

ERGODIC CONTROL OF A CLASS OF JUMP DIFFUSIONS WITH FINITE LÉVY MEASURES AND ROUGH KERNELS

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Abstract. We study the ergodic control problem for a class of jump diffusions in \mathbb{R}^d , which are controlled through the drift with bounded controls. The Lévy measure is finite, but has no particular structure—it can be anisotropic and singular. Moreover, there is no blanket ergodicity assumption for the controlled process. Unstable behavior is ‘discouraged’ by the running cost which satisfies a mild coercive hypothesis (i.e., is near-monotone). We first study the problem in its weak formulation as an optimization problem on the space of infinitesimal ergodic occupation measures, and derive the Hamilton–Jacobi–Bellman equation under minimal assumptions on the parameters, including verification of optimality results, using only analytical arguments. We also examine the regularity of invariant measures. Then, we address the jump diffusion model, and obtain a complete characterization of optimality.

Key words. controlled jump diffusions; compound Poisson process; Lévy process; ergodic control; Hamilton–Jacobi–Bellman equation

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1. Introduction. Optimal control of jump diffusions has recently attracted much attention from the control community, primarily due to its applicability to queueing networks, mathematical finance [17], image processing [23], etc. Many results for the discounted problem are available in [8], including the game theoretic setting, and different applications are discussed. However, studies of the ergodic control problem are rather scarce. Ergodic control of reflected jump diffusions over a bounded domain can be found in [33]. The ergodic control problem in \mathbb{R}^d is studied in [34], albeit under very strong blanket stability assumptions. We should also mention here the treatment of the impulse control problem in [7, 18, 31].

Our work in this paper is motivated from ergodic control problems for multiclass stochastic networks in the Halfin–Whitt regime, under service interruptions. For this model, the pure jump process driving the limiting queueing process is compound Poisson (see Theorem 3.2 in [4]), with a Lévy measure that is anisotropic, and in general, singular with respect to the Lebesgue measure. In fact, the jumps are biased towards a given direction, and thus the Lévy measure has no symmetry whatsoever. We assume that the running cost is coercive, also known as near-monotone (see (2.2)), and do not impose any blanket stability hypotheses on the controlled jump diffusion. We treat a general class of jump diffusions which is abstracted from diffusion approximations of stochastic networks, and whose controlled infinitesimal generator has the form

$$(1.1) \quad \mathcal{A}u(x, z) := \sum_{i,j} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_i b^i(x, z) \frac{\partial u}{\partial x_i}(x) + \int_{\mathbb{R}^d} (u(x+y) - u(x) - \mathbb{1}_{\{|y| \leq 1\}} \langle y, \nabla u(x) \rangle) \nu_x(dy).$$

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Here, z is a control parameter that lives in a compact metric space \mathcal{Z} , and $\nu_x(dy)$ is a finite Borel measure on \mathbb{R}^d for each x , while $x \mapsto \nu_x(A)$ is a Borel measurable function for each Borel set A . Throughout the paper, we assume that $d \geq 2$. The coefficients of \mathcal{A} are assumed to satisfy the following.

- Assumption 1.1.* (a) The matrix $a = [a^{ij}]$ is symmetric, positive definite, and locally Lipschitz continuous. The drift $b: \mathbb{R}^d \times \mathcal{Z} \rightarrow \mathbb{R}^d$ is continuous.
 (b) The map $x \mapsto \nu(x) := \nu_x(\mathbb{R}^d)$ is locally bounded.
 (c) the map $x \mapsto \nu_x(K - x)$ is bounded on \mathbb{R}^d for any fixed compact set $K \subset \mathbb{R}^d$.

The generator \mathcal{A} in (1.1) covers a variety of models of jump diffusions which appear in the literature [5, 13, 20, 21, 40]. Note also that the ‘jump rate’ $\nu(x)$ is allowed to be state dependent as in [32]. The hypotheses in Assumption 1.1 are quite general, and do not imply the existence of a controlled process with generator \mathcal{A} . Our main goal in this paper is to establish general results for ergodic control of jump diffusions governed for this class of operators. To accomplish this, we first state the ergodic control problem for the operator \mathcal{A} as a convex optimization problem over the set of infinitesimal ergodic occupation measures. We then proceed to study the ergodic Hamilton–Jacobi–Bellman (HJB) equation via analytical methods, without assuming that the martingale problem for \mathcal{A} is well posed. This of course precludes arguments that utilize stochastic representations of solutions of elliptic equations. Later, in section 4, we specialize these results to a fairly general model of controlled jump diffusions with finite Lévy measure.

It is well known that the standard method of deriving the ergodic HJB on \mathbb{R}^d is based on the vanishing discount approach, and relies crucially on structural properties that permit uniform estimates for the gradient (e.g., viscous equations in \mathbb{R}^d), or the Harnack property. Recent work on nonlocal equations has resulted in important regularity results [6, 10, 15, 16] that should prove very valuable in studying control problems. However, most of this work concerns Lévy jump processes whose kernel has a ‘nice’ density resembling that of a fractional Laplacian. For the problem at hand, even though the Lévy measure ν_x is finite, and there is a non-degenerate Wiener process component, the Lévy measure is anisotropic, and could be singular [4, Section 3.2]. As a result, there is no hope for the Harnack property for positive solutions to hold as the following example shows.

Example 1.2. Consider an operator \mathcal{A} in \mathbb{R}^2 , with a the identity matrix, $b = (3, 0)$, and $\nu = \nu_x$ a Dirac mass at $\tilde{x} = (3, 0)$. Let $f_\epsilon \in C^2(\mathbb{R}^2)$, with $\epsilon \in (0, 1)$, be defined in polar coordinates by

$$f_\epsilon(r, \theta) := -\log(r) \mathbf{1}_{\{r \geq \epsilon\}} + \left(\frac{3}{4} - \frac{r^2}{\epsilon^2} + \frac{r^4}{4\epsilon^4} - \log(\epsilon) \right) \mathbf{1}_{\{r < \epsilon\}}.$$

This function is used in [36, p. 111] to exhibit a family of positive superharmonic functions for the Laplacian that violates the Harnack property. Let u_ϵ be a function which agrees with f_ϵ on the unit ball B_1 centered at 0, and takes the values $u_\epsilon(\tilde{r}, \tilde{\theta}) = \left(\frac{4}{\epsilon^2} - \frac{4\tilde{r}^2}{\epsilon^4} + f_\epsilon(\tilde{r}, \tilde{\theta}) \right) \mathbf{1}_{\{\tilde{r} < \epsilon\}}$ on the unit ball $B_1(\tilde{x})$ centered at \tilde{x} , when expressed in polar coordinates $(\tilde{r}, \tilde{\theta})$ which are centered at \tilde{x} . Let u_ϵ take any nonnegative value elsewhere in \mathbb{R}^2 . Then u_ϵ is nonnegative on \mathbb{R}^2 and satisfies $\mathcal{A}u_\epsilon = 0$ in B . However, $\frac{u_\epsilon(0, \theta)}{u_\epsilon(e^{-1}, \theta)} = -\log(\epsilon)$, and thus the family violates the Harnack property for \mathcal{A} .

Under the general hypotheses of Assumption 1.1, even if the operator \mathcal{A} is the generator of a Markov process, the process might not be regular, or, in case it is positive recurrent, the mean hitting times to an open ball might not be locally bounded. In the latter case, it is futile to search for solutions to the ergodic HJB equation, even

in a viscosity sense. In [section 3](#), we add two hypotheses to address these pathologies. The first (see [\(H1\)](#)), is the Feller–Has’minskiĭ criterion for a diffusion process with generator \mathcal{A} to be *regular* (or *conservative*, or *non-explosive*), which requires that the equation $\mathcal{A}u - u = 0$ has no bounded positive solutions on \mathbb{R}^d . This property is equivalent to regularity, and it is clear from the proof of this equivalence in [\[26, Theorem 4.1\]](#) that the equation can be replaced by $\mathcal{A}u - \alpha u = 0$ for $\alpha > 0$. The second hypothesis, [\(H2\)](#), states that under some stationary Markov control there exists a nonnegative solution \mathcal{V} to the Lyapunov equation $\mathcal{A}\mathcal{V} \leq C\mathbf{1}_{\mathcal{B}} - \mathcal{R}$, where \mathcal{R} is the running cost, \mathcal{B} is a ball, and C is a constant. Hypothesis [\(H2\)](#) can be relaxed under certain assumptions on ν_x (see [Theorem 3.8](#)).

The paper is organized as follows. In [subsection 1.1](#) we summarize the notation we use. [Section 2](#) states the ergodic control problem, in a weak sense, as a convex optimization problem over the set of infinitesimal ergodic occupation measures for the operator \mathcal{A} , and shows that optimality is attained. Regularity properties of infinitesimal invariant measures are in [subsection 2.3](#). [Section 3](#) is devoted to the study of the HJB equation under [\(H1\)–\(H2\)](#) mentioned above. In [Section 4](#) we study a class of jump diffusions, which is abstracted from the limiting diffusions encountered in stochastic networks under service interruptions.

1.1. Notation. The standard Euclidean norm in \mathbb{R}^d is denoted by $|\cdot|$, and $\langle \cdot, \cdot \rangle$ denotes the inner product. Given two real numbers a and b , the minimum (maximum) is denoted by $a \wedge b$ ($a \vee b$), respectively. The closure, boundary, complement, and the indicator function of a set $A \subset \mathbb{R}^d$ are denoted by \bar{A} , ∂A , A^c , and $\mathbf{1}_A$, respectively. We denote by $\tau(A)$ the *first exit time* of the process X from a set $A \subset \mathbb{R}^d$, defined by $\tau(A) := \inf \{t > 0 : X_t \notin A\}$. The open ball of radius R in \mathbb{R}^d , centered at the origin, is denoted by B_R , and we let $\tau_R := \tau(B_R)$, and $\check{\tau}_R := \tau(B_R^c)$. The Borel σ -field of a topological space E is denoted by $\mathfrak{B}(E)$, and $\mathcal{P}(E)$ denotes the set of probability measures on $\mathfrak{B}(E)$.

For a domain $Q \subset \mathbb{R}^d$, the space $\mathcal{C}^k(Q)$ ($\mathcal{C}^\infty(Q)$), $k \geq 0$, refers to the class of all real-valued functions on Q whose partial derivatives up to order k (of any order) exist and are continuous, while $\mathcal{C}_c^k(Q)$ ($\mathcal{C}_b^k(Q)$) denote the subsets of $\mathcal{C}^k(Q)$, consisting of functions that have compact support (whose partial derivatives are bounded in Q). The space $L^p(Q)$, $p \in [1, \infty)$, stands for the Banach space of (equivalence classes of) measurable functions f satisfying $\int_Q |f(x)|^p dx < \infty$, and $L^\infty(Q)$ is the Banach space of functions that are essentially bounded in Q . We denote the usual norm on this space by $\|f\|_{L^p(Q)}$, $p \in [1, \infty]$. The standard Sobolev space of functions on Q whose generalized derivatives up to order k are in $L^p(Q)$, equipped with its natural norm, is denoted by $\mathcal{W}^{k,p}(Q)$, $k \geq 0$, $p \geq 1$. In general, if \mathcal{X} is a space of real-valued functions on Q , \mathcal{X}_{loc} consists of all functions f such that $f\varphi \in \mathcal{X}$ for every $\varphi \in \mathcal{C}_c^\infty(Q)$. In this manner we obtain, for example, the space $\mathcal{W}_{\text{loc}}^{2,p}(Q)$.

We adopt the notation $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$ for $i, j \in \{1, \dots, d\}$, and we often use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d .

