Potpourri of conjectures and open questions in nonlinear analysis and optimization

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Potpourri of Conjectures and Open Questions in Nonlinear Analysis and Optimization*

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Abstract. We present a collection of fourteen conjectures and open problems in the fields of nonlinear analysis and optimization. These problems can be classified into three groups: problems of pure mathematical interest, problems motivated by scientific computing and applications, and problems whose solutions are known but for which we would like to know better proofs. For each problem we provide a succinct presentation, a list of appropriate references, and a view of the state of the art of the subject.

Key words. symmetric matrices, quadratic forms, convex sets, convex functions, Legendre–Fenchel transform, variational problems, optimization

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There are no solved problems, there are only more-or-less solved problems.
–H. Poincaré

Introduction. Conjectures and open problems are motivations and driving forces in mathematical research, beside other ingredients like applications and the need to answer questions posed by users of mathematics in other sciences. Each field or subfield of mathematics has its own list of conjectures, more or less specific to the area concerned, more or less difficult to explain (due to the necessary background or technicalities), and more or less known (distinguishing characteristics include whether or not the question remains unanswered for a long time, and whether answering it may trigger new problems, solve associated questions, or open new perspectives).

In the present paper we have listed a series of conjectures and open questions in the field of nonlinear analysis and optimization, collected over recent years. Of course, they reflect the interests of the author, and a problem one mathematical researcher finds exciting and deserving of more attention could be considered just boring by another. Our list of problems can be divided into three groups, but this is not really a partition of the set: clearly a problem may belong to two different classes.

- *Problems of pure mathematical interest.* Some scientific colleagues claim that an incentive for knowing more in science is the desire "to scratch where it itches ...." Hence, some of our problems are of pure mathematical interest: the eventual answer will not revolutionize the field. Examples belonging to this category are Problems 4, 5, 6, ...

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- *Problems motivated by scientific computing and applications.* Answering these problems could provide new solution methods or algorithms or could shed light on what might be expected from the subject. Examples from our list are Problems 1, 2, 9, 10, ...
- *Problems whose solutions are known but for which we would like to have better (i.e., shorter, more natural, or more elegant) proofs.* Some colleagues expert in evaluating mathematical activities claim that two-thirds of works in mathematical research consist of synthesizing and cleaning existing results, and providing new proofs, viewpoints, etc. We also have some questions of that kind: Problems 2, 7, ...

For each of the fourteen problems described in this paper, we provide a clear-cut presentation and a list of appropriate and recent references; in short, the reader will find a summary of the state of the art and material for further investigations. For some famous and long-standing conjectures, we checked by asking specialists if they were still open. A typical answer is "the experts thought that the conjecture was plausible until they tried to prove it and couldn't; therefore now they think it is false, and can't prove it." From experience, we all know that the consensus view on a conjecture can be dramatically disproved by an original proof or a clever counterexample.

Each of the problems listed below can be read independently, according to the interest of the reader, which is why we have provided a separate list of references for each one.

A word about notation: $\langle .,. \rangle$ denotes the usual inner product in $\mathbb{R}^d$ and $\|\cdot\|$ the associated norm (same notations in a Hilbert space setting). $S_d(\mathbb{R})$ is the space of real symmetric (d,d) matrices. When a real-valued function f is differentiable at x, $\nabla f(x)$ represents the gradient (vector) of f at x.

Plan.

Problem 1. The d-step conjecture for convex polytopes.
Problem 2. Reducing the number of polynomial inequality constraints.
Problem 3. Bounding the product of the volume of a convex body by that of its polar.
Problem 4. Darboux-like properties for gradients.
Problem 5. The possible convexity of a Tchebychev set in a Hilbert space.
Problem 6. Is a set with the unique farthest point property itself a singleton?
Problem 7. Solving a Monge–Ampère-type equation on the whole space.
Problem 8. Solving an eikonal-type equation on open subsets of $\mathbb{R}^n$.
Problem 9. Convex bodies of minimal resistance.
Problem 10. J. Cheeger's geometrical optimization problem.
Problem 11. The Legendre–Fenchel transform of the product of two convex quadratic forms.
Problem 12. Simultaneous diagonalization via congruence of a finite collection of symmetric matrices.
Problem 13. Solving a system of quadratic equations.
Problem 14. Minimizing a maximum of finitely many quadratic functions.

Problem 1. The *d*-Step Conjecture for Convex Polytopes. The d-step conjecture is one of the fundamental open problems concerning the structure of convex polytopes (= convex compact polyhedra). First formulated by W. M. Hirsch in 1957, and later transformed in a d-formulation (whence the name follows; see below), the conjecture remains unsettled, though it has been proved in many special cases (for specific classes of polytopes), and counterexamples have been found for slightly stronger conjectures [Holt98]. It came from the desire to understand better the computational complexity of edge-following algorithms in linear programming.

A good way to acquaint oneself with the conjecture is section 3.3 of Ziegler's book [Zieg95]; a broad and detailed survey is provided by [Klee97]. We present it here in a succinct form.

If x and y are vertices of a convex polytope P, let $\delta_P(x,y)$ denote the smallest k such that x and y are joined by a path formed by k edges. Then, the so-called *diameter* of P is the maximum of $\delta_P(x,y)$ as x and y range over all the vertices of P. For $n > d \geq 2$, let $\Delta(d,n)$ be defined as *the maximal diameter of convex polytopes P in $\mathbb{R}^d$ with n facets* (= faces of dimension $d-1$). For example, one can easily check that $\Delta(2,n)$ is the integer part of $\frac{n}{2}$. Also, by considering the convex polytope $[-1,+1]^d$ in $\mathbb{R}^d$ (with $2d$ facets), one realizes that $\Delta(d,2d) \geq d$.

Hirsch's conjecture is as follows:

$$\text{For } n > d \geq 2, \quad \Delta(d,n) \leq n-d. \tag{1.1}$$

Although this is not obvious, the Hirsch conjecture would follow if one could prove it in the special case where $n = 2d$, that is, $\Delta(d,2d) \leq d$, which has become known as the *d-step conjecture*. Since $\Delta(d,2d) \geq d$ (see above), the bound suggested by the d-step conjecture is certainly the best possible. So the d-step conjecture is reformulated in an equivalent way:

$$\text{For } d \geq 2, \quad \Delta(d,2d) = d. \tag{1.2}$$

Among results that aim to solve these conjectures, we single out the following: Hirsch's conjecture (as stated in (1.1)) holds true for $d \leq 3$ and all n, for all pairs (n,d) having $n \leq d+5$; and the d-step conjecture (as stated in (1.2)) is true for $d \leq 5$.

