

# THE BOTTLENECK DEGREE OF ALGEBRAIC VARIETIES

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**ABSTRACT.** A bottleneck of a smooth algebraic variety  $X \subset \mathbb{C}^n$  is a pair  $(x, y)$  of distinct points  $x, y \in X$  such that the Euclidean normal spaces at  $x$  and  $y$  contain the line spanned by  $x$  and  $y$ . The narrowness of bottlenecks is a fundamental complexity measure in the algebraic geometry of data. In this paper we study the number of bottlenecks of affine and projective varieties, which we call the *bottleneck degree*. The bottleneck degree is a measure of the complexity of computing all bottlenecks of an algebraic variety, using for example numerical homotopy methods. We show that the bottleneck degree is a function of classical invariants such as Chern classes and polar classes. We give the formula explicitly in low dimension and provide an algorithm to compute it in the general case.

## 1. INTRODUCTION

In this paper we study geometric properties of algebraic varieties with applications to computational data science. Let  $f_1, \dots, f_k \in \mathbb{R}[x_1, \dots, x_n]$  be polynomials. The associated algebraic variety is the zero-set  $X \subset \mathbb{R}^n$  given by  $X = \{x \in \mathbb{R}^n : f_1(x) = \dots = f_k(x) = 0\}$ . Polynomial systems of equations arise in applications to natural science, engineering, computer science and beyond. Examples include kinematics [54], economics [44], chemistry [46], computer vision [41], machine learning [45] and optimization [43]. Polynomial systems can be analyzed naturally through the machinery of algebraic geometry. In the present study we concentrate on computing and counting so-called *bottlenecks* of an algebraic variety  $X \subset \mathbb{R}^n$ . This is the study of lines in  $\mathbb{R}^n$  orthogonal to  $X$  at two or more points. Such lines contribute to the computation of the *reach*, see Section 1.2, and may be found by solving a polynomial system (3). To be able to use the appropriate tools from algebraic geometry we often have to move from the real numbers to the algebraically closed field of complex numbers  $\mathbb{C}$ , as we illustrate below. We will see that classical invariants such as polar classes appear naturally and turn out to be essential to obtaining a closed formula for the number of bottlenecks. In our opinion this work provides yet one more illustration that classical algebraic geometry and in particular intersection theory are useful and often necessary in applications such as data science.

**1.1. Bottlenecks and optimization.** Finding lines orthogonal at two or more points is an optimization problem with algebraic constraints. The focus of this paper is to determine, or bound, the number of solutions to this optimization problem.

**Example 1.1.** Consider the ellipse  $C \subset \mathbb{R}^2$  defined by  $f = x^2 + y^2/2 - 1 = 0$ . A bottleneck on  $C$  is a pair of points  $p, q \in C$  that span a line orthogonal to  $C$  at both points. The only such lines are the  $x$ -axis and the  $y$ -axis, that is the principal axes of the ellipse, see Figure 1. A line  $l$  is orthogonal to  $C$  at a point  $p \in C$  if  $l$  is orthogonal to the tangent line  $T_p C$  at  $p$ . In other words  $l$  is the normal line  $N_p X$  at  $p$ . The direction of the normal line is given by the gradient  $\nabla f = (2x, y)$ . Consider a pair of points  $p = (x, y) \in C$  and  $q = (z, w) \in C$ . The claim that  $(p, q)$  is a bottleneck may then be expressed as

$$\begin{aligned} x - z &= \lambda 2x, \\ y - w &= \lambda y, \\ x - z &= \mu 2z, \\ y - w &= \mu w, \end{aligned}$$

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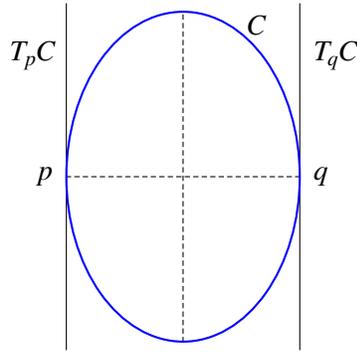


FIGURE 1. An ellipse with tangent lines and principal axes.

for some  $\lambda, \mu \in \mathbb{R}$ . These equations, together with  $x^2 + y^2/2 = 1$  and  $z^2 + w^2/2 = 1$ , constitute a polynomial system for computing bottlenecks on the curve  $C$ . Note that this is also the system we get if we apply the Lagrange multiplier method to the problem of optimizing the squared distance function  $(x - z)^2 + (y - w)^2$  subject to the constraints  $x^2 + y^2/2 - 1 = z^2 + w^2/2 - 1 = 0$ . This is thus an optimization problem and we are asking for the critical points of the distance between pairs of points on  $C$ .

Consider again an arbitrary variety  $X \subset \mathbb{R}^n$ . For convenience, we will restrict to the case where  $X$  is *smooth*, that is every point of  $X$  is a manifold point. A line is orthogonal to  $X$  if it is orthogonal to the tangent space  $T_x X \subset \mathbb{R}^n$  at  $x$ .

**Definition 1.2.** Let  $X \subset \mathbb{R}^n$  be a smooth variety. The bottlenecks of  $X$  are pairs  $(x, y)$  of distinct points  $x, y \in X$  such that the line spanned by  $x$  and  $y$  is normal to  $X$  at both points.

Equivalently one can define bottlenecks as the critical points of the squared distance function

$$(1) \quad \mathbb{R}^n \times \mathbb{R}^n : (x, y) \mapsto \|x - y\|^2,$$

subject to the constraints  $x, y \in X$  as well as the non-triviality condition  $x \neq y$ .

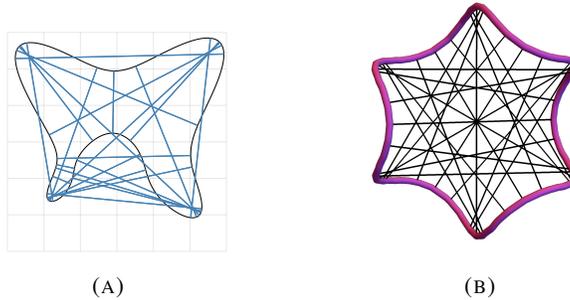


FIGURE 2. Two curves and their bottlenecks.

**Example 1.3.** Figure 2a shows a quartic curve in  $\mathbb{R}^2$  and its 22 bottleneck lines. The curve is defined by  $x^4 + y^4 + 1 - 4y - x^2y^2 - 4x^2 - x - 2y^2 = 0$ . The figure was produced by Paul Breiding and Sascha Timme using the Julia package *HomotopyContinuation.jl* [17].

As another example consider the space curve in  $\mathbb{R}^3$  defined by

$$\begin{aligned} x^3 - 3xy^2 - z &= 0, \\ x^2 + y^2 + 3z^2 - 1 &= 0. \end{aligned}$$

Figure 2b shows this curve and its 24 bottleneck lines.

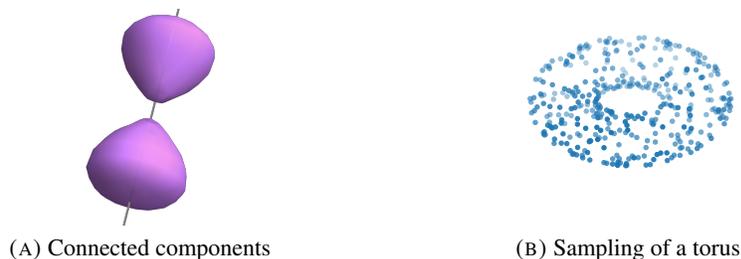


FIGURE 3

**1.2. Motivation.** The geometry of bottlenecks plays an important role in several aspects of real geometry in connection with analysis of data with support on an algebraic variety. Let  $X_{\mathbb{R}} \subset \mathbb{R}^n$  be a smooth variety which is non-empty and compact. An interesting observation is that the distance between any two distinct connected components of  $X_{\mathbb{R}}$  is realized by  $\|x - y\|$  for some bottleneck  $(x, y)$  with one point on each component. See Figure 3 for an illustration of this. The narrowest bottleneck thus bounds the smallest distance between any two connected components of  $X_{\mathbb{R}}$ . This relates bottlenecks and the so-called *reach*  $\tau_{\mathbb{R}}$  of  $X_{\mathbb{R}}$ . The number  $\tau_{X_{\mathbb{R}}}$  can be defined as the maximal distance  $r \geq 0$  such that any point  $p \in \mathbb{R}^n$  at distance less than  $r$  from  $X_{\mathbb{R}}$  has a unique closest point on  $X_{\mathbb{R}}$ .

The reach can be seen as a measure of curvature that extends to subsets of  $\mathbb{R}^n$  which are not smooth manifolds; see [30] for background and basic facts. The reach has many applications in the area of manifold reconstruction [1]. For example, it has been applied to minimax rates of convergence in Hausdorff distance [32, 40, 2]. The reach has also been applied to estimate boundary curve length and surface area [20] as well as volume [4]. Another application is dimensionality reduction via random projections [11]. In a number of papers [7, 19, 36, 53] the reach is used to bound the approximation error of such dimensionality reduction techniques. This amounts to a generalization of the Johnson-Lindenstrauss lemma [21] to higher dimensional manifolds. The reach is also used as input to standard algorithms for manifold triangulation [3, 12, 14, 13]. Another line of work seeks to compute *homology groups* of a manifold from a finite sample using techniques from persistent homology [15, 26, 37, 47]. A point cloud on a torus is illustrated in Figure 3. In this context, see also [5] on minimax rates of homology inference. The reach determines the sample size required to obtain the correct homology of the associated complex. With these applications in mind it would be useful to find efficient methods to compute the reach.

It is shown in [1] that  $\tau_{X_{\mathbb{R}}} = \min\{\rho, b\}$  where  $\rho$  is the minimal radius of curvature of a geodesic on  $X_{\mathbb{R}}$  and  $b$  is half the width of the narrowest bottleneck of  $X_{\mathbb{R}}$ . This suggests that  $\tau_{X_{\mathbb{R}}}$  could be computed with numerical methods by computing  $\rho$  and  $b$  separately. See [16], where this is carried out in detail for plane curves. We conclude that efficient methods to compute bottlenecks play a prominent role in the pursuit of efficient methods to compute the reach itself.

**1.3. Equations for bottlenecks.** We will now formulate a system of equations for bottlenecks that does not introduce auxiliary variables as in the Lagrange multiplier method. Both of these formulations are useful and the latter will be developed further in Remark 3.9.

Let  $X \subset \mathbb{R}^n$  be a smooth  $m$ -dimensional variety defined by polynomials  $f_1, \dots, f_k$ . Note that for  $x \in X$ ,  $\dim(T_x X) = \dim(X) = m$ . Here we are considering the embedded tangent space which passes through the point  $x$ . The corresponding linear space through the origin is  $(T_x X)_0 = T_x X - x$ . The orthogonal complement  $N_x X = \{z \in \mathbb{R}^n : (z - x) \perp (T_x X)_0\}$  is the normal space at  $x$  and has the complementary dimension  $n - m$ . As in the case of the ellipse in Example 1.1, the normal space is the span of the gradients  $\langle \nabla f_1, \dots, \nabla f_k \rangle$ . More precisely  $N_x X = x + \langle \nabla f_1(x), \dots, \nabla f_k(x) \rangle$ . Now, if  $x, y \in X$  are distinct then  $(x, y)$  is a bottleneck precisely when  $(y - x) \in \langle \nabla f_1(x), \dots, \nabla f_k(x) \rangle$  and  $(y - x) \in \langle \nabla f_1(y), \dots, \nabla f_k(y) \rangle$ . To formulate the equations we

define the *augmented Jacobian* to be the following matrix of size  $(k+1) \times n$ :

$$(2) \quad J(x, y) = \begin{bmatrix} y - x \\ \nabla f_1(x) \\ \vdots \\ \nabla f_k(x) \end{bmatrix},$$

where  $y - x$  is viewed as a row vector. The condition that  $y - x$  is in the span of  $\nabla f_1(x), \dots, \nabla f_k(x)$  is equivalent to saying that the matrix  $J(x, y)$  has rank less than or equal to  $n - m$ , or in other words that all  $(n - m + 1) \times (n - m + 1)$ -minors of  $J(x, y)$  vanish. There is a similar rank condition given by the  $(n - m + 1) \times (n - m + 1)$ -minors of the augmented Jacobian  $J(y, x)$  with  $x$  and  $y$  reversed. In summary, the bottlenecks of  $X$  are the non-trivial ( $x \neq y$ ) solutions to the following system of equations:

$$(3) \quad \begin{aligned} (n - m + 1) \times (n - m + 1)\text{-minors of } J(x, y) &= 0, \\ (n - m + 1) \times (n - m + 1)\text{-minors of } J(y, x) &= 0, \\ f_1(x) = \dots = f_k(x) &= 0, \\ f_1(y) = \dots = f_k(y) &= 0. \end{aligned}$$

**1.4. Counting roots, complex numbers and projective space.** In this section we will motivate the study of complex and projective bottlenecks. Let  $f_1, \dots, f_k \in \mathbb{R}[x_1, \dots, x_n]$  with corresponding variety  $X_{\mathbb{R}}$ . The system of equations  $f_1(x) = \dots = f_k(x) = 0$  may have non-real solutions  $x \in \mathbb{C}^n$ . The complex solutions are very relevant for solving polynomial systems. We can define a complex variety  $X_{\mathbb{C}} \subset \mathbb{C}^n$  given by  $X_{\mathbb{C}} = \{x \in \mathbb{C}^n : f_1(x) = \dots = f_k(x) = 0\}$ . Note that  $X_{\mathbb{C}}$  contains the real solutions  $X_{\mathbb{R}} \subseteq X_{\mathbb{C}}$ .