2. The convex analytic formulation. Define $\mathcal{L}: \mathcal{C}^2(\mathbb{R}^d) \rightarrow \mathcal{C}(\mathbb{R}^d \times \mathcal{Z})$ by

$$\mathcal{L}u(x, z) := a^{ij}(x)\partial_{ij}u(x) + \hat{b}^i(x, z)\partial_i u(x),$$

with $\hat{b}(x, z) := b(x, z) + \int_{\mathbb{R}^d} z \mathbf{1}_{\{|z| \leq 1\}} \nu_x(dz)$, and let

$$\mathcal{I}u(x) := \int_{\mathbb{R}^d} (u(x+y) - u(x)) \nu_x(dy),$$

provided that the integral is finite. Thus $\mathcal{A}u(x, z) = \mathcal{L}u(x, z) + \mathcal{I}u(x)$. With $z \in \mathcal{Z}$ treated as a parameter, we define $\mathcal{L}_z u(x) := \mathcal{L}u(x, z)$, and $\mathcal{A}_z u(x) := \mathcal{A}u(x, z)$.

Let $\mathcal{B}(\mathbb{R}^d, \mathcal{Z})$ denote the set of Borel measurable maps $v: \mathbb{R}^d \rightarrow \mathcal{Z}$. Such a map v is called a *stationary Markov control*, and we use the symbol \mathfrak{V}_{sm} to denote this class of controls. For $v \in \mathfrak{V}_{\text{sm}}$, we use the simplified notation $b_v(x) := b(x, v(x))$, and define \mathcal{A}_v , \mathcal{R}_v and ϱ_v analogously.

We augment the class \mathfrak{V}_{sm} by adopting the well-known *relaxed control* framework [2, Section 2.3]. According to this relaxation, controls take values in $\mathcal{P}(\mathcal{Z})$, the latter denoting the set of probability measures on \mathcal{Z} under the Prokhorov topology. Thus, a control $v \in \mathfrak{V}_{\text{sm}}$ may be viewed as a kernel on $\mathcal{P}(\mathcal{Z}) \times \mathbb{R}^d$, which we write as $v(dz | x)$. We extend the definition of b and \mathcal{R} , without changing the notation, i.e., we let $b_v(x) := \int_{\mathcal{Z}} b(x, z) v(dz | x)$, and analogously for \mathcal{R}_v . We endow \mathfrak{V}_{sm} with the topology that renders it a compact metric space, referred to as the *topology of Markov controls* [2, Section 2.4]. A control is said to be *precise* if it is a measurable map from \mathbb{R}^d to \mathcal{Z} , i.e., if it agrees with the definition in the preceding paragraph. It is easy to see that this relaxation preserves [Assumption 1.1](#).

2.1. The ergodic control problem for the operator \mathcal{A} . We fix a countable dense subset \mathcal{C} of $\mathcal{C}_0^2(\mathbb{R}^d)$ consisting of functions with compact supports. Here, $\mathcal{C}_0^2(\mathbb{R}^d)$ denotes the Banach space of functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ that are twice continuously differentiable and their derivatives up to second order vanish at infinity.

DEFINITION 2.1. *A probability measure $\mu_v \in \mathcal{P}(\mathbb{R}^d)$, $v \in \mathfrak{V}_{\text{sm}}$, is called infinitesimally invariant under \mathcal{A}_v if*

$$(2.1) \quad \int_{\mathbb{R}^d} \mathcal{A}_v f(x) \mu_v(dx) = 0 \quad \forall f \in \mathcal{C}.$$

If such a μ_v exists, then we say that v is a stable control, and define the (infinitesimal) ergodic occupation measure $\pi_v \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Z})$ by $\pi_v(dx, dz) := \mu_v(dx) v(dz | x)$. We denote by $\mathfrak{V}_{\text{ssm}}$, \mathcal{M} , and \mathcal{G} , the sets of stable controls, infinitesimal invariant probability measures, and ergodic occupation measures, respectively.

Remark 2.2. In [Definition 2.1](#) we select \mathcal{C} as the function space, deviating from common practice, where this is selected as $\mathcal{C}_0^\infty(\mathbb{R}^d)$, the space of smooth functions vanishing at infinity. In general, there is no uniqueness of solutions to (2.1) [39]. For the relation between infinitesimally invariant measures and invariant probability measures for diffusions we refer the reader to [14]. Note also, that as shown in [19], in order to assert that μ_v is an invariant probability measure for a Markov process with generator \mathcal{A}_v , it suffices to verify (2.1) for a dense subclass of the domain of \mathcal{A}_v consisting of functions such that the martingale problem is well posed.

It follows from [Definition 2.1](#) that $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Z})$ is an ergodic occupation measure if and only if $\int_{\mathbb{R}^d \times \mathcal{Z}} \mathcal{A}_z f(x) \pi(dx, dz) = 0$ for all $f \in \mathcal{C}$. It is also easy to show that the set of ergodic occupation measures \mathcal{G} is a closed and convex subset of $\mathcal{P}(\mathbb{R}^d \times \mathcal{Z})$ (see [2, Lemma 3.2.3]).

Let $\mathcal{R}: \mathbb{R}^d \times \mathcal{Z} \mapsto \mathbb{R}_+$ be a continuous function, which we refer to as the *running cost* function. The ergodic control problem for \mathcal{A} seeks to minimize $\pi(\mathcal{R}) = \int \mathcal{R} d\pi$ over $\pi \in \mathcal{G}$. Thus, the optimization problem is an infinite dimensional linear program. We define $\varrho_* := \inf_{\pi \in \mathcal{G}} \pi(\mathcal{R})$, and assume, of course, that this is finite. Also for $v \in \mathfrak{V}_{\text{ssm}}$, we let $\varrho_v := \pi_v(\mathcal{R})$, and we say that v is *optimal* if $\varrho_v = \varrho_*$. We seek to obtain a full characterization of optimal controls via the study of the dual problem, and this leads to the HJB equation. For more details on this linear programming formulation see Section 4 in [9].

2.2. Well posedness of the control problem. We impose a structural assumption on the running cost which renders the optimization problem well posed. We say that a function $h : \mathbb{R}^d \times \mathcal{Z} \rightarrow \mathbb{R}_+$ is *coercive relative to a constant* $c \in \mathbb{R}$, if there exists a constant $\epsilon > 0$, such that the set $\{x \in \mathbb{R}^d : \inf_{z \in \mathcal{Z}} h(x, z) \leq c + \epsilon\}$ is bounded (or empty).

Throughout the paper, we assume that the running cost is coercive relative to ϱ_* , and we fix a ball \mathcal{B}_\circ and a constant ϵ_\circ such that $\mathcal{R}(x, z) > \varrho_* + 2\epsilon_\circ$ on \mathcal{B}_\circ^c . Naturally, this property depends on ϱ_* , but note that, since $\varrho_* < \infty$, it is always satisfied if the running cost is inf-compact on $\mathbb{R}^d \times \mathcal{Z}$. Coerciveness of \mathcal{R} relative to ϱ_* is also known as *near-monotonicity* in the literature, and it is often written as

$$(2.2) \quad \liminf_{|y| \rightarrow \infty} \inf_{z \in \mathcal{Z}} \mathcal{R}(y, z) > \varrho_*.$$

We state the following theorem, which follows easily by mimicking the proofs of Lemma 3.2.11 and Theorem 3.4.5 in [2].

THEOREM 2.3. *The map $\pi \mapsto \pi(\mathcal{R})$ attains its minimum in \mathcal{G} .*

2.3. Regularity properties of infinitesimal invariant measures. In this section we establish regularity properties of the densities of infinitesimal invariant probability measures. Recall the notation $\nu(x) = \nu_x(\mathbb{R}^d)$ introduced in [Assumption 1.1](#). We need the following definition.

DEFINITION 2.4. *We decompose $\mathcal{A}_z = \tilde{\mathcal{L}}_z + \tilde{\mathcal{I}}$, with*

$$\tilde{\mathcal{L}}_z u(x) := \mathcal{L}_z u(x) - \nu(x)u(x), \quad \text{and} \quad \tilde{\mathcal{I}}u(x) := \int_{\mathbb{R}^d} u(x+y) \nu_x(dy).$$

THEOREM 2.5. *Every $\mu \in \mathcal{M}$ has a density $\phi = \phi[\mu]$ which belongs to $L_{loc}^p(\mathbb{R}^d)$ for any $p \in [1, \frac{d}{d-2})$, and is strictly positive. In addition, if ν_x is translation invariant and has compact support, then, for any $\beta \in (0, 1)$, there exists a constant $\bar{C} = \bar{C}(\beta, R)$, such that*

$$(2.3) \quad |\phi(x) - \phi(y)| \leq \bar{C} |x - y|^\beta \quad \forall x, y \in B_R.$$

Proof. As shown in [11, Theorem 2.1], if in some domain $Q \subset \mathbb{R}^d$, a probability measure μ satisfies

$$(2.4) \quad \int_Q a^{ij} \partial_{ij} f d\mu \leq C \sup_Q (|f| + |\nabla f|) \quad \forall f \in \mathcal{C}_c^\infty(Q)$$

for some constant C , then μ has a density which belongs to $L_{loc}^p(Q)$ for every $p \in [1, d')$, where $d' = \frac{d}{d-1}$. It is straightforward to verify, using [Assumption 1.1](#), that a bound of the form (2.4) holds for any $\mu \in \mathcal{M}$ on any bounded domain Q . It follows that the density ϕ of μ is in $L_{loc}^p(\mathbb{R}^d)$ for any $p \in [1, d')$, and that it is a generalized solution to the equation

$$(2.5) \quad \sum_{i,j} \int_{\mathbb{R}^d} (a^{ij}(x) \partial_j \phi(x) + (\partial_j a^{ij}(x) - \hat{b}_v^i(x)) \phi(x)) \partial_i f(x) dx \\ - \int_{\mathbb{R}^d} \nu(x) \phi(x) f(x) dx = - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x+y) \nu_x(dy) \phi(x) dx,$$

for $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. By (2.5), ϕ is a supersolution to

$$(2.6) \quad \tilde{\mathcal{L}}_v^* \phi(x) := \partial_i (a^{ij}(x) \partial_j \phi(x) + (\partial_j a^{ij}(x) - \hat{b}_v^i(x)) \phi(x)) - \nu(x) \phi(x) = 0.$$

Therefore, by the estimate for supersolutions in [22, Theorem 8.18], we deduce that $\phi \in L^p_{\text{loc}}(\mathbb{R}^d)$ for any $p \in [1, \frac{d}{d-2})$, and that it is strictly positive. Note that this theorem assumes that the supersolution is in $\mathcal{W}^{1,2}_{\text{loc}}(\mathbb{R}^d)$, but this is unnecessary. The theorem is valid for functions in $\mathcal{W}^{1,p}_{\text{loc}}(\mathbb{R}^d)$ for any $p > 1$, as seen from the results in Section 5.5 of [35], or one can use the mollifying technique in [2, Theorem 5.3.4] to show this.