Rather recently, in [Laga97], the authors reformulated the d-step conjecture in terms of an operation very familiar in numerical analysis, namely, the Gaussian elimination.

As we said earlier, the d-step conjecture remains open for $d \geq 6$. The shared view among specialists in the subject is that it is false for large d.

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Problem 2. Reducing the Number of Polynomial Inequality Constraints. A surprising and strong result by Bröcker [Broc91] and Scheiderer [Sche89] states that a closed set of the form

$$S := \left\{ x \in \mathbb{R}^d \mid P_1(x) \leq 0, \ldots, P_m(x) \leq 0 \right\}, \tag{2.1}$$

where m is a positive integer and P_i are polynomial functions of the d variables $x_1, \ldots, x_d$, can be represented by at most $\frac{d(d+1)}{2}$ polynomial inequality constraints, i.e., there exist polynomial functions $Q_1, \ldots, Q_{d(d+1)/2}$ such that

$$S = \left\{ x \in \mathbb{R}^d \mid Q_1(x) \leq 0, \ldots, Q_{d(d+1)/2} \leq 0 \right\}. \tag{2.2}$$

The result has now spread from the field of algebraic geometry to that of polynomial optimization. It is, as yet, an *existence* result, so the proofs proposed by the authors are not constructive (no explicit construction of the Q_i); in particular, the degrees of the Q_i are not controlled.

It is indeed a striking result: imagine a compact convex polyhedron in $\mathbb{R}^2$ with 10^6 vertices or boundary line segments—this can be described via only 3 polynomial inequalities. When the P_i are affine,

$$\begin{aligned} P_i(x) &= \langle a_i, x\rangle - b_i \qquad \left(a_i \in \mathbb{R}^d,\ b_i \in \mathbb{R}\right) \\ &= \sum_{j=1}^{d} (a_i)_j \, x_j - b_i, \end{aligned}$$

Grötschel and Henk [Grot03] derived some basic properties necessarily satisfied by polynomial functions Q_i, such as

$$\{Ax \le b\} = \left\{x \in \mathbb{R}^d \mid \langle a_i, x\rangle - b_i \le 0 \quad \text{for } i = 1, \ldots, m\right\} \tag{2.3}$$

$$= \left\{x \in \mathbb{R}^d \mid Q_1(x) \le 0, \ \ldots, Q_{\nu(d)}(x) \le 0\right\}, \tag{2.4}$$

and constructed (exponentially many) Q_i for which (2.4) holds true. When $d = 2$ or 3, they succeeded in achieving the "reduced" upper bound $\frac{d(d+1)}{2}$. In [Lass02], Lasserre showed that, under additional assumptions on S (like compactness), the polynomial functions Q_i in the representation (2.2) of S could be chosen as affine combinations (with coefficients P_i) of sums of squares.

Several questions arise:

- Even in the polyhedral case (2.3), how does one *explicitly construct* polynomial functions Q_i such that (2.3)–(2.4) hold true, with their number $\nu(d)$ polynomially bounded in the dimension d? A step further would be to obtain the "reduced" upper bound $\nu(d) = \frac{d(d+1)}{2}$.
- (Very likely difficult). How does one *prove constructively* the theorem of Bröcker and Scheiderer, i.e., how do we derive $Q_1, \ldots, Q_{d(d+1)/2}$ (in (2.2)) from $P_1, \ldots, P_m$ (in (2.1)) in a way amenable to an effective computation?
- The "magic" number $\frac{d(d+1)}{2}$ is the dimension of the vector space $S_d(\mathbb{R})$ of real symmetric (d, d) matrices. Knowing the recent impact of SDP optimization (optimization problems where constraints invoke the semidefiniteness of some matrices) and the relations between SDP and polynomial optimization, a question comes naturally to mind: *What is the relation between the Bröcker–Scheiderer theorem and SDP relaxations of polynomial optimization problems?*

After completing the presentation of this problem (in 2004), we became aware of a new publication on the subject [Bosse05], where further results and conjectures were laid down. If S is an n-dimensional polytope (described as in (2.3)), then $\nu(d) = 2d - 1$ is possible in (2.4) and, moreover, the authors gave an explicit description of the polynomial functions Q_i they employed (for (2.4)). They conjectured that the dimension d itself is the right value for a general upper bound $\nu(d)$ for this special case.

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Problem 3. Bounding the Product of the Volume of a Convex Body by That of Its Polar. The problem considered concerns the compact convex sets of $\mathbb{R}^d$, symmetric with respect to the origin O of $\mathbb{R}^d$ and containing O in their interior. We call $\mathcal{C}_0$ the collection of such convex bodies.

If K belongs to $\mathcal{C}_0$, so does its polar set K^0. For example, if ξ_A is the elliptic convex set associated with the symmetric positive definite (d,d) matrix A such that (its boundary is the ellipsoid associated with A)

$$\xi_A := \left\{x \in \mathbb{R}^d \mid \langle Ax, x\rangle \ \leq 1\right\},$$

then $\xi_A \in \mathcal{C}_o$ and

(i) $\xi_A^o = \xi_{A^{-1}}$;

(ii) the volume (d-dimensional Lebesgue measure) of ξ_A is $V_d \big/ \sqrt{\det A}$, where V_d is the volume of the unit Euclidean ball in $\mathbb{R}^d$ (that is, $\pi^{d/2} \big/ \Gamma\left(\frac{d}{2}+1\right)$).

As a result, $\mathrm{Vol}(\xi_A) \cdot \mathrm{Vol}(\xi_A^o) = V_d^2$.

Another basic example is provided by the unit ℓ^1 and ℓ^∞ balls. The ℓ^∞ unit ball (or unit cube) $B_\infty := [-1,+1]^d$ and the ℓ^1 unit ball (also called cross polytope) $B_1 :=$ convex hull of $\{\pm\, e_i \mid i = 1, \ldots, d\}$ belong to $\mathcal{C}_0$ and are mutually polar. Looking at their volumes, since

$$\mathrm{Vol}(B_\infty) = 2^d \quad \text{ and } \quad \mathrm{Vol}(B_1) = 2^d \big/ d\,!,$$

their product $\mathrm{Vol}(B_\infty) \cdot \mathrm{Vol}(B_1)$ equals $4^d \big/ d!$.

It was conjectured by Mahler (1939) that, in any $\mathbb{R}^d$,

$$V_d^2 \ \geq\ \mathrm{Vol}(K) \cdot \mathrm{Vol}(K^o) \geq 4^d \big/ d! \ \text{ for all } K \in \mathcal{C}_o, \tag{3.1}$$

the left inequality being characteristic of elliptic sets and the right one of ℓ^1 or ℓ^∞ balls (up to images by invertible linear mappings); cf. [Berg90].

