2008/04/07

Yet another generalization of
Postnikov's hook length formula for binary trees

Guo-Niu HAN

ABSTRACT. — We discover another one-parameter generalization of Postnikov's hook length formula for binary trees. The particularity of our formula is that the hook length h_v appears as an exponent. As an application, we derive another simple hook length formula for binary trees when the underlying parameter takes the value $1/2$.

1. Introduction

Consider the set $\mathcal{B}(n)$ of all binary trees with n vertices. For each vertex v of $T \in \mathcal{B}(n)$ the *hook length* of v, denoted by h_v, or just h for short, is the number of descendants of v (including v). The following hook length formula for binary trees

$$(1) \qquad \sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \left(1 + \frac{1}{h_v}\right) = \frac{2^n}{n!}(n+1)^{n-1}$$

was discovered by Postnikov [Po04]. Further combinatorial proofs and extensions have been proposed by several authors [CY08, GS06, MY07, Se08]. In particular, Lascoux conjectured the following one-parameter generalization

$$(2) \qquad \sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \left(x + \frac{1}{h_v}\right) = \frac{1}{(n+1)!} \prod_{k=0}^{n-1} ((n+1+k)x + n + 1 - k),$$

which was subsequently proved by Du-Liu [DL08]. The latter generalization appears to be very natural, because the *left-hand side* of (2) can be obtained from the left-hand side of (1) by replacing 1 by x.

It is also natural to look for an extension of (1) by introducing a new variable z in the *right-hand side*, namely by replacing $2^n(n+1)^{n-1}/n!$ by $2^n z(n+z)^{n-1}/n!$ It so happens that the corresponding left-hand side is also a sum on binary trees, but this time the hook length h_v appears as an exponent. The purpose of this Note is to prove the following Theorem.

Theorem 1. *For each positive integer n we have*

$$(3) \qquad \sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \frac{(z+h)^{h-1}}{h(2z+h-1)^{h-2}} = \frac{2^n z}{n!}(n+z)^{n-1}.$$
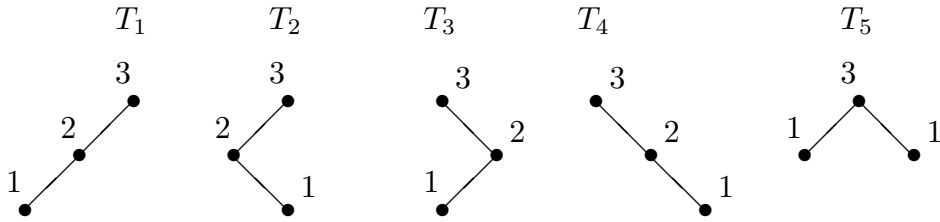
With $z = 1$ in (3) we recover Postnikov's identity (1). The following corollary is derived from our identity (3) by taking $z = 1/2$.

Corollary 2. *For each positive integer n we have*

$$(4) \qquad \sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \left(1 + \frac{1}{2h}\right)^{h-1} = \frac{(2n+1)^{n-1}}{n!}.$$

2. Proof of the Theorem

Let us take an example before proving the Theorem. There are five binary trees with $n = 3$ vertices:

The hook lengths of T_1, T_2, T_3, T_4 are all the same $1, 2, 3$; but the hook lengths of T_5 are $1, 1, 3$. The left-hand side of (3) is then equal to

$$4 \times \frac{1}{(2z)^{-1}} \cdot \frac{(z+2)^1}{2} \cdot \frac{(z+3)^2}{3(2z+1)} + \frac{1}{(2z)^{-1}} \cdot \frac{1}{(2z)^{-1}} \cdot \frac{(z+3)^2}{3(2z+1)} = \frac{2^3 z(z+3)^2}{3!}.$$

Let $y(x)$ be a formal power series in x such that

$$(5) \qquad y(x) = e^{xy(x)}.$$

By the Lagrange inversion formula $y(x)^z$ has the following explicit expansion:

$$(6) \qquad y(x)^z = \sum_{n \geq 0} z(n+z)^{n-1} \frac{x^n}{n!}.$$

Since $y^{2z} = (y^z)^2$ we have

$$(7) \qquad \sum_{n \geq 0} 2z(n+2z)^{n-1} \frac{x^n}{n!} = \left(\sum_{n \geq 0} z(n+z)^{n-1} \frac{x^n}{n!}\right)^2.$$

Comparing the coefficients of x^n on both sides of (7) yields the following Lemma.