Now suppose that ν_x is translation invariant and has compact support. Let $\widehat{\mathcal{I}}\phi(x) := \int_{\mathbb{R}^d} \phi(x-y) \nu(dy)$. Then (2.5) takes the form $\widetilde{\mathcal{L}}^*_\nu \phi(x) = -\widehat{\mathcal{I}}\phi(x)$. The operator $\widetilde{\mathcal{L}}^*_\nu$ satisfies the hypotheses of Theorem 5.5.5' in [35], which asserts that ϕ satisfies

$$(2.7) \quad \|\phi\|_{\mathcal{W}^{1,q}(B_R)} \leq \kappa(p, R) (\|\widehat{\mathcal{I}}\phi\|_{L^p(B_{2R})} + \|\phi\|_{L^1(B_{2R})}) \quad \forall p > 1,$$

with $q = q(p) := \frac{dp}{d-p}$, and a constant $\kappa(p, R)$ that depends also on d, ν , and the bounds in Assumption 1.1. Without loss of generality, suppose that ν is supported on a ball B_{R_0} . By Minkowski's integral inequality we have

$$(2.8) \quad \|\widehat{\mathcal{I}}\phi\|_{L^p(B_{2R})} \leq \nu \|\phi\|_{L^p(B_{2R+R_0})}.$$

On the other hand, by the Sobolev embedding theorem, $\mathcal{W}^{1,q}(B_R) \hookrightarrow L^r(B_R)$ is a continuous embedding for $q \leq r \leq \frac{qd}{d-q}$ and $q < d$, and $\mathcal{W}^{1,q}(B_R) \hookrightarrow \mathcal{C}^{0,r}(\overline{B_R})$ is compact for $r < 1 - \frac{d}{q}$ and $q > d$. Therefore, starting say from $p = \frac{d}{d-1}$, we deduce by repeated applications of (2.7)–(2.8), and Sobolev embedding, that $\phi \in \mathcal{W}^{1,q}_{\text{loc}}(\mathbb{R}^d)$ for any $q > 1$, which implies (2.3). \square

Remark 2.6. The assumption that ν_x is translation invariant in Theorem 2.5 is sharp. Consider a jump diffusion with $\sigma = \sqrt{2}$, $b(x) = x$, $g(x, \xi) = -x$, and $\nu = 1$. Then $\mathcal{A} = \Delta - 1 + \delta_0$, where δ_0 denotes the Dirac mass at 0. It can be easily verified that the diffusion is geometrically ergodic by employing the Lyapunov function $\mathcal{V}(x) = |x|^2$. The density of the invariant measure ϕ satisfies $\int \sum_{i,j} (\partial_i \phi)(\partial_j f) + \int \phi f = f(0)$ for all $f \in \mathcal{C}^\infty_c(\mathbb{R}^d)$, and thus it is a solution of $-\Delta \phi + \phi = \delta_0$ (viewed in the sense of distributions $\mathcal{D}'(\mathbb{R}^d)$). However, as shown in [38], every positive solution ϕ of this equation, which vanishes at infinity, satisfies $\phi(x) \sim \Gamma(x)$ as $x \rightarrow 0$, where Γ denotes the fundamental solution of $-\Delta$ in \mathbb{R}^d . Thus the density of the invariant measure in the vicinity of $x = 0$ is not any better than what is claimed in the first step in the proof, which shows that it belongs to $L^p_{\text{loc}}(\mathbb{R}^d)$ for $p < \frac{d}{d-2}$. One can select the jumps to induce multiple such singularities, and generate very pathological examples. Thus, in general, the hypothesis that ν_x is translation invariant cannot be relaxed, unless we assume that ν_x has a suitable density as shown in Corollary 2.8 below.

DEFINITION 2.7. We say that ν_x has locally compact support if there exists an increasing map $\gamma: (0, \infty) \rightarrow (0, \infty)$ such that $\nu_x(x + B^c_{\gamma(R)}) = 0$ for all $x \in B_R$. Let $\widehat{\gamma}(R) := R + \gamma(R)$. It follows from this definition that $B_{\widehat{\gamma}(R)}$ contains the support of ν_x for all $x \in B_R$.

COROLLARY 2.8. Assume that ν_x has locally compact support, and that it has a density $\psi_x \in L^{p_1}_{\text{loc}}(\mathbb{R}^d)$ for some $p_1 > \frac{d}{2}$, satisfying the following: for some $p_2 \in (1, \frac{d}{d-2})$, it holds that

$$\int_{B_{\gamma(R)}} \left(\int_{B_{\widehat{\gamma}(R)}} |\psi_x(y)|^{p_i} dy \right)^{\frac{1}{p_i-1}} dx < \infty, \quad i = 1, 2, \quad \forall R > 0.$$

Then (2.3) holds.

Proof. Note that

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x+y) \psi_x(y) dy \right) \phi(x) dx &= \int_{\mathbb{R}^d} f(z) \left(\int_{\mathbb{R}^d} \psi_{z-y}(y) \phi(z-y) dy \right) dz \\ &= \int_{\mathbb{R}^d} f(z) \left(\int_{\mathbb{R}^d} \psi_z(z-a) \phi(a) da \right) dz. \end{aligned}$$

Therefore, $\widehat{\mathcal{I}}\phi(x) = \int_{\mathbb{R}^d} \psi_a(x-a) \phi(a) da$. By the Minkowski integral inequality and the Hölder inequality, we obtain

$$\begin{aligned} \|\widehat{\mathcal{I}}h(z)\|_{L^p(B_R)} &= \left(\int_{B_R} \left(\int_{B_{\gamma(R)}} \psi_a(z-a) |h(a)| da \right)^p dz \right)^{1/p} \\ &\leq \int_{B_{\gamma(R)}} |h(a)| \left(\int_{B_R} |\psi_a(z-a)|^p dz \right)^{1/p} da \\ &\leq \|h\|_{L^p(B_{\gamma(R)})} \left(\int_{B_{\gamma(R)}} \left(\int_{B_R} |\psi_a(z-a)|^p dz \right)^{1/(p-1)} da \right)^{(p-1)/p} \\ &\leq \|h\|_{L^p(B_{\gamma(R)})} \left(\int_{B_{\gamma(R)}} \|\psi_a\|_{L^p(B_{\widehat{\gamma}(R)})}^{p/(p-1)} da \right)^{(p-1)/p}. \end{aligned}$$

Therefore, the map $\widehat{\mathcal{I}}h$ is a linear mapping from $L^{p_1}(B_{\gamma(R)}) \cup L^{p_2}(B_{\gamma(R)})$ into $L^{p_1}(B_R) \cup L^{p_2}(B_R)$ and satisfies

$$|\{x \in B_R : |\widehat{\mathcal{I}}h(x)| > t\}| \leq C \frac{\|h\|_{L^{p_i}(B_{\gamma(R)})}}{t^{p_i}}$$

for some constant C , for all $h \in L^{p_i}(B_R)$, $i = 1, 2$. Here, $|A|$ denotes the Lebesgue measure of a set A . Thus, by the Marcinkiewicz interpolation theorem, it extends to a bounded linear map from $L^p(B_{\gamma(R)})$ into $L^p(B_R)$ for any $p \in (p_1, p_2)$. The result then follows as in the proof of Theorem 2.5. \square

Remark 2.9. It is evident from Corollary 2.8 that if ν_x has locally compact support and a density $\psi_x \in L^p(\mathbb{R}^d)$ for some $p > \frac{d}{2}$, such that $x \mapsto \|\psi_x\|_{L^p(\mathbb{R}^d)}$ is locally bounded, then the density of an infinitesimal invariant measure is Hölder continuous.

3. The HJB equations. We first discuss the relationship between infinitesimal invariant probability measures and Foster–Lyapunov equations. Next, we derive the α -discounted HJB equation, and proceed to study the ergodic HJB equation using the vanishing discount approach. The treatment is analytical, and we refrain from using any stochastic representations of solutions. We state hypothesis (H1) which was discussed in section 1.

(H1) For any $v \in \mathfrak{V}_{\text{sm}}$, and $\alpha > 0$, the equation $\mathcal{A}_v u - \alpha u = 0$ has no bounded positive solution $u \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$.

3.1. On the Foster–Lyapunov equation. Consider the hypothesis:

(H2) There exist $\hat{v} \in \mathfrak{V}_{\text{sm}}$, a nonnegative $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$, an open ball $\widehat{\mathcal{B}}$, and a positive constant κ_0 such that

$$(3.1) \quad \mathcal{A}_{\hat{v}} \mathcal{V}(x) \leq \kappa_0 \mathbf{1}_{\widehat{\mathcal{B}}}(x) - \mathcal{R}_{\hat{v}}(x) \quad \forall x \in \mathbb{R}^d.$$

On the other hand, ϱ_* is finite if and only if

(H3) There exist $\hat{v} \in \mathfrak{V}_{\text{ssm}}$, and a probability measure $\mu_{\hat{v}}$ which solves (2.1), and $\mu_{\hat{v}}(\mathcal{R}_{\hat{v}}) = \int \mathcal{R}_{\hat{v}} d\mu_{\hat{v}} < \infty$.

For continuous diffusions, equivalence of (H2) and (H3) is a celebrated result of Has'minskiĭ [27]. It is pretty straightforward to show, using probabilistic arguments, that (H2) \Rightarrow (H3), and this is in fact true for a large class of Markov processes. An analytical argument for continuous diffusions can be found in the work of Bogachev and Röckner [12], under the hypothesis that $\mathcal{R}_{\hat{v}}$ is inf-compact. The argument offered by Has'minskiĭ in the proof that (H3) \Rightarrow (H2) relies crucially on the Harnack property, and therefore is not applicable for the jump diffusions considered here. In the context of general Markov processes, existence of a solution to (3.1) is related to the f -regularity of the process. For recent work on this, see [28].

In some sense, (H2) is a very mild assumption, since in any application one would first need to establish that ϱ_* is finite, and the natural venue for this is via the Foster–Lyapunov equation in (3.1). A typical example is when ν_x is translation invariant, a has sublinear growth, and for some $\theta \in [1, 2]$, $\int_{\mathbb{R}^d} |y|^\theta \nu(dy) < \infty$, $\mathcal{R}_{\hat{v}}$ grows at most as $|x|^{2(\theta-1)}$, and there exist a positive definite symmetric matrix S , and positive constants c_0 and c_1 such that $\langle b_{\hat{v}}(x), Sx \rangle \leq c_0 - c_1|x|^\theta$. Then (3.1) holds with $\mathcal{V}(x) = \langle x, Sx \rangle^{\theta/2}$. For other examples, see [4, Corollary 5.1].

Consider the class of ν_x that are either translation invariant and have compact support, or satisfy the hypotheses of Corollary 2.8, and denote it by \mathfrak{N}_0 for convenience. For $\nu_x \in \mathfrak{N}_0$, we bridge the gap between (H2) and (H3) in Theorem 3.7 by establishing the existence of a solution to the Poisson equation, and thus showing that (H3) \Rightarrow (H2), albeit for a function $\mathcal{V} \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$. This however is enough to relax (H2) in asserting the existence of a solution to the ergodic HJB for $\nu_x \in \mathfrak{N}_0$ (Theorem 3.8). Moreover, the proof of Theorem 3.8 contains an analytical argument which shows that (H2) \Rightarrow (H3), provided that $\nu_x \in \mathfrak{N}_0$, and $\mathcal{R}_{\hat{v}}$ is inf-compact.

We need the following simple assertion.

LEMMA 3.1. *Let μ_v be an infinitesimal invariant measure under $v \in \mathfrak{V}_{\text{ssm}}$. Then (2.1) holds for all $\varphi \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d) \cap \mathcal{C}_c(\mathbb{R}^d)$, $p > d$. In addition, if $\varphi \in \mathcal{W}_{\text{loc}}^{2,p}$, $p > d$, is inf-compact, and such that $\mathcal{A}_v \varphi$ is nonpositive a.e. on the complement of some ball $\mathcal{B} \subset \mathbb{R}^d$, then $\mu_v(|\mathcal{A}_v \varphi|) < \infty$.*

Proof. In the interest of simplicity, we drop the explicit dependence on v in the notation. Suppose $\varphi \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d) \cap \mathcal{C}_c(\mathbb{R}^d)$, $p > d$. Let ρ be a symmetric non-negative mollifier supported on the unit ball centered at the origin, and for $\epsilon > 0$, let $\rho_\epsilon(x) := \epsilon^{-d} \rho(\frac{x}{\epsilon})$, and $\varphi_\epsilon := \rho_\epsilon * \varphi$, where ‘ $*$ ’ denotes convolution. Then, $\mu(\mathcal{A}\varphi_\epsilon) = 0$ by (2.1). Since $\partial_{ij}\varphi_\epsilon$ converges to $\partial_{ij}\varphi$ as $\epsilon \searrow 0$ in $L^p(B_R)$ for any $p > 1$ and $R > 0$, and since μ has a density in $L_{\text{loc}}^p(\mathbb{R}^d)$ for $p < \frac{d}{d-2}$ by Theorem 2.5, it follows by Hölder’s inequality that $\int_{\mathbb{R}^d} |a^{ij}| |\partial_{ij}\varphi - \partial_{ij}\varphi_\epsilon| d\mu \rightarrow 0$ as $\epsilon \searrow 0$. Also, since $\partial_i\varphi - \partial_i\varphi_\epsilon$ converges uniformly to 0, and in view of Assumption 1.1 (b) and (c), we obtain $\mu(\hat{b}^i \partial_i \varphi_\epsilon) \rightarrow \mu(\hat{b}^i \partial_i \varphi)$, and $\mu(\mathcal{I}\varphi_\epsilon) \rightarrow \mu(\mathcal{I}\varphi)$ as $\epsilon \searrow 0$. This shows that $\mu(\mathcal{A}\varphi) = 0$.

