPOSITION 2.3. *Let K be an obstacle in $\mathbb{R}^2$ which is a finite disjoint union of strictly convex compact domains with C^3 boundaries. Then*

$$\dim(\mathbb{S}^*(\partial K) \cap \mathrm{Trap}(\Omega_K)) = 0.$$

In particular, $\mathbb{S}^*(\partial K) \cap \mathrm{Trap}(\Omega_K)$ does not contain non-trivial open subsets of $\mathbb{S}^*(\partial K)$.

Here we denote by $\dim(X)$ the *topological dimension* of a subset X of $\mathbb{R}^2$.

It turns out that for the type of obstacles considered in this paper the set $\mathrm{Trap}(\Omega_K)$ of trapped points has Lebesgue measure zero in $\mathbb{S}^*(\Omega_K)$. While formally this fact is not necessary for the proof of Theorem 1.3, we mention it here since it is a rather important feature of the billiard flow in the case considered in this paper (and also in [NS1], [St3], etc.). This appears to be accepted as a 'known fact' although we could not find a formal proof anywhere in the literature. However a simple proof follows from known facts, e.g. using the ergodicity of the so called dispersive (Sinai) billiards (see [Si1], [Si1]).

PROPOSITION 2.4. *Let K be an obstacle in $\mathbb{R}^d$ of the form (1.1). Then the set $\mathrm{Trap}(\Omega_K)$ of all of trapped points of $\mathbb{S}^*(\Omega_K)$ has Lebesgue measure zero in $\mathbb{S}^*(\Omega_K)$.*

We provide a proof of this proposition in Sect. 4 below.

3. Proof of Theorem 1.3. Assume that the obstacles K and L in $\mathbb{R}^2$ satisfy the assumptions of Theorem 1.3.

We claim that $K \subset L$. Assume this is not true and fix an arbitrary $x_0 \in \partial K$ such that $x_0 \notin L$. Let $\xi_0 \in \mathbb{S}^1$ be one of the unit vectors tangent to ∂K at x_0.

It follows from Proposition 2.2 that there exists a small $\epsilon_0 > 0$, an open neighbourhood V_0 of x_0 in ∂K, a C^2 map $V_0 \ni x \mapsto \xi(x) \in \mathbb{S}^*_x(\partial K)$ and a C^2 positive function $t(x) \in [\delta, \epsilon_0]$ on V_0 for some $\delta \in (0, \epsilon_0)$ such that

$$\Sigma = \{y(x) = x + t(x)\xi(x) : x \in V_0\}$$

is a C^2 strictly convex curve with unit normal field $\nu_\Sigma(y(x)) = \xi(x)$, $x \in V_0$. So, for any $x \in V_0$ the straight line through $y(x)$ with direction $\xi(x)$ is tangent to ∂K at x. Set $y_0 = x_0 + \epsilon_0\xi_0 \in \Sigma$.

It follows from Proposition 2.3 that for the subset

$$\Sigma' = \{y \in \Sigma : (y, \nu_\Sigma(y)) \notin \mathrm{Trap}(\Omega_K)\}$$

we have $\dim(\Sigma \setminus \Sigma') = 0$. Thus, $\dim(\Sigma') = 1$.

Next, Proposition 2.1 implies that for all but countably many $y \in \Sigma'$ the trajectories $\gamma_K(y, \nu_\Sigma(y))$ and $\gamma_L(y, \nu_\Sigma(y))$ have at most one tangency to ∂K and ∂L, respectively. For such y, since $\gamma_K(y, \nu_\Sigma(y))$ has a tangent point to ∂K, it must have exactly one tangent point to ∂K. Since the flows $\mathcal{F}_t^{(K)}$ and $\mathcal{F}_t^{(L)}$ are conjugate by assumption, $\gamma_L(y, \nu_\Sigma(y))$ also must have exactly one tangent point $z(y)$ to ∂L. More precisely, if $(y, \nu_\Sigma(y)) = \mathcal{F}_t^{(K)}(\sigma)$ for some $\sigma \in \mathbb{S}^*_+(S_0)$ and some $t > 0$, then the travelling time function t_K has a singularity at σ. Since $t_K = t_L$ on $\mathbb{S}^*_+(S_0)$ near σ, the function t_L also has a singularity at σ, so $\gamma_L(y, \nu_\Sigma(y)) = \gamma_L(\sigma)$ must have a tangent point to ∂L.