As practitioners we need tools to numerically approximate solutions to polynomial systems. A useful approach we would like to mention here is numerical homotopy methods, see for example [51, 10, 17]. These are predictor/corrector routines based on Newton's method but with probabilistic guarantees that all complex isolated solutions will be found. If a system has only finitely many solutions then the number of complex roots is an upper bound on the number of real roots. A naive approach to finding the real roots is of course to compute all complex roots and filter out the real ones. We stress this point because it illustrates how the number of complex bottlenecks (if finite) provides upper bounds on the computational complexity of real bottlenecks. It is therefore natural to explore the concept of bottlenecks in the complex setting even if one is only interested in real solutions.

An alternative approach to homotopy methods is symbolic computations via Gröbner bases, see for example [52]. Whether homotopy methods or Gröbner bases is appropriate depends on the particular system of equations at hand. See [8] for a comparison of numerical and symbolic methods for equation solving.

Let  $X \subset \mathbb{C}^n$  be a smooth variety, defined by  $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$ , with  $\dim(X) = m > 0$ . A *bottleneck* of  $X$  is defined to be a pair of distinct points  $x, y \in X$  such that the line  $\overline{xy}$  joining  $x$  and  $y$  is normal to  $X$  at both  $x$  and  $y$ . The orthogonality relation  $a \perp b$  involved in the definition of bottlenecks is given by  $\sum_{i=1}^n a_i b_i = 0$  for  $a = (a_1, \dots, a_n) \in \mathbb{C}^n$  and  $b = (b_1, \dots, b_n) \in \mathbb{C}^n$ . For a point  $x \in X$ , let  $(T_x X)_0$  denote the embedded tangent space of  $X$  translated to the origin. Then the *Euclidean normal space* of  $X$  at  $x$  is defined as  $N_x X = \{z \in \mathbb{C}^n : (z - x) \perp (T_x X)_0\}$ . A pair of distinct points  $(x, y) \in X \times X$  is thus a bottleneck exactly when  $\overline{xy} \subseteq N_x X \cap N_y X$ . Note that this is the case if and only if  $y \in N_x X$  and  $x \in N_y X$ . The *bottleneck variety* in  $\mathbb{C}^{2n}$  consists of the bottlenecks of  $X$  together with the diagonal  $\{(x, y) \in X \times X : x = y\} \subset \mathbb{C}^n \times \mathbb{C}^n$ . Just as for real varieties, the augmented Jacobian is defined by (2) and the system (3) defines the bottleneck variety of  $X$ .

In a similar manner we will define bottlenecks for projective varieties in complex projective space  $\mathbb{P}^n$ . Recall that projective space  $\mathbb{P}^n$  is obtained by gluing a hyperplane at infinity to the affine space  $\mathbb{C}^n$ . For example, the projective plane  $\mathbb{P}^2$  is the complex plane  $\mathbb{C}^2$  with an added line at infinity.

Counting the number of roots to a system of polynomials is a highly challenging problem. The simplest case is counting roots in  $\mathbb{P}^n$ . Counting roots in  $\mathbb{C}^n$  is harder and even harder is to count real roots. Consider for example a univariate polynomial  $f \in \mathbb{R}[x]$  of degree  $d$ . In this case there are always  $d$  complex roots counted with multiplicity while the number of real roots depends on the coefficients of  $f$ . Consider now the

next step of two equations  $f_1, f_2 \in \mathbb{R}[x_1, x_2]$  of degrees  $d_1$  and  $d_2$  and the corresponding intersection of two curves in  $\mathbb{C}^2$ . If the intersection is finite there can be at most  $d_1 d_2$  complex solutions. This is also the number of roots for almost all  $f_1$  and  $f_2$  of degrees  $d_1$  and  $d_2$ . In the case of real curves in  $\mathbb{R}^2$  there is no such generic root count. Also, the number of complex intersection points may be smaller than  $d_1 d_2$  as illustrated by the example of two disjoint lines defined by  $x_1 = 0$  and  $x_1 = 1$ . In contrast, the intersection of two curves in  $\mathbb{P}^2$  of degrees  $d_1$  and  $d_2$  with finite intersection always consists of  $d_1 d_2$  points counted with multiplicity. This fact is known as Bézout's theorem and it can be generalized to a system of  $n$  equations in  $n$  variables [31, Proposition 8.4]. This might seem to solve the problem, at least in  $\mathbb{P}^n$ . However, the system we want to solve might be *overdetermined* and have *excess components* of higher dimension. Both of these complications are present in the system (3). The excess component in this case consists of the discarded trivial solutions on the diagonal  $\{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : x = y\}$ . Amazingly, intersection theory provides tools to deal with these issues under certain circumstances. These tools are however often confined to the complex projective setting.

Bottlenecks for projective varieties turn out to be essential for counting bottlenecks on affine varieties. In fact, in Proposition 3.6 we reduce the affine case to the projective case by considering bottlenecks at infinity.

**1.5. Polar geometry.** Consider the ellipse  $C$  defined by  $x^2 + y^2/2 = 1$  in Example 1.1. For a point  $p \in \mathbb{R}^2$  outside the region bounded by  $C$  there are exactly two lines through  $p$  tangent to  $C$ . The two tangent points  $x, y \in C$  define what we call the first polar locus  $P_1(X, p) = \{x, y\}$ , see Figure 4. The polar locus depends on the choice of  $p$  but two polar loci  $P_1(X, p)$  and  $P_1(X, p')$  can be seen as deformations of each other by letting  $p'$  approach  $p$  along a curve. In this sense, the polar loci all represent the same *polar class*  $p_1$  on  $C$ .

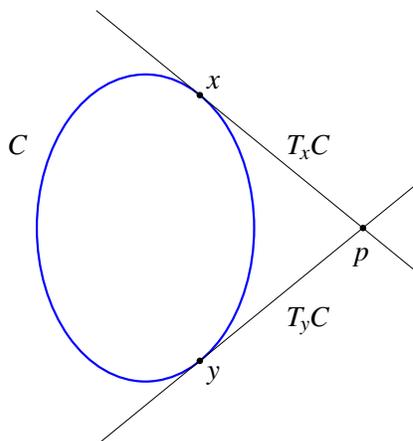


FIGURE 4. Polar locus of an ellipse.

Polar loci, also known as polar varieties, can be generalized to varieties of higher dimension and play an important role in applications of non-linear algebra. Examples include real equation solving [6], computational complexity [18], computing invariants [24, 9, 29, 23], Euclidean distance degree [25] and optimization [50].

In this paper we use polar varieties to count bottlenecks. This is done in the complex projective setting. For a smooth projective variety  $X \subset \mathbb{P}^n$  polar loci are defined using the projective tangent space  $\mathbb{T}_x X \subset \mathbb{P}^n$  at points  $x \in X$ . Consider first the case where  $X$  is a smooth hypersurface defined by a homogeneous polynomial  $f \in \mathbb{C}[x_0, \dots, x_n]$  and let  $x \in X$ . Then the hyperplane  $\mathbb{T}_x X \subset \mathbb{P}^n$  is defined by the equation  $\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i}(x) = 0$ . In general, if  $X$  is a smooth variety defined by an ideal generated by homogeneous polynomials  $f_1, \dots, f_k \in \mathbb{C}[x_0, \dots, x_n]$ , then  $\mathbb{T}_x X \subseteq \mathbb{P}^n$  is the subspace defined by the kernel of the Jacobian matrix  $\{\frac{\partial f_i}{\partial x_j}(x)\}_{i,j}$ .

For a smooth surface  $X \subset \mathbb{P}^3$  we have two polar varieties. Let  $p \in \mathbb{P}^3$  be a general point and  $l \subset \mathbb{P}^3$  a general line. Then  $P_1(X, p)$  is the set of points  $x$  such that the projective tangent plane  $\mathbb{T}_x X \subset \mathbb{P}^3$  contains  $p$ . This is a curve on  $X$ . Similarly,  $P_2(X, l) = \{x \in X : l \subseteq \mathbb{T}_x X\}$ , which is finite. We also let  $P_0(X) = X$ . More generally, an  $m$ -dimensional variety has  $m + 1$  polar varieties defined by exceptional tangent loci as

follows. Let  $X \subset \mathbb{P}^n$  be a smooth variety of dimension  $m$ . For  $j = 0, \dots, m$  and a general linear space  $V \subseteq \mathbb{P}^n$  of dimension  $n - m - 2 + j$  we define the polar locus

$$P_j(X, V) = \{x \in X : \dim(\mathbb{T}_x X \cap V) \geq j - 1\}.$$

If  $X$  has codimension 1 and  $j = 0$ , then  $V$  is the empty set using the convention  $\dim(\emptyset) = -1$ . By [31] Example 14.4.15,  $P_j(X, V)$  is either empty or of pure codimension  $j$ .

In order to link bottlenecks and polar varieties we employ the tools of intersection theory and pass from polar varieties to polar classes. For each polar variety  $P_j(X, V)$  there is a corresponding polar class  $[P_j(X, V)] = p_j$  which represents  $P_j(X, V)$  up to *rational equivalence*. For example,  $P_j(X, V)$  represents the same polar class  $p_j$ , independently of the general choice of linear space  $V$ . In a similar manner, any subvariety  $Z \subset \mathbb{P}^n$  has a corresponding rational equivalence class  $[Z]$ . We refer to Fulton's book [31] for background on intersection theory. For more details on polar classes see for example [48, 49] and [31, Example 14.4.15]. An important point is that there is a well defined multiplication of polar classes corresponding to intersection of polar varieties. This means that  $p_i p_j = [P_i(X, V) \cap P_j(X, W)]$  for  $0 \leq i, j \leq m$ . Here  $V \subset \mathbb{P}^n$  and  $W \subset \mathbb{P}^n$  are general linear spaces of dimension  $n - m - 2 + i$  and  $n - m - 2 + j$ , respectively. To express the number of bottlenecks of a variety in terms of polar classes we also need the notion of *degree of a class*. If  $Z \subset \mathbb{P}^n$  is a subvariety,  $\deg(Z)$  is the number of points of  $Z \cap L$ , where  $L \subset \mathbb{P}^n$  is a general linear space of dimension  $n - \dim(Z)$ . For the class  $[Z]$  we let  $\deg([Z]) = \deg(Z)$ .

**1.6. Results.** Let  $X \subset \mathbb{C}^n$  be a smooth variety and consider the closure  $\bar{X} \subset \mathbb{P}^n$  in projective space. For the purpose of counting bottlenecks we introduce the *bottleneck degree* of an algebraic variety. Under suitable genericity assumptions (see Definition 2.6), the bottleneck degree coincides with the number of bottlenecks.

The orthogonality relation on  $\mathbb{P}^n$  is defined via the *isotropic quadric*  $Q \subset \mathbb{P}^n$  given in homogeneous coordinates by  $\sum_0^n x_i^2 = 0$ . Varieties which are tangent to  $Q$  are to be considered degenerate in this context and we say that a smooth projective variety is in *general position* if it intersects  $Q$  transversely.

Our main result, Theorem 2.10, is a proof that the bottleneck degree of a smooth variety  $\bar{X} \subset \mathbb{P}^n$  in general position can be computed via the polar classes  $p_0, \dots, p_m$ . The arguments in the proof directly give an algorithm for expressing the bottleneck degree in terms of polar classes. We have implemented this algorithm in Macaulay2 [33] and the script is available at [22]. We give the formula for projective curves, surfaces and threefolds, with the following notation:  $h$  denotes the hyperplane class in the intersection ring of  $\bar{X}$ ,  $d = \deg(\bar{X})$  and  $\varepsilon_i = \sum_{j=0}^{m-i} \deg(p_j)$ . We also use  $\text{BND}(\bar{X})$  to denote the bottleneck degree of  $\bar{X}$ .

Curves in  $\mathbb{P}^2$ :

$$\text{BND}(\bar{X}) = d^4 - 4d^2 + 3d.$$

Curves in  $\mathbb{P}^3$ :

$$\text{BND}(\bar{X}) = \varepsilon_0^2 + d^2 - \deg(2h + 5p_1).$$

Surfaces in  $\mathbb{P}^5$ :

$$\text{BND}(\bar{X}) = \varepsilon_0^2 + \varepsilon_1^2 + d^2 - \deg(3h^2 + 6hp_1 + 12p_1^2 + p_2).$$

Threefolds in  $\mathbb{P}^7$ :

$$\text{BND}(\bar{X}) = \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 + d^2 - \deg(4h^3 + 11h^2p_1 + 4hp_1^2 + 24p_1^3 + 2hp_2 - 12p_1p_2 + 17p_3).$$

Notice that  $\varepsilon_0 = \deg(p_0) + \dots + \deg(p_m)$  is equal to the Euclidean Distance Degree of the variety.

Now consider the smooth affine variety  $X \subset \mathbb{C}^n \subset \mathbb{P}^n$  and let  $H_\infty = \mathbb{P}^n \setminus \mathbb{C}^n$  be the hyperplane at infinity. The formulas for projective varieties above have to be modified to yield the bottleneck degree  $\text{BND}(X)$  of the affine variety  $X$ . Namely, there is a contribution to  $\text{BND}(\bar{X})$  from the hyperplane section  $X \cap H_\infty$  at infinity. More precisely, we show in Proposition 3.6 that

$$\text{BND}(X) = \text{BND}(\bar{X}) - \text{BND}(\bar{X} \cap H_\infty).$$

Here we have assumed that  $X \subset \mathbb{C}^n$  is in general position in the following sense:  $\bar{X}$  and  $X \cap H_\infty$  are smooth and in general position. In the case of a plane curve  $X \subset \mathbb{C}^2$  in general position the hyperplane section  $\bar{X} \cap H_\infty$  consists of  $d$  points on the line at infinity. This results in

$$(4) \quad \text{BND}(X) = d^4 - 4d^2 + 3d - d(d-1) = d^4 - 5d^2 + 4d.$$

We end this introduction with an example illustrating the above formula for affine curves  $X \subset \mathbb{C}^2$ . Before looking at a concrete example it is worth pointing out that by convention bottlenecks are counted as ordered pairs  $(x, y) \in X \times X$ . Since  $(y, x)$  is also a bottleneck if  $(x, y)$  is a bottleneck, each unordered bottleneck pair contributes twice to the bottleneck degree.