We now turn to the second statement of the lemma. Let χ be a concave $\mathcal{C}^2(\mathbb{R}^d)$ function such that $\chi(x) = x$ for $x \leq 0$, and $\chi(x) = 1$ for $x \geq 1$. Then χ' and $-\chi''$ are nonnegative on $(0, 1)$. Define $\chi_R(x) := R + \chi(x - R)$ for $R > 0$, and observe that $\chi_R(\varphi) - R - 1$ is compactly supported by construction. We have

$$(3.2) \quad \mathcal{A}\chi_R(\varphi) = \chi'_R(\varphi) \mathcal{A}\varphi + \chi''_R(\varphi) \langle \nabla \varphi, a \nabla \varphi \rangle - (\chi'_R(\varphi) \mathcal{I}\varphi - \mathcal{I}\chi_R(\varphi)).$$

Note that the second and third terms on the right hand side of (3.2) are nonpositive. Thus, selecting R sufficiently large so that $\mathcal{A}\varphi$ is nonpositive on the complement of

B_R , and integrating with respect to μ , we first obtain $\mu((\mathcal{A}\varphi)^+) < \infty$, and using this together with (3.2) the result follows. \square

3.2. The α -discounted HJB equation. We have the following theorem.

THEOREM 3.2. *Assume (H1)–(H2). For any $\alpha \in (0, 1)$, there exists a minimal nonnegative solution $V_\alpha \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p > 1$, to the HJB equation*

$$(3.3) \quad \min_{z \in \mathcal{Z}} [\mathcal{A}_z V_\alpha(x) + \mathcal{R}(x, z)] = \alpha V_\alpha(x).$$

Moreover, $\inf_{\mathbb{R}^d} \alpha V_\alpha \leq \varrho_*$, and this infimum is attained in the set

$$\Gamma_\circ := \left\{ x \in \mathbb{R}^d : \sup_{z \in \mathcal{Z}} \mathcal{R}(x, z) \leq \varrho_* \right\}.$$

Proof. Establishing the existence of a solution is quite standard. One starts by exhibiting a solution $\psi_{\alpha,R} \in \mathcal{W}^{2,p}(B_R) \cap \mathcal{C}(\mathbb{R}^d)$ to the Dirichlet problem

$$(3.4) \quad \begin{cases} \min_{z \in \mathcal{Z}} [\mathcal{A}_z \psi_{\alpha,R}(x) + \mathcal{R}(x, z)] = \alpha \psi_{\alpha,R}(x) & x \in B_R, \\ \psi_{\alpha,R}(x) = 0 & x \in B_R^c, \end{cases}$$

for any $\alpha \in (0, 1)$ and $R > 0$.

We use Definition 2.4 to write $\mathcal{A} = \tilde{\mathcal{L}} + \tilde{\mathcal{I}}$. Applying the well-known interior estimate in [22, Theorem 9.11], for any fixed $r > 0$, we obtain

$$\|\psi_{\alpha,R}\|_{\mathcal{W}^{2,p}(B_r)} \leq C \left(\|\psi_{\alpha,R}\|_{L^p(B_{2r})} + \|\mathcal{R}_{v_\alpha} + \tilde{\mathcal{I}}\psi_{\alpha,R}\|_{L^p(B_{2r})} \right)$$

for some constant $C = C(r, p)$. Here, v_α is a measurable selector from the minimizer of the α -discounted HJB in (3.3). Using the comparison principle and (H2), it is straightforward to show that $\psi_{\alpha,R} \leq \frac{\kappa_\circ}{\alpha} + \mathcal{V}$ on \mathbb{R}^d . Thus $\{\psi_{\alpha,R}\}$ is bounded in $\mathcal{W}^{2,p}(B_r)$, uniformly in R . We then take limits as $R \rightarrow \infty$ to obtain a function $V_\alpha \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$ which solves (3.3).

Let $m_\alpha := \inf_{\mathbb{R}^d} V_\alpha$. We claim that $\alpha m_\alpha \leq \varrho_*$. Suppose on the contrary that $\alpha m_\alpha > \varrho_*$. Let $v \in \mathfrak{V}_{\text{ssm}}$. Recall the function χ in the proof of Lemma 3.1, and let $\tilde{\chi}(x) := -\chi(\frac{\varrho_*}{2} + 1 - x)$. Note that $\tilde{\chi}'' \geq 0$, and $\tilde{\chi}'(\psi_{\alpha,R})\mathcal{I}\psi_{\alpha,R} - \mathcal{I}\tilde{\chi}(\psi_{\alpha,R}) \leq 0$. Thus, using (3.4) and repeating the calculation in (3.2) we obtain

$$\mathcal{A}_v \tilde{\chi}(\psi_{\alpha,R}) \geq \tilde{\chi}'(\psi_{\alpha,R}) \mathcal{A}_v \psi_{\alpha,R} \geq \tilde{\chi}'(\psi_{\alpha,R}) (\alpha \psi_{\alpha,R} - \mathcal{R}_v).$$

It is clear that $\tilde{\chi}(\psi_{\alpha,R}) \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d) \cap \mathcal{C}_c(\mathbb{R}^d)$, for any $p > 1$. Hence, integrating with respect to μ_v , applying Lemma 3.1, and taking limits as $R \rightarrow \infty$, using monotone convergence, we obtain $\alpha m_\alpha \leq \mu_v(\alpha V_\alpha) \leq \mu_v(\mathcal{R}_v)$. Taking the infimum over $v \in \mathfrak{V}_{\text{ssm}}$ contradicts the hypothesis that $\alpha m_\alpha > \varrho_*$, and thus proves the claim.

Recall the definition ϵ_\circ in subsection 2.2. Let $\tilde{v} \in \mathfrak{V}_{\text{sm}}$ be a measurable selector from the minimizer of (3.4) and consider the Dirichlet problem

$$(3.5) \quad \begin{cases} \mathcal{A}_{\tilde{v}} \tilde{\psi}_{\alpha,R}(x) + \mathcal{R}_{\tilde{v}}(x) = \alpha \tilde{\psi}_{\alpha,R}(x) & x \in B_R, \\ \tilde{\psi}_{\alpha,R}(x) = \alpha^{-1}(\varrho_* + \epsilon_\circ) & x \in B_R^c, \end{cases}$$

for $\alpha \in (0, 1)$ and $R > 0$. Arguing as in the derivation of (3.4), it follows that $\tilde{\psi}_{\alpha,R}$ converges, as $R \rightarrow \infty$, to some $\tilde{V}_\alpha \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$ which solves $\mathcal{A}_{\tilde{v}} \tilde{V}_\alpha + \mathcal{R}_{\tilde{v}}(x) = \alpha \tilde{V}_\alpha$ on

\mathbb{R}^d . It is clear that $u = \tilde{V}_\alpha - V_\alpha$ is nonnegative and bounded. Since $\mathcal{A}_v u - \alpha u = 0$ on \mathbb{R}^d , it follows by (H1) that u cannot be strictly positive, and, in turn, by the strong maximum principle it has to be identically zero. Thus, given $\epsilon < \epsilon_0$ there exists R_ϵ such that $\min_{B_R} \alpha \tilde{\psi}_{\alpha,R} < \varrho_* + \epsilon$ for all $R > R_\epsilon$. It follows by (3.5) that $\tilde{\psi}_{\alpha,R}$ attains its minimum in the set $\Gamma_\epsilon := \{x \in \mathbb{R}^d : \mathcal{R}_v(x) \leq \varrho_* + \epsilon\}$ for all $R > R_\epsilon$, and therefore, the same applies to \tilde{V}_α . Since $\epsilon > 0$ is arbitrary, we conclude that \tilde{V}_α attains its infimum in the set $\{x \in \mathbb{R}^d : \mathcal{R}_v(x) \leq \varrho_*\} \subset \Gamma_0$, and this completes the proof. \square

3.3. The ergodic HJB equation. We start with the main convergence result of the paper which establishes solutions to the ergodic HJB via the vanishing discount method. To guide the reader, the technique of the proof consists of writing the operator in the form $\tilde{\mathcal{L}} + \tilde{\mathcal{I}}$, and obtaining estimates for supersolutions of the local operator $\tilde{\mathcal{L}}$ using the results in [3, Corollary 2.2].

THEOREM 3.3. *Grant the hypotheses of Theorem 3.2, and let V_α , $\alpha \in (0, 1)$, be the family of solutions in that theorem. Then, as $\alpha \searrow 0$, $V_\alpha - V_\alpha(0)$ converges in $\mathcal{C}^{1,r}(\bar{B}_R)$ for any $r \in (0, 1)$ and $R > 0$, to a function $V \in \mathcal{W}_{loc}^{2,p}(\mathbb{R}^d)$ for any $p > 1$, which is bounded from below in \mathbb{R}^d and solves*

$$(3.6) \quad \min_{z \in \mathcal{Z}} [\mathcal{A}_z V(x) + \mathcal{R}(x, z)] = \varrho,$$

with $\varrho = \varrho_*$. Also $\alpha V_\alpha(x) \rightarrow \varrho_*$ uniformly on compact sets. In addition, the solution of (3.6) with $\varrho = \varrho_*$ is unique in the class of functions $V \in \mathcal{W}_{loc}^{2,d}(\mathbb{R}^d)$, satisfying $V(0) = 0$, which are bounded from below in \mathbb{R}^d . For $\varrho < \varrho_*$, there is no such solution.