As for Mahler's double conjecture above, the situation today is as follows. Questions concerning the left part have been answered; see [Schn93, Grub93, Sant04] for this development. The exact right-hand bound is unknown today for $d \geq 3$, and in fact has only been known since Mahler for $d = 2$. In 1985 (see [Bour87]), Bourgain and Milman showed that there exists $c > 0$ such that

$$\mathrm{Vol}(K) \cdot \mathrm{Vol}(K^0) \ \geq\ c^d \big/ d! \quad \text{for all } K \in \mathcal{C}_o. \tag{3.2}$$

Passing from c to the conjectured value 4 is still an open problem.

A simpler (but related) question we would like to raise is that of the proof of the expression of $\mathrm{Vol}(K^0)$ in terms of the support function σ_K of K. Indeed, for $K = \xi_A$ we note that

$$\mathrm{Vol}(\xi_A^o) = \mathrm{Vol}(\xi_{A^{-1}}) = V_d \sqrt{\det\, A}, \tag{3.3}$$

while

$$\int_{\mathbb{R}^d} e^{-\sqrt{\langle A^{-1}u, u\rangle}}\, du \ =\ d!\ V_d\ \sqrt{\det\, A}. \tag{3.4}$$

Here $u \mapsto \sqrt{\langle A^{-1}u, u\rangle}$ is nothing other than the support function of ξ_A. A generalization to $K \in \mathcal{C}_o$ is as follows (mentioned in [Barv89, p. 207]):

$$\mathrm{Vol}(K^0) = \frac{1}{d!}\int_{\mathbb{R}^d} e^{-\sigma_K(u)}\,du. \tag{3.5}$$

We know of only one way of proving this, via a change of variables in properly chosen integrals. We would like a short and clear-cut *proof* of (3.5) using techniques and results from modern convex analysis.

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Problem 4. Darboux-like Properties for Gradients. An old result due to Darboux asserts that for a differentiable function $f : \mathbb{R} \to \mathbb{R}$, the image by f' of any interval $I \subset \mathbb{R}$ is an interval of $\mathbb{R}$ (even if f' is not continuous). The Darboux property fails for vector-valued functions: there are differentiable $f : \mathbb{R} \to \mathbb{R}^2$ and intervals $I \subset \mathbb{R}$ such that $f'(I)$ is not connected.

In recent years, there has been a revival of interest in Darboux-like properties for differentiable functions $f : X \to Y$. Generalizations of the aforementioned theorem can be foreseen in several directions, depending on

- the topological structure of X and Y (but we only consider here real-valued functions; see [Sain02] for some Darboux-like properties of differentiable vector-valued functions);
- the degree of smoothness of the function f;
- the kind of topological properties of $Df(C) \subset X^*$ ($C \subset X$ and Df is the differential of f) we are looking for: connectedness of $Df(C)$ whenever C is connected, its "semiclosedness"(i.e., $Df(C)$ is the closure of its interior), etc.

Here are some results achieved recently and open questions raised.

Malý published in [Maly96] the following interesting result.

Theorem 4.1. *Let X be a Banach space, and let $f : X \to \mathbb{R}$ be a (Fréchet-)differentiable function. Then, for any closed convex subset C of X with nonempty interior, the image $Df(C)$ of C by the differential Df of f is a connected subset of X^* (the topological dual space of X).*

The result does not hold true if C has an empty interior (there are counterexamples even with functions f of two variables). The first question we raise is the following: Could we replace the convexity assumption on C by one on the connectedness of C (is such a context more natural)?

Malý's result can be rephrased in terms of the so-called bump functions (f is called a "bump" when it has a nonempty bounded support); in fact, most of the results in the area we are exploring are stated for bump functions [Azag02a, Azag02b, Azag03, Borw01, Borw02, Fabi05, Gasp02, Kolar02, Kolar05, Riff89].

Theorem 4.2. *If $f : X \to \mathbb{R}$ is a differentiable bump function, then $Df(X)$ is a connected subset.*

Now we give two results by Gaspari [Gasp02]:

- If $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 bump function, then $\nabla f\left(\mathbb{R}^2\right)$ equals the closure of its interior.

  This is a specific result for functions of two variables. Question: Do we really need f to be C^2, or would just C^1 be enough? An improvement was recently made by Kolář and Kristensen [Kolar05]: the property holds true if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and the modulus of continuity $\omega(\cdot)$ of ∇f satisfies $\omega(t)/\sqrt{t} \to 0$ as $t \searrow 0$.
- Let Ω be an open connected subset of $\mathbb{R}^n$ containing O. Then there exists a (Fréchet-)differentiable bump function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f(\mathbb{R}^n) = \Omega$.

Rifford [Riff89], using tools from differential geometry, proved the following:

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^{n+1} bump function. Then $\nabla f(\mathbb{R}^n)$ equals the closure of its interior.

  Question: Is the smoothness assumption on f optimal when $n \geq 3$?

The most intriguing general question remains the following, raised in [Borw01]:

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 bump function; does $\nabla f(\mathbb{R}^n)$ equal the closure of its interior?

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Problem 5. The Possible Convexity of a Tchebychev Set in a Hilbert Space. Let $(H, \langle ., . \rangle)$ be a Hilbert space; we denote by $\| \cdot \|$ the norm derived from the inner product $\langle ., . \rangle$. Given a nonempty closed subset S of H, and any $x \in H$, we denote by $d_S(x)$ the distance from x to S, and by $P_S(x)$ the set of points in S which are "projections" of x onto S:

$$d_S(x) := \inf \{||x - s||;\ s \in S\}, \tag{5.1}$$

$$P_S(x) := \{s \in S;\ d_S(x) = ||x - s||\}. \tag{5.2}$$

The set S is said to be Tchebychev if $P_S(x)$ reduces to exactly one element for all $x \in H$. A classical result in approximation and optimization is that every closed convex set in a Hilbert space is Tchebychev. The question we pose here is, What about the converse? In other words, *Is a Tchebychev set necessarily convex*? The answer has been known to be yes if H is finite-dimensional since Bunt (1934) and Motzkin (1935). What if H is infinite-dimensional? The question was clearly stated by Klee (circa 1961) and he conjectured the answer would be no: he thought that there is an infinite-dimensional Hilbert space containing a Tchebychev nonconvex set (indeed, there are nonconvex Tchebychev sets in pre-Hilbertian spaces). Many works have been devoted to the subject in the past forty years, but the question, as posed above, is still unanswered completely. An account of the various contributions can be found in [JBHU98, Bala96, Deut01]. As far as we are concerned, we prioritize the point of view of convex and/or differential analysis.