Assume for a moment that for every $z \in \partial L$ there exists an open neighbourhood W_z of z in ∂L such that

$$W_z \cap \{z(y) : y \in \Sigma'\}$$

has topological dimension zero. Covering ∂L with a finite number of neighbourhoods W_z, it follows that Σ' has topological dimension zero – a contradiction. Thus, there exists $z_0 \in \partial L$ such that for every open neighbourhood W_0 of z_0 in ∂L the set

$$W_0 \cap \{z(y) : y \in \Sigma\}$$

has topological dimension one. Replacing y_0 (and therefore x_0 as well) by an appropriate nearby point on Σ', we may assume that $z_0 = \mathrm{pr}_1(\mathcal{F}_{t_0}^{(L)}(y_0, \nu_\Sigma(y_0)))$ for some $t_0 \in \mathbb{R}$, $t_0 \neq 0$.

We will assume that $t_0 > 0$; otherwise we just have to replace ξ_0 by $-\xi_0$ and the curve Σ by $\{x - t(x)\xi(x) : x \in V_0\}$. Let

$$\mathcal{F}_{t_0}^{(L)}(y_0, \nu_\Sigma(y_0)) = (z_0, -\zeta_0).$$

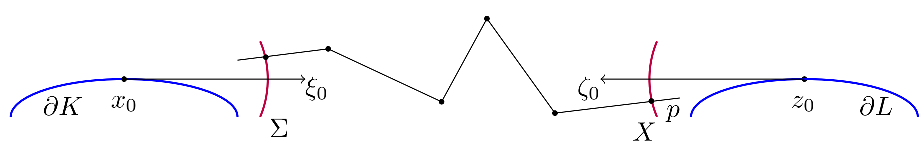

Figure 3

Using again Proposition 2.2, assuming $\epsilon_0 > 0$ is sufficiently small and shrinking the open neighbourhood W_0 of z_0 in ∂L if necessary, there exist a C^2 map

$$W_0 \ni z \mapsto \zeta(z) \in \mathbb{S}^*_z(\partial L)$$

and a C^2 positive function $s(z) \in [\delta, \epsilon_0]$ on W_0 for some $\delta \in (0, \epsilon_0)$ such that $\zeta(z_0) = \zeta_0$ and

$$X = \{p(z) = z + s(z)\,\zeta(z) : z \in W_0\}$$

is a C^2 strictly convex curve with unit normal field $\nu_X(p(z)) = \zeta(z)$, $z \in W_0$. So, for any $z \in W_0$ the straight line through $p(z)$ with direction $\zeta(z)$ is tangent to ∂L at z (see Figure 3). Set $p_0 = z_0 + \epsilon_0\zeta_0 \in X$.

We will now use a basic property of dispersive (Sinai) billiard flows concerning propagation of convex fronts. Let

$$x_1 = \mathrm{pr}_1(\mathcal{F}^{(L)}_{t_1}(x_0,\xi_0)), \ldots, x_k = \mathrm{pr}_1(\mathcal{F}^{(L)}_{t_k}(x_0,\xi_0))$$

be the common points of $\gamma_L(x_0,\xi_0)$ with ∂L (if any) with $0 < t_1 < \ldots < t_k < t_0$. Fix an arbitrary $T \in (t_k, t_0)$ close to t_k. It then follows from a well-known result of Sinai ([Si1]; see also [Si2]) that there exists an open neighbourhood Σ_0 of y_0 in Σ such that

$$Y = \{\mathrm{pr}_1(\mathcal{F}^{(L)}_T(y,\nu_\Sigma(y))) : y \in \Sigma_0\}$$

is a strictly convex curve in $\mathbb{R}^2$ with a unit normal field

$$\nu_Y(y,\nu_\Sigma(y)) = \mathrm{pr}_2(\mathcal{F}^{(L)}_T(y,\nu_\Sigma(y))).$$

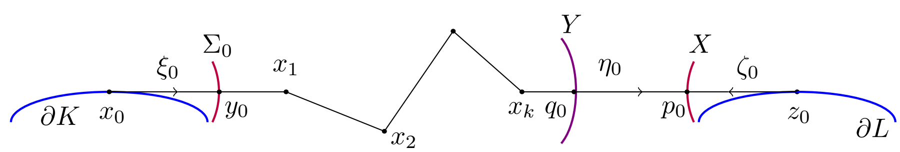

Figure 4

Set

$$q_0 = \mathrm{pr}_1(\mathcal{F}^{(L)}_T(y_0,\nu_\Sigma(y_0))) \in Y \quad , \quad \eta_0 = \nu_Y(q_0)$$

(see Figure 4). It follows from the constructions of Σ, the point $z_0 \in \partial L$, the neighbourhood W_0 and the convex fronts X and Y that for $y \in \Sigma_0 \cap \Sigma'$ the point

$$q = \mathrm{pr}_1(\mathcal{F}^{(L)}_T(y,\nu_\Sigma(y))) \in Y$$

is such that the straightline ray issued from q in direction $\nu_Y(q)$ hits X perpendicularly. However, due to the strict convexity of X and Y, this is only possible when $y = y_0$; a contradiction.