Proof. Recall the definitions of \mathcal{B}_0 and ϵ_0 in subsection 2.2. Fix an arbitrary ball $\mathcal{B} \subset \mathbb{R}^d$ such that $\mathcal{B}_0 \subset \mathcal{B}$. Since \mathcal{V} and V_α are a supersolution and subsolution of $\mathcal{A}_v u - \alpha u = -\mathcal{R}_v$ on \mathcal{B}^c by (3.1), respectively, it follows that the solution V_α of (3.3) satisfies

$$(3.7) \quad V_\alpha(x) \leq \sup_{\mathcal{B}} V_\alpha + \mathcal{V}(x) \quad \forall x \in \mathbb{R}^d.$$

By Theorem 3.2 we have $\inf_{\mathbb{R}^d} V_\alpha = \min_{\mathcal{B}_0} V_\alpha$ for all $\alpha \in (0, 1)$. For each $\alpha \in (0, 1)$, we fix some point $\hat{x}_\alpha \in \text{Arg min } V_\alpha \subset \mathcal{B}_0$. Consider the function $\varphi_\alpha := V_\alpha - V_\alpha(\hat{x}_\alpha)$. Then (3.7) implies that

$$(3.8) \quad \varphi_\alpha(x) \leq \|\varphi_\alpha\|_{L^\infty(\mathcal{B}_0)} + \mathcal{V}(x) \quad \forall x \in \mathbb{R}^d.$$

We have

$$\min_{z \in \mathcal{Z}} [\mathcal{A}_z \varphi_\alpha(x) - \alpha \varphi_\alpha(x) + \mathcal{R}(x, z)] = \alpha V_\alpha(\hat{x}_\alpha) \leq \varrho_*,$$

where the last inequality follows by Theorem 3.2. We claim that for each $R > 0$ there exists a constant κ_R such that

$$(3.9) \quad \|\varphi_\alpha\|_{L^\infty(B_R)} \leq \kappa_R \quad \forall \alpha \in (0, 1).$$

To prove the claim, let $\mathcal{B} \equiv B_R$, and D_1, D_2 be balls satisfying $\mathcal{B} \Subset D_1 \Subset D_2$. Recall Definition 2.4. For $p > 0$, let $\|u\|_{p;Q} := (\int_Q u(x) dx)^{1/p}$. Of course, this is not a norm unless $p \geq 1$, so there is a slight abuse of notation involved in this definition. Since $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$, hypothesis (H2) implies that $\tilde{\mathcal{I}}\mathcal{V} \in L_{loc}^\infty(\mathbb{R}^d)$, and the same of course

holds for φ_α by (3.8). By the local maximum principle [22, Theorem 9.20], for any $p > 0$, there exists a constant $\tilde{C}_1(p) > 0$ such that

$$\|\varphi_\alpha\|_{L^\infty(\mathcal{B})} \leq \tilde{C}_1(p) (\|\varphi_\alpha\|_{p;D_1} + \|\tilde{\mathcal{I}}\varphi_\alpha\|_{L^d(D_1)} + \|\mathcal{R}_{v_\alpha}\|_{L^d(D_1)}),$$

and by the supersolution estimate [22, Theorem 9.22], and since φ_α is nonnegative, there exist some $p > 0$ and $\tilde{C}_2 > 0$ such that $\|\varphi_\alpha\|_{p;D_1} \leq \tilde{C}_2 \varrho_* |D_2|^{1/d}$. Combining these inequalities, we obtain

$$(3.10) \quad \|\varphi_\alpha\|_{L^\infty(\mathcal{B})} \leq \tilde{C}_1(p) (\tilde{C}_2 \varrho_* |D_2|^{1/d} + \|\mathcal{R}_{v_\alpha}\|_{L^d(D_1)}) + \tilde{C}_1(p) \|\tilde{\mathcal{I}}\varphi_\alpha\|_{L^d(D_1)}.$$

Denote the first term on the right hand of (3.10) by κ_1 . By (3.8) and (3.10) we have

$$\begin{aligned} \|\varphi_\alpha\|_{L^\infty(D_2)} &\leq \|\mathcal{V}\|_{L^\infty(D_2)} + \|\varphi_\alpha\|_{L^\infty(\mathcal{B})} \\ &\leq \kappa_1 + \|\mathcal{V}\|_{L^\infty(D_2)} + \tilde{C}_1(p) \|\tilde{\mathcal{I}}\varphi_\alpha\|_{L^d(D_1)}. \end{aligned}$$

This implies that, either $\|\varphi_\alpha\|_{L^\infty(D_2)} \leq 2(\kappa_1 + \|\mathcal{V}\|_{L^\infty(D_2)})$, in which case (3.9) holds with this bound, or

$$(3.11) \quad \|\varphi_\alpha\|_{L^\infty(D_2)} \leq 2\tilde{C}_1(p) \|\tilde{\mathcal{I}}\varphi_\alpha\|_{L^d(D_1)}.$$

If (3.11) holds, then we write $\tilde{\mathcal{I}}\varphi_\alpha = \tilde{\mathcal{I}}(\mathbf{1}_{D_2}\varphi_\alpha) + \tilde{\mathcal{I}}(\mathbf{1}_{D_2^c}\varphi_\alpha)$, and use the estimate

$$\tilde{\mathcal{I}}(\mathbf{1}_{D_2^c}\varphi_\alpha)(x) \leq \|\varphi_\alpha\|_{L^\infty(\mathcal{B})} \left(\sup_{x \in D_1} \nu_x(D_2^c) \right) + \tilde{\mathcal{I}}(\mathbf{1}_{D_2^c}\mathcal{V}) \quad \forall x \in D_1,$$

which holds by (3.8), together with (3.10) and (3.11), to obtain

$$(3.12) \quad \begin{aligned} \|\tilde{\mathcal{I}}\varphi_\alpha\|_{L^\infty(D_1)} &\leq 2\tilde{C}_1(p) \|\nu\|_{L^\infty(D_1)} \|\tilde{\mathcal{I}}\varphi_\alpha\|_{L^d(D_1)} \\ &\quad + \kappa_1 \|\nu\|_{L^\infty(D_1)} + \|\tilde{\mathcal{I}}(\mathbf{1}_{D_2^c}\mathcal{V})\|_{L^\infty(D_1)}. \end{aligned}$$

We distinguish two cases from (3.12):

Case 1. Suppose that

$$(3.13) \quad \|\tilde{\mathcal{I}}\varphi_\alpha\|_{L^\infty(D_1)} \leq 4\tilde{C}_1(p) \|\nu\|_{L^\infty(D_1)} \|\tilde{\mathcal{I}}\varphi_\alpha\|_{L^d(D_1)}.$$

Let ψ_α be the solution of the Dirichlet problem

$$\tilde{\mathcal{L}}_{v_\alpha}\psi_\alpha - \alpha\psi_\alpha = -\tilde{\mathcal{I}}\varphi_\alpha \quad \text{in } D_1, \quad \text{and } \psi_\alpha = \varphi_\alpha \quad \text{on } \partial D_1.$$

Then ψ_α is nonnegative in D_1 by the strong maximum principle, and thus (3.13) together with [3, Corollary 2.2], implies that for some constant C_H we have

$$(3.14) \quad \psi_\alpha(x) \leq C_H \psi_\alpha(\hat{x}_\alpha) \quad \forall x \in \mathcal{B}, \quad \forall \alpha \in (0, 1).$$

On the other hand, $\varphi_\alpha - \psi_\alpha$ satisfies

$$(3.15) \quad \tilde{\mathcal{L}}_{v_\alpha}(\varphi_\alpha - \psi_\alpha) - \alpha(\varphi_\alpha - \psi_\alpha) = \alpha V_\alpha(\hat{x}_\alpha) - \mathcal{R}_{v_\alpha} \quad \text{in } D_1,$$

and $\varphi_\alpha - \psi_\alpha = 0$ on ∂D_1 . Thus, by the ABP weak maximum principle [22, Theorem 9.1], and since $\alpha V_\alpha(\hat{x}_\alpha) \leq \varrho_*$, we obtain from (3.15) that

$$(3.16) \quad \|\varphi_\alpha - \psi_\alpha\|_{L^\infty(D_1)} \leq C_o \quad \forall \alpha \in (0, 1),$$

for some constant C_o . Equation (3.16) implies that $\psi_\alpha(\hat{x}_\alpha) \leq C_o$. Combining (3.14) and (3.16) in the standard manner, we obtain

$$(3.17) \quad \begin{aligned} \varphi_\alpha(x) &\leq \|\varphi_\alpha - \psi_\alpha\|_{L^\infty(D_1)} + \psi_\alpha(x) \\ &\leq C_o + C_H \psi_\alpha(\hat{x}_\alpha) \leq C_o(1 + C_H) \quad \forall x \in \mathcal{B}, \quad \forall \alpha \in (0, 1). \end{aligned}$$

Case 2. Suppose that

$$\|\tilde{\mathcal{I}}\varphi_\alpha\|_{L^\infty(D_1)} \leq 2\kappa_1\|\nu\|_{L^\infty(D_1)} + 2\|\tilde{\mathcal{I}}(\mathbf{1}_{D_2^c}\mathcal{V})\|_{L^\infty(D_1)}.$$

In this case, we consider the solution $\tilde{\psi}_\alpha$ of the Dirichlet problem

$$\tilde{\mathcal{L}}_{v_\alpha}\tilde{\psi}_\alpha - \alpha\tilde{\psi}_\alpha = 0 \quad \text{in } D_1, \quad \text{and} \quad \tilde{\psi}_\alpha = \varphi_\alpha \quad \text{on } \partial D_1.$$

We have $\tilde{\psi}_\alpha(x) \leq \tilde{C}_H \tilde{\psi}_\alpha(\hat{x}_\alpha)$ for all $x \in \mathcal{B}$ and $\alpha \in (0, 1)$, for some constant \tilde{C}_H . Also,

$$(3.18) \quad \tilde{\mathcal{L}}_{v_\alpha}(\varphi_\alpha - \tilde{\psi}_\alpha) - \alpha(\varphi_\alpha - \tilde{\psi}_\alpha) = -\tilde{\mathcal{I}}\varphi_\alpha + \alpha V_\alpha(\hat{x}_\alpha) - \mathcal{R}_{v_\alpha} \quad \text{in } D_1,$$

and $\varphi_\alpha - \tilde{\psi}_\alpha = 0$ on ∂D_1 . By the ABP weak maximum principle, we obtain from (3.18) that $\|\varphi_\alpha - \tilde{\psi}_\alpha\|_{L^\infty(D_1)} \leq \tilde{C}_o$ for all $\alpha \in (0, 1)$ and for some constant \tilde{C}_o . Thus again we obtain (3.17) with constants \tilde{C}_o and \tilde{C}_H . This establishes (3.9).

It follows by (3.9) that $\bar{V}_\alpha := V_\alpha - V_\alpha(0) = \varphi_\alpha(x) - \varphi_\alpha(0)$ is locally bounded, uniformly in $\alpha \in (0, 1)$. The same applies to $\tilde{\mathcal{I}}\bar{V}_\alpha$ by (3.8) and (H2). Note that

$$\tilde{\mathcal{L}}_{v_\alpha}\bar{V}_\alpha - \alpha\bar{V}_\alpha = \alpha V_\alpha(0) - \mathcal{R}_{v_\alpha} - \tilde{\mathcal{I}}\bar{V}_\alpha \quad \text{on } \mathbb{R}^d.$$

Thus, by the interior estimate in [22, Theorem 9.11], there exists a constant $C = C(R, p)$ such that

$$\|\bar{V}_\alpha\|_{W^{2,p}(B_R)} \leq C\left(\|\bar{V}_\alpha\|_{L^p(B_{2R})} + \|\alpha V_\alpha(0) - \mathcal{R}_{v_\alpha} - \tilde{\mathcal{I}}\bar{V}_\alpha\|_{L^p(B_{2R})}\right).$$

Hence $\{\bar{V}_\alpha\}$ is bounded in $W^{2,p}(B_R)$ for any $R > 0$. A standard argument then shows that given any sequence $\alpha_n \searrow 0$, $\{\bar{V}_{\alpha_n}\}$ contains a subsequence which converges in $C^{1,r}(\bar{B}_R)$ for any $r < 1 - \frac{d}{p}$ (see, e.g., Lemma 3.5.4 in [2]). Taking limits in

$$(3.19) \quad \min_{z \in \mathcal{Z}} [\mathcal{A}_z \bar{V}_\alpha(x) - \alpha \bar{V}_\alpha(x) + \mathcal{R}(x, z)] = \alpha V_\alpha(0)$$

along this subsequence we obtain (3.6), as claimed in the statement of the theorem, for some $\varrho \in \mathbb{R}$. Since $\limsup_{\alpha \searrow 0} \alpha V_\alpha(\hat{x}_\alpha) \leq \varrho_*$, we have $\varrho \leq \varrho_*$. On the other hand, from the theory of infinite dimensional linear programming [1] it is well known that the value of the dual problem cannot be smaller than the value of the primal, hence $\varrho \geq \varrho_*$, and we have equality (see also Section 4 in [9]).