There is a function conveniently associated with S,

$$x \in H \longmapsto f_S(x) := \begin{cases} \frac{1}{2}\,\|x\|^2 & \text{if } x \in S, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.3}$$

This function f_S is lower-semicontinuous on H and convex if and only if S is convex. The Legendre–Fenchel conjugate f_S^* of f_S is easy to calculate (due to the specific calculus rules on the Hilbertian norm $\|\cdot\|$):

$$p \in H \longmapsto \varphi_S(p) := f_S^*(p) = \frac{1}{2}\left[\ \|p\|^2 - d_S^2(p)\ \right]. \tag{5.4}$$

The results proven up to now are summarized by the following scheme:

$$\begin{pmatrix} S \text{ is Tchebychev} \\ + \\ \text{some additional condition } (C) \end{pmatrix} \Rightarrow (S \text{ is convex}). \tag{5.5}$$

For example, (C) could be "S is weakly closed"; or "the mapping p_S ($p_S(x)$ stands for the only element of $P_S(x)$) enjoys some 'radial continuity property'"; or "convergent subsequences can be extracted from minimizing sequences in the definition of $d_S(x)$."

If we consider the problem from the angle of differentiability, the main known result is that if S is Tchebychev, the following statements are equivalent:

(i) d_S^2 (or φ_S) is Gâteaux-differentiable on H;

(ii) d_S^2 (or φ_S) is Fréchet-differentiable on H;

(iii) S is convex.

So, the following central question remains: *Does the Tchebychev property of S imply that d_S^2 (or the convex function φ_S) is Gâteaux-differentiable on H?*

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Problem 6. Is a Set with the Unique Farthest Point Property Itself a Singleton? One of the oldest questions (dating back to the 1960s) in real analysis and approximation, related to Problem 5, is the so-called farthest point conjecture, which is formulated as follows: Given a bounded closed subset S of a normed vector space X, consider the set-valued mapping Q_S, which assigns to $x \in X$ the points in S which are farthest from x; *if $Q_S(x)$ contains one and only one element for all $x \in X$, could we assert that S itself is a singleton?* More than one hundred papers have been devoted to this question since its formulation, positively answering the conjecture in rather general situations (such as when S is compact, if X is finite-dimensional, when X is a particular normed vector space, etc.) but not in all of them.

The problem can be broached from the point of view of convex and/or differential analysis; it reduces to answering a question on the Fréchet-differentiability of an ad hoc convex function. Since the problem, as originally posed by Klee (circa 1961), is still unanswered in a Hilbert space setting, we consider it in such a context.

For a nonempty bounded closed subset S of a Hilbert space $(H, \langle .,. \rangle)$, for all $x \in H$ let

$$\Delta_S(x) := \sup \{\| x - s \|; \quad s \in S\}, \tag{6.1}$$

$$Q_S(x) := \{s \in S; \quad \Delta_S(x) = \| x - s \|\}, \tag{6.2}$$

where $|| \cdot ||$ denotes the Hilbertian norm built up from the inner product $\langle .,. \rangle$.

Two functions are conveniently associated with S (cf. [JBHU06, section 4.2]):

$$x \in H \longmapsto g_S(x) := \begin{cases} -\frac{1}{2} ||x||^2 & \text{if } x \in -S, \\ +\infty & \text{otherwise;} \end{cases} \tag{6.3}$$

$$x \in H \longmapsto \theta_S(x) := \frac{1}{2} \| x \|^2 - \sigma_{-S}(x), \tag{6.4}$$

where σ_{-S} denotes the support function of $-S$ (i.e., $\sigma_{-S}(x) := \sup_{\sigma \in -S} \langle x, \sigma \rangle$). This function θ_S is finite and continuous on H.

The convexity of θ_S (or of g_S) gives the answer to the farthest point conjecture; indeed, the following are equivalent:

(i) g_S is convex;
(ii) θ_S is convex;
(iii) S is a singleton.

What does the Legendre–Fenchel transform bring to the understanding of the conjecture? First of all, the Legendre–Fenchel conjugates of g_S and θ_S are

$$p \in H \longmapsto \psi_S(p) := g_S^*(p) = \frac{1}{2} \left[\Delta_S^2(p) - \| p \|^2 \right];$$

$$\theta_S^*(p) = \frac{1}{2} \Delta_S^2(p).$$

Second, the Fréchet-differentiability of the Legendre–Fenchel conjugate h^* of (an arbitrary function) h induces the convexity of h (see [JBHU06, section 4.3] and the references therein). So, the farthest point conjecture boils down to the following question:

$$\left(\begin{array}{l} Q_S(x) \text{ is a singleton} \\ \text{for all } x \in H \end{array} \right) \overset{?}{\Rightarrow} \left(\begin{array}{l} \Delta_S^2 \text{ is a Fréchet-differentiable} \\ (\text{convex}) \text{ function on } H \end{array} \right).$$

Finally, it is interesting to draw a parallel between Problem 5 and Problem 6 by comparing their definitions and results:

	Problem 5	Problem 6
Assumption:	$P_S(x)$ is a singleton for all $x \in H$.	$Q_S(x)$ is a singleton for all $x \in H$.
Functions involved:	f_S d_S^2 $\varphi_S = \frac{1}{2}\left[\|\cdot\|^2 - d_S^2\right]$	g_S Δ_S^2 $\psi_S = \frac{1}{2}\left[\Delta_S^2 - \|\cdot\|^2\right]$
Desired conclusion:	S is convex	S is a singleton
Key:	differentiability of φ_S $\left(\text{or } d_S^2\right)$	differentiability of ψ_S $\left(\text{or } \Delta_S^2\right)$.

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Problem 7. Solving a Monge–Ampère-Type Equation on the Whole Space. Monge–Ampère equations are nonlinear partial differential equations (PDEs) of the form

$$\det\left(\nabla^2 f\right) = g \quad \text{on } \Omega, \tag{7.1}$$

where $\nabla^2 f$ denotes the Hessian matrix of the desired "smooth" solutions f, g is a given function, and Ω an open subset of $\mathbb{R}^d$. There are various interpretations of the wording "f is a solution of (7.1)"; the regularity of solutions f depends heavily on the behavior of f at the boundary of Ω; for an account of the huge amount of literature devoted to this subject, consult [Bake94, Guti01, Cafa04] and [Vill03, Chapter 4].

Here we consider a particular case: Ω is the whole of $\mathbb{R}^d$, so that no behavior of f on the boundary of Ω interferes; this "rigidity" imposed on the problem leads to the following result.