**Example.** Consider the Trott curve  $X \subset \mathbb{C}^2$  defined by the equation

$$144(x_1^4 + x_2^4) - 225(x_1^2 + x_2^2) + 350x_1^2x_2^2 + 81.$$

This nonsingular quartic curve is notable because all 28 bitangents are real.

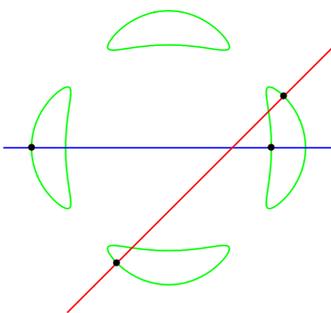


FIGURE 5. The quartic Trott curve depicted with two bottleneck pairs and their corresponding normal lines.

The bottleneck pairs  $\{(x_1, x_2), (y_1, y_2)\}$  are the off-diagonal solutions to the following set of four equations, which are the equations of the bottleneck ideal described in (3). The first two imply that each point is on the curve and the second two imply that each point is on the normal line to the curve at the other point:

$$\begin{aligned} 144(x_1^4 + x_2^4) - 225(x_1^2 + x_2^2) + 350x_1^2x_2^2 + 81 &= 0 \\ 144(y_1^4 + y_2^4) - 225(y_1^2 + y_2^2) + 350y_1^2y_2^2 + 81 &= 0 \\ x_1(-576x_1^2 - 700x_2^2 + 450)(y_2 - x_2) &= x_2(576x_2^2 + 700x_1^2 - 450)(x_1 - y_1) \\ y_1(-576y_1^2 - 700y_2^2 + 450)(x_2 - y_2) &= y_2(576y_2^2 + 700y_1^2 - 450)(y_1 - x_1). \end{aligned}$$

For a general enough affine plane curve of degree 4, (4) gives a bottleneck degree of 192. This is in fact the number of bottlenecks of the Trott curve. It was verified in Macaulay2 by creating the ideal of the four equations above and then saturating to remove the diagonal.

In this example, the 192 bottlenecks correspond to  $192/2=96$  bottleneck pairs. In particular, the real part of the Trott curve intersects the  $x$ - and  $y$ -axis each 4 times and in each case the relevant axis is the normal line to the curve at the intersection, leading to six bottleneck pairs on each axis.

The paper is naturally divided between the treatment of the projective case (Section 2) and the affine case (Section 3). Section 4 provides a small library of examples.

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## 2. PROJECTIVE VARIETIES

**2.1. Notation and background in intersection theory.** Below we introduce the Chow group of a subscheme of complex projective space  $\mathbb{P}^n$  and present the double point formula from intersection theory. The reason for considering schemes and not only algebraic varieties is that isolated bottlenecks are counted with multiplicity and similar considerations should be made for higher dimensional bottleneck components. Specifically, the double point class defined below is a push forward of the double point scheme and the latter carries multiplicity information. In the end we only study bottlenecks on algebraic varieties and little is lost if the reader wishes to think of varieties in place of schemes.

The notation used in this paper will closely follow that of Fulton's book [31]. Let  $X \subseteq \mathbb{P}^n$  be a closed  $m$ -dimensional subscheme. We use  $A_k(X)$  to denote the group of  $k$ -cycles on  $X$  up to rational equivalence and  $A_*(X) = \bigoplus_{k=0}^m A_k(X)$  denotes the Chow group of  $X$ . For a subscheme  $Z \subseteq X$  we have an associated cycle class  $[Z] \in A_*(X)$ . Also, for a zero cycle class  $\alpha \in A_0(X)$  we have the notion of degree, denoted  $\deg(\alpha)$ , which counts the number of points with multiplicity of a 0-cycle representing  $\alpha$ .

Suppose now that  $X \subseteq \mathbb{P}^n$  is a smooth variety of dimension  $m$ . In this case we will also consider the intersection product on  $A_*(X)$  which makes it into a ring. For  $\alpha, \beta \in A_*(X)$  we denote their intersection product by  $\alpha\beta$  or  $\alpha \cdot \beta$ . Now let  $\alpha \in A_k(X)$  with  $k > 0$  and consider the hyperplane class  $h \in A_{m-1}(X)$  induced by the embedding  $X \subseteq \mathbb{P}^n$ . In this paper, we define  $\deg(\alpha) = \deg(h^k \alpha)$ . This means that if  $\alpha$  is represented by a subvariety  $Z \subseteq X$ , then  $\deg(\alpha)$  is the degree of  $Z$ . For a cycle class  $\alpha \in A_*(X)$ , we will use  $(\alpha)_k$  to denote the homogeneous piece of  $\alpha$  of codimension  $k$ , that is  $(\alpha)_k$  is the projection of  $\alpha$  to  $A_{m-k}(X)$ . Finally, for  $i = 0, \dots, m$ ,  $c_i(T_X)$  denotes the  $i$ -th Chern class of the tangent bundle of  $X$  and  $c(T_X) = c_0(T_X) + \dots + c_m(T_X)$  denotes the total Chern class.

Now let  $X$  and  $Y$  be subschemes of projective space. A map  $f : X \rightarrow Y$  gives rise to a push forward group homomorphism  $f_* : A_*(X) \rightarrow A_*(Y)$  and if  $X$  and  $Y$  are smooth varieties we also have a pull-back ring homomorphism  $f^* : A_*(Y) \rightarrow A_*(X)$ .

Let  $f : X \rightarrow Y$  be a morphism of smooth projective varieties. Let  $x \in A_k(X)$ ,  $y \in A_l(Y)$  satisfy  $k+l = \dim(Y)$ . By the projection formula [31, Proposition 8.3 (c)],  $f_*(f^*(y) \cdot x) = y \cdot f_*(x)$ . In particular  $\deg(y \cdot f_*(x)) = \deg(f_*(f^*(y) \cdot x)) = \deg(f^*(y) \cdot x)$ . This relation is used many times in the sequel.

Now let  $f : X \rightarrow Y$  be a map of smooth projective varieties with  $\dim(X) = k$  and  $\dim(Y) = 2k$ . Let  $f \times f : X \times X \rightarrow Y \times Y$  be the induced map, let  $\text{Bl}_{\Delta_X}(X \times X)$  be the blow-up of  $X \times X$  along the diagonal  $\Delta_X \subset X \times X$  and let  $bl : \text{Bl}_{\Delta_X}(X \times X) \rightarrow X \times X$  be the blow-up map. Consider the map  $h = (f \times f) \circ bl : \text{Bl}_{\Delta_X}(X \times X) \rightarrow Y \times Y$  and the inverse image scheme  $h^{-1}(\Delta_Y)$  of the diagonal  $\Delta_Y \subset Y \times Y$ . Then the exceptional divisor  $bl^{-1}(\Delta_X)$  is a subscheme of  $h^{-1}(\Delta_Y)$  and its residual scheme in  $h^{-1}(\Delta_Y)$  is called the *double point scheme* of  $f$  and is denoted  $\tilde{D}(f)$ . The exceptional divisor  $bl^{-1}(\Delta_X)$  may be interpreted as the projectivized tangent bundle  $\mathbb{P}(T_X)$ . The support of the double point scheme  $\tilde{D}(f)$  consists of the pairs of distinct points  $(x, y) \in X \times X \subset \text{Bl}_{\Delta_X}(X \times X) \setminus bl^{-1}(\Delta_X)$  such that  $f(x) = f(y)$  together with the tangent directions in  $\mathbb{P}(T_X)$  where the differential  $df : T_X \rightarrow T_Y$  vanishes, see [42, Remark 14]. There is also an associated residual intersection class  $\bar{\mathbb{D}}(f) \in A_0(\tilde{D}(f))$  defined in [31, Theorem 9.2]. If  $\tilde{D}(f)$  has dimension 0, as expected, then  $\bar{\mathbb{D}}(f) = [\tilde{D}(f)]$ . Let  $\eta : \tilde{D}(f) \rightarrow X$  be the map induced by  $bl$  and the projection  $X \times X \rightarrow X$  onto the first factor. Then the *double point class*  $\mathbb{D}(f) \in A_0(X)$  is defined by  $\mathbb{D}(f) = \eta_*(\bar{\mathbb{D}}(f))$ . By the double point formula, [31, Theorem 9.3],

$$\mathbb{D}(f) = f^* f_*[X] - (c(f^* T_Y) c(T_X)^{-1})_k.$$

**2.2. The conormal variety.** Let  $X \subset \mathbb{P}^n$  be a smooth variety of dimension  $m$ . Recall that  $H^0(X, \mathcal{O}_X(1)) \cong \mathbb{C}^{n+1}$  is the vector space parameterizing the hyperplane sections of the embedding  $X \subseteq \mathbb{P}^n \cong \mathbb{P}(H^0(X, \mathcal{O}_X(1)))$ .

Consider the surjective linear map:

$$\text{jet}_x : H^0(X, \mathcal{O}_X(1)) \rightarrow H^0(\mathcal{O}_X(1) \otimes \mathcal{O}_X/m_x^2) \cong \mathbb{C}^{m+1},$$

where  $m_x$  is the maximal ideal at  $x$ . Roughly speaking this map assigns to a global section  $s$  the  $(m+1)$ -tuple  $(s(x), \dots, \frac{\partial s}{\partial x_i}(x), \dots)$ , where  $(x_1, \dots, x_m)$  is a system of coordinates around  $x$ . We also have that

$$\mathbb{T}_x X = \mathbb{P}(\text{im}(\text{jet}_x)) \cong \mathbb{P}^m.$$

Let  $N_{X/\mathbb{P}^n}$  be the normal bundle of  $X$  in  $\mathbb{P}^n$  and let  $N_{X/\mathbb{P}^n}^\vee$  be its dual. The fibers of the dual normal bundle at  $x$  are given by the kernel of the map  $\text{jet}_x : \ker(\text{jet}_x) \cong N_{X/\mathbb{P}^n, x}^\vee \otimes \mathcal{O}_X(1)_x$ . The projective tangent spaces at points  $x \in X$  glue together to form the first jet bundle  $\mathbb{J}$  with fiber  $\mathbb{J}_x = H^0(\mathcal{O}_X(1) \otimes \mathcal{O}_X/m_x^2)$ , inducing the exact sequence of vector bundles:

$$(5) \quad 0 \rightarrow N_{X/\mathbb{P}^n}^\vee \otimes \mathcal{O}_X(1) \rightarrow X \times H^0(\mathcal{O}_X(1)) \rightarrow \mathbb{J} \rightarrow 0$$

The projectivized bundle of the conormal bundle is called the *conormal variety*:

$$\mathcal{C}_X = \mathbb{P}(N_{X/\mathbb{P}^n}^\vee) \cong \mathbb{P}(N_{X/\mathbb{P}^n}^\vee \otimes \mathcal{O}_X(1)) \subset \mathbb{P}^n \times (\mathbb{P}^n)^*$$

where  $\mathbb{P}(N_{X/\mathbb{P}^n}^\vee)$  denotes the projectivized conormal bundle of  $X$  in  $\mathbb{P}^n$ , see [31, Example 3.2.21] for more details. From the exact sequence (5) it follows that the conormal variety consists of pairs of points  $x \in X$  and hyperplanes in  $\mathbb{P}^n$  that contain the projective tangent space  $\mathbb{T}_x X$ .

**2.3. Bottleneck degree.** Let  $X \subset \mathbb{P}^n$  be a smooth variety of dimension  $m < n$  and consider the conormal variety  $\mathcal{C}_X = \mathbb{P}(N_{X/\mathbb{P}^n}^\vee) \subset \mathbb{P}^n \times (\mathbb{P}^n)^*$  introduced above. We use  $\mathcal{O}(1)$  to denote the dual of the tautological line bundle on  $\mathcal{C}_X$ , see [31, Appendix B.5.1 and B.5.5], and  $\xi = c_1(\mathcal{O}(1))$  denotes the first Chern class of  $\mathcal{O}(1)$ . Also, let  $\pi : \mathcal{C}_X \rightarrow X$  be the projection. Note that  $\dim(\mathcal{C}_X) = n - 1$ .

*Remark 2.1.* In the sequel we will compute the degrees of zero cycle classes in  $A_0(\mathbb{P}(N_{X/\mathbb{P}^n}^\vee))$ . By [31, Theorem 3.3 (b)],  $A_0(X) \cong A_0(\mathbb{P}(N_{X/\mathbb{P}^n}^\vee))$  via the map  $\alpha \mapsto \xi^{n-m-1} \pi^* \alpha$ . This means that every element of  $A_0(\mathbb{P}(N_{X/\mathbb{P}^n}^\vee))$  can be written uniquely in the form  $\xi^{n-m-1} \pi^* \alpha$  where  $\alpha \in A_0(X)$ , leading to a degree formula:

$$\deg(\xi^{n-m-1} \pi^* \alpha) = \deg(\alpha).$$

Also, by [31, Remark 3.2.4]

$$(6) \quad \xi^{n-m} + c_1(\pi^* N_{X/\mathbb{P}^n}^\vee) \xi^{n-m-1} + \dots + c_{n-m}(\pi^* N_{X/\mathbb{P}^n}^\vee) = 0.$$

Hence, given a zero cycle class  $Z \in A_0(\mathbb{P}(N_{X/\mathbb{P}^n}^\vee))$  of the form  $Z = \xi^i \pi^* \beta$  where  $i > n - m - 1$  and  $\beta \in A_{i-(n-m-1)}(X)$  we may use (6) to write  $Z$  as  $\xi^{n-m-1} \pi^* \alpha$  for some  $\alpha \in A_0(X)$ . More generally, consider a 0-cycle class  $Z \in A_0(\mathbb{P}(N_{X/\mathbb{P}^n}^\vee))$  which is a polynomial in  $\xi$  and pull-backs of classes on  $X$ ,  $Z = \sum_{i=0}^l \xi^i \pi^* \beta_i$ . Then  $\xi^i \pi^* \beta_i = 0$  for  $i < n - m - 1$  and  $\beta_i \in A_{i-(n-m-1)}(X)$  for  $i \geq n - m - 1$ . Again we can use the relation (6) to write  $Z$  as  $\xi^{n-m-1} \pi^* \alpha$  for some  $\alpha \in A_0(X)$ . This may be done in practice by applying the function `pseudoRemainder` in `Macaulay2` [33] to  $Z$  and the left hand side of (6). We will make use of this to compute bottleneck degrees in Algorithm 1.