This proves that we must have $K \subset L$.

Using a similar argument we derive that $L \subset K$, as well. Therefore $K = L$. ■

4. On the set of trapped points. Here we prove Proposition 2.4.

Assume again that K is an obstacle in $\mathbb{R}^d$ $(d \geq 2)$ of the form (1.1) where K_i are strictly convex disjoint domains in $\mathbb{R}^d$ with C^3 smooth boundaries ∂K_i. Let λ be the Lebesgue measure on $\mathbb{S}^*(\mathbb{R}^d)$. Let S_0 be a large sphere in $\mathbb{R}^d$ as in Sect. 1, and let μ be the *Liouville measure* on $\mathbb{S}^*_+(S_0)$ defined by

$$d\mu = d\rho(q)d\omega_q |\langle \nu(q), v\rangle|,$$

where ρ is the measure on S_0 determined by the Riemannian metric on S_0 and ω_q is the Lebesgue measure on the $(d-2)$-dimensional sphere $\mathbb{S}_q(S_0)$ (see e.g. Sect. 6.1 in [CFS]).

We will need the following generalisation of Santalo's formula proved in [St4]. In fact, the latter deals with general billiard flows on Riemannian manifolds (under some natural assumptions), however here we will restrict ourselves to the case considered in Sect. 1.

THEOREM 4.1. *([St4]) Let K be as above. Then for every λ-measurable function*

$$f : \mathbb{S}^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K) \longrightarrow \mathbb{C}$$

such that $|f|$ is integrable we have

$$\int_{\mathbb{S}^*(\Omega_K)\setminus\mathrm{Trap}(\Omega_K)} f(x)\, d\lambda(x) = \int_{\mathbb{S}^*_+(S_0)\setminus\mathrm{Trap}(\Omega_K)} \left(\int_0^{t_K(x)} f(\mathcal{F}_t^{(K)}(x))\, dt \right) d\mu(x).$$

As we mentioned earlier $\mathrm{Trap}(\Omega_K) \cap \mathbb{S}^*_+(S_0)$ has Lebesgue measure zero in $\mathbb{S}^*_+(S_0)$ (see Theorem 1.6.2 in [LP]; see also Proposition 2.3 in [St2] for a more rigorous proof). Using this and the above theorem with $f = 1$ gives the following.

COROLLARY 4.2. *([St4]) Let K be as above. Then*

$$\lambda(\mathbb{S}^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K)) = \int_{\mathbb{S}^*_+(S_0)\setminus\mathrm{Trap}(\Omega_K)} t_K(x)\, d\mu(x).$$

That is,

$$\lambda(\mathrm{Trap}(\Omega_K)) = \lambda(\mathbb{S}^*(\Omega_K)) - \int_{\mathbb{S}^*_+(S_0)} t_K(x)\, d\mu(x).$$

Proof of Proposition 2.4. We can regard K as a subset of a domain Q in $\mathbb{R}^d$ with a piecewise smooth boundary which is strictly convex inwards[1] (see Figure 5). Consider the billiard flow ϕ_t on $\mathbb{S}^*(Q)$. It is well-known (see [CFS]) that ϕ_t preserves the Lebesgue measure λ (restricted to $\mathbb{S}^*(Q)$). Moreover ϕ_t is ergodic with respect to λ ([Si1], [Si2]).