Theorem 7.1. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a C^2 convex function satisfying*

$$\det\left(\nabla^2 f(x)\right) = 1 \quad \textit{for all } x \in \mathbb{R}^d. \tag{7.2}$$

Then f is a quadratic function.

This result was proved for $d = 2$ by Jörgens in 1954 (using techniques and results from complex analysis), then for some values of d (3, 5, for example) by Calabi [Cala58] and Pogorelov (1964). The general result (for any dimension d), as stated in Theorem 7.1, is due to Pogorelov [Pogo72]. The question we pose now is this: *How do we prove Theorem* 7.1 *using results from modern convex analysis?* Several reasons make this question plausible:

- The "rigidity condition" (7.2) is global; as a consequence, the set of solutions is invariant under the action of rotations (as changes of variables).
- The eigenvalues of $\nabla^2 f(x)$, ranked like

  $$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x),$$

  are continuous functions (of $x \in \mathbb{R}^d$); the condition imposed on them is

  $$\prod_{i=1}^{d} \lambda_i(x) = 1 \text{ for all } x \in \mathbb{R}^d, \tag{7.3}$$

  and the expected result is that the λ_i do not depend on x.

- Without loss of generality, we may assume that

$$f(0) = 0, \quad \nabla f(0) = 0. \tag{7.4}$$

Hence, the convex function f is positive on $\mathbb{R}^d$. The first step then would be to prove that f is 1-coercive on $\mathbb{R}^d$, that is, $f(x)/\| x \| \to +\infty$ as $\| x \| \to +\infty$.

Now the Legendre–Fenchel transformation $f \longmapsto f^*$ of convex functions f enters into the picture. There are precise relations between the gradient vectors and Hessian matrices of f and f^* (see [JBHU96, Chapter X] and [Seeg92], for example). Here is one of them [JBHU96, p. 89]: Let $f : \mathbb{R}^d \to \mathbb{R}$ be convex, twice differentiable, and 1-coercive on $\mathbb{R}^d$; assume moreover that $\nabla^2 f(x)$ is positive definite for all $x \in \mathbb{R}^d$. Then f^* enjoys the same properties and we have the following parameterizations:

(7.5)
- $\nabla f\left(\mathbb{R}^d\right) = \mathbb{R}^d$;
- $\nabla f^*(p) = x$ and $\nabla^2 f^*(p) = \left[\nabla^2 f(x)\right]^{-1}$ at $p = \nabla f(x)$.

Thus, f^* enjoys the same properties as f, notably the "rigidity condition" (7.2). But there are quite a few classes of convex functions stable under the Legendre–Fenchel transformation—in fact, $\frac{1}{2} \| \cdot \|^2$ is the only convex function satisfying $f^* = f$. So, we are not far from quadratic convex functions $x \mapsto \frac{1}{2}\langle Ax, x\rangle$, with A symmetric positive definite. *How should this problem be concluded?*

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Problem 8. Solving an Eikonal-Type Equation on Open Subsets of $\mathbb{R}^n$. Let Ω be an open subset of $\mathbb{R}^n$. We are interested in (classical) solutions of the PDE $\| \nabla f \| = 1$ on Ω, i.e., C^1-smooth functions $f : \Omega \to \mathbb{R}$ satisfying

$$\| \nabla f(x) \| = 1 \text{ for all } x \in \Omega \tag{8.1}$$

($\| \cdot \|$ denotes the usual Euclidean norm on $\mathbb{R}^n$). We could also add the requirement that f be continuous on the closure of Ω, but in the present context no condition is imposed on the boundary of Ω. PDEs of the (8.1) type are called eikonal and have roots in geometrical optics. There are several possible definitions of "f is a solution of (8.1)": classical, generalized, or viscosity solutions [Cann04, Kruz75, Mant03]. We exclusively consider classical solutions here.

To take a first example, consider $\Omega = (a, b) \subset \mathbb{R}$ and look for $f : [a, b] \to \mathbb{R}$, continuous on $[a, b]$, differentiable on (a, b), and satisfying $|f'(x)| = 1$ for all $x \in (a, b)$;

it is then easy to prove that f is affine on $[a,b]$, i.e.,

$$f(x) = f(a) \pm (x-a) \text{ for all } x \in [a,b].$$

Let us mention some known results.

- If Ω is the whole of $\mathbb{R}^n$, the "rigidity" imposed on the problem (8.1) (like in the Monge–Ampère equation; see Problem 7) means that the only solutions are *affine* functions on $\mathbb{R}^n$, i.e.,

  $$x \in \mathbb{R}^n \mapsto f(x) = \langle \mu, x \rangle + \nu,$$

  with $\mu \in \mathbb{R}^n$, $\| \mu \| = 1$, and $\nu \in \mathbb{R}$.
  This can be proved by using techniques from differential equations [Epre98] or applying Cauchy characteristics [Brya82, Theorem 2.4].
- If $\Omega = \mathbb{R}^n \setminus \{a\}$, where $a \in \mathbb{R}^n$, there is an additional solution to (8.1), namely,

  $$x \in \mathbb{R}^n \mapsto g(x) = g(a) \pm \| x - a \| . \tag{8.2}$$

- If $\Omega \neq \mathbb{R}^n$, only partial answers to our problem are known. For example, if C is any closed convex set contained in Ω^c (the complementary set of Ω), then classical results from convex analysis tell us that the distance function d_C is a solution of (8.1) [JBHU79, Poly84].
  Indeed, distance functions d_C to C, as well as signed distance functions Δ_C,

  $$x \in \mathbb{R}^n \longmapsto \Delta_C(x) := d_C(x) - d_{C^c}(x) \tag{8.3}$$

  (studied in [JBHU79] from the convex and nonsmooth analysis viewpoints; see also [Delf94]), seem to play a key role in answering our questions. Take, for example, $\Omega = \{x \in \mathbb{R}^n \,;\, x \neq 0 \text{ and } \| x \| \neq 1\}$. Then, the functions $x \longmapsto \| x \|$ (distance to a single point set contained in Ω^c), but also $x \longmapsto \Delta_{\overline{B}(0,1)}$ (the signed distance associated with the closed unit ball $\overline{B}(0,1)$) are solutions of (8.1).

