

COUPLING TECHNIQUES FOR NONLINEAR HYPERBOLIC EQUATIONS. I. SELF-SIMILAR DIFFUSION FOR THIN INTERFACES

BENJAMIN BOUTIN^{1,2}, FRÉDÉRIC COQUEL², AND PHILIPPE G. LEFLOCH²

ABSTRACT. We investigate various analytical and numerical techniques for the coupling of nonlinear hyperbolic systems and, in particular, we introduce here an augmented formulation which allows for the modeling of the dynamics of interfaces between fluid flows. The main technical difficulty to be overcome lies in the possible resonance effect when wave speeds coincide and global hyperbolicity is lost. As a consequence, non-uniqueness of weak solutions is observed for the initial value problem which need to be supplemented with further admissibility conditions. This first paper is devoted to investigating these issues in the setting of self-similar vanishing viscosity approximations to the Riemann problem for general hyperbolic systems. Following earlier works by Joseph, LeFloch, and Tzavaras, we establish an existence theorem for the Riemann problem under fairly general structural assumptions on the nonlinear hyperbolic system and its regularization. Our main contribution consists of nonlinear wave interaction estimates for solutions which apply to resonant wave patterns.

1. INTRODUCTION

This is the first part of a series devoted to analytical and numerical techniques relevant for the coupling of nonlinear hyperbolic systems. We mainly discuss an augmented formulation which allows for the modeling of the dynamics of interfaces between fluid flows. The main technical difficulty overcome here for the Riemann problem (that is, a Cauchy problem with piecewise constant initial data) lies in the possible resonance effect when wave speeds coincide and global hyperbolicity is lost. As a consequence, non-uniqueness of weak solutions is observed for the initial value problem which need to be supplemented with further admissibility conditions. In the present paper, we restrict attention to self-similar vanishing viscosity approximations to the Riemann problem for general hyperbolic systems.

Specifically, we are interested in the following class of nonlinear hyperbolic systems of $(N + 1)$ partial differential equations

$$(1.1) \quad \begin{aligned} A_0(u, v) \partial_t u + A_1(u, v) \partial_x u &= 0, \\ \partial_t v &= 0, \end{aligned}$$

where the vector-valued field $u = u(t, x) \in \mathbb{R}^N$ and the scalar function $v = v(t, x) \in \mathbb{R}$ (with $x \in \mathbb{R}$ and $t \geq 0$) are the main unknowns of the theory. We assume that the first set of equations in (1.1) forms a strictly hyperbolic system but admits one wave speed that changes sign, so that the overall system (1.1) is only weakly hyperbolic. Specifically, the mappings A_0, A_1 are assumed to be smooth, matrix-valued maps such that A_0 is invertible so that the first set of equations in (1.1) is formally equivalent to the following nonconservative system with variable coefficients:

$$(1.2) \quad \partial_t u + A_0(u, v)^{-1} A_1(u, v) \partial_x u = 0.$$

It is assumed that the product matrix $A_0(u, v)^{-1} A_1(u, v)$ admits real and distinct eigenvalues denoted by $\lambda_i(u, v)$, $1 \leq i \leq N$. Finally, it is assumed that one eigenvalue λ_m may take values

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¹ Institut de Recherche Mathématiques de Rennes (IRMAR), Université de Rennes 1, 263 Av. General Leclerc, 35042 Rennes. Email: benjamin.boutin@univ-rennes1.fr

² Laboratoire Jacques-Louis Lions & Centre National de la Recherche Scientifique, Université Pierre et Marie Curie (Paris 6), 4 Place Jussieu, 75252 Paris, France.

Blog: philippelefloch.org. Email: coquel@ann.jussieu.fr, contact@philippelefloch.org

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about the origin. For instance, there might exist a state $(u^*, v^*) \in \mathbb{R}^N \times \mathbb{R}$ such that the matrix $A_1(u^*, v^*)$ is non-invertible, with

$$(1.3) \quad \lambda_m(u^*, v^*) = 0.$$

In view of the above assumption, the system (1.1) is called a *weakly hyperbolic system*. Our objective, precisely, is to study this *resonant regime*.

Recall that Dafermos [12, 13, 11] advocated the use of self-similar regularizations in order to capture the whole wave fan structure of weak solutions to the Riemann problem. This consists in searching for self-similar solutions depending only on the variable $\xi := x/t$ and, then, introducing a self-similar regularization of the given hyperbolic system. Specifically, for the problem of coupling under consideration in this paper we propose, in the variable (x, t) , to regularize (1.1) in the form

$$\begin{aligned} A_0(u^\epsilon, v^\epsilon) \partial_t u^\epsilon + A_1(u^\epsilon, v^\epsilon) \partial_x u^\epsilon &= \epsilon t \partial_x (B_0(u^\epsilon, v^\epsilon) \partial_x u^\epsilon), \\ \partial_t v^\epsilon &= \epsilon^p t \partial_{xx} v^\epsilon, \end{aligned}$$

where $\epsilon > 0$ is a small parameter and $B_0 = B_0(u, v)$ is a given matrix referred to as the viscosity matrix and $p > 0$ is a real parameter. In the self-similar variable ξ , the equations satisfied by the viscous solutions $(u^\epsilon, v^\epsilon) = (u^\epsilon(\xi), v^\epsilon(\xi))$ read (with $\xi \in \mathbb{R}$)

$$(1.4) \quad \begin{aligned} (-\xi A_0(u^\epsilon, v^\epsilon) + A_1(u^\epsilon, v^\epsilon)) u_\xi^\epsilon &= \epsilon (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi, \\ -\xi v_\xi^\epsilon &= \epsilon^p v_{\xi\xi}^\epsilon. \end{aligned}$$

Our objective is to study the existence and regularity of solutions to (1.4) and to rigorously justify the passage to the limit $\epsilon \rightarrow 0$. Following earlier works by Joseph, LeFloch, and Tzavaras (see references below), we are going to establish a uniform (ϵ -independent) bound on the total variation $TV(u^\epsilon)$, and an existence theorem under fairly general structural assumptions on the hyperbolic system and its regularization.

This general strategy was proposed in the case $B_0 = I$ and $A_0 = I$ by Tzavaras [34] (for conservative systems) and extended by LeFloch and Tzavaras [30] (for non-conservative systems). The technique was further developed by Joseph and LeFloch in the series of papers [22, 23, 24, 25, 26, 27]. For the purpose of the present paper, we will especially build on [27] where a general technique to derive interaction estimates was introduced and general matrices B_0 were dealt with. For other results on self-similar limits including viscosity-capillarity terms and large data, we refer to pioneering works by Slemrod [16], Slemrod and Tzavaras [33], Fan and Slemrod [16].

The coupling that we are studying in the present work may be non-conservative in nature (cf. Section 2 for details). In contrast, for coupling techniques based on systems in conservation form, a large literature is available; see for instance [1, 3, 9, 8, 14, 31].

An outline of this paper follows. In Section 2, we present a general approach involving the coupling of nonlinear hyperbolic systems. In Section 3 we discuss the case of scalar-valued unknowns u , and establish a general existence theorem for the viscous self-similar Riemann problem (1.4) when $N = 1$. As $\epsilon \rightarrow 0$, we prove that this smooth solution converges to an entropy solution, at least in each half-space $x < 0$ and $x > 0$. This global existence result requires no smallness assumption on the data nor on the coupling of the two models.

The core part of this paper is contained in Sections 4 and 5 which cover general systems of N equations. Imposing a natural smallness condition on the Riemann data and the coupling of the two models, we establish the existence of smooth solutions to the viscous problem, even in the presence of a resonance effect. In Section 4 we derive the main estimates on the total variation while, in Section 5, we justify the limit $\epsilon \rightarrow 0$. We refer to the forthcoming works [5, 6, 7] for further investigation of these solutions and related issues.

2. A FORMULATION OF THE COUPLING OF NONLINEAR HYPERBOLIC SYSTEMS

Before we can state our new formulation based on an augmented system, we start by briefly explaining the formulation based on a fixed interface. The weakly hyperbolic problem mentioned in the introduction arises, in particular, via the following coupling technique. Consider two strictly

hyperbolic systems posed in half-spaces:

$$(2.1) \quad \begin{aligned} \partial_t w + \partial_x f_-(w) &= 0, & x < 0, & t > 0, \\ \partial_t w + \partial_x f_+(w) &= 0, & x > 0, & t > 0, \end{aligned}$$

where the flux f_{\pm} are given smooth maps defined on open subsets $\Omega_{\pm} \subset \mathbb{R}^N$ and the unknown of the problem is $w = w(t, x) \in \Omega_- \cup \Omega_+$. In addition to initial data, a certain coupling condition must be prescribed at the (fixed) interface $\{x=0\}$. This problem can be regarded as a boundary and initial value problem within each half-problem, and the fundamental question is how to formulate a physically relevant boundary condition so that the global problem is well-posed. One natural requirement, following Godlewski and Raviart [19, 18], is imposing the continuity condition

$$(2.2) \quad \theta_-(w(0-, t)) = \theta_+(w(0+, t)), \quad t > 0,$$

where θ_-, θ_+ are two invertible maps in \mathbb{R}^N , with inverses

$$\gamma_- := \theta_-^{-1}, \quad \gamma_+ := \theta_+^{-1}.$$

These functions precisely provide the necessary freedom to handle various types of couplings. For example, by choosing $\theta_- = \theta_+ = \text{Id}$ one imposes the continuity of the variable w at the interface, while by choosing $\theta_{\pm} = f_{\pm}$ one imposes the continuity of the flux at the interface (so that the general problem is conservative).

Recalling Dubois and LeFloch's theory [15] of the initial and boundary value problem for nonlinear hyperbolic systems, it is clear that the condition (2.2) is realistic only when the boundary is *not characteristic*, that is when all eigenvalues are bounded away from 0. In the latter case, instead, following the weak formulation of the boundary conditions proposed in [15] generalized, for the coupling problem, by Godlewski and Raviart [19, 18] and Ambroso et al. [2], we impose that the interface condition is satisfied *in a weak sense, only* and, specifically, takes the form

$$(2.3) \quad \begin{aligned} w(0+, t) &\in \Phi_+(\theta_+ \circ \theta_-^{-1}(w(0-, t))), \\ w(0-, t) &\in \Phi_-(\theta_- \circ \theta_+^{-1}(w(0+, t))). \end{aligned}$$

where $\Phi_+(b_+)$ (resp. $\Phi_-(b_-)$) is the Dubois-LeFloch's set of admissible traces of the associated Riemann solutions

$$\begin{aligned} \Phi_+(b_+) &:= \{R_+(0+; b_+, a), \quad a \in \Omega_+\}, \\ \Phi_-(b_-) &:= \{R_-(0-; a, b_-), \quad a \in \Omega_-\}. \end{aligned}$$

Here, $R = R_+(x/t; b_+, a)$ denotes the solution of the Riemann problem

$$\partial_t R + \partial_x f_+(R) = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad R(x, 0) = \begin{cases} b_+, & x < 0, \\ a, & x > 0, \end{cases}$$

and similarly $R = R_-(x/t; a, b_-)$ is the solution of

$$\partial_t R + \partial_x f_-(R) = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad R(x, 0) = \begin{cases} a, & x < 0, \\ b_-, & x > 0. \end{cases}$$

Yet, when $f_- \neq f_+$, the question of the existence and uniqueness of weak solutions satisfying (2.3) is a challenging issue. In the present work, we propose to *reformulate the above problem* by "removing" the interface and defining a new problem posed on the whole space \mathbb{R} .

We proceed as follows. First of all, we define the *new variables*

$$(2.4) \quad u_- := \theta_-(w), \quad u_+ := \theta_+(w), \quad u := \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases}$$

and we rewrite the half-space problems in the (conservative) form

$$(2.5) \quad \partial_t \gamma_{\pm}(u) + \partial_x f_{\pm}(\gamma_{\pm}(u)) = 0, \quad \pm x > 0, \quad t > 0,$$

or equivalently in the (nonconservative) form

$$(2.6) \quad (D_u \gamma_{\pm}(u)) \partial_t u + (D_{\gamma} f_{\pm})(\gamma_{\pm}(u)) (D_u \gamma_{\pm}(u)) \partial_x u = 0, \quad \pm x > 0, \quad t > 0.$$

The coupling condition becomes

$$(2.7) \quad \begin{aligned} u(0+, t) &\in \Psi_+(u(0-, t)), \\ u(0-, t) &\in \Psi_-(u(0+, t)), \end{aligned}$$

where $\Psi_+(b)$ (and similarly $\Psi_-(b)$) is the following set of admissible trace at $\xi = 0+$

$$\Psi_+(b) = \{R_+(0+, b, u_+), u_+ \in \Omega\},$$

and $u = R_+(\cdot, b, u_+)$ is the self-similar solution of the following Cauchy problem

$$\partial_t \gamma_+(u) + \partial_x f_+(\gamma_+(u)) = 0, \quad x \in \mathbb{R}, t > 0, \quad u(x, 0) = \begin{cases} b, & x < 0, \\ u_+, & x > 0. \end{cases}$$

In absence of a resonance phenomenon, this reformulation allows us to simply impose the *continuity of u at interface*

$$(2.8) \quad u(0-, t) = u(0+, t).$$

Second, we propose to replace the problem (2.5)-(2.7) by the new problem (already mentioned in the introduction)

$$(2.9) \quad \begin{aligned} A_0(u, v) \partial_t u + A_1(u, v) \partial_x u &= 0, \\ \partial_t v &= 0, \end{aligned}$$

which is a nonlinear hyperbolic system in nonconservative form [28, 29] and where $v : [0, +\infty) \times \mathbb{R} \rightarrow [-1, 1]$ will be referred to as the *color function*. We arrange that regions where $v = -1$ correspond to the left-hand half-problem while regions where $v = 1$ correspond to the right-hand half-problem, by requiring the following consistency property on A_0, A_1 :

$$(2.10) \quad \begin{aligned} A_0(u, \pm 1) &= D_u \gamma_{\pm}(u), \\ A_1(u, \pm 1) &= D_\gamma f_{\pm}(\gamma_{\pm}(u)) D_u \gamma_{\pm}(u). \end{aligned}$$

and assuming the existence of a function $C = C(u, v)$ so that

$$A_0(u, v) = D_u C(u, v),$$

and $C(u, \pm 1) = \gamma_{\pm}(u)$. For $j = 0, 1$, by definition, the matrices $A_j(u, v)$ should smoothly connect $A_j(u, -1)$ to $A_j(u, 1)$ as v describes the interval $[-1, 1]$. Moreover A_0 must be invertible and $A_0^{-1} A_1$ have real and distinct eigenvalues, extending here the strict hyperbolicity of the original hyperbolic half-problems.