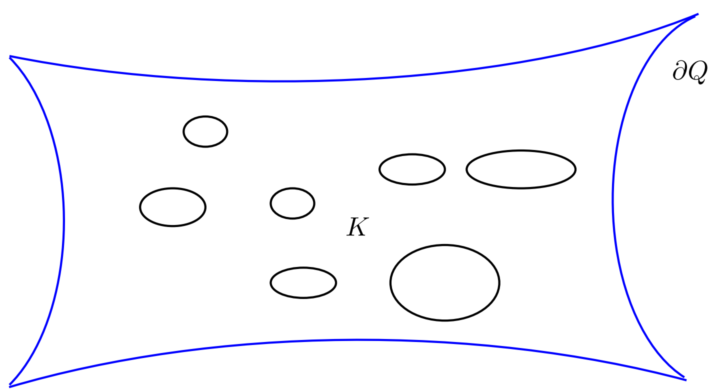

Figure 5

Let T be the set of points $x \in \mathbb{S}^*(\Omega_K)$ such that the billiard trajectory $\gamma_K(x)$ is trapped in both directions. Then Corollary 4.2 and the fact mentioned above that $\mathrm{Trap}(\Omega_K) \cap \mathbb{S}^*_+(S_0)$ has Lebesgue measure zero in $\mathbb{S}^*_+(S_0)$ imply that $\mathrm{Trap}(\Omega_K) \setminus T$ has Lebesgue measure zero in $\mathbb{S}^*(\Omega_K)$. So, it is enough to prove that $\lambda(T) = 0$.

[1] Or as a domain on the flat d-dimensional torus $\mathbb{T}^d$. Both embeddings will produce the required result.

The billiard flow ϕ_t coincides with the flow $\mathcal{F}_t^{(K)}$ on the set T, and T is an invariant set with respect to $\mathcal{F}_t^{(K)}$, and so with respect to ϕ_t. Clearly T is a proper subset of $S^*(Q)$ and $S^*(Q) \setminus T$ has positive measure. Now the ergodicity of ϕ_t implies that $\lambda(T) = 0$. ■

5. Appendix. Here we prove Proposition 2.3 using part of the argument in the proof of Proposition 5.5 in [St2].

It is enough to prove that every $x_0 \in \partial K$ has an open neighbourhood V_0 in ∂K such that $\dim(\mathbb{S}^*(V_0) \cap \mathrm{Trap}(\Omega_K)) = 0$.

Let $x_0 \in \partial K$. As in the proof of Theorem 1.3, it follows from Proposition 2.2 that there exists a small $\epsilon_0 > 0$, an open neighbourhood V_0 of x_0 in ∂K, a C^2 map $V_0 \ni x \mapsto \xi(x) \in \mathbb{S}^*_x(\partial K)$ and a C^2 positive function $t(x) \in [\delta, \epsilon_0]$ on V_0 for some $\delta \in (0, \epsilon_0)$ such that

$$\Sigma = \{y(x) = x + t(x)\xi(x) : x \in V_0\}$$

is a C^2 strictly convex curve with unit normal field $\nu_\Sigma(y(x)) = \xi(x)$, $x \in V_0$. Set

$$\widetilde{\Sigma} = \{(y, \nu_\Sigma(y)) : y \in \Sigma\}.$$

It follows from Proposition 2.1(c) that there exists a countable subset $X' = \{Q_i\}$ of $\mathbb{S}^*(\partial K)$ such that for any $\sigma \in S^*(\partial K) \setminus X'$, the trajectory $\gamma_K(\sigma)$ has at most one tangency to ∂K, and therefore it has exactly one tangency to ∂K.

Let X_0 the set of those $\sigma \in \widetilde{\Sigma} \cap \mathrm{Trap}(\Omega_K)$ such that the trajectory $\gamma_K^+(\sigma)$ has no tangencies to ∂K. Set $F = \{1, 2, \dots, k_0\}$, and consider

$$\widetilde{F} = \prod_{r=1}^{\infty} F$$

with the product topology. It is well known that $\dim(\widetilde{F}) = 0$ and therefore every subspace of $\widetilde{F}$ has topological dimension zero (cf. e.g. [HW] or [E]). Consider the map $f : X_0 \longrightarrow \widetilde{F}$, defined by

$$f(\sigma) = (i_1, i_2, \dots, i_n, \dots),$$

where the nth reflection point of $\gamma_K^+(\sigma)$ belongs to ∂K_{i_n} for all $n = 1, 2, \dots$. Clearly, the map f is continuous and it follows from [St1] that f is injective, so it defines a homeomorphism $f : X_0 \longrightarrow f(X_0)$. Thus, X_0 is homeomorphic to a subspace of $\widetilde{F}$ and therefore $\dim(X_0) = 0$.

Now the Sum Theorem for dim (cf. [HW] or [E]) shows that $\dim(X' \cup X_0) = 0$. Since $S^*(V_0) \cap \mathrm{Trap}(\Omega_K)$ is naturally homeomorphic to $X' \cup X_0$, it follows that

$$\dim(S^*(V_0) \cap \mathrm{Trap}(\Omega_K)) = 0.$$

This proves the proposition. ■

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