We will consider  $\mathcal{C}_X$  as a subvariety of  $\mathbb{P}^n \times \mathbb{P}^n$  as follows. Fix coordinates on  $\mathbb{P}^n$  induced by the standard basis of  $\mathbb{C}^{n+1}$ . Then identify  $\mathbb{P}^n$  with  $(\mathbb{P}^n)^*$  via the isomorphism  $L : \mathbb{P}^n \rightarrow (\mathbb{P}^n)^*$  which sends a point  $(a_0, \dots, a_n) \in \mathbb{P}^n$  to the hyperplane  $\{(x_0, \dots, x_n) \in \mathbb{P}^n : a_0 x_0 + \dots + a_n x_n = 0\}$ . Define  $a \perp b$  by  $\sum_{i=0}^n a_i b_i = 0$  for  $a = (a_0, \dots, a_n), (b_0, \dots, b_n) \in \mathbb{P}^n$ . For a point  $p \in X$  we denote by  $(\mathbb{T}_p X)^\perp$  the orthogonal complement of the projective tangent space of  $X$  at  $p$ . The span  $\langle p, (\mathbb{T}_p X)^\perp \rangle$  of  $p$  and  $(\mathbb{T}_p X)^\perp$  is called the Euclidean normal space of  $X$  at  $p$  and is denoted  $N_p X$ . The Euclidean normal space is intrinsically related to the conormal variety as:

$$\mathcal{C}_X = \{(p, q) \in \mathbb{P}^n \times \mathbb{P}^n : p \in X, q \in (\mathbb{T}_p X)^\perp\}.$$

**Definition 2.2.** We say that a smooth variety  $X \subset \mathbb{P}^n$  is in *general position* if  $\mathcal{C}_X$  is disjoint from the diagonal  $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$ .

Let  $Q \subset \mathbb{P}^n$  be the *isotropic quadric*, which is defined by  $\sum_{i=0}^n x_i^2 = 0$ . If  $p \in X \cap Q$  is that such that  $\mathbb{T}_p X \subseteq \mathbb{T}_p Q$ , then  $(p, p) \in \mathcal{C}_X$ . Conversely, if  $(p, p) \in \mathcal{C}_X$ , then  $p \in X \cap Q$  and  $\mathbb{T}_p X \subseteq \mathbb{T}_p Q$ . In other words,  $X$  is in general position if and only if  $X$  intersects the isotropic quadric transversely.

Suppose that  $X$  is in general position. We then have a map

$$(7) \quad f : \mathcal{C}_X \rightarrow \text{Gr}(2, n+1) : (p, q) \mapsto \langle p, q \rangle,$$

from  $\mathcal{C}_X$  to the Grassmannian of lines in  $\mathbb{P}^n$ . The map sends a pair  $(p, q)$  to the line spanned by  $p$  and  $q$ . For the remainder of the paper,  $f$  will be used to denote this map associated to a variety  $X$ . To simplify notation we will also let  $G = \text{Gr}(2, n+1)$ .

Note that for  $p \in X$ , the map  $f$  restricted to the fiber  $\{(p', q') \in \mathcal{C}_X : p' = p\}$  parameterizes lines in the Euclidean normal space  $N_p X$  passing through  $p$ .

**Example 2.3.** In the case where  $X \subset \mathbb{P}^n$  is a smooth hypersurface,  $\mathcal{C}_X \cong X$  via the projection on the first factor of  $\mathbb{P}^n \times (\mathbb{P}^n)^*$ . Consider a general curve  $X \subset \mathbb{P}^2$  of degree  $d$  defined by a polynomial  $F \in \mathbb{C}[x, y, z]$ . For  $u \in \{x, y, z\}$ , let  $F_u = \frac{\partial F}{\partial u}$ . In this case  $G = (\mathbb{P}^2)^*$  and the map  $f : X \rightarrow (\mathbb{P}^2)^*$  defined above is given by  $(x, y, z) \mapsto (yF_z - zF_y, zF_x - xF_z, xF_y - yF_x)$ . Note that  $f(p) = N_p X$  is the Euclidean normal line to  $X$  at  $p$ .

Returning to a smooth  $m$ -dimensional variety  $X \subset \mathbb{P}^n$  in general position, consider the projection  $\eta : \mathcal{C}_X \times \mathcal{C}_X \rightarrow X \times X$  and the incidence correspondence

$$I(X) = \eta(\{(u, v) \in \mathcal{C}_X \times \mathcal{C}_X : f(u) = f(v)\}).$$

Pairs  $(x, y) \in I(X) \subset X \times X$  with  $x \neq y$  are called *bottlenecks* of  $X$ . The following lemma relates this definition of bottlenecks to the one given for affine varieties in Section 1. For  $x \in X$ , recall the definition of the Euclidean normal space  $N_x X = \langle x, (\mathbb{T}_x X)^\perp \rangle$ , where  $\mathbb{T}_x X$  denotes the projective tangent space of  $X$  at  $x$ .

**Lemma 2.4.** *Let  $X \subset \mathbb{P}^n$  be a smooth variety in general position. For a pair of distinct points  $x, y \in X$ ,  $(x, y)$  is a bottleneck if and only if  $y \in N_x X$  and  $x \in N_y X$ .*

*Proof.* By definition  $(x, q) \in \mathcal{C}_X \subset \mathbb{P}^n \times \mathbb{P}^n$  if and only if  $x \in X$  and  $q \in (\mathbb{T}_x X)^\perp$ . Hence, for  $(x, q) \in \mathcal{C}_X$ , the line  $\langle x, q \rangle$  is contained in  $N_x X$ . Now, if  $(x, y) \in X \times X$  is a bottleneck, then  $(x, q), (y, q') \in \mathcal{C}_X$  for some  $q, q' \in \mathbb{P}^n$  with  $\langle x, q \rangle = \langle y, q' \rangle$ . Hence  $y \in \langle x, q \rangle \subseteq N_x X$ . In the same way  $x \in N_y X$ . To see the converse let  $x, y \in X$  be distinct points such that  $y \in N_x X$  and  $x \in N_y X$ . Since  $y \in N_x X$ ,  $y \in \langle x, q \rangle$  for some  $q \in (\mathbb{T}_x X)^\perp$ . Then  $(x, q) \in \mathcal{C}_X$  and  $q \neq x$  since  $X$  is in general position. This implies that  $\langle x, y \rangle = \langle x, q \rangle$ . In the same way,  $x \in N_y X$  implies that  $(y, q') \in \mathcal{C}_X$  for some  $q' \in \mathbb{P}^n$  with  $\langle x, y \rangle = \langle y, q' \rangle$ . Since  $\langle x, q \rangle = \langle y, q' \rangle$ ,  $(x, y)$  is a bottleneck.  $\square$

Applying the double point formula to the map  $f$  we obtain

$$\mathbb{D}(f) = f^* f_* [\mathcal{C}_X] - (c(f^* T_G) c(T_{\mathcal{C}_X})^{-1})_{n-1},$$

where  $\mathbb{D}(f)$  is the double point class of  $f$ .

**Definition 2.5.** Let  $X \subset \mathbb{P}^n$  be a smooth variety in general position. We call  $\deg(\mathbb{D}(f))$  the *bottleneck degree* of  $X$  and denote it by  $\text{BND}(X)$ .

The bottleneck degree is introduced to count bottlenecks on  $X$  but there are some issues that need to be considered. The first issue is that there might be higher dimensional components worth of bottlenecks. In this case the bottleneck degree assigns multiplicities to these components which contribute to the bottleneck degree. We will not pursue this aspect of bottlenecks in this paper even though it is an interesting topic. Consider now a smooth variety  $X \subset \mathbb{P}^n$  in general position with only finitely many bottlenecks. As mentioned in Section 2.1, the double point scheme of  $f$  contains not only bottlenecks but also the tangent directions in  $\mathbb{P}(T_{\mathcal{C}_X})$  where the differential of  $f$  vanishes. This motivates the following definition of *bottleneck regular* varieties. As we shall see in Proposition 2.8, the bottleneck degree is equal to the number of bottlenecks counted with multiplicity in this case.

**Definition 2.6.** We will call a smooth variety  $X \subset \mathbb{P}^n$  *bottleneck regular* (BN-regular) if

- (1)  $X$  is in general position,
- (2)  $X$  has only finitely many bottlenecks and
- (3) the differential  $df_p : T_p \mathcal{C}_X \rightarrow T_{f(p)} G$  of the map  $f$  has full rank for all  $p \in \mathcal{C}_X$ .

**Proposition 2.7.** *Assume  $X$  is BN-regular. Let  $X \subset \mathbb{P}^a \subseteq \mathbb{P}^b$  be a smooth variety where  $\mathbb{P}^a \subseteq \mathbb{P}^b$  is a coordinate subspace. If  $X$  is in general position with respect to  $\mathbb{P}^a$  then  $X$  is in general position with respect to  $\mathbb{P}^b$  and the bottleneck degree is independent of the choice of ambient space.*

*Proof.* For  $c = a, b$ , let  $\mathcal{C}^c$  denote the conormal variety with respect to the embedding  $X \subset \mathbb{P}^c$ . The embedding  $\mathbb{P}^a \subseteq \mathbb{P}^b$  induces an embedding  $\mathcal{C}^a \subseteq \mathcal{C}^b$ . Similarly for  $c = a, b$ , let  $f_c : \mathcal{C}^c \rightarrow \text{Gr}(2, c+1)$  be the map given by  $(p, q) \mapsto \langle p, q \rangle$  and let  $\Delta_c \subset \mathbb{P}^c \times \mathbb{P}^c$  be the diagonal. Suppose that  $(p, q) \in \mathcal{C}^b \cap \Delta_b$ . Since  $p \in X \subset \mathbb{P}^a$ , we have that  $q = p \in \mathbb{P}^a$  and  $(p, q) \in \mathcal{C}^a \cap \Delta_a = \emptyset$ . Hence  $X$  is in general position with respect to  $\mathbb{P}^b$ .

We will consider  $\mathbb{D}(f_a)$  as a cycle class on  $\mathcal{C}^b$  via the inclusion  $\mathcal{C}^a \subseteq \mathcal{C}^b$ . We will show that  $\tilde{D}(f_a) = \tilde{D}(f_b)$ . Since  $X$  is BN-regular  $[\tilde{D}(f_c)] = \mathbb{D}(f_c)$  for  $c = a, b$ . It follows that  $\mathbb{D}(f_a) = \mathbb{D}(f_b)$ , which in turn implies that the bottleneck degree is independent of the choice of ambient space. Note that  $\tilde{D}(f_a) \subseteq \tilde{D}(f_b)$ .

We will first show that the differential  $df_b : T_{\mathcal{C}^b} \rightarrow T_G$  has full rank outside  $T_{\mathcal{C}^a}$ . This implies that  $\tilde{D}(f_b) \setminus \tilde{D}(f_a)$  consists of pairs  $x, y \in \mathcal{C}^b$  with  $x \neq y$  and  $f_b(x) = f_b(y)$ . Suppose that  $x = (p, q) \in \mathcal{C}^b$  and  $v \in T_x \mathcal{C}^b$  is a non-zero tangent vector such that  $(df_b)_x(v) = 0$ . Let  $D \subset \mathbb{C}$  be the unit disk and let  $P, Q : D \rightarrow \mathbb{C}^{b+1} \setminus \{0\}$  be smooth analytic curves such that the induced curve  $D \rightarrow \mathbb{P}^n \times \mathbb{P}^n$  is contained in  $\mathcal{C}^b$ , passes through  $x = (p, q)$  at  $0 \in D$  and has tangent vector  $v$  there. In other words  $P(0) \in p$  and  $Q(0) \in q$  are representatives of  $p$  and  $q$ . We need to show that  $Q(0), Q'(0) \in \mathbb{C}^{a+1}$ . Since  $(df_b)_x(v) = 0$ , we have by [35, Example 16.1] that  $P'(0), Q'(0) \in \langle P(0), Q(0) \rangle$ . Suppose first that  $P'(0)$  and  $P(0)$  are independent. Then  $Q(0), Q'(0) \in \langle P(0), P'(0) \rangle$  and  $\langle P(0), P'(0) \rangle \subseteq \mathbb{C}^{a+1}$  since  $X \subset \mathbb{P}^a$ . Now suppose that  $P'(0)$  is a multiple of  $P(0)$ . Since  $v \neq 0$ ,  $Q(0)$  and  $Q'(0)$  are independent and  $Q'(0)$  corresponds to a point  $q' \in \mathbb{P}^n$ . That  $P(0)$  and  $P'(0)$  are dependent implies that  $(p, q') \in \mathcal{C}^b$ . Moreover,  $P(0) \in \langle Q(0), Q'(0) \rangle$  by above and hence  $p \in \langle q, q' \rangle$ . It follows that  $(p, p) \in \mathcal{C}^b$ , which contradicts that  $X$  is in general position.