The system (2.9) is then supplemented with the initial data

$$(2.11) \quad \begin{aligned} u_0(x, 0) = u_0(x) &=: \theta_{\pm}(w_0(x)), \quad \pm x > 0 \\ v_0(x, 0) = v_0(x) &:= \pm 1, \quad \pm x > 0, \end{aligned}$$

for some given data u_0 .

We are especially interested in the case that the interface is characteristic for some state value (u^*, v^*) , that is, when the matrix $A_1(u^*, v^*)$ admits the eigenvalue 0 and (2.8) need not be satisfied as an equality, in general, so that the weak formulation above is necessary.

3. EXISTENCE THEORY FOR SCALAR CONSERVATION LAWS

3.1. Riemann problem with diffusion. In the present section, we restrict attention to scalar equations and provide a rather complete study of the problem described in the introduction. Note that the problem under consideration is nonconservative in nature, and reduces to a conservative system when the component v takes the values ± 1 . As explained earlier, we search for a function $u = u(\xi)$ obtained as the limit of smooth approximations u^ϵ, v^ϵ to

$$(3.1) \quad \begin{aligned} (-\xi A_0(u^\epsilon, v^\epsilon) + A_1(u^\epsilon, v^\epsilon)) u_\xi^\epsilon &= \epsilon (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi, \\ -\xi v_\xi^\epsilon &= \epsilon^p v_{\xi\xi}^\epsilon, \end{aligned}$$

supplemented with Riemann initial data

$$(3.2) \quad \begin{aligned} u^\epsilon(-\infty) &= u_L, & u^\epsilon(+\infty) &= u_R, \\ v^\epsilon(-\infty) &= -1, & v^\epsilon(+\infty) &= 1. \end{aligned}$$

In (3.1), the maps A_0 and A_1 are now smooth *scalar-valued* functions, which satisfy the following consistency condition with the underlying hyperbolic coupling problem determined by the functions γ_\pm and f_\pm : there exist constants c_1, c_2, c_3 , such that

$$(3.3) \quad 0 < c_1 \leq A_0(u, v), \quad 0 < c_2 \leq B_0(u, v) \leq c_3,$$

and

$$(3.4) \quad \begin{aligned} A_0(u, -1) &= \gamma'_-(u), & A_0(u, 1) &= \gamma'_+(u), \\ A_1(u, -1) &= (f_- \circ \gamma_-)'(u), & A_1(u, 1) &= (f_+ \circ \gamma_+)'(u). \end{aligned}$$

We set $\Omega := [\min(u_L, u_R), \max(u_L, u_R)]$, and introduce the Lipschitz constants ω_0, ω_1 of A_0, A_1 , respectively, i.e.

$$|A_j(\tilde{u}, \tilde{v}) - A_j(u, v)| \leq \omega_j (|\tilde{u} - u| + |\tilde{v} - v|)$$

for all $(\tilde{u}, \tilde{v}), (u, v) \in \Omega \times [-1, 1]$ and $j = 0, 1$.

The first equation in (3.1) can be equivalently rewritten as

$$(3.5) \quad (-\xi + \lambda(u^\epsilon, v^\epsilon))G(u^\epsilon, v^\epsilon)B_0(u^\epsilon, v^\epsilon)u_\xi^\epsilon = \epsilon (B_0(u^\epsilon, v^\epsilon)u_\xi^\epsilon)_\xi,$$

where λ and G are defined by

$$\lambda(u, v) := \frac{A_1(u, v)}{A_0(u, v)}, \quad G(u, v) := \frac{A_0(u, v)}{B_0(u, v)}.$$

Furthermore, our assumptions imply that (for some $\Lambda > 0$)

$$|\lambda(u, v)| \leq \Lambda, \quad (u, v) \in \Omega \times [-1, 1],$$

which expresses the property of finite speed of propagation for the underlying hyperbolic problem.

Given any $M > \Lambda$, we will study first the problem on the bounded interval $[-M, M]$ with the boundary conditions in (3.2) imposed at the end points $\pm M$. Later, we will let M tend to infinity.

Proposition 3.1 (Existence for Riemann problem with diffusion). *For each $\epsilon > 0$ the problem (3.1) admits a smooth solution $(u^\epsilon, v^\epsilon) \in C^0([-M, M], \Omega \times [-1, 1])$ (space of continuous functions) given by the implicit formula:*

$$(3.6) \quad \begin{aligned} u^\epsilon(\xi) &= u_L + (u_R - u_L) \frac{\int_{-M}^{\xi} e^{-h^\epsilon(u^\epsilon; \zeta)/\epsilon} B_0(u^\epsilon, v^\epsilon)^{-1} d\zeta}{\int_{-M}^M e^{-h^\epsilon(u^\epsilon; \zeta)/\epsilon} B_0(u^\epsilon, v^\epsilon)^{-1} d\zeta}, \\ v^\epsilon(\xi) &= -1 + 2 \frac{\int_{-\infty}^{\xi} e^{-\zeta^2/2\epsilon^p} d\zeta}{\int_{-\infty}^{+\infty} e^{-\zeta^2/2\epsilon^p} d\zeta}, \end{aligned}$$

with

$$h^\epsilon(u^\epsilon; \xi) := \int_{\alpha}^{\xi} (\zeta - \lambda(u^\epsilon, v^\epsilon)) G(u^\epsilon, v^\epsilon) d\zeta.$$

Moreover, these solutions u^ϵ and v^ϵ are monotone, bounded, and continuous, and have uniformly bounded total variation:

$$TV(u^\epsilon) \leq |u_R - u_L|, \quad TV(v^\epsilon) \leq 2.$$

Proof. Solving the second equation in (3.1) is immediate. On the other hand, we can rewrite the problem (3.1) as

$$\begin{aligned} B_0(u^\epsilon, v^\epsilon)u_\xi^\epsilon &= \varphi, \\ (-\xi + \lambda(u^\epsilon, v^\epsilon)) G(u^\epsilon, v^\epsilon) \varphi &= \epsilon\varphi_\xi. \end{aligned}$$

Given $\tilde{u} \in C^0([-M, M], \Omega)$ we consider the solution $u^\epsilon(\tilde{u}; \xi)$ of the “linearized” problem

$$\begin{aligned} B_0(\tilde{u}, v^\epsilon)u_\xi^\epsilon &= \varphi, \\ (-\xi + \lambda(\tilde{u}, v^\epsilon)) G(\tilde{u}, v^\epsilon) \varphi &= \epsilon\varphi_\xi, \end{aligned}$$

together with the boundary conditions

$$u^\epsilon(-M) = u_L, \quad u^\epsilon(M) = u_R.$$

The solution is explicitly given by

$$(3.7) \quad \begin{aligned} u^\epsilon(\tilde{u}; \xi) &= u_L + (u_R - u_L) \frac{\int_{-M}^{\xi} \varphi(\tilde{u}; \zeta) B_0(\tilde{u}, v^\epsilon)^{-1} d\zeta}{\int_{-M}^M \varphi(\tilde{u}; \zeta) B_0(\tilde{u}, v^\epsilon)^{-1} d\zeta}, \\ \varphi(\tilde{u}; \xi) &= \exp(-h^\epsilon(\tilde{u}; \xi)/\epsilon), \\ h^\epsilon(\tilde{u}; \xi) &= \int_\alpha^\xi (\zeta - \lambda(\tilde{u}, v^\epsilon)) G(\tilde{u}, v^\epsilon) d\zeta, \end{aligned}$$

in which $\alpha \in [-M, M]$ is arbitrary. The above formulas determine a map T^ϵ that takes $\tilde{u} \in C^0([-M, M], \Omega)$ to the function $u^\epsilon(\tilde{u}; \cdot) \in C^0([-M, M], \Omega)$. We need to find a fixed point of T^ϵ .

The uniform bounds on $\lambda(\tilde{u}, v)$ and $G(\tilde{u}, v)$ (for any $\tilde{u} \in C^0([-M, M], \Omega)$ and $v \in [-1, 1]$) allow us to choose $\alpha_\epsilon \in [-M, M]$ so that

$$h^\epsilon(\tilde{u}; \xi) \geq 0, \quad \xi \in [-M, M]; \quad h^\epsilon(\tilde{u}; \alpha_\epsilon) = 0.$$

Consequently, for all $\xi \in [-M, M]$ and for some constant c_4 we have

$$0 \leq h^\epsilon(\tilde{u}; \xi) \leq c_4,$$

so that

$$c_3^{-1} \exp(-c_4/\epsilon) \leq \varphi(\tilde{u}, \xi) B_0(\tilde{u}, v^\epsilon)^{-1} d\xi \leq c_2^{-1}.$$

We also obtain the uniform bound

$$\begin{aligned} \left| \frac{d}{d\xi} u_\epsilon(\tilde{u}; \xi) \right| &\leq |u_R - u_L| \frac{\varphi(\tilde{u}; \xi) B_0(\tilde{u}, v^\epsilon)^{-1}}{\int_{-M}^M \varphi(\tilde{u}; \zeta) B_0(\tilde{u}, v^\epsilon)^{-1} d\zeta} \\ &\leq |u_R - u_L| \frac{c_2^{-1}}{2M c_3^{-1} \exp(-c_4/\epsilon)} \\ &\leq |u_R - u_L| \frac{c_3 \exp(c_4/\epsilon)}{2M c_2}. \end{aligned}$$

The bound above being independent of \tilde{u} , we deduce that the family T_ϵ is equicontinuous and its image is relatively compact in $C^0([-M, M], \Omega)$. Since this image is a convex closed subset of the Banach space $C^0([-M, M], \Omega)$, Schauder’s fixed point theorem applies and ensures that T^ϵ admits a fixed point. Hence, there exists $u^\epsilon \in C^0([-M, M], \Omega)$ such that $T^\epsilon(u^\epsilon) = u^\epsilon$, and the representation formula (3.6) holds. The uniform total variation bounds follow directly from (3.6). \square

3.2. Passage to the limit. Using the notation in Proposition 3.1, we continue with the following two lemmas.

Lemma 3.2 (Existence of a pointwise limit). *After extracting a subsequence if necessary, the sequence u^ϵ converges pointwise to a limiting function u lying in the space $BV([-M, M])$ (of all functions with bounded variation):*

$$u^\epsilon(\xi) \rightarrow u(\xi), \quad \xi \in [-M, M],$$

which satisfies, in the sense of distributions,

$$(3.8) \quad \begin{aligned} -\xi \frac{d}{d\xi} \gamma_-(u) + \frac{d}{d\xi} f_-(\gamma_-(u)) &= 0, & \xi < 0, \\ -\xi \frac{d}{d\xi} \gamma_+(u) + \frac{d}{d\xi} f_+(\gamma_+(u)) &= 0, & \xi > 0. \end{aligned}$$

Lemma 3.3 (Entropy inequalities). *The limit u given by Lemma 3.2 also satisfies, in the sense of distributions,*

$$(3.9) \quad \begin{aligned} -\xi \frac{d}{d\xi} \eta(\gamma_-(u)) + \frac{d}{d\xi} q_-(\gamma_-(u)) &\leq 0, & \xi < 0, \\ -\xi \frac{d}{d\xi} \eta(\gamma_+(u)) + \frac{d}{d\xi} q_+(\gamma_+(u)) &\leq 0, & \xi > 0, \end{aligned}$$

for all convex entropy functions η and associated entropy flux $q'_\pm = \eta' f'_\pm$.

Proof of Lemma 3.2. In view of Lemma 3.1, Helly's compactness theorem applies and, as $\epsilon \rightarrow 0$, ensures the existence of a pointwise limit (u, v) with bounded variation. Fix $\theta \in (0, M)$ and let $\phi \in C_0^\infty((\theta, M))$ be a compactly supported test-function. In the integral form, (3.1) becomes

$$-\int_0^M \xi A_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon \phi d\xi + \int_0^M A_1(u^\epsilon, v^\epsilon) u_\xi^\epsilon \phi d\xi = \epsilon \int_0^M (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi \phi d\xi,$$

that is

$$-\int_0^M \xi A_0(u^\epsilon, 1) u_\xi^\epsilon \phi d\xi + \int_0^M A_1(u^\epsilon, 1) u_\xi^\epsilon \phi d\xi + \Omega^\epsilon = \epsilon \int_0^M (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi \phi d\xi,$$

where

$$\Omega^\epsilon := \int_0^M \xi (A_0(u^\epsilon, 1) - A_0(u^\epsilon, v^\epsilon)) u_\xi^\epsilon \phi d\xi - \int_0^M (A_1(u^\epsilon, 1) - A_1(u^\epsilon, v^\epsilon)) u_\xi^\epsilon \phi d\xi.$$

Using (3.3) and (3.4), we can write

$$-\int_0^M \xi \frac{d}{d\xi} \gamma_+(u^\epsilon) \phi d\xi + \int_0^M \frac{d}{d\xi} f_+(\gamma_+(u^\epsilon)) \phi d\xi + \Omega^\epsilon = \epsilon \int_0^M (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi \phi d\xi.$$

The term Ω^ϵ vanishes with ϵ , since

$$\begin{aligned} |\Omega^\epsilon| &\leq M \int_\theta^M \omega_0 |1 - v^\epsilon| |u_\xi^\epsilon| |\phi| d\xi + \int_\theta^M \omega_1 |1 - v^\epsilon| |u_\xi^\epsilon| |\phi| d\xi \\ &\leq (M \omega_0 + \omega_1) |1 - v^\epsilon(\theta)| \|\phi\|_\infty TV(u^\epsilon), \end{aligned}$$

where the total variation term $TV(u^\epsilon)$ remains bounded and $|1 - v^\epsilon(\theta)|$ tends to 0. On the other hand, we have

$$\begin{aligned} \left| \epsilon \int_0^M (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi \phi d\xi \right| &= \left| \epsilon \int_0^M (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon) \phi_\xi d\xi \right| \\ &\leq \epsilon \|\phi_\xi\|_\infty c_3 TV(u^\epsilon), \end{aligned}$$

which also converges to zero. As ϵ tends to 0, we conclude that

$$\int_0^M \left(-\xi \frac{d}{d\xi} \gamma_+(u) + \frac{d}{d\xi} f_+(\gamma_+(u)) \right) \phi d\xi = 0,$$

which is the first condition in (3.8). The same arguments apply on the interval $[-M, 0]$, by using test-functions supported in the interval $(-M, \theta)$, with $\theta < 0$. \square