Suppose now that $\tilde{V} \in W_{\text{loc}}^{2,d}(\mathbb{R}^d)$ is bounded from below in \mathbb{R}^d , and satisfies

$$(3.20) \quad \min_{z \in \mathcal{Z}} [\mathcal{A}_z \tilde{V}(x) + \mathcal{R}(x, z)] = \varrho_*.$$

Let $\tilde{v} \in \mathfrak{V}_{\text{sm}}$ be an a.e. measurable selector from the minimizer of (3.20). Define $\tilde{V}^\epsilon := (1 + \epsilon)\tilde{V}$, $\epsilon > 0$. Arguing as in the derivation of (3.8), it is clear that this equation holds with \mathcal{V} replaced by \tilde{V}^ϵ . Translate \tilde{V}^ϵ by an additive constant until it touches φ_α at some point from above. Since

$$\mathcal{A}_{\tilde{v}}(\tilde{V}^\epsilon - \varphi_\alpha) - \alpha(\tilde{V}^\epsilon - \varphi_\alpha) \leq (1 + \epsilon)\varrho_* - \alpha\varphi_\alpha(\hat{x}_\alpha) - \mathcal{R}_{\tilde{v}},$$

taking first limits as $\alpha \searrow 0$, and then as $\epsilon \searrow 0$, we obtain $\mathcal{A}_v(\tilde{V} - V) \leq 0$, and conclude that $\tilde{V} = V$ by the strong maximum principle.

It is evident from the uniqueness of the solution, that the limit of (3.19) is independent of the subsequence $\alpha_n \searrow 0$ chosen. It is also clear that $\alpha V_\alpha(x) \rightarrow \varrho_*$ as $\alpha \searrow 0$, uniformly on compact sets. This completes the proof. \square

Remark 3.4. If ν_x is translation invariant and has compact support, and \mathcal{R} and b are locally Hölder continuous in x , then $\tilde{I}V$ is locally Hölder continuous, and thus the solution V in Theorem 3.3 is in $\mathcal{C}^{2,r}(\mathbb{R}^d)$ for some $r \in (0, 1)$ by elliptic regularity [22, Theorem 9.19].

3.3.1. Verification of optimality. We start with the following theorem.

THEOREM 3.5. *Assume the hypotheses of Theorem 3.3. If $v \in \mathfrak{V}_{\text{ssm}}$ is optimal, then it satisfies*

$$(3.21) \quad b_v^i(x) \partial_i V(x) + \mathcal{R}_v(x) = \inf_{z \in \mathcal{Z}} [b^i(x, z) \partial_i V(x) + \mathcal{R}(x, z)] \quad \text{a.e. } x \in \mathbb{R}^d.$$

In addition, provided V is inf-compact, any stable $v \in \mathfrak{V}_{\text{ssm}}$ which satisfies (3.21) is necessarily optimal.

Proof. Suppose not. Then there exists some ball \mathcal{B} such that

$$(3.22) \quad h(x) := \left(b_v^i(x) \partial_i V(x) + \mathcal{R}_v(x) - \inf_{z \in \mathcal{Z}} [b^i(x, z) \partial_i V(x) + \mathcal{R}(x, z)] \right) \mathbb{1}_{\mathcal{B}}(x)$$

is a nontrivial nonnegative function. Since $\partial_i V_\alpha$ converges uniformly to $\partial_i V$ as $\alpha \searrow 0$ on compact sets by Theorem 3.3, it follows that if we define h_α as the right hand side of (3.22), but with V replaced by V_α , then $h - h_\alpha$ converges to 0 a.e. in \mathcal{B} , and also $\mu_v(|h - h_\alpha|) \rightarrow 0$ as $\alpha \searrow 0$, since μ_v has a density in $L_{\text{loc}}^p(\mathbb{R}^d)$ for some $p > 1$. We have $\mathcal{A}_v V_\alpha \geq \alpha V_\alpha + h_\alpha - \mathcal{R}_v$ a.e. on \mathbb{R}^d by the definition of h_α . With $\psi_{\alpha,R}$ the solution in (3.4), and $m_\alpha = \inf_{\mathbb{R}^d} V_\alpha$, and define $\check{\psi}_{\alpha,R} := \psi_{\alpha,R} - m_\alpha$. Repeating the above argument, there exists $\check{h}_{\alpha,R}$ supported on \mathcal{B} such that $\mu_v(|\check{h}_{\alpha,R} - h_\alpha|) \rightarrow 0$ as $R \rightarrow \infty$, and $\mathcal{A}_v \check{\psi}_{\alpha,R} \geq \alpha \check{\psi}_{\alpha,R} + \check{h}_{\alpha,R} - \mathcal{R}_v$. We apply the function $\check{\chi}(x) := -\chi(\frac{\varrho_*}{2} + 1 - x)$, with χ as defined in the proof of Lemma 3.1, and repeat the argument in Theorem 3.2, also letting $R \rightarrow \infty$, to obtain $\mu_v(\mathcal{R}_v) \geq \mu_v(\alpha V_\alpha) + \mu_v(h_\alpha)$. By the proof of Theorem 3.3 $\inf_{\mathbb{R}^d} \alpha V_\alpha \rightarrow \varrho_*$ as $\alpha \searrow 0$. Thus, taking limits as $\alpha \searrow 0$, we obtain $\mu_v(h) \leq 0$, and since μ_v has everywhere positive density, this implies $h = 0$ a.e.

The second assertion of the theorem is easily established by the argument in the proof of Lemma 3.1, using the function χ_R . \square

Remark 3.6. If we impose the additional assumption that the coefficients a and b have at most affine growth, and that $\nu_x(B_R - x)$ vanishes as $|x| \rightarrow \infty$, for any ball B_R , then it is standard to show that the solution V in Theorem 3.3 is inf-compact, so that the second assertion of Theorem 3.5 applies. However, this leaves open the question whether a $v \in \mathfrak{V}_{\text{sm}}$ that satisfies (3.21) is necessarily stable. We provide a partial answer to this in Theorem 3.8 below.

Recall Definition 2.7. We impose additional assumptions on ν_x to establish existence of solutions to the Poisson equation.

THEOREM 3.7. *We assume (H1) and one of the following:*

- (a) $\nu = \nu_x$ is translation invariant and has compact support.
- (b) ν_x has locally compact support and satisfies the hypotheses of Corollary 2.8.

Let $\hat{v} \in \mathfrak{V}_{\text{ssm}}$ be such that $\mathcal{R}_{\hat{v}}$ is coercive relative to $\varrho_{\hat{v}}$. Then, up to an additive constant, there exists a unique $\hat{V} \in \mathcal{W}_{\text{loc}}^{2,d}(\mathbb{R}^d)$ which is bounded from below in \mathbb{R}^d , and satisfies

$$(3.23) \quad \mathcal{A}_{\hat{v}} \hat{V}(x) + \mathcal{R}_{\hat{v}}(x) = \beta \quad \forall x \in \mathbb{R}^d,$$

for some $\beta = \varrho_{\hat{v}}$. For $\beta < \varrho_{\hat{v}}$, there is no such solution.

Proof. For $n \in \mathbb{N}$, let $\mathcal{R}^n = n \wedge \mathcal{R}$ denote the n -truncation of the running cost. It is clear that \mathcal{R}^n is coercive relative to $\varrho_{\hat{v}}$ for all $n > \varrho_{\hat{v}}$. Consider the α -discounted problem in [Theorem 3.2](#). The Dirichlet problem in [\(3.4\)](#) is now a linear problem, and we let $\hat{\psi}_{\alpha,R}^n$ denote the corresponding solution. It is clear that $\|\hat{\psi}_{\alpha,R}^n\|_{L^\infty(\mathbb{R}^d)} \leq \frac{n}{\alpha}$, and this is inherited by the function \hat{V}_α^n at the limit $R \rightarrow \infty$. Thus, by the proof of [Theorem 3.2](#), \hat{V}_α^n is in $\mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$ for any $p \geq 1$, and satisfies $\mathcal{A}_{\hat{v}} \hat{V}_\alpha^n + \mathcal{R}_{\hat{v}}^n = \alpha \hat{V}_\alpha^n$. Repeating the argument in the proof of [Theorem 3.3](#), the infimum of \hat{V}_α^n over \mathbb{R}^d is attained in a ball \mathcal{B}_o as defined in [subsection 2.2](#) (relative to $\varrho_{\hat{v}}$), and if $\hat{x}_\alpha^n \in \mathcal{B}_o$ denotes a point where the infimum is attained, then $\alpha \hat{V}_\alpha^n(\hat{x}_\alpha^n) \leq \varrho_{\hat{v}}$. With $\varphi_\alpha^n := \hat{V}_\alpha^n - \hat{V}_\alpha^n(\hat{x}_\alpha^n)$, we write the equation as

$$(3.24) \quad \begin{aligned} \tilde{\mathcal{L}}_{\hat{v}} \varphi_\alpha^n(x) - \alpha \varphi_\alpha^n(x) &= \alpha \hat{V}_\alpha^n(\hat{x}_\alpha^n) - \mathcal{R}_{\hat{v}}^n(x) - \tilde{\mathcal{I}} \varphi_\alpha^n(x) \\ &\leq \varrho_{\hat{v}} - \mathcal{R}^n(x) - \tilde{\mathcal{I}} \varphi_\alpha^n(x) \quad \text{a.e. } x \in \mathbb{R}^d. \end{aligned}$$

We express [\(3.24\)](#) in divergence form as

$$\partial_j (a^{ij} \partial_i \varphi_\alpha^n) + (\hat{b}^i - \partial_i a^{ij}) \partial_j \varphi_\alpha^n - \nu \varphi_\alpha^n \leq \varrho_{\hat{v}} - \mathcal{R}_{\hat{v}} - \tilde{\mathcal{I}} \varphi_\alpha^n,$$

and apply [\[22, Theorem 8.18\]](#) to obtain $\|\varphi_\alpha^n\|_{L^p(B_{2R}(x_0))} \leq \varrho_{\hat{v}} \kappa_{p,R}$ for some constant $\kappa_{p,R}$, for any $p \in (1, \frac{d}{d-2})$. Therefore, $\inf_{B_{2R}(x_0) \setminus B_R(x_0)} \varphi_\alpha^n$ is bounded over $\alpha \in (0, 1)$ and $n \geq \varrho_{\hat{v}}$. Thus, we can select some $x'_0 \in B_{2R}(x_0) \setminus B_R(x_0)$ satisfying $\sup_n \varphi_\alpha^n(x'_0) < \infty$, and repeat the procedure to show by induction that φ_α^n is locally bounded in L^p for any $p \in (1, \frac{d}{d-2})$, uniformly over $\alpha \in (0, 1)$ and $n \geq \varrho_{\hat{v}}$.