2

Lemma 3. *We have*

$$(8) \qquad \frac{2z(n+2z)^{n-1}}{n!} = \sum_{k=0}^{n} \frac{z(k+z)^{k-1}}{k!} \times \frac{z(n-k+z)^{n-k-1}}{(n-k)!}.$$

Proof of the Theorem. Let

$$P(n) = \sum_{T \in \mathcal{B}(n)} \prod_{v \in T} \frac{(z+h)^{h-1}}{h(2z+h-1)^{h-2}}.$$

With each binary tree $T \in \mathcal{B}(n)$ $(n \geq 1)$ we can associate a triplet (T', T'', u), where $T' \in \mathcal{B}(k)$ $(0 \leq k \leq n-1)$, $T'' \in \mathcal{B}(n-1-k)$ and u is a vertex of hook length $h_u = n$. Hence

$$(9) \qquad P(n) = \sum_{k=0}^{n-1} P(k)P(n-1-k) \times \frac{(z+n)^{n-1}}{n(2z+n-1)^{n-2}}.$$

It is routine to verify that $P(n) = 2^n z(z+n)^{n-1}/n!$ for $n = 1, 2, 3$. Suppose that $P(k) = 2^k z(z+k)^{k-1}/k!$ for $k \leq n-1$. From identity (9) and Lemma 3 we have

$$P(n) = \sum_{k=0}^{n-1} \frac{2^k z(z+k)^{k-1}}{k!} \times \frac{2^{n-k-1}z(z+n-k-1)^{n-k-2}}{(n-k-1)!}$$

$$\times \frac{(z+n)^{n-1}}{n(2z+n-1)^{n-2}}$$

$$= \frac{2^n z}{n!}(z+n)^{n-1}.$$

By induction, formula (3) is true for any positive integer n. $\quad$ □

3. Concluding and Remarks

The right-hand sides of (3) and (4) have been studied by other authors [GS06, DL08, MY07], but our formula has the following two major differences: (i) the hook length h_v appears as an exponent; (ii) the underlying set remains the set of binary trees, whereas in the above mentioned papers the summation has been changed to the set of m-ary trees or plane forests. It is interesting to compare Corollary 2 with the following results obtained by Du and Liu [DL08]. Note that the right-hand sides of formulas (4), (10) and (11) are all identical!

Proposition 4. *For each positive integer n we have*

$$\text{(10)} \qquad \sum_{T \in \mathcal{T}(n)} \prod_{v \in I(T)} \left(\frac{2}{3} + \frac{1}{3h}\right) = \frac{(2n+1)^{n-1}}{n!},$$

where $\mathcal{T}(n)$ is the set of all 3-ary trees with n internal vertices and $I(T)$ is the set of all internal vertices of T.

Proposition 5. *For each positive integer n we have*

$$\text{(11)} \qquad \sum_{T \in \mathcal{F}(n)} \prod_{v \in (T)} \left(2 - \frac{1}{h}\right) = \frac{(2n+1)^{n-1}}{n!},$$

where $\mathcal{F}(n)$ is the set of all plane forests with n vertices.

References

[CY08] Chen, William Y.C.; Yang, Laura L.M., On Postnikov's hook length formula for binary trees, *European Journal of Combinatorics*, in press, 2008.

[DL08] Du, Rosena R. X.; Liu, Fu, (k,m)-Catalan Numbers and Hook Length Polynomials for Plane Trees, *European J. Combin*, **28** (2007), pp. 1312–1321.

[GS06] Gessel, Ira M.; Seo, Seunghyun, A refinement of Cayley's formula for trees, *arXiv:math.CO/0507497*.

[MY07] Moon, J. W.; Yang, Laura L. M., Postnikov identities and Seo's formulas, *Bull. Inst. Combin. Appl.*, **49** (2007), pp. 21–31.

[Po04] Postnikov, Alexander, Permutohedra, associahedra, and beyond, *arXiv:math. CO/0507163*, 2004.

[Se08] Seo, Seunghyun, A combinatorial proof of Postnikov's identity and a generalized enumeration of labeled trees, *arXiv:math.CO/0409323*.

I.R.M.A. UMR 7501
Université Louis Pasteur et CNRS,
7, rue René-Descartes
F-67084 Strasbourg, France
guoniu@math.u-strasbg.fr