Proof of Lemma 3.3. Fix $\theta > 0$ and let $\phi \in C_0^\infty((\theta, M))$ be a non-negative test-function. Multiplying (3.1) by $\eta'(C(u^\epsilon, 1))\phi$, we get

$$(3.10) \quad \begin{aligned} & - \int_0^M \xi A_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon \eta'(C(u^\epsilon, 1)) \phi \, d\xi + \int_0^M A_1(u^\epsilon, v^\epsilon) u_\xi^\epsilon \eta'(C(u^\epsilon, 1)) \phi \, d\xi \\ & = \epsilon \int_0^M (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi \eta'(C(u^\epsilon, 1)) \phi \, d\xi. \end{aligned}$$

Observing that

$$\begin{aligned} & \left| \int_0^M \xi (A_0(u^\epsilon, v^\epsilon) - A_0(u^\epsilon, 1)) u_\xi^\epsilon \eta'(C(u^\epsilon, 1)) \phi \, d\xi \right| \\ & \leq M \omega_0 |1 - v^\epsilon(\theta)| \, TV(u^\epsilon) \|\eta'(C(\cdot, 1))\|_\infty \|\phi\|_\infty, \end{aligned}$$

and similarly for the coefficient A_1 , we see that the left-hand side of (3.10) is equivalent (modulo terms that tend to zero with ϵ) to

$$\begin{aligned} & - \int_0^M \xi \partial_u(C(u^\epsilon, 1)) u_\xi^\epsilon \eta'(C(u^\epsilon, 1)) \phi \, d\xi + \int_0^M A_1(u^\epsilon, 1) u_\xi^\epsilon \eta'(C(u^\epsilon, 1)) \phi \, d\xi \\ & = - \int_0^M \xi \frac{d}{d\xi} \eta(\gamma_+(u^\epsilon)) \phi \, d\xi + \int_0^M \frac{d}{d\xi} q_+(\gamma_+(u^\epsilon)) \phi \, d\xi. \end{aligned}$$

On the other hand, the right-hand side of (3.10) can be rewritten in the form

$$\begin{aligned} & \epsilon \int_0^M (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi \eta'(C(u^\epsilon, 1)) \phi \, d\xi \\ & = -\epsilon \int_0^M B_0(u^\epsilon, v^\epsilon) (u_\xi^\epsilon)^2 A_0(u^\epsilon, 1) \eta''(C(u^\epsilon, 1)) \phi \, d\xi \\ & \quad - \epsilon \int_0^M B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon \eta'(C(u^\epsilon, 1)) \phi_\xi \, d\xi, \end{aligned}$$

in which the first term is non-positive and the second one tends to 0. Thus, letting $\epsilon \rightarrow 0$ we obtain

$$- \int_0^M \xi \frac{d}{d\xi} \eta(\gamma_+(u)) \phi \, d\xi + \int_0^M \frac{d}{d\xi} q_+(\gamma_+(u)) \phi \, d\xi \leq 0,$$

which yields the second identity in the statement of the lemma. The derivation of the first identity in the half-space $\xi < 0$ is completely similar. \square

3.3. Riemann problem for the hyperbolic coupling problem. In view of the boundary condition (3.2), it is natural to extend u by

$$(3.11) \quad u(\xi) = \begin{cases} u_L, & \xi \leq -M, \\ u_R, & \xi \geq M. \end{cases}$$

The conclusions in Lemmas 3.2 and 3.3 then clearly hold on the intervals $(-\infty, -M)$ and $(M, +\infty)$. In addition, an interface condition for the solution u at the end points $\xi = -M$ and $\xi = M$ is now derived, which is necessary to ensure that (3.8) and (3.9) extend (in the sense of distributions) to $(0, +\infty)$ and $(-\infty, 0)$.

Lemma 3.4. *As ϵ tends to 0, the solution u^ϵ converges uniformly toward u_R (repectively u_L) on the interval $(\Lambda, M]$ (resp. $(-M, -\Lambda]$).*

Proof. Let $\xi \in (\frac{\Lambda+M}{2}, M)$ be given. According to (3.6) we have

$$\begin{aligned} |u^\epsilon(\xi) - u_R| &= |u_L - u_R| \frac{\int_{\xi}^M e^{-h^\epsilon(u^\epsilon)/\epsilon} B_0(u^\epsilon, v^\epsilon)^{-1} d\xi}{\int_{-M}^M e^{-h^\epsilon(u^\epsilon)/\epsilon} B_0(u^\epsilon, v^\epsilon)^{-1} d\xi} \\ &\leq |u_L - u_R| \frac{\int_{\frac{\Lambda+M}{2}}^M e^{-h^\epsilon(u^\epsilon)/\epsilon} B_0(u^\epsilon, v^\epsilon)^{-1} d\xi}{\int_{-M}^M e^{-h^\epsilon(u^\epsilon)/\epsilon} B_0(u^\epsilon, v^\epsilon)^{-1} d\xi}. \end{aligned}$$

We use here the constant $\alpha \in [-\Lambda, \Lambda]$ as the lower integration bound for both h^ϵ and the function ($\xi \in \mathbb{R}$)

$$h(\xi) := \int_{\alpha}^{\xi} (\zeta - \lambda(u(\zeta), v(\zeta))) G(u(\zeta), v(\zeta)) d\zeta,$$

hence $h \geq 0$. Moreover, $h^\epsilon(u^\epsilon, \cdot)$ converges uniformly to h , with

$$\begin{aligned} &|h^\epsilon(u^\epsilon, \xi) - h(\xi)| \\ &= \left| \int_{\alpha}^{\xi} (\lambda(u^\epsilon(\zeta), v^\epsilon(\zeta)) - \lambda(u(\zeta), v(\zeta))) G(u^\epsilon(\zeta), v^\epsilon(\zeta)) d\zeta \right| \\ &\leq \left(\frac{1}{c} \omega_1 + \|A_1\|_{\infty} \frac{\omega_0}{c^2} \right) \|G\|_{\infty} \int_{-M}^M (|u^\epsilon(\zeta) - u(\zeta)| + |v^\epsilon(\zeta) - v(\zeta)|) d\zeta \\ &\leq \left(\frac{1}{c} \omega_1 + \|A_1\|_{\infty} \frac{\omega_0}{c^2} \right) \|G\|_{\infty} (\|u^\epsilon - u\|_{L^1} + \|v^\epsilon - v\|_{L^1}). \end{aligned}$$

The uniform convergence of $h^\epsilon(u^\epsilon, \cdot)$ toward a positive continuous function h such that $h(\alpha) = 0$ insures that there exists $\epsilon_0 > 0$ together with $A > B > 0$ and $\eta > 0$ such that, for all $\epsilon < \epsilon_0$,

$$\begin{aligned} h^\epsilon(u^\epsilon, \xi) &\geq A, & \frac{\Lambda + M}{2} &\leq \xi \leq M, \\ h^\epsilon(u^\epsilon, \xi) &\leq B, & \xi &\in [-M, M], \quad |\xi - \alpha| \leq \eta. \end{aligned}$$

Thus, we deduce that

$$\begin{aligned} |u^\epsilon(\xi) - u_R| &\leq |u_R - u_L| \frac{\frac{M-\Lambda}{2} e^{-A/\epsilon} c_2^{-1}}{\eta e^{-B/\epsilon} c_3^{-1}} \\ &= |u_R - u_L| \frac{M - \Lambda}{2\eta} \frac{c_3}{c_2} e^{-(A-B)/\epsilon}, \end{aligned}$$

and u^ϵ converges uniformly toward u_R on the open interval $(\frac{\Lambda+M}{2}, M)$. The same argument leads to the uniform convergence of u^ϵ toward u_L on $(-M, \frac{\Lambda-M}{2})$. \square

We summarize the results established in Lemmas 3.2, 3.3, and 3.4, as follows.

Theorem 3.5 (The Riemann problem for the coupling of two scalar equations). *Up to extracting a subsequence, the solution u^ϵ to (3.1)-(3.2) converges pointwise to a function $u \in BV(\mathbb{R})$,*

$$u^\epsilon(\xi) \rightarrow u(\xi), \quad \xi \in \mathbb{R},$$

which satisfies the conservation laws and entropy inequalities

$$\begin{aligned} -\xi \frac{d}{d\xi} \gamma_-(u) + \frac{d}{d\xi} f_-(\gamma_-(u)) &= 0, & \xi < 0, \\ -\xi \frac{d}{d\xi} \gamma_+(u) + \frac{d}{d\xi} f_+(\gamma_+(u)) &= 0, & \xi > 0, \end{aligned}$$

and

$$\begin{aligned} -\xi \frac{d}{d\xi} \eta(\gamma_-(u)) + \frac{d}{d\xi} q_-(\gamma_-(u)) &\leq 0, & \xi < 0, \\ -\xi \frac{d}{d\xi} \eta(\gamma_+(u)) + \frac{d}{d\xi} q_+(\gamma_+(u)) &\leq 0, & \xi > 0, \end{aligned}$$

for all convex entropy pairs, together with the boundary conditions

$$u(-\infty) = u_L, \quad u(+\infty) = u_R.$$

Equivalently, in terms of the function w in (2.4), we have established

$$\begin{aligned} -\xi \frac{d}{d\xi} w + \frac{d}{d\xi} f_-(w) &= 0, & -\xi \frac{d}{d\xi} \eta(w) + \frac{d}{d\xi} q_-(w) &\leq 0, & \xi < 0, \\ -\xi \frac{d}{d\xi} w + \frac{d}{d\xi} f_+(w) &= 0, & -\xi \frac{d}{d\xi} \eta(w) + \frac{d}{d\xi} q_+(w) &\leq 0, & \xi > 0, \end{aligned}$$

with

$$w(-\infty) = w_L, \quad w(+\infty) = w_R.$$

We have thus established that the interface problem admits a solution which has bounded variation.

4. EXISTENCE THEORY FOR SYSTEMS

4.1. Terminology and notation. We will now generalize the technique developed in Tzavaras [34] and Joseph and LeFloch [22, 25], and cover the class of nonconservative and resonant systems (1.4) under consideration. We follow closely the notation and presentation in [25].

Specifically we consider the *diffusive Riemann problem* (1.4) with Riemann data u_L, u_R , and establish that, provided $u_L, u_R \in \mathcal{B}(\delta_1)$ with a sufficiently small $\delta_1 < \delta_0$ and under some structural hypotheses on the matrix fields A_0, A_1 , this problem admits a smooth, self-similar solution, $u_\epsilon = u_\epsilon(x/t) \in \mathcal{B}(\delta_0)$ and $v_\epsilon = v_\epsilon(x/t) \in [-1, 1]$, which has uniformly bounded total variation

$$(4.1) \quad TV(u_\epsilon) + TV(v_\epsilon) \leq C,$$

for some uniform $C > 0$. Solutions to (1.4) will be decomposed in terms of “wave strengths” of the associated Riemann problem (1.1). The uniform estimate (4.1) is the key to the convergence analysis as $\epsilon \rightarrow 0$, and the proof of the existence of the Riemann solution to the underlying hyperbolic problem, discussed in the following section.

We are interested in solutions u taking values in a small neighborhood of a given state (normalized to be the origin without loss of generality), that is, in the ball $\Omega := \mathcal{B}(\delta_0)$ with (small) radius δ_0 . For each $u \in \mathcal{B}(\delta_0)$ and $v \in [-1, 1]$, let $\lambda_1(u, v) < \dots < \lambda_N(u, v)$ be the real and distinct eigenvalues of the $N \times N$ matrix

$$A(u, v) := A_1(u, v) A_0(u, v)^{-1},$$

and let $l_1(u, v), \dots, l_N(u, v)$ and $r_1(u, v), \dots, r_N(u, v)$ be basis of left- and right-eigenvectors, respectively, normalized so that $l_i(u, v) \cdot r_j(u, v) = 0$ if $i \neq j$ and $l_i(u, v) \cdot r_i(u, v) = 1$.