Now let  $(x, y) \in \tilde{D}(f_b)$  with  $x \neq y$  and  $f_b(x) = f_b(y)$ . If  $x, y \in \mathcal{C}^a$  then  $(x, y) \in \tilde{D}(f_a)$  so assume that  $x \notin \mathcal{C}^a$ . Let  $x = (p_1, q_1)$  and  $y = (p_2, q_2)$  with  $(p_i, q_i) \in X \times \mathbb{P}^b$ . Since  $\langle p_1, q_1 \rangle = \langle p_2, q_2 \rangle$  and because this line intersects  $\mathbb{P}^a$  in exactly one point  $p \in \mathbb{P}^a$ , we have that  $p_1 = p_2 = p$ . Moreover,  $p \in \langle q_1, q_2 \rangle$  and hence  $(p, p) \in \mathcal{C}^a$  contradicting that  $X$  is in general position. This means that  $\tilde{D}(f_b) \subseteq \tilde{D}(f_a)$  and hence  $\tilde{D}(f_a) = \tilde{D}(f_b)$ .  $\square$

If  $X \subset \mathbb{P}^n$  is BN-regular, then the double point scheme  $\tilde{D}(f)$  is finite and in one-to-one correspondence with the bottlenecks of  $X$  through the projection  $\eta : \mathcal{C}_X \times \mathcal{C}_X \rightarrow X \times X$ . Using the scheme-structure of  $\tilde{D}(f)$  we assign a multiplicity to each bottleneck. With notation as in Section 2.1,  $[\tilde{D}(f)] = \tilde{\mathbb{D}}(f)$  and we therefore have the following.

**Proposition 2.8.** *If  $X \subset \mathbb{P}^n$  is BN-regular, then  $\text{BND}(X)$  is equal to the number of bottlenecks of  $X$  counted with multiplicity.*

*Remark 2.9.* Recalling the notation from above,  $\mathcal{O}(1)$  denotes the dual of the tautological line bundle on the conormal variety  $\mathcal{C}_X$ ,  $\pi : \mathcal{C}_X \rightarrow X$  is the projection and  $\xi = c_1(\mathcal{O}(1))$ . The bottleneck degree depends on the Chern classes of  $\mathcal{C}_X$  and below we shall relate these to the Chern classes of  $X$ , the hyperplane class and  $\xi$ . By [31, Example 3.2.11] we have that  $c(T_{\mathcal{C}_X}) = c(\pi^* T_X) c(\pi^* N_{X/\mathbb{P}^n}^\vee \otimes \mathcal{O}(1))$ . Since the rank of  $N_{X/\mathbb{P}^n}^\vee$  is  $n - m$  we have by [31, Remark 3.2.3] that

$$\begin{aligned} c(\pi^* N_{X/\mathbb{P}^n}^\vee \otimes \mathcal{O}(1)) &= \sum_{i=0}^{n-m} c_i(\pi^* N_{X/\mathbb{P}^n}^\vee) (1 + \xi)^{n-m-i} \\ &= \sum_{i=0}^{n-m} (-1)^i \pi^* c_i(N_{X/\mathbb{P}^n}) (1 + \xi)^{n-m-i}. \end{aligned}$$

Note also that  $c_i(N_{X/\mathbb{P}^n}) = 0$  for  $i > m = \dim(X)$ . Moreover, the normal bundle  $N_{X/\mathbb{P}^n}$  is related to the tangent bundles  $T_X$  and  $T_{\mathbb{P}^n}$  by the exact sequence

$$0 \rightarrow T_X \rightarrow i^* T_{\mathbb{P}^n} \rightarrow N_{X/\mathbb{P}^n} \rightarrow 0,$$

where  $i : X \rightarrow \mathbb{P}^n$  is the inclusion. It follows that  $c(N_{X/\mathbb{P}^n}) = c(i^*T_{\mathbb{P}^n})c(T_X)^{-1}$ . Also,  $c(T_{\mathbb{P}^n}) = (1 + H)^{n+1}$  where  $H \in A_{n-1}(\mathbb{P}^n)$  is the hyperplane class.

For  $n-1 \geq a \geq b \geq 0$ , define the Schubert class  $\sigma_{a,b} \in A_*(G)$  as the class of the locus  $\Sigma_{a,b} = \{l \in G : l \cap A \neq \emptyset, l \subset B\}$  where  $A \subset B \subseteq \mathbb{P}^n$  is a general flag of linear spaces with  $\text{codim}(A) = a+1$  and  $\text{codim}(B) = b$ . In the case  $b=0$  we use the notation  $\sigma_{a,0} = \sigma_a$ . See [27] for basic properties of Schubert classes. In particular we will make use of the relations  $\sigma_1^2 = \sigma_{1,1} + \sigma_2$  if  $n \geq 3$  and  $\sigma_{a+c,b+c} = \sigma_{c,c}\sigma_{a,b}$  for  $n-1 \geq a \geq b \geq 0$  and  $0 \leq a+c \leq n-1$ . Also,  $\sigma_{n-1-i,i} \cdot \sigma_{n-1-j,j} = 0$  for  $0 \leq i, j \leq \lfloor \frac{n-1}{2} \rfloor$  if  $i \neq j$  and  $\sigma_{n-1-i,i}^2$  is the class of a point. In Algorithm 1 below we will need to express the total Chern class  $c(T_G)$  of the Grassmannian as a polynomial in Schubert classes. To do this we apply the routine `chern` from the `Macaulay2` package `Schubert2` [34].

We will recall the definition of the polar classes  $p_0, \dots, p_m \in A_*(X)$  of  $X$ . For a general linear space  $V \subseteq \mathbb{P}^n$  of dimension  $n-m-2+j$  we have that  $p_j$  is the class represented by the polar locus

$$P_j(X, V) = \{x \in X : \dim(\mathbb{T}_x X \cap V) \geq j-1\}.$$

If  $X$  has codimension 1 and  $j=0$ , then  $V$  is the empty set using the convention  $\dim(\emptyset) = -1$ . By [31, Example 14.4.15],  $P_j(X, V)$  is either empty or of pure codimension  $j$  and

$$(8) \quad p_j = \sum_{i=0}^j (-1)^i \binom{m-i+1}{j-i} h^{j-i} c_i(T_X),$$

where  $h \in A_{n-1}(X)$  is the hyperplane class. Moreover, the polar loci  $P_j(X, V)$  are reduced, see [48]. Inverting the relationship between polar classes and Chern classes we get

$$(9) \quad c_j(T_X) = \sum_{i=0}^j (-1)^i \binom{m-i+1}{j-i} h^{j-i} p_i.$$

We will examine an alternative interpretation of polar classes via the conormal variety  $\mathcal{C}_X$ . This will help us to determine the class of  $\mathcal{C}_X$  in  $A_*(\mathbb{P}^n \times \mathbb{P}^n)$ . Recall that the polar loci  $P_j(X, V)$  are either empty or of codimension  $j$ . It follows that for a generic point  $x \in P_j(X, V)$ ,  $\mathbb{T}_x X$  intersects  $V$  in exactly dimension  $j-1$ , that is  $\dim(\mathbb{T}_x X \cap V) = j-1$ . Let  $0 \leq i \leq m$  and let  $\hat{V}, W \subseteq \mathbb{P}^n$  be general linear spaces with  $\dim(\hat{V}) = i+1$  and  $\dim(W) = n-i$ . Recall the fixed isomorphism  $L : \mathbb{P}^n \rightarrow (\mathbb{P}^n)^*$  and let  $V \subset \mathbb{P}^n$  be the intersection of all hyperplanes in  $L(\hat{V})$ . Note that  $\dim(V) = n-2-i$ . Now consider the intersection  $J = \mathcal{C}_X \cap (W \times \hat{V}) \subseteq \mathbb{P}^n \times \mathbb{P}^n$ . Then  $J$  is finite and we have the projection map  $\pi_J : J \rightarrow P_{m-i}(X, V) \cap W$ . Now,  $\pi_J$  is bijective onto  $P_{m-i}(X, V) \cap W$  because given  $x \in P_{m-i}(X, V) \cap W$ ,  $\dim(\mathbb{T}_x X \cap V) = m-i-1$  and therefore the span of  $\mathbb{T}_x X$  and  $V$  is the unique hyperplane containing  $\mathbb{T}_x X$  and  $V$ . Let  $\alpha, \beta \in A_{2n-1}(\mathbb{P}^n \times \mathbb{P}^n)$  be the pullbacks of the hyperplane class of  $\mathbb{P}^n$  under the two projections and consider  $[\mathcal{C}_X]$  as an element of  $A_*(\mathbb{P}^n \times \mathbb{P}^n)$ . Then  $[W \times \hat{V}] = \alpha^i \beta^{n-1-i}$  and  $\deg([\mathcal{C}_X] \cdot \alpha^i \beta^{n-1-i}) = \deg(J) = \deg(p_{m-i})$ . Note that  $[\mathcal{C}_X] \cdot \alpha^i = 0$  if  $i > m$  since  $\alpha$  is the pullback of a divisor on  $\mathbb{P}^n$ .

**Theorem 2.10.** *Let  $X \subset \mathbb{P}^n$  be a smooth  $m$ -dimensional variety in general position. Let  $h = \pi^*(h_X) \in A_*(\mathcal{C}_X)$  where  $h_X \in A_*(X)$  is the hyperplane class and  $\pi : \mathcal{C}_X \rightarrow X$  is the projection. We use  $\mathcal{O}(1)$  to denote the dual of the tautological line bundle on  $\mathcal{C}_X$  and  $\xi$  to denote its first Chern class. Also  $\alpha, \beta \in A_{2n-1}(\mathbb{P}^n \times \mathbb{P}^n)$  denote the pullbacks of the hyperplane class of  $\mathbb{P}^n$  under the two projections. Let  $k = \min\{\lfloor \frac{n-1}{2} \rfloor, m\}$  and for  $i = 0, \dots, k$ , put  $\varepsilon_i = \sum_{j=r_i}^{m-i} \deg(p_j)$  where  $r_i = \max\{0, m-n+1+i\}$ . Then the following holds:*

$$(10) \quad [\mathcal{C}_X] = \sum_{i=0}^m \deg(p_{m-i}) \alpha^{n-i} \beta^{1+i},$$

$$(11) \quad f^*(\sigma_{a,b}) = \sum_{i=0}^{a-b} h^{b+i} (\xi - h)^{a-i},$$

$$(12) \quad f_*[\mathcal{C}_X] = \sum_{i=0}^k \varepsilon_i \sigma_{n-1-i,i},$$

$$(13) \quad \deg(f^* f_* [\mathcal{C}_X]) = \sum_{i=0}^k \varepsilon_i^2.$$

Hence

$$\text{BND}(X) = \sum_{i=0}^k \varepsilon_i^2 - \deg(B_{m,n}),$$

for some polynomial  $B_{m,n}$  in the polar classes and the hyperplane class of  $X$ .

*Proof.* To show (10), note that  $\text{codim}(\mathcal{C}_X) = 2n - (n-1) = n+1$  and write  $[\mathcal{C}_X] = \sum_{i=0}^{n-1} d_i \alpha^{n-i} \beta^{1+i}$  for some  $d_i \in \mathbb{Z}$ . Let  $0 \leq i \leq n-1$ . Because  $d_i = \deg([\mathcal{C}_X] \cdot \alpha^i \beta^{n-1-i})$ , it follows that:

$$d_i = \deg(p_{m-i}) \text{ if } 0 \leq i \leq m \text{ and } d_i = 0 \text{ if } i > m.$$

Let  $bl : \text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \rightarrow \mathbb{P}^n \times \mathbb{P}^n$  be the blow-up of  $\mathbb{P}^n \times \mathbb{P}^n$  along the diagonal  $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$  and let  $E = bl^{-1}(\Delta)$ , the exceptional divisor. The map  $\mathbb{P}^n \times \mathbb{P}^n \setminus \Delta \rightarrow \text{Gr}(2, n+1)$ , which sends a pair of points  $(p, q)$  to the line spanned by  $p$  and  $q$ , extends to a map  $\gamma : \text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \rightarrow \text{Gr}(2, n+1)$ , see [39]. The theorem in Appendix B paragraph 3 of [39], with  $X = \mathbb{P}^N$  in the notation used there, states that

$$\gamma^*(\sigma_a) = \sum_{i=0}^a bl^* \alpha^i bl^* \beta^{a-i} + \sum_{i=0}^{a-1} (-1)^{i+1} \binom{a+1}{i+2} bl^* \alpha^{a-1-i} [E]^{i+1}.$$

Consider  $\mathcal{C}_X$  as a subvariety of  $\text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n)$  and let  $i : \mathcal{C}_X \rightarrow \text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n)$  be the embedding. Then  $i^* bl^* \alpha = h$  and by [31, Example 3.2.21],  $\xi - h = i^* bl^* \beta$ . Moreover, since  $X$  is in general position,  $i^*[E] = 0$ . Using  $f = \gamma \circ i$ , we get that  $f^*(\sigma_a) = i^* \gamma^*(\sigma_a) = \sum_{i=0}^a h^i (\xi - h)^{a-i}$ . In particular,  $f^*(\sigma_1) = \xi$ , which proves (11) in the case  $n = 2$ . If  $n \geq 3$ , we have by above that  $f^*(\sigma_2) = \xi^2 - h\xi + h^2$ . Moreover  $\sigma_{1,1} = \sigma_1^2 - \sigma_2$ , and hence  $f^*(\sigma_{1,1}) = h(\xi - h)$ . Since  $\sigma_{b,b} = \sigma_{1,1}^b$  we get  $f^*(\sigma_{b,b}) = h^b (\xi - h)^b$ . Finally, using  $\sigma_{a,b} = \sigma_{b,b} \sigma_{a-b}$  we get that  $f^*(\sigma_{a,b}) = h^b (\xi - h)^b \sum_{i=0}^{a-b} h^i (\xi - h)^{a-b-i}$ , which gives (11).