Question: *What are the solutions of* (8.1)*? Are they necessarily of the form* $\pm d_S + r$ *(for various* S *and* r*) on each connected component of* Ω*?*

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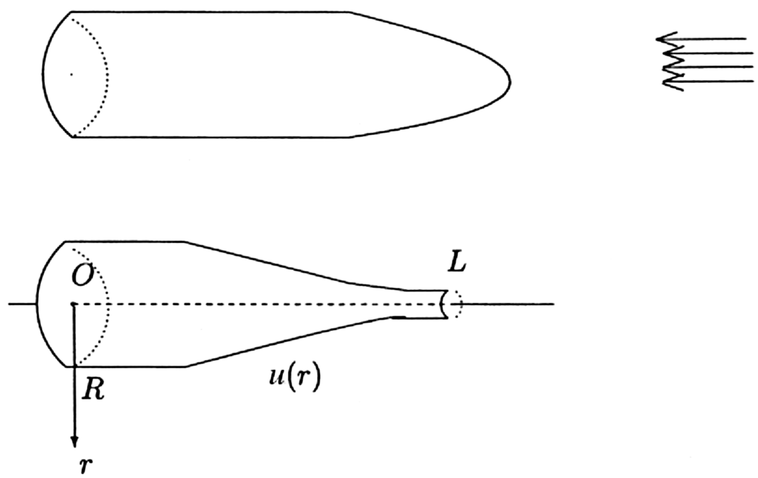

Fig. 9.1

Problem 9. Convex Bodies of Minimal Resistance. In 1686, Newton posed the following question: Among bodies with the same base (say, a disk of radius $R > 0$), limited in its length (an upper bound $L > 0$ is given), what is the shape of the body offering the least resistance (at a constant speed) in a fluid (with given specific physical properties)?

Newton considered only solids of revolution (designed by the graphs of functions $r \mapsto u(r)$ rotated around the horizontal axis) and, provided that some assumptions on the physical context of the problem were made, he was led to the following one-dimensional variational problem (see Figure 9.1):

$$(\mathcal{P}) \qquad \begin{cases} \text{Minimize } J(u) := \displaystyle\int_0^R \frac{r}{1+|\dot{u}(r)|^2}\, dr, \\ u(0) = L, \ u(R) = 0, \\ \dot{u}(r) \le 0 \text{ on } [0, R]. \end{cases}$$

The solution proposed by Newton is as in Figure 9.2, with a (surprising) flat piece at the end of the body. One of the standard ways of solving $\mathcal{P}$ nowadays is via results from optimal control ($\dot{u} = v$ is the control variable and $V = \mathbb{R}_-$ is the set of admissible controls). For historical accounts of this problem and classical ways of solving it, see [Tikh90, eighth story].

Most mathematicians considered Newton's problem of the body of minimal resistance to be solved. It is indeed the case (this has been proved rigorously) if one supposes the *radial* symmetry of the considered bodies. However, it has been recently shown (in [Guas96], for example) that there exist nonradial convex bodies for which the resistance is less than the minimum obtained in Newton's (radial) case. This discovery boosted new research works on the subject; see [Butta93, Broc96, Lach00] and the website of Lachand-Robert (http://www.lama.univ-savoie.fr/∼lachand/),[1] where

[1] T. Lachand-Robert died tragically in an accident in his home (February 2006); he had just turned thirty nine. Problems 9 and 10 are dedicated to his memory.

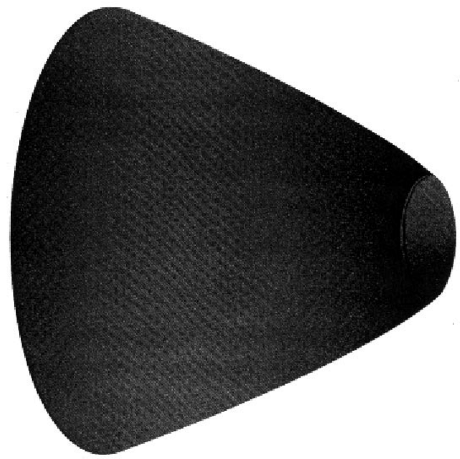

Fig. 9.2

most of the appropriate references and papers can be found. From the mathematical viewpoint, the variational problem posed is as follows:

$$(\mathcal{P}') \qquad \begin{cases} \text{Minimize } \; J(u) := \displaystyle\int_{B(0,R)} \frac{1}{1+\| \nabla u(x) \|^2} \, dx \\ \text{over } \; C, \end{cases}$$

where $C := \left\{ u \in W^{1,\infty}_{loc} \mid 0 \leq u \leq L, u \text{ concave} \right\}$ (u is a function of two variables x).

The shape condition on u (u must be concave) is strong enough to produce a compactness assumption which implies the existence of a solution in $(\mathcal{P}')$. Newton's case corresponded to the same variational problem but with a smaller constraint set,

$$C_{rad} := \{ u \in C \mid u \text{ is radially symmetric} \} .$$

As we mentioned before, the new fact here is that

$$\inf_{u \in C} J(u) < \inf_{u \in C_{rad}} J(u) . \tag{9.1}$$

To summarize recent works on this subject and still open questions, let us say that we are faced with a strange mathematical situation:

- A variational problem $(\mathcal{P}')$ which *does have* solutions (minimizers are unknown, except in some very particular situations).
- There are *infinitely many solutions* to $(\mathcal{P}')$ (a solution to $(\mathcal{P}')$ is nonradial necessarily, so that rotating it around the axis provides another solution).
- (Usual) numerical methods cannot solve $(\mathcal{P}')$ (the concavity of the desired object is a constraint very difficult to handle in a numerical procedure).

What is conjectured is that the optimal solutions in $(\mathcal{P}')$ are diamond-like bodies (with a number of flat pieces). Indeed, as for minimal surfaces, there is no subset on which a minimizer in $(\mathcal{P}')$ is strictly convex; in particular, wherever the Gaussian curvature is finite, it is null.

Numerical profiles recently obtained by ad hoc methods [Lach04] are better than any previously conjectured optimal shapes.

As a whole, the theoretical characterizations of the solutions in ($\mathcal{P}'$) as well as their effective numerical approximation remain open challenging problems.

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Problem 10. J. Cheeger's Geometrical Optimization Problem. Consider the following geometrical optimization problem in $\mathbb{R}^2$: given an open nonempty bounded set $\Omega \subset \mathbb{R}^2$, solve the following minimization problem:

$$(\mathcal{P}) \qquad \begin{cases} \text{Minimize} \quad \dfrac{\text{perimeter of } X}{\text{area of } X}, \\ X \text{ open and simply connected}, \\ X \subset \bar{\Omega}. \end{cases}$$

If the class of admissible domains X is restricted to smoothly bounded and simply connected domains which are compactly contained in Ω, then ($\mathcal{P}$) is known as the Cheeger problem for Ω; cf. [Chee70]. The (positive) minimal value in ($\mathcal{P}$) is achieved by a subset of $\bar{\Omega}$ which touches the boundary of Ω. The generalization of ($\mathcal{P}$) to the d-dimensional setting is easy to imagine: given an enclosing open bounded set $\Omega \subset \mathbb{R}^d$, what are the subsets X of $\bar{\Omega}$ (if any) which minimize the ratio (surface area of X) over (volume of X)?