By reducing δ_0 if necessary, we may assume that the wave speeds $\lambda_i(u, v)$ are sufficiently close to the constants $\lambda_i(0, 0)$ and, in particular, are uniformly separated for all $u \in \mathcal{B}(\delta_0)$ in the sense that, for some constants

$$-M < \underline{\Lambda}_1 < \overline{\Lambda}_1 < \underline{\Lambda}_2 < \dots < \underline{\Lambda}_N < \overline{\Lambda}_N < M,$$

$$(4.2) \quad \underline{\Lambda}_i := \lambda_i(0, 0) - O(\delta_0), \quad \overline{\Lambda}_i := \lambda_i(0, 0) + O(\delta_0)$$

and

$$(4.3) \quad \underline{\Lambda}_i \leq \lambda_i(u, v) \leq \overline{\Lambda}_i, \quad u \in \mathcal{B}(\delta_0), v \in [-1, 1].$$

Let m be the index associated with the resonant wave, i.e. such that λ_m may change sign. In addition, for δ_0 sufficiently small the vectors $r_i(u, v)$ are sufficiently close to $r_i(0, 0)$ and we assume that

$$(4.4) \quad \begin{aligned} l_i(u_1, v) \cdot r_i(u_2, v) &\geq 1 - \delta_0, & u_1, u_2 \in \mathcal{B}(\delta_0), v \in [-1, 1], \\ |l_i(u_1, v) \cdot r_j(u_2, v)| &\leq \delta_0, & u_1, u_2 \in \mathcal{B}(\delta_0), i \neq j, v \in [-1, 1]. \end{aligned}$$

In (1.4), the matrix $B_0 = B_0(u, v)$ is assumed to be non-degenerate and depend smoothly upon u and v . We treat the case that the diffusion matrix $B(u, v) := B_0(u, v) A_0(u, v)^{-1}$ is sufficient close to the identity matrix, that is, for some given matrix norm and $\eta > 0$ sufficiently small,

$$(4.5) \quad \sup_{\substack{u \in \mathcal{B}(\delta_0) \\ v \in [-1, 1]}} |B(u, v) - I| \leq \eta.$$

To handle arbitrary diffusion matrices $B(u, v)$, we follow Joseph and LeFloch [27] and introduce the *generalized eigenvalue problem*:

$$(4.6) \quad \begin{aligned} (-\xi \text{Id} + A(u, v)) \widehat{r}_i(u, v, \xi) &= \mu_i(u, v, \xi) B(u, v) \widehat{r}_i(u, v, \xi), \\ \widehat{l}_i(u, v, \xi) \cdot (-\xi \text{Id} + A(u, v)) &= \mu_i(u, v, \xi) \widehat{l}_i(u, v, \xi) \cdot B(u, v). \end{aligned}$$

with unknowns the vectors $\widehat{r}_i(u, v, \xi)$, $\widehat{l}_i(u, v, \xi)$ and the scalars $\mu_i(u, v, \xi)$. We impose the following normalization to generalized left- and right-eigenvectors:

$$\begin{aligned} \widehat{r}_i(u, v, \xi) \cdot \widehat{r}_i(u, v, \xi) &= 1, \\ \widehat{l}_i(u, v, \xi) \cdot B(u, v) \widehat{r}_j(u, v, \xi) &= 0 \quad \text{if } i \neq j, \\ \widehat{l}_i(u, v, \xi) \cdot B(u, v) \widehat{r}_i(u, v, \xi) &= 1. \end{aligned}$$

Multiplying the first equation in (4.6) on the left by $\widehat{r}_i(u, v, \xi)$ and rearranging terms, we get

$$(4.7) \quad \mu_i(u, v, \xi) = (-\xi + \widehat{\lambda}_i(u, v, \xi)) d_i(u, v, \xi),$$

where

$$(4.8) \quad \begin{aligned} \widehat{\lambda}_i(u, v, \xi) &:= \widehat{r}_i(u, v, \xi) \cdot A(u, v) \widehat{r}_i(u, v, \xi), \\ 1/d_i(u, v, \xi) &:= \widehat{r}_i(u, v, \xi) \cdot B(u, v) \widehat{r}_i(u, v, \xi). \end{aligned}$$

Clearly, in the special case where $B(u, v) = I$, we find

$$\mu_i(u, v, \xi) = -\xi + \lambda_i(u, v), \quad \widehat{r}_i(u, v, \xi) = r_i(u, v), \quad \widehat{l}_i(u, v, \xi) = l_i(u, v).$$

So, by continuity, when B gets closer to the identity matrix, the coefficients $d_i(u, v, \xi)$ and $\widehat{\lambda}_i(u, v, \xi)$ get closer to 1 and $\lambda_i(u, v)$, respectively. In consequence, under the assumption $|B(u, v) - I| < \eta$ with η sufficiently small and by increasing the gaps $\overline{\Lambda}_i - \underline{\Lambda}_i$ if necessary, we can always assume that

$$\begin{aligned} \underline{\Lambda}_i - O(\eta) &\leq \widehat{\lambda}_i(u, v, \xi) \leq \overline{\Lambda}_i + O(\eta), \\ 1 - O(\eta) &\leq d_i(u, v, \xi) \leq 1 + O(\eta) \end{aligned}$$

for $u \in \mathcal{B}(\delta_0)$, $v \in [-1, 1]$, $\xi \in [-M, M]$. The following property was pointed out in [27].

Lemma 4.1. *The ξ -derivatives of the generalized eigenvectors and eigenvalues satisfy*

$$(4.9) \quad \begin{aligned} |\partial_\xi \widehat{r}_i(u, v, \xi)| &= O(\eta), \\ \partial_\xi \mu_i(u, v, \xi) &= -1 + O(\eta). \end{aligned}$$

Finally, we introduce the coefficient

$$(4.10) \quad \nu := \sup \left| \widehat{l}_i \cdot \partial_v (B \widehat{r}_j) \right|$$

allows one to measure how closed the left-hand and right-hand hyperbolic models are and, from now on, this coefficient is assumed to be sufficiently small.

Example 4.2 (p-system). *The following example illustrates the meaning of ν . Consider the coupling between two systems of two conservation laws, specifically p-systems, with two different pressure laws $p_\pm = p_\pm(\tau)$*

$$\begin{aligned} \partial_t \tau - \partial_x V &= 0, \\ \partial_t V + \partial_x p_\pm(\tau) &= 0, \quad \tau > 0, \quad V \in \mathbb{R}, \end{aligned}$$

in which the associated flux F_\pm have Jacobian matrices

$$\nabla F_\pm = \begin{pmatrix} 0 & -1 \\ p'_\pm(\tau) & 0 \end{pmatrix}.$$

Suppose that the coupling is based on $B = \text{Id}$ and the average matrix

$$A(\tau, v) = \frac{1+v}{2} \nabla F_+ + \frac{1-v}{2} \nabla F_- = \begin{pmatrix} 0 & -1 \\ \frac{1+v}{2} p'_+(\tau) + \frac{1-v}{2} p'_-(\tau) & 0 \end{pmatrix}.$$

Then, the eigenvalues and eigenvectors of the system are ($j = 1, 2$ and \pm corresponding to the two wave families)

$$\begin{aligned} \lambda_j(\tau, v) &= \pm \sqrt{-\frac{1+v}{2} p'_+(\tau) - \frac{1-v}{2} p'_-(\tau)}, \\ \widehat{r}_j(\tau, v) &= \left(1, \mp \sqrt{-\frac{1+v}{2} p'_+(\tau) - \frac{1-v}{2} p'_-(\tau)} \right), \\ \widehat{l}_j(\tau, v) &= \left(\pm \sqrt{-\frac{1+v}{2} p'_+(\tau) - \frac{1-v}{2} p'_-(\tau)}, 1 \right), \end{aligned}$$

and a tedious calculation yields

$$\begin{aligned} \left| \widehat{l}_i \cdot \partial_v(B \widehat{r}_j) \right| &\leq \frac{|p'_+(\tau) - p'_-(\tau)|}{2 \sqrt{-\frac{1+v}{2} p'_+(\tau) - \frac{1-v}{2} p'_-(\tau)}} \\ &\leq \frac{|p'_+(\tau) - p'_-(\tau)|}{2 \min \left(\sqrt{-p'_+(\tau)}, \sqrt{-p'_-(\tau)} \right)}, \end{aligned}$$

so that

$$\nu \leq C \sup |p'_+ - p'_-|.$$

4.2. Equations satisfied by the characteristic coefficients. We supplement (cf. (1.4))

$$(4.11) \quad \begin{aligned} (-\xi A_0(u^\epsilon, v^\epsilon) + A_1(u^\epsilon, v^\epsilon)) u_\xi^\epsilon &= \epsilon (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi, \\ -\xi v_\xi^\epsilon &= \epsilon^p v_{\xi\xi}^\epsilon \end{aligned}$$

with the following boundary conditions, inherited from Riemann initial data

$$(4.12) \quad \begin{aligned} u^\epsilon(-M) &= u_L, & u^\epsilon(M) &= u_R, \\ v^\epsilon(-M) &= -1, & v^\epsilon(M) &= 1. \end{aligned}$$

We describe now an asymptotic expansion for the solution of the diffusive Riemann problem (4.11)-(4.12). To handle an arbitrary diffusion matrix, the decomposition must be based on the modified eigenvectors (4.6). We first solve explicitly the equation concerning v^ϵ :

$$(4.13) \quad \psi^\epsilon(\xi) := v_\xi^\epsilon(\xi) = 2 \frac{e^{-\frac{\xi^2}{2\epsilon^p}}}{\int_{-M}^M e^{-\frac{x^2}{2\epsilon^p}} dx}.$$

Remark 4.3. In the limiting case $p = +\infty$, we would formally get a Dirac mass solution $\psi = v_\xi = 2\delta_{\xi=0}$ and the transition from $v_L = -1$ to $v_R = 1$ at the interface would then be discontinuous.

The main equation (4.11) now reads

$$(4.14) \quad (-\xi I + A(u^\epsilon, v^\epsilon)) A_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon = \epsilon (B(u^\epsilon, v^\epsilon) A_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi.$$

To deal with this equation, we introduce a decomposition of the vector $A_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon(\xi)$ on the basis of eigenvectors $\widehat{r}_j(u^\epsilon(\xi), v^\epsilon(\xi), \xi)$, that is,

$$(4.15) \quad \begin{aligned} A_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon(\xi) &= \sum_{j=1}^N a_j^\epsilon(\xi) \widehat{r}_j(u^\epsilon, v^\epsilon, \xi), \\ a_j^\epsilon(\xi) &= \widehat{l}_j(u^\epsilon, v^\epsilon, \xi) B(u^\epsilon, v^\epsilon) A_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon(\xi), \end{aligned}$$

where the functions a_j^ϵ are referred to as the *characteristic coefficients*. Removing the explicit dependence in ϵ , the right-hand side of (4.14) takes the form

$$\begin{aligned} (B(u, v) A_0(u, v) u_\xi)_\xi &= \sum_j a'_k B(u, v) \widehat{r}_j(u, v, \cdot) \\ &\quad + \sum_{j,k} a_j a_k D_u(B \widehat{r}_j)(u, v, \cdot) A_0(u, v)^{-1} \widehat{r}_k(u, v, \cdot) \\ &\quad + \sum_j a_j \partial_v(B \widehat{r}_j)(u, v, \cdot) v_\xi + \sum_j a_j B(u, v) \partial_\xi \widehat{r}_j(u, v, \cdot). \end{aligned}$$

Therefore, given any solution to (4.11), we obtain

$$\begin{aligned} &\sum_j \left(\epsilon a'_j B(u, v) \widehat{r}_j(u, v, \cdot) - a_j (-\xi + A(u, v)) \widehat{r}_j(u, v, \cdot) \right) \\ &= -\epsilon \sum_{j,k} a_j a_k D_u(B \widehat{r}_j)(u, v, \cdot) A_0(u, v)^{-1} \widehat{r}_k(u, v, \cdot) - \epsilon \sum_j a_j \partial_v(B \widehat{r}_j)(u, v, \cdot) \psi \\ &\quad - \epsilon \sum_j a_j B(u, v) \partial_\xi \widehat{r}_j(u, v, \cdot). \end{aligned}$$

Now, multiplying the above equations by each vector $\widehat{l}_i(u, v, \cdot)$ for $i = 1, \dots, N$ and relying on the equation (4.6), we arrive at a *coupled system of N differential equations* for the characteristic coefficients a_i :

$$(4.16a) \quad a'_i - \frac{\mu_i(u, v, \cdot)}{\epsilon} a_i = \eta L_i(u, v, \cdot) + Q_i(u, v, \cdot) + S_i(u, v, \cdot),$$

where the linear, quadratic, and source terms are defined by

$$(4.16b) \quad \begin{aligned} L_i(u, v, \cdot) &:= \sum_j \pi_{ij}(u, v, \cdot) a_j, & Q_i(u, v, \cdot) &:= \sum_{j,k} \kappa_{ijk}(u, v, \cdot) a_j a_k, \\ S_i(u, v, \cdot) &:= \sum_j \sigma_{ij}(u, v, \cdot) a_j \psi, \end{aligned}$$

respectively, with

$$(4.16c) \quad \begin{aligned} \pi_{ij}(u, v, \cdot) &:= -\eta^{-1} (\widehat{l}_i \cdot B \partial_\xi \widehat{r}_j)(u, v, \cdot), \\ \kappa_{ijk}(u, v, \cdot) &:= -(\widehat{l}_i \cdot D_u(B \widehat{r}_j) A_0^{-1} \widehat{r}_k)(u, v, \cdot), \\ \sigma_{ij}(u, v, \cdot) &:= (\widehat{l}_i \cdot \partial_v(B \widehat{r}_j))(u, v, \cdot). \end{aligned}$$

The main equation (4.11) is therefore completely equivalent to (4.15)-(4.16). In view of (4.9) and (4.10), (4.16) takes the form

$$(4.17) \quad a'_i - \frac{1}{\epsilon} \mu_i(u, v, \cdot) a_i = O(\eta) \sum_j |a_j| + O(1) \sum_{j,k} |a_j| |a_k| + O(\nu) \sum_j |a_j| |\psi|.$$

Consider the principal part of (4.16), that is, given some function $u = u(y)$ (which at this stage need not to be a solution to (4.16)) let us consider the following *decoupled system of N linear equations*:

$$(4.18) \quad \varphi_i^{*'} - \frac{\mu_i(u, v, \cdot)}{\epsilon} \varphi_i^* = 0, \quad i = 1, \dots, N.$$

The general solution is (up to a multiplicative constant)

$$(4.19) \quad \varphi_i^* := \frac{e^{-g_i/\epsilon}}{\int_{-M}^M e^{-g_i/\epsilon} dy}, \quad g_i(y) := -\int_{\rho_i}^y \mu_i(u, v, \cdot)(x) dx,$$

where the constants ρ_i will be chosen so that the functions g_i are non-negative (cf. Section 4.3). Clearly, the functions φ_i^* are strictly positive and their integral over \mathbb{R} equals 1.

We will search for the general solutions a_i of (4.16a) in the form

$$(4.20) \quad a_i = \tau_i \varphi_i^* + \theta_i$$

where τ_i refers to a wave strength and with θ_i small w.r.t τ_i . When solving the equation (4.16a) for a given right-hand side and considering again first-order terms, we are naturally led to consider the following *linear wave coefficients* $J_{j \rightarrow i}$, the *quadratic wave coefficients* $F_{j,k \rightarrow i}$ and the *resonant quadratic coefficients* $J_{j \rightarrow i}^\psi$ defined by

$$(4.21) \quad J_{j \rightarrow i}(y) := \varphi_i^*(y) \int_{c_i}^y \frac{\varphi_j^*(x)}{\varphi_i^*(x)} dx,$$

$$(4.22) \quad F_{j,k \rightarrow i}(y) := \varphi_i^*(y) \int_{c_i}^y \frac{\varphi_j^* \varphi_k^*}{\varphi_i^*} dx,$$

$$(4.23) \quad J_{j \rightarrow i}^\psi(y) := \varphi_i^*(\xi) \int_{c_i}^\xi \psi(x) \frac{\varphi_j^*(x)}{\varphi_i^*(x)} dx$$

for some constants $c_i \in [\underline{\Delta}_i, \overline{\Lambda}_i]$ independent of ϵ . By studying these coefficients, we will gain useful information on the possible growth of the total variation of solutions to (4.16): roughly speaking, $J_{j \rightarrow i}$ bounds the influence of the j -th family on the i -th family, $F_{j,k \rightarrow i}$ bounds the contribution on the i -th family due to waves of the j -th and k -th characteristic families and $J_{j \rightarrow i}^\psi$ bounds the influence of the j -th family on the i -th family “through” the coupling wave ψ .