For (12), note first that

$$(14) \quad \gamma^*(\sigma_{a,b}) = \gamma^*(\sigma_{b,b}) \gamma^*(\sigma_{a-b}) = \sum_{i=0}^{a-b} bl^* \alpha^{b+i} bl^* \beta^{a-i} + R,$$

where  $R = [E] \cdot \delta$  for some  $\delta \in A_*(\text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n))$ . Also,  $f_*[\mathcal{C}_X] = \sum_{i=0}^s e_i \sigma_{n-1-i,i}$  where  $e_i = \deg(f_*[\mathcal{C}_X] \cdot \sigma_{n-1-i,i})$  and  $s = \lfloor \frac{n-1}{2} \rfloor$ . Since  $\gamma$  restricts to  $f$  on  $\mathcal{C}_X$ ,  $f_*[\mathcal{C}_X] = \gamma_*[\mathcal{C}_X]$  and  $e_i = \deg(\gamma_*[\mathcal{C}_X] \cdot \sigma_{n-1-i,i}) = \deg([\mathcal{C}_X] \cdot \gamma^* \sigma_{n-1-i,i})$  by the projection formula. Here  $[\mathcal{C}_X]$  denotes the class of  $\mathcal{C}_X$  on  $\text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n)$  and  $[\mathcal{C}_X] \cdot R = 0$  since  $X$  is in general position. Moreover, by (10) we have that  $[\mathcal{C}_X] = bl^*(\sum_{l=0}^m \deg(p_{m-l}) \alpha^{n-l} \beta^{l+1})$ . Using (14) we get

$$[\mathcal{C}_X] \cdot \gamma^* \sigma_{n-1-i,i} = bl^* \left( \sum_{l=0}^m \deg(p_{m-l}) \alpha^{n-l} \beta^{l+1} \right) \cdot bl^* \left( \sum_{j=0}^{n-1-2i} \alpha^{i+j} \beta^{n-1-i-j} \right).$$

It follows that  $[\mathcal{C}_X] \cdot \gamma^* \sigma_{n-1-i,i} = 0$  if  $i > m$ . For  $i \leq m$ , we get

$$[\mathcal{C}_X] \cdot \gamma^* \sigma_{n-1-i,i} = bl^*(\alpha^n \beta^n) \sum_{j=0}^t \deg(p_{m-(i+j)}),$$

where  $t = \min\{m-i, n-1-2i\}$ . Hence  $e_i = 0$  for  $i > m$  and  $e_i = \varepsilon_i$  otherwise, which gives (12).

To show (13), let  $0 \leq i \leq k$  and note that by the projection formula

$$\varepsilon_i = \deg(f_*[\mathcal{C}_X] \cdot \sigma_{n-1-i,i}) = \deg([\mathcal{C}_X] \cdot f^* \sigma_{n-1-i,i}) = \deg(f^* \sigma_{n-1-i,i}).$$

Hence applying  $f^*$  to (12) gives (13).

Since the intersection ring of  $\text{Gr}(2, n+1)$  is generated by  $\sigma_{a,b}$  as a group, we may express  $c(f^* T_G)$  as a polynomial in  $\xi$  and  $h$  by (11). Moreover,  $c(T_{\mathcal{C}_X})$  is a polynomial in pullbacks of polar classes,  $h$  and  $\xi$  by (9) and Remark 2.9. It follows from Remark 2.1 that  $(c(f^* T_G) c(T_{\mathcal{C}_X})^{-1})_{n-1} = \xi^{n-m-1} \pi^* B_{m,n}$  for some

polynomial  $B_{m,n}$  in polar classes and the hyperplane class of  $X$ . Also by Remark 2.1,  $\deg(\xi^{n-m-1}\pi^*B_{m,n}) = \deg(B_{m,n})$ .  $\square$

Note that the polynomials  $B_{m,n}$  in Theorem 2.10 only depend on  $n$  and  $m$ . Combining Theorem 2.10, Remark 2.9 and Remark 2.1 gives an algorithm to compute polynomials  $B_{m,n}$  as in Theorem 2.10. We will now give a high level description of this algorithm. It has been implemented in Macaulay2 [33] and is available at [22].

We will use the notation in Theorem 2.10. In addition, we use  $p_1, \dots, p_m$  to denote the polar classes of  $X$  and  $c_1, \dots, c_m$  to denote the Chern classes of  $X$ . Also  $i: X \rightarrow \mathbb{P}^n$  denotes the inclusion and  $h_X$  is the hyperplane class on  $X$ . The algorithm makes use of the routines `pseudoRemainder` from [33] and `chern` from [34].

The input to the algorithm are integers  $0 < m < n$  and the output is a polynomial  $B_{m,n}$  in  $p_1, \dots, p_m, h_X$  such that  $(c(f^*T_G)c(T_{\mathcal{E}_X})^{-1})_{n-1} = \xi^{n-m-1}\pi^*B_{m,n}$ .

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**Algorithm 1** Algorithm to compute polynomial  $B_{m,n}$  in  $p_1, \dots, p_m, h_X$  as in Theorem 2.10

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**Input:** Integers  $0 < m < n$ .

**Output:** Polynomial  $B_{m,n}$ .

Invert  $c(T_X): c(T_X)^{-1} = 1 - \delta + \delta^2 + \dots + (-1)^m \delta^m$  where  $\delta = c(T_X) - 1$ .

Let  $c(i^*T_{\mathbb{P}^n}) = (1 + h_X)^{n+1}$ .

Compute  $c(N_{X/\mathbb{P}^n}) = c(i^*T_{\mathbb{P}^n})c(T_X)^{-1}$ .

Compute  $c(\pi^*N_{X/\mathbb{P}^n}^\vee \otimes \mathcal{O}(1)) = \sum_{j=0}^{n-m} (-1)^j \pi^*c_j(N_{X/\mathbb{P}^n})(1 + \xi)^{n-m-j}$ .

Compute  $c(T_{\mathcal{E}_X}) = c(\pi^*T_X)c(\pi^*N_{X/\mathbb{P}^n}^\vee \otimes \mathcal{O}(1))$ .

Invert  $c(T_{\mathcal{E}_X}): c(T_{\mathcal{E}_X})^{-1} = 1 - \delta + \delta^2 + \dots + (-1)^{n-1} \delta^{n-1}$  where  $\delta = c(T_{\mathcal{E}_X}) - 1$ .

Apply `chern` to express  $c(T_G)$  as a polynomial in Schubert classes  $\sigma_{a,b}$ .

Apply the substitution (11) to express  $c(f^*T_G) = f^*c(T_G)$  as a polynomial in  $\xi$  and  $\pi^*h_X$ .

Compute  $(c(f^*T_G)c(T_{\mathcal{E}_X})^{-1})_{n-1}$ .

Let  $R = \xi^{n-m} - c_1(\pi^*N_{X/\mathbb{P}^n})\xi^{n-m-1} + \dots + (-1)^{n-m}c_{n-m}(\pi^*N_{X/\mathbb{P}^n})$ .

Let  $P$  be the output of `pseudoRemainder` applied to  $(c(f^*T_G)c(T_{\mathcal{E}_X})^{-1})_{n-1}$  and  $R$ .

Let  $\hat{B}_{m,n}$  be  $P$  divided by  $\xi^{n-m-1}$  and with  $\pi^*c_1, \dots, \pi^*c_m, \pi^*h_X$  replaced by  $c_1, \dots, c_m, h_X$ .

Replace  $c_1, \dots, c_m$  by  $p_1, \dots, p_m$  using (9) on  $\hat{B}_{m,n}$  to acquire  $B_{m,n}$ .

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**Corollary 2.11.** *Let  $X \subset \mathbb{P}^n$  be a smooth variety in general position. Let  $d = \deg(X) = \deg(p_0)$ ,  $\varepsilon_i = \sum_{j=0}^{m-i} \deg(p_j)$  with  $m = \dim(X)$  and  $h \in A_{m-1}(X)$  the hyperplane class. The following holds:*

(1) *If  $X$  is a curve in  $\mathbb{P}^2$  then*

$$\text{BND}(X) = d^4 - 4d^2 + 3d.$$

(2) *If  $X$  is a curve in  $\mathbb{P}^3$  then*

$$\text{BND}(X) = \varepsilon_0^2 + d^2 - 5 \deg(p_1) - 2d,$$

*where  $\varepsilon_0 = d + \deg(p_1)$  is the Euclidean distance degree of  $X$ .*

(3) *If  $X$  is a surface in  $\mathbb{P}^5$  then*

$$\text{BND}(X) = \varepsilon_0^2 + \varepsilon_1^2 + d^2 - \deg(3h^2 + 6hp_1 + 12p_1^2 + p_2).$$

(4) *If  $X$  is a threefold in  $\mathbb{P}^7$  then*

$$\text{BND}(X) = \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 + d^2 - \deg(4h^3 + 11h^2p_1 + 4hp_1^2 + 24p_1^3 + 2hp_2 - 12p_1p_2 + 17p_3).$$

*Proof.* The formulas are acquired by applying Algorithm 1, which has been implemented in Macaulay2 [33] and is available at [22].

For illustrative purposes we will carry out the computation for curves in  $\mathbb{P}^3$ . By the double point formula

$$\mathbb{D}(f) = f^*f_*[\mathcal{E}_X] - (c(f^*T_G)c(T_{\mathcal{E}_X})^{-1})_2,$$

and using Theorem 2.10 we get that  $\deg(f^*f_*[\mathcal{C}_X]) = \varepsilon_0^2 + \varepsilon_1^2 = \varepsilon_0^2 + d^2$ . Moreover,  $c(f^*T_G) = f^*c(T_G) = 1 + 4f^*\sigma_1 + 7(f^*\sigma_2 + f^*\sigma_{1,1})$  and  $f^*\sigma_1 = \xi$ ,  $f^*\sigma_2 = \xi^2 - \xi h + h^2 = \xi^2 - \xi h$  and  $f^*\sigma_{1,1} = h\xi$  by Theorem 2.10.

Let  $c_1$  denote the first Chern class of  $X$ . To compute  $c(T_{\mathcal{C}_X})$  we follow the steps of Remark 2.9. First of all  $\pi^*c(N_{X/\mathbb{P}^n}^\vee) = 1 - 4h + \pi^*c_1$ . Hence we get that  $c(\pi^*N_{X/\mathbb{P}^n}^\vee \otimes \mathcal{O}(1)) = (1 + \xi)^2 + (1 + \xi)(-4h + \pi^*c_1)$ . Moreover, by (6),  $\xi^2 = -\pi^*c_1(N_{X/\mathbb{P}^n}^\vee)\xi = (4h - \pi^*c_1)\xi$  and hence  $c(\pi^*N_{X/\mathbb{P}^n}^\vee \otimes \mathcal{O}(1)) = 1 + 2\xi + (-4h + \pi^*c_1)$ . This means that

$$\begin{aligned} c(T_{\mathcal{C}_X}) &= c(\pi^*N_{X/\mathbb{P}^n}^\vee \otimes \mathcal{O}(1))c(\pi^*T_X) = (1 + 2\xi - 4h + \pi^*c_1)(1 + \pi^*c_1) \\ &= 1 + 2\xi - 4h + 2\pi^*c_1 + 2\xi\pi^*c_1. \end{aligned}$$

Hence  $c(T_{\mathcal{C}_X})^{-1} = 1 - 2\xi + 4h - 2\pi^*c_1 + 2\xi\pi^*c_1$ . It follows that:

$$(c(f^*T_G)c(T_{\mathcal{C}_X})^{-1})_2 = ((1 + 4f^*\sigma_1 + 7(f^*\sigma_2 + f^*\sigma_{1,1}))(1 - 2\xi + 4h - 2\pi^*c_1 + 2\xi\pi^*c_1))_2.$$

Multiplying out and using the expressions for  $f^*\sigma_2$  and  $f^*\sigma_{1,1}$  above we get

$$\begin{aligned} (c(f^*T_G)c(T_{\mathcal{C}_X})^{-1})_2 &= 7f^*\sigma_2 + 7f^*\sigma_{1,1} + 4f^*\sigma_1(-2\xi + 4h - 2\pi^*c_1) + 2\xi\pi^*c_1 \\ &= 7(\xi^2 - \xi h) + 7h\xi + 4\xi(-2\xi + 4h - 2\pi^*c_1) + 2\xi\pi^*c_1. \end{aligned}$$

Simplifying the last expression results in the following formulas:

$$\begin{aligned} (c(f^*T_G)c(T_{\mathcal{C}_X})^{-1})_2 &= -\xi^2 + 16\xi h - 6\xi\pi^*c_1 \\ &= -(4h - \pi^*c_1)\xi + 16\xi h - 6\xi\pi^*c_1 \\ &= \xi(12h - 5\pi^*c_1). \end{aligned}$$

Finally, using Remark 2.1, we get  $\deg((c(f^*T_G)c(T_{\mathcal{C}_X})^{-1})_2) = 12d - 5\deg(c_1) = 2d + 5\deg(p_1)$ , since  $\deg(p_1) = 2d - \deg(c_1)$ . This shows the claim about  $\text{BND}(X)$  for a smooth curve  $X \subset \mathbb{P}^3$  in general position.