Next, we apply successively the Calderón–Zygmund estimate [\[22, Theorem 9.11\]](#) to the non-divergence form of the equation in [\(3.24\)](#) which states that

$$\|\varphi_\alpha^n\|_{\mathcal{W}^{2,p}(B_R)} \leq C \left(\|\varphi_\alpha^n\|_{L^p(B_{2R})} + \|\alpha V_\alpha(\hat{x}_\alpha^n) - \mathcal{R}_{\hat{v}}^n - \tilde{\mathcal{I}} \varphi_\alpha^n\|_{L^p(B_{2R})} \right).$$

We start with the L^p estimate, say with $p = \frac{d}{d-r}$ for $r \in (1, 2)$. If (a) holds, then $\|\tilde{\mathcal{I}} \varphi_\alpha^n\|_{L^p(B_R(x))} \leq \nu \|\varphi_\alpha^n\|_{L^p(B_{R+R_o}(x))}$ by the Minkowski integral inequality, where R_o is such that the support of ν is contained in B_{R_o} , while in case (b) we use the technique in the proof of [Corollary 2.8](#). Using the compactness of the embedding $\mathcal{W}^{2,p}(B_R) \hookrightarrow L^q(B_R)$ for $p \leq q < \frac{pd}{d-2p}$, we choose $q = \frac{pd}{d-rp}$ to improve the estimate to a new $p = \frac{d}{d-2r}$. Continuing in this manner, in at most $d-1$ steps we obtain

$$\sup_{n \geq \varrho_{\hat{v}}} \sup_{\alpha \in (0,1)} \|\varphi_\alpha^n\|_{\mathcal{W}^{2,p}(B_R)} < \infty$$

for any $p > d$ and $R > 0$. Letting first $n \rightarrow \infty$, and then $\alpha \searrow 0$, along an appropriate subsequence, we obtain a solution to [\(3.23\)](#) as claimed. The rest follow as in the proof of [Theorem 3.3](#). \square

THEOREM 3.8. *Grant the hypotheses of [Theorem 3.7](#). Then the conclusions of [Theorem 3.3](#) hold. Moreover, provided V is inf-compact, a control $v \in \mathfrak{V}_{\text{sm}}$ is optimal if and only if it satisfies (3.21).*

Proof. Note that the only place we use the assumption $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ in the proof of [Theorem 3.3](#) is to assert that $\tilde{\mathcal{I}}\mathcal{V} \in L_{\text{loc}}^\infty(\mathbb{R}^d)$. Thus, under (a), or (b) of [Theorem 3.7](#), if we select $\hat{v} \in \mathfrak{V}_{\text{ssm}}$ such that $\varrho_{\hat{v}} \leq \varrho_* + \epsilon_0$, then the Poisson equation in (3.23) can be used in lieu (H2), and the conclusions of [Theorem 3.3](#) follow. We next show that any $v \in \mathfrak{V}_{\text{sm}}$ which satisfies (3.21) is stable. We adapt the technique which is used in [12, Theorem 1.2] for a local operator, to construct an infinitesimal invariant measure μ_v . Let $\tilde{\mathcal{L}}_v^*$ be the operator in (2.6), and set $\hat{\mathcal{I}}u(x) := \int_{\mathbb{R}^d} u(x-y)\nu(dy)$ if ν_x is translation invariant; otherwise, under hypothesis (b) of [Theorem 3.7](#), we define $\hat{\mathcal{I}}u(x) := \int_{\mathbb{R}^d} \psi_{x-y}(y)u(x-y)dy$. Consider the solution ϕ_k of the Dirichlet problem $\tilde{\mathcal{L}}_v^*\phi_k + \hat{\mathcal{I}}\phi_k = 0$ on B_k , with ϕ_k equal to a positive constant c_k on B_k^c .

Concerning the solvability of the Dirichlet problem, note that for $f \in L^2(B_k)$, the problem $\tilde{\mathcal{L}}_v^*u = -\hat{\mathcal{I}}f$ on B_k , with $f = u = c_k$ on B_k^c , has a unique solution $u \in W^{2,2}(B_k)$, which obeys the estimate $\|u\|_{W^{2,2}(B_k)} \leq \kappa(1 + \|u\|_{L^2(B_k)} + \|\hat{\mathcal{I}}f\|_{L^2(B_k)})$ for some constant κ . Thus we can combine [Corollary 2.8](#), the compactness of the embedding $W^{2,2}(B_R) \hookrightarrow L^q(B_R)$ for $q = \frac{2d}{d-1}$, and the Leray–Schauder fixed point theorem to assert the existence of a solution $\phi_k \in W^{2,2}(B_k)$ as claimed in the preceding paragraph. The solutions ϕ_k are nonnegative by the weak maximum principle [22, Theorem 8.1]. We choose the constant c_k so that $\int_{B_k} \phi_k(x)dx = 1$.

We improve the regularity of ϕ_k by following the proofs of [Theorem 2.5](#) and [Corollary 2.8](#), and show that for any $n > 0$, there exists $N(n) \in \mathbb{N}$ such that the sequence $\{\phi_k : k > N(n)\}$ is Hölder equicontinuous on the ball B_n . Let $R = R(n) > 0$ be such that $V(x) > R + 1$ on B_n^c . It is always possible to select such $R(n)$ in a manner that $R(n) \rightarrow \infty$ as $n \rightarrow \infty$ by the assumption that V is inf-compact. Employing the function $\chi_R(V)$ as in the proof of [Lemma 3.1](#) and using (3.6), it follows that $\int_{B_{R(n)}} \mathcal{R}_v(x)\phi_k(x)dx \leq \varrho_*$ for all $k > N(n)$ and $n \in \mathbb{N}$. This implies that $\int_{B_0} \phi_k(x)dx \geq \frac{2\epsilon_0}{\varrho_* + 2\epsilon_0}$ for all large enough k . By the Arzelà–Ascoli theorem combined with Fatou’s lemma, ϕ_k converges along a subsequence to some positive, locally Hölder continuous $\phi \in L^1(\mathbb{R}^d)$ uniformly on compact sets, which is a generalized solution of (2.5), and thus satisfies $\int_{\mathbb{R}^d} f(x)\phi(x)dx = 0$ for all $f \in \mathcal{C}$. Thus, after normalization, ϕ is the density of an infinitesimal invariant measure. Therefore, $v \in \mathfrak{V}_{\text{ssm}}$, and the rest follows by [Theorem 3.5](#). \square

4. A jump diffusion model. In this section, we consider a jump diffusion process $X = \{X_t : t \geq 0\}$ in \mathbb{R}^d , $d \geq 2$, defined by the Itô equation

$$(4.1) \quad dX_t = b(X_t, Z_t)dt + \sigma(X_t)dW_t + dL_t, \quad X_0 = x \in \mathbb{R}^d.$$

Here, $W = \{W_t, t \geq 0\}$ is a d -dimensional standard Wiener process, and $L = \{L_t, t \geq 0\}$ is a Lévy process such that $dL_t = \int_{\mathbb{R}_*^m} g(X_{t-}, \xi)\tilde{\mathcal{N}}(dt, d\xi)$, where $\tilde{\mathcal{N}}$ is a martingale measure in $\mathbb{R}_*^m = \mathbb{R}^m \setminus \{0\}$, $m \geq 1$, corresponding to a standard Poisson random measure \mathcal{N} . In other words, $\tilde{\mathcal{N}}(t, A) = \mathcal{N}(t, A) - t\Pi(A)$ with $\mathbb{E}[\mathcal{N}(t, A)] = t\Pi(A)$ for any $A \in \mathfrak{B}(\mathbb{R}^m)$, with Π a σ -finite measure on \mathbb{R}_*^m , and g a measurable function.

The processes W and \mathcal{N} are defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. Assume that the initial condition X_0 , W_0 , and $\mathcal{N}(0, \cdot)$ are mutually independent. The control process $Z = \{Z_t, t \geq 0\}$ takes values in a compact, metrizable space \mathcal{Z} , is \mathfrak{F}_t -adapted, and *non-anticipative*: for $s < t$, $(W_t - W_s, \mathcal{N}(t, \cdot) - \mathcal{N}(s, \cdot))$ is independent

of

$\mathfrak{F}_s :=$ the completion of $\sigma\{X_0, Z_r, W_r, \mathcal{N}(r, \cdot) : r \leq s\}$ relative to $(\mathfrak{F}, \mathbb{P})$.

Such a process Z is called an *admissible control* and we denote the set of admissible controls by \mathfrak{Z} .

4.1. The ergodic control problem for the jump diffusion. Let $\mathcal{R} : \mathbb{R}^d \times \mathbb{Z} \mapsto \mathbb{R}_+$ denote the running cost function, which is assumed to satisfy (2.2).

For an admissible control process $Z \in \mathfrak{Z}$, we consider the *ergodic cost* defined by

$$\tilde{\varrho}_Z(x) := \limsup_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}_x^Z \left[\int_0^T \mathcal{R}(X_t, Z_t) dt \right].$$

Here \mathbb{E}_x^Z denotes the expectation operator corresponding to the process controlled under Z , with initial condition $X_0 = x \in \mathbb{R}^d$. The ergodic control problem seeks to minimize the ergodic cost over all admissible controls. We define $\tilde{\varrho}_*(x) := \inf_{Z \in \mathfrak{Z}} \tilde{\varrho}_Z(x)$. As we show in Theorem 4.3, this infimum is realized with a stationary Markov control, and $\tilde{\varrho}_*(x) = \varrho_*$, with ϱ_* as defined in subsection 2.1, so it does not depend on x .

4.2. Assumptions on the parameters and the running cost. We impose the following set of assumptions on the data which guarantee the existence of a solution to the Itô equation (4.1) (see, e.g., [2, 21]). These augment and replace Assumption 1.1, and are assumed throughout this section by default. In these hypotheses, C_R is a positive constant, depending on $R \in (0, \infty)$. Also $a := \frac{1}{2} \sigma \sigma'$, $\mathbb{R}_*^m := \mathbb{R}^m \setminus \{0\}$, and $\|M\| := (\text{trace } MM')^{1/2}$ denotes the Hilbert–Schmidt norm of a $d \times k$ matrix M for $d, k \in \mathbb{N}$.

$$\begin{aligned} & |b(x, z) - b(y, z)|^2 + \|\sigma(x) - \sigma(y)\|^2 + \int_{\mathbb{R}_*^m} |g(x, \xi) - g(y, \xi)|^2 \Pi(d\xi) \\ & + |\mathcal{R}(x, z) - \mathcal{R}(y, z)|^2 \leq C_R |x - y|^2 \quad \forall x, y \in B_R, \quad \forall z \in \mathbb{Z}, \\ & \langle x, b(x, z) \rangle^+ + \|\sigma(x)\|^2 + \int_{\mathbb{R}_*^m} |g(x, \xi)|^2 \Pi(d\xi) \leq C_1 (1 + |x|^2) \quad \forall (x, z) \in \mathbb{R}^d \times \mathbb{Z}, \\ & \sum_{i,j} a^{ij}(x) \zeta_i \zeta_j \geq (C_R)^{-1} |\zeta|^2 \quad \forall \zeta \in \mathbb{R}^d, \quad \forall x \in B_R. \end{aligned}$$

The measure ν_x in (1.1) then takes the form $\nu_x(A) = \Pi(\{\xi \in \mathbb{R}_*^m : g(x, \xi) \in A\})$, and it clearly satisfies $\int_{\mathbb{R}^d} |y|^2 \nu_x(dy) < C_R |x|^2$. Note that for this model $\nu = \nu_x(\mathbb{R}^d)$ is constant. It is evident that if $g(x, \xi)$ does not depend on x , then ν_x is translation invariant.

4.3. Existence of solutions. For any admissible control Z_t , the Itô equation in (4.1) has a unique strong solution [21], is right-continuous w.p.1, and is a strong Feller process. On the other hand, if Z_t is a Markov control, i.e., if it takes the form $Z_t = v(t, X_t)$ for some Borel measurable function $v : \mathbb{R}_+ \times \mathbb{R}^d$, then it follows from the results in [24] that, under the assumptions in subsection 4.2, the diffusion

$$(4.2) \quad d\tilde{X}_t = b(\tilde{X}_t, v(t, \tilde{X}_t)) dt + \sigma(\tilde{X}_t) dW_t, \quad X_0 = x \in \mathbb{R}^d$$

has a unique strong solution. As shown in [40], since the Lévy measure is finite, the solution of (4.1) can be constructed in a piecewise fashion using the solution of

(4.2) (see also [30]). It thus follows that, under a Markov control, (4.2) has a unique strong solution. In addition, its transition probability has positive mass.