4.3. Linearized wave measures. The results of this section were established earlier in Joseph and LeFloch [22] and are presented for the convenience of the reader. We fix some speed range $[\lambda_{\min}, \lambda_{\max}]$ and analyze the formula (4.19) within this speed range. We introduce first the space of functions that are “almost linear at infinity”. Observe the coefficients μ_i will belong to such spaces in their respective speed ranges $[\underline{\Delta}_i, \overline{\Lambda}_i]$.

Definition 4.4. *A function $h : [-M, M] \mapsto \mathbb{R}$ is said to belong to the class \mathcal{L} of almost linear functions if there exists two functions $d, \lambda \in L^\infty([-M, M], \mathbb{R})$, and two positive reals d_{\min}, d_{\max} such that*

$$h(x) = d(x)(\lambda(x) - x), \quad x \in [-M, M]$$

and

$$\begin{aligned} 0 < d_{\min} &\leq d(x) \leq d_{\max}, \\ -M < \lambda_{\min} &\leq \lambda(x) \leq \lambda_{\max} < M. \end{aligned}$$

Lemma 4.5. *Let $h : [-M, M] \mapsto \mathbb{R}$ be a function of class \mathcal{L} and, given $y \in [-M, M]$, set*

$$g(x) = - \int_y^x h(x') dx', \quad x \in [-M, M].$$

Then, g is Lipschitz continuous and achieves its global minimum at some (non-unique) point $\rho \in [\lambda_{\min}, \lambda_{\max}]$ such that $\lambda(\rho) = \rho$.

Proof. Using definition 4.4 of the class \mathcal{L} we get

$$\begin{aligned} h(x) &> 0, & x < \lambda_{\min}, \\ h(x) &< 0, & x > \lambda_{\max}, \end{aligned}$$

g being continuous, decreasing on $[-M, \lambda_{\min}]$ and increasing on $[\lambda_{\max}, M]$ (since $g' = -h$), we deduce g achieves its global minimum at some (non-unique) point $\rho \in [\lambda_{\min}, \lambda_{\max}]$. Moreover $h(\rho) = -g' = 0$, that means $\lambda(\rho) = \rho$. \square

We now introduce a new notation suggested by the formula (4.19) and derive useful algebraic properties. For $x, y \in [-M, M]$ and $h \in L^1([-M, M], \mathbb{R})$, we set

$$\begin{aligned}\phi(y, x; h) &:= \exp\left(\frac{1}{\epsilon} \int_y^x h(x') dx'\right), \\ I(y; h) &:= \int_{-M}^M \phi(y, x'; h) dx', \quad \varphi(x; h) := \frac{\phi(y, x; h)}{I(y; h)}.\end{aligned}$$

That is, $x \mapsto \varphi(x; h)$ is a solution (with unit mass) to the differential equation

$$\varphi' - \frac{h}{\epsilon} \varphi = 0.$$

Lemma 4.6 (Algebraic properties of the mapping ϕ). *For general functions h, \tilde{h} in $L^1([-M, M], \mathbb{R})$ and $x, y, z \in [-M, M]$ the following algebraic properties hold:*

$$\begin{aligned}\frac{\phi(x, y; h)}{\phi(x, y; \tilde{h})} &= \phi(x, y; h - \tilde{h}), \\ \frac{\phi(x, y; h)}{\phi(z, y; h)} &= \frac{\phi(y, z; h)}{\phi(y, x; h)} = \phi(x, z; h), \\ \phi(x, y; h) &= \phi(y, x; -h) = \frac{1}{\phi(y, x; h)}.\end{aligned}$$

In view of these properties, the definition of $\varphi(x; h)$ is checked to be independent of the variable y . When the function h belongs to the class \mathcal{L} , by taking $y = \rho$ given by Lemma 4.5 we obtain a negative argument in the exponential defining ϕ and that argument vanishes at points where g is minimized.

Now, fix μ_1, μ_2 in the class \mathcal{L} , and let $\varphi_1 := \varphi(\cdot; \mu_1)$ and $\varphi_2 := \varphi(\cdot; \mu_2)$ be solutions to the differential equation associated with μ_1 and μ_2 , respectively. Denote by ρ_1 and ρ_2 the minimization points of the associated functions as defined in Lemma 4.5. Moreover, fix some yet unspecified scalar $c \in [\lambda_{\min}, \lambda_{\max}]$ which we take to be independent of ϵ . We then want to control the linear coefficient

$$(4.24a) \quad J_{\varphi_2 \rightarrow \varphi_1}(y) = \varphi_1(y) \int_c^y \frac{\varphi_2(x)}{\varphi_1(x)} dx$$

characterising the first order (linear) influence of φ_2 on φ_1 . Using Lemma 4.6 we find those two following equivalent expressions, both useful in the sequel according to the sign of $\mu_2 - \mu_1$,

$$\begin{aligned}J_{\varphi_2 \rightarrow \varphi_1}(y) &= \frac{I(\rho_1; \mu_1)}{I(\rho_2; \mu_2)} \phi(\rho_2, \rho_1; \mu_2) \varphi_1(y) \int_c^y \phi(\rho_1, x; \mu_2 - \mu_1) dx, \\ J_{\varphi_2 \rightarrow \varphi_1}(y) &= \varphi_2(y) \int_c^y \phi(x, y; \mu_1 - \mu_2) dx.\end{aligned}$$

In order to estimate those coefficients, we also need for more information on the asymptotical behavior of appearing quantities as ϵ tends to 0. The two following lemmas will give it.

Lemma 4.7 (Asymptotic behavior of ϕ). *Let $[x, y]$ be an interval of $[-M, M]$, (with $x < y$), and let h be a continuous function on $[x, y]$. If h is strictly positive bounded, say we have $h(x') \geq h_{\min} > 0$ on $[x, y]$, then the following integral is at most linear in ϵ*

$$\int_x^y \phi(x', y, -h) dx' \leq \frac{\epsilon}{h_{\min}},$$

Lemma 4.8 (Asymptotic behavior of $I(\rho; h)$). *For a function h in the class \mathcal{L} , and ρ defined in Lemma 4.5, the integrals $I(\rho; h)$ satisfy*

$$c\epsilon \leq I(\rho; h) \leq 2M.$$

Proof of Lemma 4.7. Let us suppose $h(x') \geq h_{\min} > 0$. Then, for all $x < x' < y$, we have

$$-\int_{x'}^y h(t) dt \leq -h_{\min}(y - x'),$$

thus

$$\int_x^y \phi(x', y; h) dx' \leq \int_x^y e^{-h_{\min}(y-x')/\epsilon} dx' \leq \frac{\epsilon}{h_{\min}} \int_0^\infty e^{-x'} dx' \leq \frac{\epsilon}{h_{\min}},$$

and the result follows. \square

Proof of Lemma 4.8. By the definition of ρ , the argument of the exponential defining ϕ is everywhere nonpositive so $\phi \leq 1$ and $I(\rho; h) \leq 2M$. Moreover at the point ρ , the primitive of h is locally Lipschitz continuous, so there exists a sufficiently small η and a constant $c > 0$ such that

$$0 \leq -\int_\rho^x h(x') dx' \leq \frac{1}{c}|x - \rho|, \quad |x - \rho| < \eta.$$

Then

$$I(\rho; h) \geq \int_{-\eta}^\eta e^{-\frac{1}{c\epsilon}|x|} dx = 2c\epsilon \int_0^{\frac{\eta}{c\epsilon}} e^{-x'} dx' \geq c\epsilon.$$

\square

Note that in Lemma 4.7, if $h_{\min} = 0$, then the considered integral remains bounded as ϵ vanishes.

The following result Lemma 4.9 shows that, on any compact subset of the complement set $[\lambda_{\min}, \lambda_{\max}]^c$, the mass of the linearized wave measures tends to zero. In the limit, all the mass of the wave measure $\varphi(\cdot; h)$ is concentrated on the interval $[\lambda_{\min}, \lambda_{\max}]$.

Lemma 4.9 (Behavior of linearized wave measures). *For all h in the class \mathcal{L} the function $\varphi(\cdot; h)$ satisfies the estimates*

$$0 \leq \varphi(x; h) \leq O(1/\epsilon) \begin{cases} e^{-(x-\lambda_{\min})^2 d_{\min}/2\epsilon}, & -M < x < \lambda_{\min}, \\ 1, & x \in [\lambda_{\min}, \lambda_{\max}], \\ e^{-(x-\lambda_{\max})^2 d_{\min}/2\epsilon}, & \lambda_{\max} < x < M. \end{cases}$$

Proof. For $x \geq \lambda_{\max}$ we have, with ρ defined through Lemma 4.5, and d and λ given in Definition 4.4

$$\begin{aligned} -\int_\rho^x h(y) dy &= \int_{\lambda_{\max}}^x d(y) (y - \lambda(y)) dy + \int_{\rho_i}^{\lambda_{\max}} d(y) (y - \lambda(y)) dy \\ &\geq \int_{\lambda_{\max}}^x d(y) (y - \lambda_{\max}) dy \geq \frac{(x - \lambda_{\max})^2 d_{\min}}{2}, \end{aligned}$$

while a similar argument for $x < \lambda_{\min}$ gives

$$-\int_\rho^x h(y) dy \geq \int_{\lambda_{\min}}^x d(y) (y - \lambda_{\min}) dy \geq \frac{(x - \lambda_{\min})^2 d_{\min}}{2}.$$

Finally, the desired conclusion follows from definition of $\varphi(\cdot; h)$ and from Lemma 4.8. \square

4.4. Wave coefficients. We rely on the earlier work by Joseph and LeFloch [22], in Lemma 4.10. Our new contribution is about the case of resonant wave coefficients treated in Lemma 4.12.

Lemma 4.10 (Estimates of the wave coefficients). *The linear coefficients $J_{j \rightarrow i}$ defined in (4.21) satisfy the estimate*

$$(4.26) \quad |J_{j \rightarrow i}(y)| \leq \begin{cases} O(\epsilon) (\varphi_i^*(y) + \varphi_j^*(y)), & i \neq j, \\ 2M \varphi_i^*(y), & i = j, \end{cases}$$

for all $i, j = 1, \dots, N$ and $y \in [-M, M]$. Moreover, the quadratic wave coefficients defined in (4.22) satisfy

$$(4.27) \quad |F_{j,k \rightarrow i}(y)| \leq C (\varphi_i^*(y) + \varphi_j^*(y) + \varphi_k^*(y)).$$

Lemma 4.11. *By choosing δ small enough, for all $i \neq j$ there exists positive constants C and D independent of ϵ such that*

$$(4.28) \quad \left\| \frac{\varphi_i^*}{\varphi_j^*} \right\|_{L^\infty([\underline{\Delta}_j, \bar{\Lambda}_j])} \leq C e^{-D/\epsilon}.$$

Lemma 4.12 (Resonant quadratic coefficients). *Given $\psi \in L^1$ and $i \neq j$, then one has*

$$\left| J_{j \rightarrow i}^\psi(y) \right| \leq O(1) \|\psi\|_1 (\varphi_j^*(y) + \varphi_i^*(y)), \quad \epsilon > 0.$$

Proof of Lemma 4.10. The case $i = j$ is obvious, so we only need to consider the case $i \neq j$. For definiteness we suppose that $j > i$, the proof for $j < i$ being similar.

First, using Lemmas 4.7 and 4.8, we get in the region $y > c_i$:

$$\begin{aligned} \left| \varphi_i^*(y) \int_{c_i}^y \frac{\varphi_j^*(x)}{\varphi_i^*(x)} dx \right| &= \varphi_j^*(y) \int_{c_i}^y \phi(x, y; \mu_i - \mu_j) dx \\ &\leq \frac{\epsilon}{\underline{\Delta}_j - \bar{\Lambda}_i} \varphi_j^*(y). \end{aligned}$$

On the other hand, in the region $y < c_i$ we have

$$\begin{aligned} \left| \varphi_i^*(y) \int_{c_i}^y \frac{\varphi_j^*(x)}{\varphi_i^*(x)} dx \right| &= \varphi_i^*(y) \frac{I_i}{I_j} \phi(\rho_j, \rho_i; \mu_j) \int_y^{c_i} \phi(\rho_i, x, \mu_j - \mu_i) dx \\ &= \varphi_i^*(y) \frac{I_i}{I_j} \phi(\rho_j, \rho_i; \mu_j) \phi(\rho_i, c_i; \mu_j - \mu_i) \int_y^{c_i} \phi(c_i, x, \mu_j - \mu_i) dx. \end{aligned}$$

But, by Lemma 4.7 we have

$$\int_y^{c_i} \phi(c_i, x, \mu_j - \mu_i) dx \leq \frac{\epsilon}{\underline{\Delta}_j - \bar{\Lambda}_i}$$

and, by an easy computation,

$$\begin{aligned} \phi(\rho_j, \rho_i; \mu_j) &\leq e^{-(\underline{\Delta}_j - \rho_i)^2 / (2\epsilon(1+\eta))}, \\ \phi(\rho_i, c_i; \mu_j - \mu_i) &\leq e^{(\bar{\Lambda}_j - \underline{\Delta}_i) |c_i - \rho_i| / \epsilon}, \\ I_i / I_j &= O(1/\epsilon). \end{aligned}$$

Using these observations we also get for all $y < c_i$

$$\left| \varphi_i^*(y) \int_{c_i}^y \frac{\varphi_j^*(x)}{\varphi_i^*(x)} dx \right| \leq C e^{-\beta_{ij}/\epsilon} \varphi_i^*(y),$$

with

$$\begin{aligned} \beta_{ij} &= -(\bar{\Lambda}_j - \underline{\Delta}_i) |c_i - \rho_i| + \frac{(\underline{\Delta}_j - \rho_i)^2}{2(1+\eta)} \\ &\geq -(\bar{\Lambda}_j - \underline{\Delta}_i) (\bar{\Lambda}_i - \underline{\Delta}_i) + \frac{1}{2(1+\eta)} (\underline{\Delta}_j - \bar{\Lambda}_i)^2 \end{aligned}$$

When δ_0 tends to 0, $(\bar{\Lambda}_j - \underline{\Delta}_i)$ remains bounded, $(\bar{\Lambda}_i - \underline{\Delta}_i)$ vanishes, while $(\underline{\Delta}_j - \bar{\Lambda}_i)$ tends to $\lambda_j(0, 0) - \lambda_i(0, 0) \neq 0$, so assuming δ_0 small enough, we can suppose each quantity β_{ij} is positive. The first desired result (4.26) therefore follows.