Known results on this problem concern essentially the two-dimensional case:

- When Ω is convex, there is an unique convex minimizer X in ($\mathcal{P}$), denoted by X_Ω. The explicit form of this minimizer X_Ω is known only in particular cases: when Ω is a disk or an annulus (in which case $X_\Omega = \Omega$), and when Ω is a triangle or a rectangle (in which case X_Ω is obtained from Ω by "rounding the corners").
- Still when Ω is convex, a constructive algorithm for numerically approximating X_Ω is provided in [Kawo06]; this paper contains all the references to recent works on the subject in the two-dimensional case.

  In higher dimensions, $d \geq 3$, *all the questions related to the optimization problem* ($\mathcal{P}$) *are unanswered*, among them:
  - If Ω is convex, is there an unique solution to ($\mathcal{P}$)?
  - Still with Ω convex, are all the minimizers in ($\mathcal{P}$) convex? (What is known is that there exists at least one convex minimizer.)

In [Lach04], an algorithm is proposed (and tested on examples) to approximate a convex solution X_Ω of ($\mathcal{P}$) when Ω is convex in $\mathbb{R}^3$. This algorithm also serves for other optimization problems among convex bodies, like Newton's problem of the body of minimal resistance (see Problem 9 above).

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Problem 11. The Legendre–Fenchel Transform of the Product of Two Convex Quadratic Forms. Let A and B be two symmetric positive definite (n, n) matrices, and let q_A (resp., q_B) be the associated quadratic form on $\mathbb{R}^n$, that is,

$$x \in \mathbb{R}^n \longmapsto q_A(x) := \frac{1}{2}\langle Ax, x\rangle. \tag{11.1}$$

It is well known that q_A is strictly (even strongly) convex on $\mathbb{R}^n$, and that its Legendre–Fenchel conjugate is

$$s \in \mathbb{R}^n \longmapsto q_A^*(s) = \frac{1}{2}\langle A^{-1}s, s\rangle. \tag{11.2}$$

Now we turn our attention to the product function

$$x \in \mathbb{R}^n \longmapsto f(x) := q_A(x)\ q_B(x). \tag{11.3}$$

The function f is clearly C^∞ on $\mathbb{R}^n$; the second-order Taylor–Young development of f at $x \in \mathbb{R}^n$ yields the Hessian matrix of f at x:

$$\nabla^2 f(x) = q_B(x) A + q_A(x) B + Axx^T B + Bxx^T A. \tag{11.4}$$

Unfortunately, $\nabla^2 f(x)$ is not positive semidefinite for all $x \in \mathbb{R}^n$, even if $B = A^{-1}$ (contrary to what is stated in [JBHU01, Exercise 18, p. 120]). However, from the variational analysis point of view, it is interesting to pose the following question: *What is the Legendre–Fenchel conjugate of* f?

Most likely, q_A^* and q_B^* play a part in the expression of $f^*(s)$. A (usable) answer to the question above would provide interesting (possibly unknown) inequalities involving convex quadratic forms.

A particular instance of the problem is when $B = A^{-1}$. Then the Legendre–Fenchel transform of $f(x) = q_A(x)\, q_{A^{-1}}(x)$ would allow us to recover inequalities like that of Kantorovich [Huang05]:

$$\begin{cases} \| x \|^4 \ \leq\ \langle Ax, x\rangle\langle A^{-1}x, x\rangle \ \leq\ \dfrac{1}{4}\left(\sqrt{\dfrac{\lambda_1}{\lambda_n}} + \sqrt{\dfrac{\lambda_n}{\lambda_1}}\right)^2 \| x \|^4 \\ \text{for all } x \in \mathbb{R}^n, \end{cases} \tag{11.5}$$

where λ_1 (resp., λ_n) denotes the largest (resp., the smallest) eigenvalue of A.

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Problem 12. Simultaneous Diagonalization via Congruence of a Finite Collection of Symmetric Matrices. A collection of m symmetric (n,n) matrices $\{A_1, A_2, \ldots, A_m\}$ is said to be *simultaneously diagonalizable via congruence* if there exists a nonsingular matrix P such that each of the $P^T A_i P$ is diagonal. Simultaneous diagonalization via congruence corresponds to transforming the quadratic forms q_i on $\mathbb{R}^n$ associated with the A_i's (i.e., $q_i(x) = \langle Ax, x\rangle$ for $x \in \mathbb{R}^n$) into linear combinations of squares by a single change of variable; it is a more accessible property than the (usual) diagonalization via similarity (beware of confusion between these two types of reduction of matrices).

Here as in the next two problems, $\succ 0$ (resp., $\succeq 0$) means positive definite (resp., positive semidefinite).

We recall two results pertaining to the case where *two* symmetric (n,n) matrices are involved.

- If

  (12.1)
  $$\text{there exists } \mu_1,\ \mu_2 \in \mathbb{R} \text{ such that } \mu_1 A + \mu_2 B \succ 0 \text{ (for example, if } A \succ 0),$$

  then

  (12.2) *A and B are simultaneously diagonalizable via congruence.*

  This is a classical result in matrix analysis; see section 7.6 of [Horn85], for example.

- *Let $n \geq 3$. If*

  $$\begin{pmatrix} \langle Ax, x\rangle = 0 \\ \textit{and} \\ \langle Bx, x\rangle = 0 \end{pmatrix} \Rightarrow (x = 0)\,, \tag{12.3}$$

  then (12.2) *holds true.*

  This result is proposed on pp. 272–280 of [Greub76]; the proof, due to Milnor, clearly shows that the assumption $n \geq 3$ on the dimension of the underlying space is essential.

Actually, statements (12.1) and (12.2) above are equivalent whenever $n \geq 3$; this was proved by Finsler (circa 1936) and rediscovered by Calabi (1964). A very good account of Finsler–Calabi-type results, including the historical developments, remains the survey paper by Uhlig [Uhlig79].