Of fundamental importance in the study of functionals of X is Itô's formula. For $f \in \mathcal{C}^2(\mathbb{R}^d)$ and Z_s an admissible control, it holds that

$$(4.3) \quad f(X_t) = f(X_0) + \int_0^t \mathcal{A}f(X_s, Z_s) ds + \mathcal{M}_t \quad \text{a.s.},$$

with \mathcal{A} as in (1.1), and

$$(4.4) \quad \mathcal{M}_t := \int_0^t \langle \nabla f(X_s), \sigma(X_s) dW_s \rangle + \int_0^t \int_{R_*^m} \left(f(X_{s-} + g(X_{s-}, \xi)) - f(X_{s-}) \right) \tilde{\mathcal{N}}(ds, d\xi)$$

is a local martingale. Krylov's extension of Itô's formula [29, p. 122] shows that (4.3) is valid for functions f in the local Sobolev space $\mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p \geq d$.

Recall that, in the context of diffusions, a control $v \in \mathfrak{V}_{\text{sm}}$ is called *stable* if the process X under v is positive Harris recurrent. This is of course equivalent to the existence of an invariant probability measure for X , and it follows by the Theorem in [19] that μ_v is an invariant probability measure for the diffusion if and only if it is infinitesimally invariant for the operator \mathcal{A} in the sense of (2.1). Thus the two notions of stable controls agree.

4.4. Existence of an optimal stationary Markov control.

DEFINITION 4.1. For $Z \in \mathfrak{Z}$ and $x \in \mathbb{R}^d$, we define the mean empirical measures $\{\bar{\zeta}_{x,t}^Z : t > 0\}$, and (random) empirical measures $\{\zeta_t^Z : t > 0\}$, by

$$(4.5) \quad \bar{\zeta}_{x,t}^Z(f) = \int_{\mathbb{R}^d \times \mathcal{Z}} f(x, z) \bar{\zeta}_{x,t}^Z(dx, dz) := \frac{1}{t} \int_0^t \mathbb{E}_x^Z \left[\int_{\mathcal{Z}} f(X_s, z) Z_s(dz) \right] ds,$$

and $\zeta_{x,t}^Z$ as in (4.5) but without the expectation \mathbb{E}_x^Z , respectively, for all $f \in \mathcal{C}_b(\mathbb{R}^d \times \mathcal{Z})$.

We let $\overline{\mathbb{R}}^d$ denote the one-point compactification of \mathbb{R}^d , and we view $\mathbb{R}^d \subset \overline{\mathbb{R}}^d$ via the natural imbedding. As a result, $\mathcal{P}(\mathbb{R}^d \times \mathcal{Z})$ is viewed as a subset of $\mathcal{P}(\overline{\mathbb{R}}^d \times \mathcal{Z})$. Let $\bar{\mathcal{G}}$ denote the closure of \mathcal{G} in $\mathcal{P}(\overline{\mathbb{R}}^d \times \mathcal{Z})$.

LEMMA 4.2. Almost surely, every limit $\hat{\zeta} \in \mathcal{P}(\overline{\mathbb{R}}^d \times \mathcal{Z})$ of ζ_t^Z as $t \rightarrow \infty$ takes the form $\hat{\zeta} = \delta\zeta' + (1 - \delta)\zeta''$ for some $\delta \in [0, 1]$, with $\zeta' \in \mathcal{G}$ and $\zeta''(\{\infty\} \times \mathcal{Z}) = 1$. The same claim holds for the mean empirical measures, without the qualifier 'almost surely'.

Proof. Write $\hat{\zeta} = \delta\zeta' + (1 - \delta)\zeta''$ for some $\zeta' \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Z})$, and $\zeta''(\{\infty\} \times \mathcal{Z}) = 1$. For $f \in \mathcal{C}$, applying Itô's formula, we obtain

$$\frac{f(X_t) - f(X_0)}{t} = \frac{1}{t} \int_0^t \mathcal{A}_{Z_s} f(X_s) ds + \frac{1}{t} \mathcal{M}_t,$$

where \mathcal{M}_t is given in (4.4). As shown in the proof of [2, Lemma 3.4.6], we have $\frac{1}{t} \int_0^t \langle \nabla f(X_s), \sigma(X_s) dW_s \rangle \rightarrow 0$ a.s. as $t \rightarrow \infty$.

Define

$$(4.6) \quad M_{1,t} := \int_0^t \int_{R_*^m} \left(f(X_{s-} + g(X_{s-}, \xi)) - f(X_{s-}) \right) \mathcal{N}(ds, d\xi),$$

and $M_{2,t}$ analogously by replacing $\mathcal{N}(ds, d\xi)$ by $\Pi(d\xi) ds$ in (4.6). Note that the second integral in (4.4), denoted as M_t , is a square integrable martingale, and takes the form $M_t = M_{1,t} - M_{2,t}$. Since f is bounded on \mathbb{R}^d and Π is a finite measure, we have $\langle M_1 \rangle_t \leq C_1 \mathcal{N}(t, \mathbb{R}_*^m)$, and $\langle M_2 \rangle_t \leq C_2 t$ for some positive constants C_1 and C_2 . Since $\langle M \rangle_t \leq \langle M_1 \rangle_t + \langle M_2 \rangle_t$, then by Proposition 7.1 in [37] we obtain $\limsup_{t \rightarrow \infty} \frac{\langle M \rangle_t}{t} < \infty$ a.s. For the discrete parameter square-integrable martingale $\{M_n : n \in \mathbb{N}\}$, it is well-known that $\lim_{n \rightarrow \infty} \frac{M_n}{\langle M \rangle_n} = 0$ a.s. on the event $\{\langle M \rangle_\infty = \infty\}$. Thus, we obtain

$$(4.7) \quad \lim_{n \rightarrow \infty} \frac{M_n}{n} = 0 \quad \text{a.s.}$$

on the event $\{\langle M \rangle_\infty = \infty\}$. Since f is bounded, then for some constant $C > 0$, we have

$$(4.8) \quad \sup_{t \in [n, n+1]} \frac{|M_t - M_n|}{n} \leq \frac{C}{n} (\mathcal{N}(n+1, \mathbb{R}_*^m) - \mathcal{N}(n, \mathbb{R}_*^m) + 1) \xrightarrow{n \rightarrow \infty} 0,$$

and (4.7)–(4.8) imply that $\lim_{t \rightarrow \infty} \frac{1}{t} M_t \rightarrow 0$ a.s. on the event $\{\langle M \rangle_\infty = \infty\}$.

Next, we examine convergence on the event $\{\langle M \rangle_\infty < \infty\}$. It is well-known that a square-integrable martingale $\{M_n : n \in \mathbb{N}\}$ with quadratic variation $\langle M \rangle$ satisfies $\{\langle M \rangle_\infty < \infty\} \subset \{M_n \rightarrow \}$ a.s., where we write $\{M_n \rightarrow \}$ for the event on which (M_n) converges to a real-valued limit [25, Theorem 2.15]. Thus (4.7) holds on the event $\{\langle M \rangle_\infty < \infty\}$, and it then follows by (4.8) that $\lim_{t \rightarrow \infty} \frac{1}{t} M_t \rightarrow 0$ a.s.

Thus we have shown that $\lim_{t \rightarrow \infty} \frac{1}{t} M_t \rightarrow 0$ a.s., and the claims of the lemma then follow as in the proof of [2, Theorem 3.4.7]. \square

THEOREM 4.3. *There exists an optimal control $v \in \mathfrak{V}_{\text{ssm}}$ for the ergodic problem. In addition, every stationary Markov optimal control v_* is in $\mathfrak{V}_{\text{ssm}}$, and is pathwise optimal in somewhat stronger sense, i.e., it satisfies*

$$(4.9) \quad \liminf_{T \rightarrow \infty} \frac{1}{T} \left[\int_0^T \mathcal{R}(X_t, Z_t) dt \right] \geq \limsup_{T \rightarrow \infty} \frac{1}{T} \left[\int_0^T \mathcal{R}(X_t, v_*(X_t)) dt \right] = \varrho_*$$

a.s. for any admissible control Z_t .

Proof. Define $\hat{\varrho}_* := \inf_{\pi \in \mathcal{G}} \pi(\mathcal{R})$. Following the proof of [2, Theorem 3.4.5], we have $\hat{\varrho}_* = \pi_{v_*}(\mathcal{R})$ for some $v_* \in \mathfrak{V}_{\text{ssm}}$. Also, (4.9) holds by Lemma 4.2 and the proof in [2, Theorem 3.4.7]. \square

4.5. The ergodic HJB equation. We summarize the results in the following theorem.

THEOREM 4.4. *We assume (H2) for some $\hat{v} \in \mathfrak{V}_{\text{sm}}$. Then we have the following:*

- (a) *There exists a unique function $V \in \mathcal{W}_{\text{loc}}^{2,p}(\mathbb{R}^d)$, $p > d$, with $V(0) = 0$, which is bounded from below in \mathbb{R}^d and solves $\min_{z \in \mathcal{Z}} [\mathcal{A}_z V(x) + \mathcal{R}(x, z)] = \varrho$, with $\varrho = \varrho_*$. For $\varrho < \varrho_*$, there is no such solution. Moreover, if ν_x has locally compact support (see Definition 2.7), then $V \in \mathcal{C}^2(\mathbb{R}^d)$.*
- (b) *A control $v \in \mathfrak{V}_{\text{sm}}$ is optimal if and only if it satisfies*

$$(4.10) \quad b_v^i(x) \partial_i V(x) + \mathcal{R}_v(x) = \inf_{z \in \mathcal{Z}} [b^i(x, z) \partial_i V(x) + \mathcal{R}(x, z)] \quad \text{a.e. } x \in \mathbb{R}^d.$$

- (c) *The solution V has the stochastic representation*

$$V(x) = \lim_{r \searrow 0} \inf_{v \in \mathfrak{V}_{\text{ssm}}} \mathbb{E}_x^v \left[\int_0^{\tau_r} (\mathcal{R}_v(X_t) - \varrho_*) dt \right].$$

Proof. Under the assumptions in subsection 4.2, it is straightforward to establish Theorem 3.2. Thus, part (a) follows from Theorem 3.3 and Remark 3.4. Using the Itô formula, one can readily show that any v which satisfies (4.10) is stable and optimal. The necessity part of (b) follows by Theorem 3.5. Part (c) can be established by following the proof of Lemma 3.6.9 in [2]. \square

5. Concluding remarks. The results in this paper extend naturally to models under uniform stability, in which case, of course, we do not need to assume that \mathcal{R} is coercive. Suppose that there exist nonnegative functions $\Psi \in C^2(\mathbb{R}^d)$, and $h: \mathbb{R}^d \times \mathcal{Z}$, with $h \geq 1$ and locally bounded, satisfying

$$(5.1) \quad \mathcal{A}_z \Psi(x) \leq \kappa \mathbf{1}_{\mathcal{B}}(c) - h(x, z) \quad \forall (x, z) \in \mathbb{R}^d \times \mathcal{Z},$$

for some constant κ and a ball $\mathcal{B} \subset \mathbb{R}^d$. In addition, suppose that either \mathcal{R} is bounded, or that $|\mathcal{R}|$ grows slower than h . Under (5.1), the jump diffusion is positive recurrent under any stationary Markov control, and the collection of ergodic occupation measures is tight. Using Ψ as a barrier, all the results in section 4 can be readily obtained, and moreover, for any $v \in \mathfrak{V}_{\text{sm}}$, the Poisson equation $\mathcal{A}_v \Phi = \mathcal{R}_v - \varrho_v$ has a solution in $W_{\text{loc}}^{2,p}(\mathbb{R}^d)$, for any $p > 1$, which is unique, up to an additive constant, in the class of functions Φ which satisfy $|\Phi| \leq C(1 + h_v)$ for some constant C .

We have not considered allowing the jumps to be control dependent, primarily because this is not manifested in the queueing network model motivating this work, but also because this would require us to introduce various assumptions on the regularity of the jumps and the Lévy measure (see, e.g., [34]). This, however, is an interesting problem for future work.

In conclusion, what we aimed for in this work, was to study the ergodic control problem for jump diffusions controlled through the drift via analytical methods, and under minimal assumptions on the (finite) Lévy measure and the parameters.

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