Using the inequality $|ab| \leq (a^2 + b^2)/2$ we have

$$|F_{j,k \rightarrow i}| \leq \frac{1}{2} (|F_{j,j \rightarrow i}| + |F_{k,k \rightarrow i}|).$$

So, we only need to consider the coefficients of the form $F_{j \rightarrow i}$, that is,

$$G_{i,j}(y) := \frac{\phi(\rho_i, y; \mu_i)}{I_j^2} \left| \int_{c_i}^y \frac{\phi(\rho_j, x; \mu_j)^2}{\phi(\rho_i, x; \mu_i)} dx \right|.$$

To avoid any distinction between the cases $y > c_i$ and $y < c_i$, the integral would be noted $\int_{[c_i, y]}$ (this is allowed by the positivity of the integrand). Clearly, when $j = i$ we have

$$G_{i,i} \leq \varphi_i,$$

since

$$\int_{[c_i, y]} \phi(\rho_i, x; \mu_i) dx \leq I_i.$$

So, we now suppose $i \neq j$, then

$$\begin{aligned} G_{i,j}(y) &= \frac{\phi(\rho_i, y; \mu_i)}{I_j^2} \int_{[c_i, y]} \phi(\rho_i, x; -\mu_i) \phi(\rho_j, x; \mu_j)^2 dx \\ &= \frac{\phi(\rho_j, y; \mu_j)}{I_j^2} \int_{[c_i, y]} \phi(x, y; \mu_i - \mu_j) \phi(\rho_j, x; \mu_j) dx \\ &\leq \frac{\phi(\rho_j, y; \mu_j)}{I_j^2} \int_{[c_i, y]} \phi(x, y; \mu_i - \mu_j) dx, \end{aligned}$$

thanks to the judicious choice of ρ_j that gives $\phi(\rho_j, x; \mu_j) \leq 1$. Note the important use of this L^∞ estimate that will default in the future. Finally we have

$$G_{i,j}(y) \leq \frac{1}{I_j} J_{j \rightarrow i}(y)$$

which, together with (4.26) and Lemma 4.8 completes this proof. \square

Proof of Lemma 4.11. Fix $y \in [\underline{\Lambda}_j, \bar{\Lambda}_j]$ and recall that

$$\varphi_i^*(y) \leq \frac{C}{\epsilon} \exp\left(\frac{1}{\epsilon} \int_{\rho_i}^y (\lambda_i(t) - t) dt\right).$$

If $y \geq \rho_i$ then we find

$$\begin{aligned} \int_{\rho_i}^y (\lambda_i(t) - t) dt &\leq \int_{\rho_i}^y (\bar{\Lambda}_i - t) dt \leq \frac{1}{2} ((\bar{\Lambda}_i - \rho_i)^2 - (\bar{\Lambda}_i - y)^2) \\ &\leq \frac{1}{2} ((\bar{\Lambda}_i - \underline{\Lambda}_i)^2 - (\bar{\Lambda}_i - y)^2), \end{aligned}$$

while, if $y \leq \rho_i$,

$$\begin{aligned} \int_{\rho_i}^y (\lambda_i(t) - t) dt &= \int_y^{\rho_i} (t - \lambda_i(t)) dt \leq \int_y^{\rho_i} (t - \underline{\Lambda}_i) dt \\ &\leq \frac{1}{2} ((\underline{\Lambda}_i - \rho_i)^2 - (\underline{\Lambda}_i - y)^2) \leq \frac{1}{2} ((\underline{\Lambda}_i - \bar{\Lambda}_i)^2 - (\underline{\Lambda}_i - y)^2). \end{aligned}$$

If $i < j$ then for all $\epsilon > 0$, $y \leq \rho_i$ and

$$\int_{\rho_i}^y (\lambda_i(t) - t) dt \leq \frac{1}{2} ((\bar{\Lambda}_i - \underline{\Lambda}_i)^2 - (\bar{\Lambda}_i - \underline{\Lambda}_j)^2),$$

while if $j < i$ then for all $\epsilon > 0$, $y \geq \rho_i$ and

$$\int_{\rho_i}^y (\lambda_i(t) - t) dt \leq \frac{1}{2} ((\bar{\Lambda}_i - \underline{\Lambda}_i)^2 - (\underline{\Lambda}_i - \bar{\Lambda}_j)^2),$$

In all cases, we thus can write

$$\varphi_i^*(y) \leq \frac{C}{\epsilon} \exp\left(\frac{1}{2\epsilon} (\ell_i^2 - \Delta_{ij}^2)\right),$$

where $\ell_i = \bar{\Lambda}_i - \underline{\Lambda}_i$ represents the width of the i -th wave and Δ_{ij} the gap between two different waves $\Delta_{ij} = \min(|\bar{\Lambda}_i - \underline{\Lambda}_j|, |\bar{\Lambda}_j - \underline{\Lambda}_i|)$.

Moreover, we have

$$\frac{1}{\varphi_j^*(y)} \leq 2M \exp\left(-\frac{1}{\epsilon} \int_{\rho_j}^y (\lambda_j(t) - t) dt\right) \leq 2M e^{(\bar{\Lambda}_j - \underline{\Lambda}_j)^2/\epsilon} \leq 2M e^{\ell_j^2/\epsilon},$$

so finally,

$$\frac{\varphi_i^*(y)}{\varphi_j^*(y)} \leq 2MC\epsilon^{-1} e^{\frac{\ell_j^2 + \ell_i^2 - \Delta_{ij}^2}{\epsilon}}.$$

Choosing δ_0 small enough, we can assure the positivity of all quantities $\Delta_{ij}^2 - \ell_j^2 + \ell_i^2$, and the result follows. The multiplicative coefficient ϵ^{-1} can be absorbed by reducing the exponential factor. \square

Proof of Lemma 4.12. Using integration by part and denoting Ψ an anti-derivative of ψ , (with so $\|\Psi\|_\infty \leq \|\psi\|_1 < \infty$), we obtain

$$\begin{aligned} J_{j \rightarrow i}^\psi(y) &= \varphi_j^*(y) \left[\Psi(x) \frac{\varphi_j^*(x)}{\varphi_i^*(x)} \right]_{c_i}^y - \varphi_i^*(y) \int_{c_j}^y \Psi(x) \frac{d}{dx} \left(\frac{\varphi_j^*(x)}{\varphi_i^*(x)} \right) dx \\ &= \varphi_j^*(y) \Psi(y) - \varphi_i^*(y) \Psi(c_i) \frac{\varphi_j^*(c_i)}{\varphi_i^*(c_i)} - \varphi_i^*(y) \int_{c_i}^y \Psi(x) \frac{d}{dx} \left(\frac{\varphi_j^*(x)}{\varphi_i^*(x)} \right) dx. \end{aligned}$$

The explicit formula for φ_k^* gives

$$\begin{aligned} \frac{d}{dx} \left(\frac{\varphi_j^*(x)}{\varphi_i^*(x)} \right) &= \frac{d}{dx} \left(\frac{I_i}{I_j} \exp \left(\frac{1}{\epsilon} \int_{\rho_j}^x (\lambda_j(t) - t) dt - \frac{1}{\epsilon} \int_{\rho_i}^x (\lambda_j(t) - t) dt \right) \right) \\ &= \frac{1}{\epsilon} (\lambda_j(x) - \lambda_i(x)) \frac{\varphi_j^*(x)}{\varphi_i^*(x)} \end{aligned}$$

and, consequently,

$$\left| J_{j \rightarrow i}^\psi(y) \right| \leq \|\Psi\|_\infty \left(\varphi_j^*(y) + \varphi_i^*(y) \left\| \frac{\varphi_j^*}{\varphi_i^*} \right\|_{L^\infty([\Delta_i, \bar{\lambda}_i])} + \frac{\|\lambda_i - \lambda_j\|_\infty}{\epsilon} \left| \varphi_i^*(y) \int_{c_i}^y \frac{\varphi_j^*(x)}{\varphi_i^*(x)} dx \right| \right).$$

Thus, by Lemma 4.11 and Lemma 4.10 on binary terms for different wave families,

$$|J_{j \rightarrow i}(y)| \leq O(\epsilon) (\varphi_j^*(y) + \varphi_i^*(y)),$$

we get

$$\left| J_{j \rightarrow i}^\psi(y) \right| \leq \|\psi\|_1 \left(\varphi_j^*(y) + \varphi_i^*(y) C e^{-D/\epsilon} + O(1) (\varphi_j^*(y) + \varphi_i^*(y)) \right).$$

\square

Remark 4.13. The method just employed could be used to establish Lemma 4.10 directly, noting that $\|\varphi_k^*\|_{L^1} = 1$ and $J_{j \rightarrow i}^{\varphi_k^*} = F_{j,k \rightarrow i}$, and on the other hand, $\|\varphi_j^*\|_{L^1} = 1$ and $J_{k \rightarrow i}^{\varphi_j^*} = F_{j,k \rightarrow i}$. We deduce

$$|F_{j,k \rightarrow i}(y)| \leq O(1) (\varphi_j^*(y) + \varphi_i^*(y)),$$

$$|F_{j,k \rightarrow i}(y)| \leq O(1) (\varphi_k^*(y) + \varphi_i^*(y)),$$

and (4.27) follows.

5. CONSTRUCTION OF THE ENTROPY SOLUTION

5.1. Correction vector for a given strength. Let $C_0(\mathbb{R})$ be the space of all continuous functions that decay to zero as $|\xi| \rightarrow +\infty$, and define the following weighted sup-norm for $\theta \in [C_0(\mathbb{R})]^N$ as

$$\|\theta\| = \sum_{k=1}^N \sup_{\xi \in \mathbb{R}} \frac{|\theta_k(\xi)|}{\sum_{h=1}^N \varphi_h^*(\xi)}.$$

Thus we search for θ in the Banach space

$$(5.1) \quad E = \left\{ \theta = (\theta_1, \dots, \theta_N) \in [C_0(\mathbb{R})]^N : \|\theta\| < \infty \right\}.$$

For $\delta > 0$, consider the ball with radius δ

$$B_\delta := \{ \tau \in \mathbb{R}^N : |\tau| \leq \delta \},$$

and for $\tau \in B_\delta$

$$(5.2) \quad \mathcal{F} := \left\{ \theta \in E : |\theta_k(\xi)| \leq A(\eta|\tau| + |\tau|^2 + \nu|\tau|) \sum_h \varphi_h^*(\xi), k = 1, \dots, N \right\},$$

where A is a positive constant to be chosen later and $\nu := \|\sigma\|_\infty$.

The set \mathcal{F} is a closed bounded subset of E in the weighted norm $\|\cdot\|$. The quadratic quantity $|\tau|^2$ is already present in Tzavaras [34] and comes from quadratic coefficients between the $\tau_i \varphi_i^*$ waves. Note however that the presence of the coupling wave ψ (of unit total mass), with its strength ν , enforces the subset \mathcal{F} to contain correction waves that comes from coefficients associated with ψ and the $\tau_i \varphi_i^*$, and of strength at most $\nu|\tau|$ relative to the φ_h^* .

Now, we will define, for a given strength τ the correction $\theta(\tau; \cdot)$. Let define the map T that takes $u \in \bar{\Omega}$ where

$$\bar{\Omega} := \{u \in C^0([-M, M]), \|u(\cdot) - u_L\|_\infty \leq \varsigma\}, \tau \in B_\delta$$

and $\theta \in \mathcal{F}$ to the vector-valued function $T(u, \tau, \theta)$ whose components are given by ($k = 1, \dots, N$)

$$(5.3) \quad \begin{aligned} & T_k(u, \tau, \theta)(\xi) \\ &= \eta \varphi_k^*(\xi) \int_{c_k}^\xi \frac{1}{\varphi_k^*(x)} \sum_i \pi_{ik}(x) \left(\tau_i \varphi_i^*(x) + \theta_i(x) \right) dx \\ &+ \varphi_k^*(\xi) \int_{c_k}^\xi \frac{1}{\varphi_k^*(x)} \sum_{i,j} \kappa_{ijk}(x) \left(\tau_i \varphi_i^*(x) + \theta_i(x) \right) \left(\tau_j \varphi_j^*(x) + \theta_j(x) \right) dx \\ &+ \varphi_k^*(\xi) \int_{c_k}^\xi \frac{1}{\varphi_k^*(x)} \sum_i \sigma_{ik}(x) \left(\tau_i \varphi_i^*(x) + \theta_i(x) \right) \psi(x) dx. \end{aligned}$$

Lemma 5.1 (Contraction property). *There exists positive constants A, η, δ_0 and ν such that for $\delta < \delta_0$:*

- (1) $T : \bar{\Omega} \times B_\delta \times \mathcal{F} \rightarrow \mathcal{F}$ is well defined.
- (2) There exists $0 < \alpha < 1$ such that

$$\|T(u, \tau, \theta) - T(u, \tau, \hat{\theta})\| \leq \alpha \|\theta - \hat{\theta}\|, \quad \theta, \hat{\theta} \in \mathcal{F},$$

and for any $u \in \bar{\Omega}, \tau \in B_\delta$. Therefore $T(u, \tau, \cdot) : \mathcal{F} \rightarrow \mathcal{F}$ is a uniform contraction.

- (3) There exists a positive constant C , depending on μ but independent of δ , such that

$$\|T(u, \tau, \theta) - T(u, \hat{\tau}, \theta)\| \leq C(\eta + \nu + \delta) |\tau - \hat{\tau}|, \quad \tau, \hat{\tau} \in B_\delta$$

and for any $u \in \bar{\Omega}, \theta \in \mathcal{F}$.

We deduce from this lemma the following existence result of a correction $\theta(\tau; \cdot)$.