In the case of a general plane curve  $X \subset \mathbb{P}^2$  we have that  $\deg(c_1) = 2 - 2g$  where  $g = (d-1)(d-2)/2$  is the genus of  $X$ . It follows that  $\deg(p_1) = d^2 - d$  and  $\varepsilon_0 = d^2$  and  $\text{BND}(X) = d^4 - 4d^2 + 3d$ .  $\square$

*Remark 2.12.* The formulas in Corollary 2.11 are given for specific ambient dimensions  $n$ . For example, Corollary 2.11 (2) is for curves in  $\mathbb{P}^3$  and one may ask if the same formula is valid for curves in  $\mathbb{P}^4$ . For the formulas given in Corollary 2.11 we have checked that they are valid for any ambient dimension  $n \leq 30$  (excluding the case  $X = \mathbb{P}^n$ ). This was done using the Macaulay2 implementation [22].

Consider now the general case of a smooth  $m$ -dimensional variety  $X \subset \mathbb{P}^n$ . Combining Algorithm 1 and Theorem 2.10 we get an algorithm that, for any given  $m$  and  $n$ , computes the bottleneck degree of a smooth  $m$ -dimensional variety  $X \subset \mathbb{P}^n$  in general position. The result is a formula that expresses the bottleneck degree in terms of polar classes of  $X$ . Now, if we let  $n = 2m + 1$  we get a formula for each  $m$ . It is our belief that through projection arguments one can show that this formula is in fact valid in any ambient dimension  $n > m$ . Thus we conjecture that the formula in terms of polar classes only depends on the dimension  $m$ .

### 3. THE AFFINE CASE

In this section we define bottlenecks for affine varieties and show how they may be counted using the bottleneck degrees of projective varieties.

Let  $X \subset \mathbb{C}^n$  be a smooth affine variety of dimension  $m$ . Consider coordinates  $x_0, \dots, x_{n-1}$  given by the standard basis on  $\mathbb{C}^n$  and the usual embedding  $\mathbb{C}^n \subset \mathbb{P}^n$  with coordinates  $x_0, \dots, x_n$  on  $\mathbb{P}^n$ . Let  $H_\infty = \mathbb{P}^n \setminus \mathbb{C}^n$  be the hyperplane at infinity defined by  $x_n = 0$ . Also consider the closure  $\bar{X} \subset \mathbb{P}^n$  and the intersection  $\bar{X}_\infty = \bar{X} \cap H_\infty$ . We consider  $\bar{X}_\infty$  a subvariety of  $\mathbb{P}^{n-1} \cong H_\infty$ .

**Definition 3.1.** A smooth affine variety  $X \subset \mathbb{C}^n$  is in *general position* if  $\bar{X}_\infty$  is smooth and both  $\bar{X}$  and  $\bar{X}_\infty$  are in general position.

Assume that  $X$  is in general position. Let  $v : \mathbb{P}^n \setminus \{o\} \rightarrow H_\infty$  be the projection from the point  $o = (0, \dots, 0, 1)$ . If  $(p, q) \in \mathcal{C}_{\bar{X}}$  then  $q \neq o$  since  $\bar{X}_\infty$  is smooth. Also,  $p \neq v(q)$  since  $\bar{X}_\infty$  is in general position. Therefore we can define a map

$$g : \mathcal{C}_{\bar{X}} \rightarrow \text{Gr}(2, n+1) : (p, q) \mapsto \langle p, v(q) \rangle,$$

mapping a pair  $(p, q) \in \mathcal{C}_{\bar{X}}$  to the line spanned by  $p$  and  $v(q)$ . For the remainder of this section we will use  $g$  to denote this map associated to a variety  $X$ . In the following lemma we show that for  $x \in X$  the fiber  $F_x = \{(x', q) \in \mathcal{C}_{\bar{X}} : x' = x\}$  together with the map  $g$  parameterize lines in the Euclidean normal space  $N_x X$  passing through  $x$ . Recall that for  $x \in X \subset \mathbb{C}^n$ ,  $(T_x X)_0$  denotes the embedded tangent space translated to the origin and the Euclidean normal space at  $x$  is given by  $N_x X = \{z \in \mathbb{C}^n : (z - x) \in (T_x X)_0^\perp\}$ .

**Lemma 3.2.** *Let  $X \subset \mathbb{C}^n$  be a smooth variety in general position. As above we consider  $\mathbb{C}^n \subset \mathbb{P}^n$  and  $X \subset \bar{X} \subset \mathbb{P}^n$ . Let  $x \in X$  and  $F_x = \{(x', q) \in \mathcal{C}_{\bar{X}} : x' = x\}$ . Then the map  $u \mapsto g(u) \cap \mathbb{C}^n$  on  $F_x$  defines a one-to-one correspondence between  $F_x$  and the set of lines in  $N_x X$  passing through  $x$ .*

*Proof.* Let  $(x, q) \in \mathcal{C}_{\bar{X}}$  with  $q = (q_1, \dots, q_{n+1})$  and  $x = (x_1, \dots, x_n, 1)$  where  $(x_1, \dots, x_n) \in X \subset \mathbb{C}^n$ . The line  $\langle x, v(q) \rangle \cap \mathbb{C}^n$  expressed in coordinates on  $\mathbb{C}^n$  is given by  $\{(x_1, \dots, x_n) + a(q_1, \dots, q_n) : a \in \mathbb{C}\}$ . To show that this line is normal to  $X$  at  $x$  we need to show that  $(q_1, \dots, q_n) \in (T_x X)_0^\perp \subset \mathbb{C}^n$ . Let  $(v_1, \dots, v_n) \in (T_x X)_0$ . Then  $(x_1 + v_1, \dots, x_n + v_n) \in T_x X$  where  $T_x X \subset \mathbb{C}^n$  is the embedded tangent space of  $X$  at  $x$ . This means that  $(x_1 + v_1, \dots, x_n + v_n, 1) \in \mathbb{T}_x \bar{X} \subset \mathbb{P}^n$  and hence  $\sum_{i=1}^n (x_i + v_i)q_i + q_{n+1} = 0$ . Since  $(x_1, \dots, x_n, 1) \in \mathbb{T}_x \bar{X}$  we have that  $\sum_{i=1}^n x_i q_i + q_{n+1} = 0$ . It follows that  $\sum_{i=1}^n v_i q_i = 0$  and we have shown  $(q_1, \dots, q_n) \in (T_x X)_0^\perp$ .

Now let  $x \in X \subset \bar{X}$  with  $x = (x_1, \dots, x_n, 1)$  and consider a line in  $N_x X$  through  $x$ . In coordinates on  $\mathbb{C}^n$  the line is given by  $\{(x_1, \dots, x_n) + a(q_1, \dots, q_n) : a \in \mathbb{C}\}$  for some  $0 \neq (q_1, \dots, q_n) \in (T_x X)_0^\perp \subseteq \mathbb{C}^n$ . Note that  $(q_1, \dots, q_n)$  is unique up to scaling. We need to show that there is a unique  $q_{n+1} \in \mathbb{C}$  such that  $(x, q) \in \mathcal{C}_{\bar{X}}$  where  $q = (q_1, \dots, q_n, q_{n+1})$ . Since  $x \in \mathbb{T}_x \bar{X}$ , a necessary condition on  $q_{n+1} \in \mathbb{C}$  is that  $\sum_{i=1}^n x_i q_i + q_{n+1} = 0$ . Accordingly we let  $q_{n+1} = -\sum_{i=1}^n x_i q_i$ . It remains to show  $(x, q) \in \mathcal{C}_{\bar{X}}$ . Since  $\{(v_1, \dots, v_n, v_{n+1}) \in \mathbb{T}_x \bar{X} : v_{n+1} \neq 0\} \subset \mathbb{T}_x \bar{X}$  is a dense subset, it is enough to show that for all  $(v_1, \dots, v_n, 1) \in \mathbb{T}_x \bar{X}$  we have that  $\sum_{i=1}^n v_i q_i + q_{n+1} = 0$ . Note that  $(v_1, \dots, v_n) \in T_x X \subset \mathbb{C}^n$  and  $(v_1 - x_1, \dots, v_n - x_n) \in (T_x X)_0$ . Hence  $\sum_{i=1}^n (v_i - x_i)q_i = 0$ . It follows that  $\sum_{i=1}^n v_i q_i + q_{n+1} = \sum_{i=1}^n x_i q_i + q_{n+1} = 0$ .  $\square$

Consider the projection  $p : \mathcal{C}_{\bar{X}} \rightarrow \bar{X}$ . A *bottleneck* of the affine variety  $X$  is a pair of distinct points  $x, y \in X \subset \bar{X}$  such that there exists  $u, v \in \mathcal{C}_{\bar{X}}$  with  $p(u) = x$ ,  $p(v) = y$  and  $g(u) = g(v)$ . We will now show that this definition of bottlenecks is equivalent to the definition given in Section 1 in terms of Euclidean normal spaces.

**Lemma 3.3.** *Let  $X \subset \mathbb{C}^n$  be a smooth variety in general position. A pair of distinct points  $(x, y) \in X \times X$  is a bottleneck if and only if the line in  $\mathbb{C}^n$  joining  $x$  and  $y$  is contained in  $N_x X \cap N_y X$ .*

*Proof.* If  $(x, y) \in X \times X$  is a bottleneck, then there are  $u, v \in \mathcal{C}_{\bar{X}}$  with  $g(u) = g(v)$ ,  $p(u) = x$  and  $p(v) = y$ . The line  $g(u) \cap \mathbb{C}^n$  in  $\mathbb{C}^n$  thus contains  $x$  and  $y$  and it is contained in  $N_x X \cap N_y X$  by Lemma 3.2. For the converse, let  $x, y \in X$  be distinct such that the line  $l \subset \mathbb{C}^n$  joining  $x$  and  $y$  is contained in  $N_x X \cap N_y X$ . By Lemma 3.2 there are  $u, v \in \mathcal{C}_{\bar{X}}$  with  $l = g(u) \cap \mathbb{C}^n$ ,  $l = g(v) \cap \mathbb{C}^n$ ,  $p(u) = x$  and  $p(v) = y$ . It follows that  $g(u) = g(v)$  and  $(x, y)$  is a bottleneck.  $\square$

The map  $g$  can have double points that do not correspond to actual bottlenecks of  $X$  since we require that  $x, y \in X$  lie in the affine part. Note however that if  $u, v \in \mathcal{C}_{\bar{X}}$  with  $g(u) = g(v)$  and  $p(u) \in \bar{X}_\infty$ , then  $p(v) \in \bar{X}_\infty$  as well. Therefore the extraneous double point pairs of  $g$  are in one-to-one correspondence with double point pairs of the map

$$g_\infty : \mathcal{C}_{\bar{X}_\infty} \rightarrow \text{Gr}(2, n) : (p, q) \mapsto \langle p, q \rangle.$$

Here  $\mathcal{C}_{\bar{X}_\infty}$  is defined with respect to the embedding  $\bar{X}_\infty \subset \mathbb{P}^{n-1}$ . This leads us to consider the double point classes  $\mathbb{D}(g)$ ,  $\mathbb{D}(g_\infty)$  of  $g$  and  $g_\infty$  and define the bottleneck degree of  $X$  as the difference of the degrees of these classes.

**Definition 3.4.** Let  $X \subset \mathbb{C}^n$  be a smooth variety in general position. The *bottleneck degree* of  $X$  is  $\deg(\mathbb{D}(g)) - \deg(\mathbb{D}(g_\infty))$  and is denoted by  $\text{BND}(X)$ .

**Example 3.5.** Consider a general plane curve  $X \subset \mathbb{C}^2$  of degree  $d$  defined by a polynomial  $F \in \mathbb{C}[x, y]$ . Then  $\bar{X} \subset \mathbb{P}^2$  is defined by the homogenization  $\bar{F} \in \mathbb{C}[x, y, z]$  of  $F$  with respect to  $z$ . We may assume that  $\bar{X}$  is smooth. The map  $g : \bar{X} \rightarrow (\mathbb{P}^2)^*$  is given by  $(x, y, z) \mapsto (-z\bar{F}_y, z\bar{F}_x, x\bar{F}_y - y\bar{F}_x)$ . It maps a point  $p \in X$  to the closure of the normal line  $N_p X \subset \mathbb{C}^2$  in  $\mathbb{P}^2$ . The bottlenecks of  $X$  are the pairs  $(p, q) \in X \times X$  with  $p \neq q$  and  $N_p X = N_q X$ . We shall now consider the other double point pairs of  $g$ , that is distinct points  $p, q \in \bar{X}$  such that  $g(p) = g(q)$  and  $p$  or  $q$  lies on the line at infinity  $H_\infty$ . The latter corresponds to the point  $(0, 0, 1) \in (\mathbb{P}^2)^*$ . If  $p = (x, y, z) \in H_\infty \cap \bar{X}$ , that is  $z = 0$ , then  $g(p) = (0, 0, 1)$ . Conversely, if  $q \in \bar{X}$  and  $g(q) = (0, 0, 1)$  then  $q \in H_\infty$  since  $q$  is a point on the line  $g(q)$ . The extraneous double points of  $g$  are thus exactly the  $d$  points  $\bar{X}_\infty$ , the intersection of  $\bar{X}$  with the line at infinity. This gives rise to  $d(d-1)$  extraneous double point pairs at infinity.