When more than two symmetric matrices are involved, none of the two aforementioned results remains true. This is related to the convexity or nonconvexity of the (image) cone

$$\mathcal{K} := \{(\langle A_1 x, x\rangle, \langle A_2 x, x\rangle, \ldots, \langle A_m x, x\rangle) \mid x \in \mathbb{R}^n\}$$

(see [JBHU02] for developments on this topic). So, the following problem remains a strong one: *Find sensible and "palpable" conditions on $A_1, A_2, \ldots, A_m$ ensuring they are simultaneously diagonalizable via congruence.*

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Problem 13. Solving a System of Quadratic Equations. As stated in the previous problem, when A and B are two symmetric (n,n) matrices *with* $n \geq 3$, the following are equivalent:

(13.1) 0 *is the only solution of the system of two quadratic equations* $\langle Ax,\ x\rangle = 0$ *and* $\langle Bx,\ x\rangle = 0$;

$$\mu_1 A + \mu_2 B \succ 0 \quad \text{for some } \mu_1, \mu_2 \in \mathbb{R}. \tag{13.2}$$

(13.2) does not necessarily result from (13.1) when $n = 2$; a simple counterexample is provided by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; indeed, (13.1) holds true, but there is no way to have $\mu_1 A + \mu_2 B \succ 0$. One way of explaining this shortcoming is the lack of convexity of the image set

$$B := \{(\langle Ax, x\rangle, \langle Bx, x\rangle) \mid \| x \| = 1\}.$$

Brickman's theorem (1961) asserts that B is convex for $n \geq 3$ (see [JBHU02]); thus, "separating" the origin $(0,0)$ from B in $\mathbb{R}^2$ leads to the missing implication (13.1) $\Rightarrow$(13.2).

When m (≥ 3) symmetric matrices $A_1, A_2, \ldots, A_m$ are involved, the condition

$$\sum_{i=1}^{m} \mu_i A_i \succ 0 \text{ for some } \mu_1, \ldots, \mu_m \in \mathbb{R} \tag{13.3}$$

indeed ensures that

$$(\langle A_i x, x\rangle = 0 \text{ for all } i = 1, \ldots, m) \Rightarrow (x = 0). \tag{13.4}$$

But this is too strong a sufficient condition, by far. So, the following problem is posed: *How do we express equivalently (or give "mild" sufficient conditions for) the assertion* (13.4) *in terms of the* A_i*'s?*

REFERENCE

[JBHU02] J.-B. Hiriart-Urruty and M.Torki, *Permanently going back and forth between the "quadratic world" and the "convexity world" in optimization*, Appl. Math. Optim., 45 (2002), pp. 169–184.

Problem 14. Minimizing a Maximum of Finitely Many Quadratic Functions. Condition (13.1) in the previous problem could be formulated in a variational fashion as follows:

$$\max \{|\langle Ax, x\rangle|\,,\ |\langle Bx, x\rangle|\} > 0 \quad \text{for all } x \neq 0 \text{ in } \mathbb{R}^n. \tag{14.1}$$

By strengthening this inequality (removing the absolute values in (14.1)), it is possible to obtain a "unilateral" version of the equivalence between (13.1) and (13.2) in the previous problem, valid now for any n. This is the so-called Yuan's lemma [Yuan90, JBHU02]:

Let A and B be two symmetric (n, n) matrices. Then the following are equivalent:

$$
\text{(14.2)} \qquad
\begin{aligned}
&\bullet \ \max \ \{\langle Ax, x\rangle, \langle Bx, x\rangle\} \ > 0 \textit{ for all } x \neq 0 \textit{ in } \mathbb{R}^n \\
&\quad (\textit{resp., } \ \geq 0 \ \textit{ for all } x \in \mathbb{R}^n)\,; \\
&\bullet \ \textit{there exist } \mu_1 \geq 0,\ \mu_2 \geq 0,\ \mu_1 + \mu_2 = 1 \ \textit{ such that} \\
&\quad \mu_1 A + \mu_2 B \succ 0 \ (\textit{resp., } \succeq 0).
\end{aligned}
$$

So, the "unilateral" version of the question posed at the end of the presentation of the previous problem is as follows: *Let* $A_1,\ A_2, \ldots,\ A_m$ *be* m *symmetric* (n, n) *matrices. How do we express equivalently (or give "mild" sufficient conditions for) the following inequality in terms of the* A_i*'s:*

$$
\text{(14.3)} \qquad
\begin{aligned}
&\max \ \{\langle A_1 x, x\rangle, \langle A_2 x, x\rangle, \ldots, \langle A_m x, x\rangle\} > 0 \textit{ for all } x \neq 0 \textit{ in } \mathbb{R}^n \\
&(\textit{resp., } \ \geq 0 \textit{ for all } x \in \mathbb{R}^n)?
\end{aligned}
$$

If a convex combination of the A_i's is positive definite (resp., positive semidefinite), then we clearly have a sufficient condition for securing (14.3), but not a necessary one (at least for $m \ \geq 3$).

It is important to note that (14.3) is related to the fields of necessary/sufficient conditions in nonsmooth optimization: The function $x \in \mathbb{R}^n \mapsto q(x) := \max\{\langle A_i x, x\rangle \mid i = 1, \ldots, m\}$ is globally minimized at $\bar{x} = 0$ (under assumption (14.3)); however, expressing some generalized second-order necessary condition for minimality for q at $\bar{x} = 0$ does not give any interesting information about the A_i's.

A more general question behind this is that of "second-order approximation models" or "generalized Hessian operators" for nonsmooth functions of the form $f = \max \ \{f_1, f_2, \ldots, f_m\}$, where the f_i's are C^2 functions on $\mathbb{R}^n$. For the first-order approximation model around $\bar{x}$ or minimality conditions at $\bar{x}$, there is no ambiguity:

$$
\partial f\,(\bar{x}) := \text{convex hull of all the } \nabla f_i\,(\bar{x}) \ \text{ for which } f_i\,(\bar{x}) = f\,(\bar{x})
$$

is an appropriate "multigradient" playing the role of a substitute for the gradient of f at $\bar{x}$. As for the second-order approximation model of f around $\bar{x}$, the Hessian matrices useful when moving in a direction $d \in \mathbb{R}^n$ are those corresponding to the $i = 1, \ldots, m$ for which $f_i\,(\bar{x}) = f\,(\bar{x})$ and, moreover,

$$
\max\left\{\langle \nabla f_i\,(\bar{x})\,, d\rangle \mid f_i\,(\bar{x}) = f\,(\bar{x})\right\} \ = \ \langle \nabla f_i\,(\bar{x})\,, d\rangle.
$$

So, in spite of numerous new ideas and works on the subject, no "generalized Hessian operator" stands out. Actually, in recent years, further different directions have been taken in order to handle second-order approximation models and resulting minimization algorithms for functions like max $\{f_i, \ldots, f_m\}$ [JBHU96, Rock98].

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