Proposition 5.2. *Given $u \in \bar{\Omega}, \tau \in B_\delta$, there exists a unique $\theta(\tau; \cdot) \in \mathcal{F}$, i.e. in the class of functions satisfying*

$$(5.4) \quad |\theta_k(\tau; \cdot)| \leq A(\eta|\tau| + |\tau|^2 + \nu|\tau|) \sum_h \varphi_h^*, \quad |\tau| \leq \delta, k = 1, \dots, N,$$

solution of the fixed point equation $T(u, \tau, \theta) = \theta$. Moreover, there exists a constant C independent of δ such that

$$(5.5) \quad |\theta_k(\tau; \cdot) - \theta_k(\hat{\tau}; \cdot)| \leq C(\eta + \nu + \delta) |\tau - \hat{\tau}| \sum_h \varphi_h^*, \quad \tau, \hat{\tau} \in B_\delta.$$

Proof of Lemma 5.1. The main difficulty is handling the coupling wave $\nu\psi$. First of all, we show T keeps the subset \mathcal{F} stable, using the definitions (5.2), (5.3), and the different definitions of the

coefficients (4.21), (4.22) and (4.23), we get

$$\begin{aligned}
& |T_k(u, \tau, \theta)(\xi)| \\
& \leq \eta \|\pi\|_\infty \sum_i J_{i \rightarrow k}(\xi) \left(|\tau| + A(\eta|\tau| + |\tau|^2 + \nu|\tau|) \right) \\
& \quad + \|\kappa\|_\infty \sum_{ij} F_{ij \rightarrow k}(\xi) \left(|\tau|^2 + 2|\tau|A(\eta|\tau| + |\tau|^2 + \nu|\tau|) + A^2(\eta|\tau| + |\tau|^2 + \nu|\tau|)^2 \right) \\
& \quad + \|\sigma\|_\infty \sum_i J_{i \rightarrow k}^\psi(\xi) \left(|\tau| + A(\eta|\tau| + |\tau|^2 + \nu|\tau|) \right).
\end{aligned}$$

Thus, by using Lemmas 4.10 and 4.12 we get

$$\begin{aligned}
& |T_k(u, \tau, \theta)(\xi)| \\
& \leq \eta \|\pi\|_\infty C_1 N^2 (|\tau| + A(\eta|\tau| + |\tau|^2 + \nu|\tau|)) \sum_h \varphi_h^*(\xi) \\
& \quad + \|\kappa\|_\infty C_2 N^4 \left(|\tau|^2 + 2A|\tau|(\eta|\tau| + |\tau|^2 + \nu|\tau|) + A^2(\eta|\tau| + |\tau|^2 + \nu|\tau|)^2 \right) \sum_h \varphi_h^*(\xi) \\
& \quad + \|\sigma\|_\infty C_3 N^2 (|\tau| + A(\eta|\tau| + |\tau|^2 + \nu|\tau|)) \sum_h \varphi_h^*(\xi) \\
& \leq C(1 + A(\eta + \nu + \delta))^2 (\eta|\tau| + |\tau|^2 + \nu|\tau|) \sum_h \varphi_h^*(\xi),
\end{aligned}$$

where C is a constant depending only on N the dimension of the space, on $\|\pi\|_\infty$, $\|\kappa\|_\infty$ and of the constants C_1, C_2, C_3 . A necessary condition to get the stability of the subset \mathcal{F} by $T(u, \tau, \cdot)$ is also

$$C(1 + A(\eta + \nu + \delta))^2 \leq A$$

A way to get this inequality is for example, fixing $A = 4C$, to choose η, ν and δ together such that $\eta + \nu + \delta \leq 1/4C$.

Now, T is an uniform contraction relative to the variable $\theta \in \mathcal{F}$, since (from similar arguments)

$$\begin{aligned}
& |T_k(u, \tau, \theta)(\xi) - T_k(u, \tau, \hat{\theta})(\xi)| \\
& \leq \eta \|\pi\|_\infty \sum_j \|\theta - \hat{\theta}\| \sum_i J_{i \rightarrow k}(\xi) \\
& \quad + \|\kappa\|_\infty \sum_{ij} \left(2|\tau| \|\theta - \hat{\theta}\| \sum_l F_{il}^k(\xi) + 2A(\eta|\tau| + |\tau|^2 + \nu|\tau|) \|\theta - \hat{\theta}\| \sum_{lm} F_{lm \rightarrow k}(\xi) \right) \\
& \quad + \|\sigma\|_\infty \sum_j \|\theta - \hat{\theta}\| \sum_i J_{i \rightarrow k}^\psi(\xi).
\end{aligned}$$

In view of Lemmas 4.10 and 4.12 we obtain

$$\begin{aligned}
& |T_k(u, \tau, \theta)(\xi) - T_k(u, \tau, \hat{\theta})(\xi)| \\
& \leq \eta \|\pi\|_\infty C_1 N^2 \|\theta - \hat{\theta}\| \sum_h \varphi_h^*(\xi) \\
& \quad + \|\kappa\|_\infty C_2 N^4 (|\tau| + A(\eta|\tau| + |\tau|^2 + \nu|\tau|)) \|\theta - \hat{\theta}\| \sum_h \varphi_h^*(\xi) \\
& \quad + \|\sigma\|_\infty C_3 N^2 \|\theta - \hat{\theta}\| \sum_h \varphi_h^*(\xi) \\
& \leq C(\eta + \nu + \delta)(1 + A\delta) \|\theta - \hat{\theta}\| \sum_h \varphi_h^*(\xi).
\end{aligned}$$

Finally, we obtain the item (2) of Proposition 5.1 with $\alpha = C(\eta + \nu + \delta)(1 + A\delta)$ by choosing $\eta + \nu + \delta$ sufficiently small to assure that $\alpha < 1$.

Finally, we have to check the Lipschitz continuity of T in the variable τ .

$$\begin{aligned} & |T_k(u, \tau, \theta)(\xi) - T_k(u, \hat{\tau}, \theta)(\xi)| \\ & \leq \eta \|\pi\|_\infty |\tau - \hat{\tau}| \sum_i J_{i \rightarrow k}(\xi) \\ & \quad + \|\kappa\|_\infty \sum_{ij} \left(|\tau_i \hat{\tau}_j - \tau_j \hat{\tau}_i| F_{ij \rightarrow k}(\xi) + 2|\tau - \hat{\tau}| A\delta(\eta + \delta + \nu) \sum_l F_{il \rightarrow k}(\xi) \right) \\ & \quad + \|\sigma\|_\infty |\tau - \hat{\tau}| \sum_i J_{i \rightarrow k}^{tb}(\xi). \end{aligned}$$

By using Lemmas 4.10 and 4.12 we get

$$\begin{aligned} & |T_k(u, \tau, \theta)(\xi) - T_k(u, \hat{\tau}, \theta)(\xi)| \\ & \leq \eta \|\pi\|_\infty C_1 N |\tau - \hat{\tau}| \sum_h \varphi_h^*(\xi) + \|\kappa\|_\infty C_2 N^2 (2\delta |\tau - \hat{\tau}| \\ & \quad + 2AN\delta |\tau - \hat{\tau}|) \sum_h \varphi_h^*(\xi) + \|\sigma\|_\infty C_3 N |\tau - \hat{\tau}| \sum_h \varphi_h^*(\xi) \\ & \leq C(\eta + \nu + \delta) |\tau - \hat{\tau}| \sum_h \varphi_h^*(\xi), \end{aligned}$$

and the item (3) of Proposition 5.1 follows. \square

Proof of Proposition 5.2. The inequality (5.4) is a direct consequence of the contraction mapping theorem, that previous proposition ensures to apply ($\alpha < 1$). Let $u \in \bar{\Omega}$ be given, and $\tau, \hat{\tau} \in B_\delta$, then

$$\begin{aligned} \theta(\tau) - \theta(\hat{\tau}) &= T(u, \tau, \theta(\tau)) - T(u, \hat{\tau}, \theta(\hat{\tau})) \\ &= (T(u, \tau, \theta(\tau)) - T(u, \tau, \theta(\hat{\tau}))) + (T(u, \tau, \theta(\hat{\tau})) - T(u, \hat{\tau}, \theta(\hat{\tau}))), \end{aligned}$$

thus

$$\begin{aligned} \|\theta(\tau) - \theta(\hat{\tau})\| &\leq \|T(u, \tau, \theta(\tau)) - T(u, \tau, \theta(\hat{\tau}))\| + \|T(u, \tau, \theta(\hat{\tau})) - T(u, \hat{\tau}, \theta(\hat{\tau}))\| \\ &\leq \alpha \|\theta(\tau) - \theta(\hat{\tau})\| + C(\eta + \nu + \delta) |\tau - \hat{\tau}|. \end{aligned}$$

Hence,

$$\|\theta(\tau) - \theta(\hat{\tau})\| \leq \frac{C}{1 - \alpha} (\eta + \nu + \delta) |\tau - \hat{\tau}|,$$

and (5.5) ensues. \square

5.2. Strength vector for given Riemann data. Fix a left-state vector $u_L \in \mathbb{R}^N$ and $u \in \bar{\Omega} = \{u \in C^0([-M, M]), \|u(\cdot) - u_L\|_\infty \leq \varsigma\}$. Being given $\tau \in B_\delta$, we previously constructed a unique $\theta(\tau, \cdot) \in \mathcal{F}$ such that $T(u, \tau, \theta) = \theta$. The question is now to link the vector $\tau \in B_\delta$ to the boundary data u_L, u_R . Consider the following operator:

$$S(\tau) := u_L + A_0(u, v)^{-1} \sum_k \int_{-M}^M [\tau_k \varphi_k^*(\xi) + \theta_k(\tau, \xi)] \hat{r}_k(u(\xi), v(\xi), \xi) d\xi.$$

Lemma 5.3. *There exist constants $\delta, r > 0$ such that the operator $P : B_r(u_L) \times \bar{\Omega} \times B_\delta \rightarrow B_\delta$ defined by*

$$P(u_R, u, \tau) = A_0(u, v)(u_R - u_L) - \sum_k \int_{-M}^M \theta_k(\tau, \xi) \hat{r}_k(u(\xi), v(\xi), \xi) d\xi.$$

satisfies the contraction property (for some $0 < \alpha < 1$)

$$|P(u_R, u, \tau) - P(u_R, u, \hat{\tau})| \leq \alpha |\tau - \hat{\tau}|, \quad \tau, \hat{\tau} \in B_\delta$$

for any $u_R \in B_r(u_L), u \in \bar{\Omega}$.

Proposition 5.4. *Given $u \in \bar{\Omega}$, there exist positive constants r and δ such that the following holds:*

- (1) *For all $u_R \in B_r(u_L)$ there exists a unique solution of the equation $S(\tau) = u_R$ with $\tau \in B_\delta$.*

(2) For each $u \in \bar{\Omega}$ and $\epsilon > 0$ the inverse map $S^{-1} : B_r(u_L) \rightarrow B_\delta$ is well defined and satisfies

$$(5.6) \quad |S^{-1}(u_R)| \leq \gamma |u_R - u_L|,$$

where γ is a constant depending on ς but independent of $u \in \bar{\Omega}$ and ϵ .

Proof of Lemma 5.3. Letting $u_R \in B_r(u_L)$, $u \in \bar{\Omega}$ and $\tau \in B_\delta$, one has

$$\begin{aligned} |P(u_R, u, \cdot)| &\leq \|A_0\| |u_R - u_L| + N^2 R A(\eta) |\tau| + |\tau|^2 + \nu |\tau| \\ &\leq (\|A_0\| r + RN^2 A(\eta \delta + \delta^2 + \nu \delta)). \end{aligned}$$

Hence, one has the inclusion $P(u_R, u, B_\delta) \subset B_\delta$ provided

$$(\|A_0\| r + RN^2 A(\eta \delta + \delta^2 + \nu \delta)) \leq \delta,$$

that is by choosing r , η , δ and ν such that

$$\begin{aligned} RN^2 A(\eta + \delta + \nu) \delta &\leq \delta/2, \\ \|A_0\| \beta r &\leq \delta/2, \end{aligned}$$

that is to say

$$\begin{aligned} \eta + \delta + \nu &\leq 1/2RN^2 A, \\ r &\leq \delta/2\|A_0\|. \end{aligned}$$

Given τ and $\hat{\tau}$ in B_δ we have

$$\begin{aligned} P(u_R, u, \tau) - P(u_R, u, \hat{\tau}) &= \sum_k \int_{-M}^M [\theta_k(\xi, \tau) - \theta_k(\xi, \hat{\tau})] \hat{r}_k(u(\xi), v(\xi), \xi) d\xi, \\ |P(u_R, u, \tau) - P(u_R, u, \hat{\tau})| &\leq \sum_k \int_{-M}^M |\theta_k(\xi, \tau) - \theta_k(\xi, \hat{\tau})| |\hat{r}_k(u(\xi), v(\xi), \xi)| d\xi \\ &\leq RNC(\eta + \delta + \nu) |\tau - \hat{\tau}| \sum_k \int_{-M}^M \varphi_k^*(\xi) d\xi \\ &\leq N^2 C(\eta + \delta + \nu) |\tau - \hat{\tau}|. \end{aligned}$$

Provided

$$(5.7) \quad \alpha := N^2 C(\eta + \delta + \nu) < 1,$$

the map $P(u_R, u, \cdot)$ is a uniform contraction on B_δ . □

Proof of Proposition 5.4. Let u_L be fixed. The equation $S(\tau) = u_R$ takes the form

$$\begin{aligned} A_0(u, v)(u_R - u_L) &= \sum_k \tau_k \int_{-M}^M \varphi_k^*(\xi) \hat{r}_k(u(\xi), v(\xi), \xi) d\xi + \sum_k \int_{-M}^M \theta_k(\tau, \xi) \hat{r}_k(u(\xi), v(\xi), \xi) d\xi, \end{aligned}$$

in other words τ solves the equation

$$(5.8) \quad A_0(u, v)(u_R - u_L) = C(u, v) \tau + \sum_k \int_{-M}^M \theta_k(\tau, \xi) \hat{r}_k(u(\xi), v(\xi), \xi) d\xi,$$

where $C(u, v)$ is the matrix whose k -th columns is given by

$$\int_{-M}^M \varphi_k^*(\xi) \hat{r}_k(u(\xi), v(\xi), \xi) d\xi, \quad k = 1, \dots, N.$$

This matrix has the important property it is invertible for any $u \in \bar{\Omega}$ and the inverse matrix $C(u, v)^{-1}$ is uniformly bounded (cf [34])

$$(5.9) \quad |C(u, v)^{-1}| \leq \beta, \quad u \in \bar{\Omega}.$$

In order to solve the equation $S(\tau) = u_R$, observe that solutions of (5.8) are also fixed points of the map $\tau \mapsto C(u, v)^{-1} P(u_R, u, \tau)$, whose existence are ensured by Lemma 5.3. As a consequence,

given $u_R \in B_r(u_L)$, there exists a unique fixed point τ of $P(u_R, u, \cdot)$ in the ball B_δ . Moreover it also satisfies

$$|\tau| \leq |A_0(u, v)C(u, v)^{-1}| |u_R - u_L| + |C(u, v)^{-1}| \sum_k \int_{-M}^M |\theta_k(\xi, \tau)| |\widehat{r}_k(u(\xi), v(\xi), \xi)| d\xi,$$

thus

$$\begin{aligned} |\tau| &\leq \|A_0\| \beta |u_R - u_L| + \beta R A N^2 (\eta |\tau| + |\tau|^2 + \nu |\tau|) \\ &\leq \|A_0\| \beta |u_R - u_L| + 1/2 |\tau|. \end{aligned}$$

Thus, $|\tau| \leq 2\|A_0\| \beta |u_R - u_L|$, which finally implies (5.6). \square

5.3. Riemann problem. We search for a solution of (4.14) under the form (4.15) satisfying the boundary conditions (4.12) and where v is known by (4.13).