**Proposition 3.6.** For a smooth affine variety  $X \subset \mathbb{C}^n$  in general position,

$$\text{BND}(X) = \text{BND}(\bar{X}) - \text{BND}(\bar{X}_\infty).$$

*Proof.* By definition  $\text{BND}(\bar{X}_\infty) = \deg(\mathbb{D}(g_\infty))$  and so it remains to prove that  $\text{BND}(\bar{X}) = \deg(\mathbb{D}(g))$ . In other words we need to show that  $\deg(\mathbb{D}(f)) = \deg(\mathbb{D}(g))$  where  $f$  is the map

$$f : \mathcal{C}_{\bar{X}} \rightarrow \text{Gr}(2, n+1) : (p, q) \mapsto \langle p, q \rangle.$$

By the double point formula it is enough to show that  $f^* f_* [\mathcal{C}_{\bar{X}}] = g^* g_* [\mathcal{C}_{\bar{X}}]$  and  $c(f^* T_G) = c(g^* T_G)$  where  $G = \text{Gr}(2, n+1)$ . Since the Schubert classes generate  $A_*(G)$  as a group the equality  $c(f^* T_G) = c(g^* T_G)$  would follow after showing that  $f^* \sigma_{a,b} = g^* \sigma_{a,b}$ . We will do this first. As in the proof of Theorem 2.10, let  $bl : \text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \rightarrow \mathbb{P}^n \times \mathbb{P}^n$  be the blow-up of  $\mathbb{P}^n \times \mathbb{P}^n$  along the diagonal  $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$  and let  $E = bl^{-1}(\Delta)$ . Let  $\alpha, \beta \in A_*(\mathbb{P}^n \times \mathbb{P}^n)$  be the pullbacks of the hyperplane class of  $\mathbb{P}^n$  under the two projections and let  $\gamma$  be as in Theorem 2.10. By (14)

$$\gamma^*(\sigma_{a,b}) = \sum_{i=0}^{a-b} bl^* \alpha^{b+i} bl^* \beta^{a-i} + R,$$

where  $R = [E] \cdot \delta$  for some  $\delta \in A_*(\text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n))$ . Let  $i : \mathcal{C}_{\bar{X}} \rightarrow \text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n)$  be the map induced by the inclusion  $\mathcal{C}_{\bar{X}} \subset \mathbb{P}^n \times \mathbb{P}^n$  and let  $j : \mathcal{C}_{\bar{X}} \rightarrow \text{Bl}_\Delta(\mathbb{P}^n \times \mathbb{P}^n)$  be induced by the map  $\mathcal{C}_{\bar{X}} \rightarrow \mathbb{P}^n \times \mathbb{P}^n : (p, q) \mapsto (p, v(q))$ , where  $v : \mathbb{P}^n \setminus \{o\} \rightarrow H_\infty$  is the linear projection. Note that  $f = \gamma \circ i$  and  $g = \gamma \circ j$ . The map  $bl \circ i$  is the identity on  $\mathcal{C}_{\bar{X}}$  and  $bl \circ j : \mathcal{C}_{\bar{X}} \rightarrow \mathbb{P}^n \times \mathbb{P}^n$  is the map  $(p, q) \mapsto (p, v(q))$ . It follows that  $i^* bl^* \alpha = j^* bl^* \alpha$  and  $i^* bl^* \beta = j^* bl^* \beta$ . Since  $\bar{X}$  and  $\bar{X}_\infty$  are in general position,  $i^* R = j^* R = 0$ , and we conclude that  $f^* \sigma_{a,b} = g^* \sigma_{a,b}$ .

Now write  $f_* [\mathcal{C}_{\bar{X}}] = \sum_i e_i \sigma_{n-1-i, i}$  and  $g_* [\mathcal{C}_{\bar{X}}] = \sum_i e'_i \sigma_{n-1-i, i}$  where  $e_i, e'_i \in \mathbb{Z}$ . Note that  $e_i = \deg(f_* [\mathcal{C}_{\bar{X}}] \cdot \sigma_{n-1-i, i}) = \deg([\mathcal{C}_{\bar{X}}] \cdot f^* \sigma_{n-1-i, i})$  and the same way  $e'_i = \deg([\mathcal{C}_{\bar{X}}] \cdot g^* \sigma_{n-1-i, i})$ . Since  $f^* \sigma_{n-1-i, i} = g^* \sigma_{n-1-i, i}$ , we have that  $e_i = e'_i$  for all  $i$ . It follows that

$$f^* f_* [\mathcal{C}_{\bar{X}}] = \sum_i e_i f^* \sigma_{n-1-i, i} = \sum_i e'_i g^* \sigma_{n-1-i, i} = g^* g_* [\mathcal{C}_{\bar{X}}].$$

□

**Example 3.7.** For a general curve  $X \subset \mathbb{C}^2$  of degree  $d$  we have

$$\text{BND}(X) = d^4 - 5d^2 + 4d.$$

Namely, the bottleneck degree of  $\bar{X}$  is given in Corollary 2.11 and putting this together with Proposition 3.6 and Example 3.5 we get  $\text{BND}(X) = d^4 - 4d^2 + 3d - d(d-1) = d^4 - 5d^2 + 4d$ .

Hence a general line in  $\mathbb{C}^2$  has no bottlenecks, as one might expect. For  $d = 2$  we get that a general conic has 4 bottlenecks, which corresponds to 2 pairs of points with coinciding normal lines. These lines can be real: Consider the case of a general real ellipse and its two principal axes, see Figure 6.

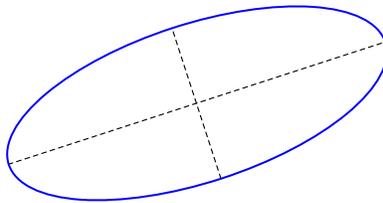


FIGURE 6. Ellipse with principal axes.

*Remark 3.8.* Let  $X$  be a smooth affine variety in general position. As we have seen, the bottleneck degrees of  $\bar{X}$  and  $\bar{X}_\infty$  are functions of the polar numbers of these varieties. If  $i : \bar{X}_\infty \rightarrow \bar{X}$  is the inclusion then the relation between the polar classes of  $\bar{X}$  and those of its hyperplane section  $\bar{X}_\infty$  is  $p_j(\bar{X}_\infty) = i^* p_j(\bar{X})$ . This is straightforward to verify using for example the adjunction formula [31, Example 3.2.12] and the relation between polar classes and Chern classes (8). This means that the polar numbers of  $\bar{X}$  and  $\bar{X}_\infty$  are the same in the sense that  $\deg(p_1(\bar{X}_\infty)^{a_1} \dots p_{m-1}(\bar{X}_\infty)^{a_{m-1}}) = \deg(p_1(\bar{X})^{a_1} \dots p_{m-1}(\bar{X})^{a_{m-1}})$  for any  $a_1, \dots, a_{m-1} \in \mathbb{N}$  such that  $\sum_{j=1}^{m-1} j \cdot a_j \leq m-1$ . As a consequence, Proposition 3.6 may be used to express the bottleneck degree of  $X \subset \mathbb{C}^n$  in terms of the polar numbers of its closure  $\bar{X} \subset \mathbb{P}^n$ .

*Remark 3.9.* Let  $g_1, \dots, g_k \in \mathbb{C}[x_1, \dots, x_n]$  be a system of polynomials of degrees  $d_1, \dots, d_k$  which define a complete intersection  $X \subset \mathbb{C}^n$ . Suppose that the bottlenecks of  $X$  are known. If  $X$  is general enough to have the maximal number of bottlenecks, we may compute the isolated bottlenecks of any other complete intersection  $Y \subset \mathbb{C}^n$  defined by polynomials  $f_1, \dots, f_k \in \mathbb{C}[x_1, \dots, x_n]$  of the same degrees  $d_1, \dots, d_k$ . We propose to do this by a parameter homotopy from  $X$  to  $Y$ . For background on homotopy methods see for example [51]. Suppose that both  $X$  and  $Y$  are smooth. Let  $h_i(x) = (1-t)f_i(x) + \gamma g_i(x)$  where  $\gamma \in \mathbb{C}$  is random and  $t$  is the homotopy parameter. The homotopy paths are tracked from the bottlenecks of  $X$  at  $t = 1$  to the bottlenecks at  $Y$  at  $t = 0$ . Introduce new variables  $y_1, \dots, y_n$  and  $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k$ . The parameter homotopy is then the following square system of equations in  $2(n+k)$  variables:

$$\begin{aligned} h_1(x) &= \dots = h_k(x) = 0, \\ h_1(y) &= \dots = h_k(y) = 0, \\ y - x &= \sum_{i=1}^k \lambda_i \nabla h_i(x), \\ y - x &= \sum_{i=1}^k \mu_i \nabla h_i(y). \end{aligned}$$

For the starting points of the homotopy we need the bottleneck pairs  $(x, y)$  of  $X$ . To find the  $\lambda_1, \dots, \lambda_k$  and  $\mu_1, \dots, \mu_k$  corresponding to a bottleneck pair  $(x, y)$  one would need to solve the linear systems  $y - x = \sum_{i=1}^k \lambda_i \nabla g_i(x)$  and  $y - x = \sum_{i=1}^k \mu_i \nabla g_i(y) = 0$ .

Along similar lines, [28] presents an efficient homotopy to compute bottlenecks of affine varieties.

#### 4. EXAMPLES

**Example 4.1.** Consider the space curve in  $\mathbb{C}^3$  given by the intersection of these two hypersurfaces:

$$\begin{aligned} x^3 - 3xy^2 - z &= 0 \\ x^2 + y^2 + 3z^2 - 1 &= 0. \end{aligned}$$

As computed in Macaulay2, the ideal of the bottleneck variety (with the diagonal removed) associated to this affine curve has dimension 0 and degree 480. The curve is the complete intersection of two surfaces of degrees  $d_1 = 2$  and  $d_2 = 3$ .

Now consider a smooth complete intersection curve  $X \subset \mathbb{C}^3$  cut out by surfaces of degree  $d_1$  and  $d_2$ . Assume that  $X$  is in general position. By Corollary 2.11 the bottleneck degree of  $\bar{X}$  is given by  $\varepsilon_0^2 + d^2 - 5 \deg(p_1) - 2d$ , where  $\varepsilon_0 = d + \deg(p_1)$  and  $d = d_1 d_2$ . Using for example the adjunction formula, [31, Example 3.2.12], one can see that  $c_1(T_X) = (4 - d_1 - d_2)h$ , where  $h \in A_0(X)$  is the hyperplane class. Also, by (8) we have  $p_1 = 2h - c_1(T_X) = (d_1 + d_2 - 2)h$ . Thus  $\deg(p_1) = (d_1 + d_2 - 2)d_1 d_2$ . By Proposition 3.6,

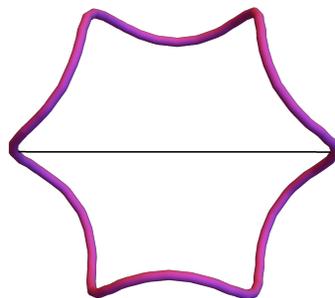


FIGURE 7. A space curve of degree 6 shown with one of its bottleneck lines joining the points  $(0, -1, 0)$  and  $(0, 1, 0)$ .

to obtain the bottleneck degree of the affine variety  $X$  we subtract  $\text{BND}(\bar{X}_\infty)$  from  $\text{BND}(\bar{X})$ . In this case, we have

$$\text{BND}(\bar{X}_\infty) = d_1 d_2 (d_1 d_2 - 1).$$

We obtain the following formula for the bottleneck degree of a smooth complete intersection curve  $X \subset \mathbb{C}^3$  in general position:

$$\text{BND}(X) = d_1^4 d_2^2 + 2d_1^3 d_2^3 + d_1^2 d_2^4 - 2d_1^3 d_2^2 - 2d_1^2 d_2^3 + d_1^2 d_2^2 - 5d_1^2 d_2 - 5d_1 d_2^2 + 9d_1 d_2.$$

Substituting  $d_1 = 2$  and  $d_2 = 3$ , we obtain  $\text{BND}(X) = 480$ , in agreement with the Macaulay2 computation for the sextic curve above.

**Example 4.2.** Let  $X \subset \mathbb{C}^3$  be a general surface of degree  $d$ . Then

$$\text{BND}(X) = d^6 - 2d^5 + 3d^4 - 15d^3 + 26d^2 - 13d.$$

To see this use Proposition 3.6. Apply Corollary 2.11 to get  $\text{BND}(\bar{X})$  and the bottleneck degree of the planar curve  $\bar{X}_\infty$ .

**Example 4.3.** Consider the quartic surface  $X \subset \mathbb{C}^3$  defined by the equation

$$(0.3x^2 + 0.5z + 0.3x + 1.2y^2 - 1.1)^2 + (0.7(y - 0.5x)^2 + y + 1.2z^2 - 1)^2 = 0.3.$$

For a general quartic surface in  $\mathbb{C}^3$ , the bottleneck degree is 2220 by Example 4.2. In this case,  $\text{BND}(X) = 1390$  was found using the Julia package *HomotopyContinuation.jl* [17]. Among the 1390 solutions are 49 distinct real bottleneck pairs. The quartic with its bottlenecks is shown in Figure 8.

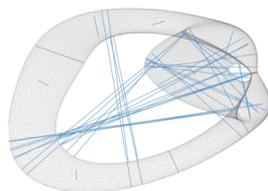


FIGURE 8. The quartic surface of Example 4.3 shown with its real bottleneck lines. The shortest bottleneck line is shown in red. The figure was produced by Sascha Timme using the Julia package *HomotopyContinuation.jl* [17].

**Example 4.4.** Consider the ellipsoid  $X \subset \mathbb{C}^3$  defined by the equation

$$36x^2 + 9y^2 + 4z^2 = 36.$$

For a general quadric surface in  $\mathbb{C}^3$ , the bottleneck degree is 6 by Example 4.2. In this case, there are indeed three bottleneck pairs, all with real coordinates. The pairs occur on each of the coordinate axes, at  $\{(-1, 0, 0), (1, 0, 0)\}$ ,  $\{(0, -2, 0), (0, 2, 0)\}$  and  $\{(0, 0, -3), (0, 0, 3)\}$ .

If we set two axes to be the same length, as in the equation of the spheroid

$$4x^2 + y^2 + z^2 = 4,$$

then there is only one isolated bottleneck pair:  $\{(-1, 0, 0), (1, 0, 0)\}$ . The rest of the bottlenecks are part of an infinite locus. Intersecting with the plane  $\{x = 0\}$  which is normal to the spheroid, we obtain the circle  $\{y^2 + z^2 = 4\}$  and every antipodal pair of points of the circle is a bottleneck.

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