Theorem 5.5 (Uniform estimates and existence result). *There exists a solution $u^\epsilon \in \bar{\Omega}$ of the problem (4.12)-(4.14) satisfying, for some constant K independent of ϵ ,*

$$(5.10) \quad \begin{aligned} TV(u^\epsilon) &\leq K |u_R - u_L|, \\ \epsilon |u_\xi^\epsilon| &\leq K. \end{aligned}$$

After extracting a subsequence if necessary, this result provides us with a solution with bounded total variation.

Proof. $\varsigma > 0$ is chosen so that conditions of eigenvalue separation are fulfilled on $\bar{\Omega}$. Fix u_L and $u \in \bar{\Omega}$. For ϵ fixed, we construct z as

$$z(\xi) = u_L + A_0(u, v)^{-1} \int_{-M}^\xi \sum_{j=1}^n (\tau_j \varphi_j^*(\zeta) + \theta_j(\tau; \zeta)) \widehat{r}_j(u(\zeta), v(\zeta), \zeta) d\zeta$$

by following steps:

- (1) Each φ_j^* is constructed as the fundamental wave measure from (4.18), recalling

$$\varphi_j^{*'} - \frac{\mu_j(u, v, \cdot)}{\epsilon} \varphi_j^* = 0.$$

- (2) For each τ small enough we can get, through Proposition 5.2, a correction $\theta(\tau, \cdot)$ so that $a_j = \tau_j \varphi_j^* + \theta_j$ is solution of (4.16a)

$$a_j' - \frac{\mu_j(u, v, \cdot)}{\epsilon} a_j = \eta L_j(u, v, \cdot) + Q_j(u, v, \cdot) + S_j(u, v, \cdot).$$

- (3) The vector of strength τ is then chosen, through Proposition 5.4, as a solution of $S(\tau) = u_R \in B_r(u_L)$. This way the solution \tilde{u} of

$$A_0(u, v) \tilde{u}_\xi(\xi) = \sum_j a_j(\xi) \widehat{r}_j(u, v, \xi)$$

satisfying $\tilde{u}(-M) = u_L$, satisfies moreover $\tilde{u}(M) = u_R$.

These steps allow us to construct an operator $\mathcal{T} : \bar{\Omega} \rightarrow E, u \mapsto z$, and $\mathcal{T}(u) = z \in \bar{\Omega}$. We only need to get a fixed point result on \mathcal{T} to get the solution u of the whole problem, and then sufficiently strong estimates to ensures existence of the limit as ϵ tends to 0. \square

Lemma 5.6. *The function v^ϵ converges toward the sign function (denoted by sgn) and, more precisely, for all $c > 0$*

$$(5.11) \quad \|v^\epsilon - \text{sgn}\|_{L^\infty(\mathbb{R} \setminus [-c, c])} = o(\epsilon).$$

Proof. Indeed the formula (4.13) implies v^ϵ takes the form

$$v^\epsilon(\xi) = -1 + 2 \frac{\int_{-M}^\xi e^{-\frac{x^2}{2\epsilon^p}} dx}{\int_{-M}^M e^{-\frac{x^2}{2\epsilon^p}} dx}.$$

Fix $\xi > c > 0$, so that

$$\begin{aligned} |v^\epsilon(x) - 1| &\leq 2 \frac{\int_c^M e^{-\frac{x^2}{2\epsilon^p}} dx}{\int_{-M}^M e^{-\frac{x^2}{2\epsilon^p}} dx} \\ &\leq 2 \frac{M e^{-\frac{c^2}{2\epsilon^p}}}{\epsilon^{p/2} \int_{-M/\epsilon^{p/2}}^{M/\epsilon^{p/2}} e^{-\frac{y^2}{2}} dy} \leq \frac{C}{\epsilon^{p/2}} e^{-\frac{c^2}{2\epsilon^p}}. \end{aligned}$$

For $\xi < -c < 0$, by the same procedure, we get

$$|v^\epsilon(x) + 1| \leq \frac{C}{\epsilon^{p/2}} e^{-\frac{c^2}{2\epsilon^p}},$$

the lemma is therefore proved. \square

Theorem 5.7 (Convergence to an entropy solution). *The sequence u^ϵ converges pointwise toward $u \in BV$, satisfying*

$$(5.12) \quad \begin{aligned} -\xi \frac{d}{d\xi} \gamma_-(u) + \frac{d}{d\xi} f_-(\gamma_-(u)) &= 0, & \text{in } \xi < 0, \\ -\xi \frac{d}{d\xi} \gamma_+(u) + \frac{d}{d\xi} f_+(\gamma_+(u)) &= 0, & \text{in } \xi > 0. \end{aligned}$$

Let $\eta_\pm = \eta_\pm(u) \in \mathbb{R}^N$ be two entropy functions compatible with the viscosity matrix in the sense that

$$\nabla^2 \eta_\pm(u) B_{0\pm}(u) \geq 0, \quad u \in \Omega.$$

Then following entropy inequalities are satisfied,

$$(5.13) \quad \begin{aligned} -\xi \frac{d}{d\xi} \eta_-(\gamma_-(u)) + \frac{d}{d\xi} q_-(\gamma_-(u)) &\leq 0, & \text{in } \xi < 0, \\ -\xi \frac{d}{d\xi} \eta_+(\gamma_+(u)) + \frac{d}{d\xi} q_+(\gamma_+(u)) &\leq 0, & \text{in } \xi > 0. \end{aligned}$$

Proof. Let Φ be a neighborhood of 0, the solution u^ϵ of (4.11) is obtained as a smooth function so that, under the consistency hypothesis (2.10), we have outside Φ

$$\begin{aligned} A_0(u^\epsilon, v^0) \frac{du^\epsilon}{d\xi} &= \frac{d}{d\xi} \gamma_\pm(u^\epsilon), \\ A_1(u^\epsilon, v^0) \frac{du^\epsilon}{d\xi} &= \frac{d}{d\xi} f_\pm(\gamma_\pm(u^\epsilon)). \end{aligned}$$

Let $\phi \in C_c^\infty(\mathbb{R}_- \setminus \Phi)$ be a test-function with a compact support included in $\mathbb{R}_- \setminus \Phi$, then (4.11) implies

$$\begin{aligned} &\int_{\mathbb{R}} \left(-\xi \frac{d}{d\xi} \gamma_-(u^\epsilon) + \frac{d}{d\xi} f_-(\gamma_-(u^\epsilon)) \right) \phi \, d\xi \\ &= \int_{\mathbb{R}} -\xi (A_0(u^\epsilon, v^0) - A_0(u^\epsilon, v^\epsilon)) u_\xi^\epsilon \phi \, d\xi \\ &\quad + \int_{\mathbb{R}} (A_1(u^\epsilon, v^0) - A_1(u^\epsilon, v^\epsilon)) u_\xi^\epsilon \phi \, d\xi + \int_{\mathbb{R}} \epsilon (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi \phi \, d\xi. \end{aligned}$$

Moreover, thanks to Lemma 5.6 and using Lipschitz continuity properties, we have

$$\begin{aligned} \left| \int_{\mathbb{R}} -\xi (A_0(u^\epsilon, v^0) - A_0(u^\epsilon, v^\epsilon)) u_\xi^\epsilon \phi \, d\xi \right| &\leq o(\epsilon) \text{Lip}(A_0) \|\xi \phi\|_\infty TV(u^\epsilon), \\ \left| \int_{\mathbb{R}} (A_1(u^\epsilon, v^0) - A_1(u^\epsilon, v^\epsilon)) u_\xi^\epsilon \phi \, d\xi \right| &\leq o(\epsilon) \text{Lip}(A_1) \|\phi\|_\infty TV(u^\epsilon), \end{aligned}$$

and

$$\begin{aligned} \epsilon \left| \int_{\mathbb{R}} (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi \phi \, d\xi \right| &= \epsilon \left| \int_{\mathbb{R}} B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon \phi_\xi \, d\xi \right| \\ &\leq \epsilon \|B_0\|_\infty \|\phi_\xi\|_\infty TV(u^\epsilon). \end{aligned}$$

Thus, as ϵ tends to 0, we get the weak formulation for the limit u

$$\int_{\mathbb{R}} \left(-\xi \frac{d}{d\xi} \gamma_-(u) + \frac{d}{d\xi} f_-(\gamma_-(u)) \right) \varphi \, d\xi = 0.$$

By a similar method we get for $\varphi \in C_c^\infty(\mathbb{R}_+ \setminus \Phi)$

$$\int_{\mathbb{R}} \left(-\xi \frac{d}{d\xi} \gamma_+(u) + \frac{d}{d\xi} f_+(\gamma_+(u)) \right) \varphi \, d\xi = 0.$$

Entropy inequalities are obtained by first using the consistency hypothesis (2.10), that give outside Φ

$$\begin{aligned} \nabla \eta_{\pm}(u^\epsilon) \cdot A_0(u^\epsilon, v^0) u_\xi^\epsilon &= \frac{d}{d\xi} \eta_{\pm}(\gamma_{\pm}(u^\epsilon)), \\ \nabla \eta_{\pm}(u^\epsilon) \cdot A_1(u^\epsilon, v^0) u_\xi^\epsilon &= \frac{d}{d\xi} q_{\pm}(\gamma_{\pm}(u^\epsilon)). \end{aligned}$$

Let $\phi \in C_c^\infty(\mathbb{R}_- \setminus \Phi)$ be a non-negative test function with a compact support included in $\mathbb{R}_- \setminus \Phi$, then (4.11) implies

$$\begin{aligned} &\int_{\mathbb{R}} \left(-\xi \frac{d}{d\xi} \eta_-(\gamma_-(u^\epsilon)) + \frac{d}{d\xi} q_-(\gamma_-(u^\epsilon)) \right) \phi \, d\xi \\ &= \int_{\mathbb{R}} -\xi \nabla \eta_-(u^\epsilon) \cdot (A_0(u^\epsilon, v^0) - A_0(u^\epsilon, v^\epsilon)) u_\xi^\epsilon \phi \, d\xi \\ &\quad + \int_{\mathbb{R}} \nabla \eta_-(u^\epsilon) \cdot (A_1(u^\epsilon, v^0) - A_1(u^\epsilon, v^\epsilon)) u_\xi^\epsilon \phi \, d\xi \\ &\quad + \int_{\mathbb{R}} \epsilon \nabla \eta_-(u^\epsilon) \cdot (B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon)_\xi \phi \, d\xi. \end{aligned}$$

With similar arguments as previously, the first and the second terms of right hand side tends to 0 as ϵ tends to 0. Moreover, after reporting the ξ -derivative on $\phi \nabla \eta_-(u^\epsilon)$, the last term

$$\begin{aligned} &-\int_{\mathbb{R}} \epsilon \phi_\xi \nabla \eta_-(u^\epsilon) \cdot B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon \, d\xi - \int_{\mathbb{R}} \epsilon \phi \nabla^2 \eta_-(u^\epsilon) \cdot B_0(u^\epsilon, v^0) |u_\xi^\epsilon|^2 \, d\xi \\ &+ \int_{\mathbb{R}} \epsilon \phi \nabla^2 \eta_-(u^\epsilon) \cdot (B_0(u^\epsilon, v^0) - B_0(u^\epsilon, v^\epsilon)) |u_\xi^\epsilon|^2 \, d\xi \end{aligned}$$

satisfies the estimates

$$\left| \int_{\mathbb{R}} \epsilon \phi_\xi \nabla \eta_-(u^\epsilon) \cdot B_0(u^\epsilon, v^\epsilon) u_\xi^\epsilon \, d\xi \right| \leq K \epsilon TV(u^\epsilon)$$

and

$$\begin{aligned} &\left| \int_{\mathbb{R}} \epsilon \phi \nabla^2 \eta_-(u^\epsilon) \cdot (B_0(u^\epsilon, v^0) - B_0(u^\epsilon, v^\epsilon)) |u_\xi^\epsilon|^2 \, d\xi \right| \\ &\leq K \text{Lip}(B_0) \|\nabla^2 \eta_-\|_\infty \|\phi\|_\infty TV(u^\epsilon) \|v^\epsilon - v^0\|_{L^\infty(\mathbb{R} \setminus \Phi)}. \end{aligned}$$

However the quantity $\int_{\mathbb{R}} \epsilon \phi \nabla^2 \eta_-(u^\epsilon) \cdot B_0(u^\epsilon, v^0) |u_\xi^\epsilon|^2 \, d\xi$ is not guaranteed to vanish as ϵ tends to 0, but it converges toward a positive value under the hypothesis $\nabla^2 \eta_{\pm} B_{0\pm}(u) \geq 0$. The following weak formulation of the entropy inequality on \mathbb{R}_- follows:

$$\int_{\mathbb{R}} \left(-\xi \frac{d}{d\xi} \eta_-(\gamma_-(u)) + \frac{d}{d\xi} q_-(\gamma_-(u)) \right) \phi \, d\xi \leq 0.$$

Similar arguments lead to the entropy inequality on \mathbb{R}_+ . \square

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